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Z-EQUILIBRIA IN BI-MATRIX GAMES WITH UNCERTAIN PAYOFFS

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Abstract. The concept of Z-equilibrium has been introduced by Zhuk-ovskii (Mathematical Methods in Operations Research. Bulgarian Academy of Sciences, Sofia (1985) 103–195) for games in normal form. This concept is always Pareto optimal and individually rational for the players. Moreover, Pareto optimal Nash equilibria are Z-equilibria. We consider a bi-matrix game whose payoffs are uncertain variables. By appropriate ranking criteria of Liu uncertainty theory, we introduce some concepts of equilibrium based on Z-equilibrium for such games. We provide sufficient conditions for the existence of the introduced concepts. Moreover, using mathematical programming, we present a procedure for their computation. A numerical example is provided for illustration.

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1. Introduction

The bi-matrix game is one of the fundamental models in game theory. It is a two-persons game in normal form, which is specified by a finite set of pure strategies for each player. A player is allowed to randomize his strategy according to a probability distribution on his pure strategy set, which defines a mixed strategy for the player. A Nash equilibrium is a strategy profile that is immune against unilateral deviation of players. Nash [30] showed that a bi-matrix game has at least one Nash equilibrium in mixed strategies.

However, in many games, pure strategy Nash equilibrium does not exist or is not Pareto optimal. The concept of Z-equilibrium has been introduced by Zhukovskii [45] as an alternative solution to Nash equilibrium for such games as it always exists in finite games and it is always Pareto optimal as well. The following property is another difference between these two concepts: for each deviation of a player from her/his Z-equilibrium strategy, the other player has a specific punishing strategy, whereas in Nash equilibrium, each player punishes the deviating player just by staying in Nash equilibrium.

Keywords. Bi-matrix game, Pareto optimal, uncertainty theory, Z-equilibrium.

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Z-equilibrium in continuous and deterministic games has been studied in details by Zhukovskii [45] and Zhukovskii and Tchickry [46]. Recently, Ferhat and Radjef [11] have generalized *Z*-equilibrium to the multiple criteria games in mixed strategies *i.e.* games where the players have multiple payoff functions. Furthermore, Bouchama *et al.* [4] established the equivalence between the concept of solution of a constraint satisfaction problem and the *Z*-equilibrium of its associated game. They also proved the existence of a *Z*-equilibrium in pure strategies for a finite normal game. Another direction of research related to the study of *Z*-equilibrium is the investigation of this concept in games involving uncertainty. Indeed, in real-world games, players are generally not able to evaluate precisely the payoffs results from the implementation of a strategy profile because of lack of information about other players' behavior or about the environment. This lack of precision (indeterminacy) may be modeled by different ways: probabilistic tools [13], ambiguity [10], fuzzy set theory [44], D-S-theory [39], Liu uncertainty theory [20], etc.

Harsanyi [13] studied the imprecision of probabilistic nature in games by developing Bayesian games. Just to mention a few works on Nash equilibrium in bayesian games [14], Jackson *et al.* [15], [38, 40, 41]. Fuzzy sets theory was introduced in non-cooperative game theory by Butnariu [5,6]. Nash equilibrium in games with fuzzy payoffs received a great attention in the literature. Without being exhaustive, we mention the works of Nishizaki and Sakawa [32], Maeda [25], Chunqiao and Qiang [7], Roy [35], Larbani [17], Das and Roy [8,9], Bandyopadhyay *et al.* [3], Roy and Mula [36,37] and Mula *et al.* [29]. Further, Xiong *et al.* [43] used D-S-Theory to study a game with ambiguous payoff and played with ambiguity. Gao [12] studied Nash equilibrium in bi-matix games using Liu uncertainty theory. Besides the mentionned works, some other approaches for dealing with uncertainty in normal form games are considered in the literature such as robustness by Aghassi and Bertsimas [1] and Perchet [34], uncertainty aversion in Klibanoff [16], ambiguity in Bade [2] and ambiguous games introduced in Xiong *et al.* [43]. All the mentionned works are related to Nash equilibrium and its properties.

The first work on *Z*-equilibrium in games involving uncertainty appeared in Larbani and Lebbah [18]. This work investigated games with partially uncertain payoffs. The uncertainty appears in the form of a parameter in the payoff functions, $f_i(x, y), i = 1, 2, \ldots, n$, where y is an unknown parameter and x is the strategy profile. The introduced concept is called ZS-equilibrium. Some sufficient conditions for its existence are established. Recently, this work was generalized to games where the players can form coalitions, the introduced concept is called ZP-equilibrium Nessah *et al.* [31].

A part from the mentioned works on games with *partially uncertain* payoff functions, there is no work investigating *Z*-equilibrium in games where the payoff functions are *completely uncertain*. The contribution of this paper is to fill partially this gap. We investigate the basic class of bi-matrix games with uncertain payoffs only. In dealing with the uncertain payoffs, we use the Liu uncertainty theory Liu [20]. This theory offers powerful techniques and methods for handling subjective uncertainty that cannot be modeled with fuzziness. It has been successfully used in many areas of research and application. Just to mention few related works Liu [20, 21, 24], Li and Liu [19] and Liu [24]. It is interesting to note that Liu uncertainty theory has been used in the investigation of Nash equilibrium in bi-matrix games by Gao [12].

Precisely, we investigate a bi-matrix game whose payoffs are uncertain variables. Using the Liu theory [20], we introduce concepts of equilibrium for this game based on the *Z*-equilibrium. Then we establish sufficient conditions for their existence and present a procedure for their computation.

The rest of the paper is organized as follows. In Section 2, we discuss the concept of *Z*-equilibrium in bi-matrix game whose payoffs are characterized by real numbers and give some of its properties. In Section 3, we recall some basic concepts and results of Liu uncertainty theory. In Section 4, we present our extensions of the *Z*-equilibrium to uncertain bi-matrix games and give sufficient conditions for their existence. Next, in Section 5, using mathematical programming tools, we present a procedure for computing the equilibria introduced in Section 4. Finally, we provide a numerical example in Section 6. The last section concludes the paper.

2. *Z*-EQUILIBRIUM

2.1. Definitions and notations

– A 2-person strategic game G is given as follows

$$G = \langle N, X_1 \times X_2, (U_1(x_1,x_2), U_2(x_1,x_2))\rangle,$$

where $N = \{1,2\}$ is the set of players; $X_i \subset \mathbb{R}^{n_i}$, $n_i > 0, X_i$ the set of strategies of the ith player $i = 1,2$. $U_i : X_1 \times X_2 \longrightarrow \mathbb{R}$, is the utility function of the ith player. Assume that the aim of each player is to maximize her/his utility function.

– A bi-criteria maximization problem (BMP) is denoted by

$$\langle H, (F_1, F_2)\rangle,$$

where $H \subset \mathbb{R}^m$ and $F_l : H \longrightarrow \mathbb{R}$, $l = 1,2$.

– We will use the following relation in $\mathbb{R}^2$, for all $(\overline{r}_1,\ \overline{r}_2)\ (r_1,\ r_2) \in \mathbb{R}^2$,

$$(\overline{r}_1,\ \overline{r}_2) \leq (r_1,\ r_2) \Longleftrightarrow (\forall k \in \{1,2\},\quad \overline{r}_k \leq r_k \quad \text{and} \quad \exists l \in \{1,2\},\quad \overline{r}_l < r_l).$$

We recall the concept of Pareto optimal solution.

Definition 2.1. A feasible solution $x^* \in H$ is said to be Pareto optimal to the (BMP) $\langle H, (F_1, F_2)\rangle$ if there is no feasible solution $x \in H$ such that

$$(F_1(x^*), F_2(x^*)) \leq (F_1(x), F_2(x)).$$

Pareto optimality means that it is impossible to improve any one objective function without sacrificing on the other one.

Definition 2.2. $x^* \in X_1 \times X_2$ is said to be a *Z*-equilibrium for the game G if and only if the following two conditions hold:

(1) $\begin{cases} \forall x_1 \in X_1, \quad \exists x_2 \in X_2,\ U_1(x_1,x_2) \leq U_1(x^*); \\ \forall x_2 \in X_2, \quad \exists x_1 \in X_1,\ U_2(x_1,x_2) \leq U_2(x^*); \end{cases}$

(2) x^* is Pareto optimal solution of the (BMP) $\langle X_1 \times X_2, (U_1, U_2)\rangle$.

As mentioned in the introduction, Nash equilibrium may not exist in pure strategies in finite games, while *Z*-equilibrium always exists. To guarantee the existence of Nash equilibrium, mixed strategies have to be considered. In the continuous games case, Nash equilibrium exists if the sets of strategies are compact and convex, and the payoff functions are continuous and quasi-concave with respect to players' strategies. *Z*-equilibrium exists under weaker conditions than of Nash's: only compactness of strategy sets and continuity of players' payoff functions are required Zhukovskii [45]. When one of these two conditions is not satisfied, *Z*-equilibrium existence is not guaranteed. The following example shows this fact.

Example 2.3. Consider the strategic game $\langle\{1,2\},\ X_1 \times X_2,\ (U_1, U_2)\rangle$, where $X_1 = [0,1],\ X_2 = [0,1],\ U_1 = x_1x_2 - x_1$ and

$$U_2(x_1,x_2) = \begin{cases} \frac{x_1}{x_2} + \frac{1}{x_2} & \text{if} \quad (x_1,x_2) \in [0,1]\times]0,1] \\ 1 & \text{if} \quad x_2 = 0, x_1 \in [0,1] \end{cases}.$$

We have $\sup\limits_{x_2 \in X_2} \inf\limits_{x_1 \in X_1} U_2(x_1,x_2) = +\infty$. Thus, $\forall (x_1,x_2) \in X_1 \times X_2,\ U_2(x_1,x_2) < \sup\limits_{x_2 \in X_2} \inf\limits_{x_1 \in X_1} U_2(x_1,x_2)$. There is no *Z*-equilibrium for this game.

The existence of *Z*-equilibrium in this game fails because of the non-continuity of the function U_2.

2.2. Z-equilibrium in bi-matrix games

To help the reader understand the Z-equilibrium, we recall and compare the concepts of Nash equilibrium Nash [30] with Z-equilibrium Zhukovskii [45] for bi-matrix games.

A bi-matrix game can be represented by a pair of two matrices (A, B) where A and B are $m \times n$ matrices of payoffs to the row player I and column player II, respectively, with $A = (a_{ij})_{(i,j)\in\{1,\ldots,m\}\times\{1,\ldots,n\}}$ and $B = (b_{ij})_{(i,j)\in\{1,\ldots,m\}\times\{1,\ldots,n\}}$.

Assume that $X = \{1, 2, \ldots, m\}$ and $Y = \{1, 2, \ldots, n\}$ are the set of pure strategies for player I and player II, respectively. The aim of each player in this game, which we denote

$$G1 = \langle \{I, II\}, X \times Y, (A, B)\rangle,$$

is to maximize his payoff.

Definition 2.4. A pair $(k^*, l^*) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$ is a Nash equilibrium of the bi-matrix game $G1$, if

$$\begin{cases} \forall i \in \{1, \ldots, m\},\ a_{il^*} \le a_{k^*l^*}; \\ \forall j \in \{1, \ldots, n\},\ b_{k^*j} \le b_{k^*l^*}. \end{cases}$$

Definition 2.5. A pair $(k^*, l^*) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$ is a Z-equilibrium of the bi-matrix game $G1$, if it satisfies the following two conditions:

(1) $\begin{cases} \forall i \in \{1, \ldots, m\}, & \exists j \in \{1, \ldots, n\},\ a_{ij} \le a_{k^*l^*}; \\ \forall j \in \{1, \ldots, n\}, & \exists i \in \{1, \ldots, m\},\ b_{ij} \le b_{k^*l^*}; \end{cases}$

(2) (k^*, l^*) is a Pareto optimal solution to the (BMP)$\langle X \times Y,\ (F_1, F_2)\rangle$, where $F_1(i, j) = a_{ij},\ F_2(i, j) = b_{ij}$, for all $(i, j) \in X \times Y = \{1, \ldots, m\} \times \{1, \ldots, n\}$.

Remark 2.6. Condition (1) of Definition 2.5 guarantees the stability of Z-equilibrium. Indeed, for each deviation i (resp. j) of the row player (resp. column player) from his Z-equilibrium strategy, the other player has a counter strategy j (resp. i) to punishes him, such that

$$\begin{array}{ll} \forall i \in \{1, \ldots, m\}, & \exists j \in \{1, \ldots, n\}, a_{ij} \le a_{k^*l^*} \\ (\text{resp.} \quad \forall j \in \{1, \ldots, n\}, & \exists i \in \{1, \ldots, m\}, b_{ij} \le b_{k^*l^*}). \end{array}$$

Note that in Nash equilibrium, the response k^* (resp. l^*) of the other player is the same against every deviation i (resp. j) of the row (resp. column) player: $\forall i \in \{1, \ldots, n\}, a_{il} \le a_{k^*l^*}$ (resp. $\forall j \in \{1, \ldots, n\}, b_{k^*j} \le b_{k^*l^*}$).

Therefore, Z-equilibrium is said to be an "active" equilibrium and Nash equilibrium is said to be "passive" Zhukovskii [45]. In Nash equilibrium a player punishes the deviating player just by staying in Nash equilibrium, while in Z-equilibrium each player has a specific punishing or counter strategy for each deviation of the other player.

Remark 2.7. As in Z-equilibrium for each deviation of a player from Z-equilibrium strategy there is a specific punishment strategy by the other player, the implementation of this concept in real-world games should be in two stages:

Stage 1. Z-equilibria are determined or computed. The players conduct a pre-play round of discussions or negotiations to select a Z-equilibrium and identify the punishment strategies against all possible deviations by all players.

Stage 2. The game is played, the strategies are revealed. Should any player deviate from the selected Z-equilibrium, he is punished by the other player using the known and appropriate strategy.

From this process it appears that when Nash equilibrium does not exist or it is dominated or there are many Nash equilibria in a game, Z-equilibrium could be a suitable solution as it is Pareto optimal and it can be stabilized through punishments. Indeed, when there is no-self-enforcing equilibrium (Nash equilibrium), it is wiser and advisable for players to have pre-play negotiations to avoid that some or all players end up selecting strategies that harm them and could be avoided.

Remark 2.8. *Z*-equilibrium is always Pareto optimal for the players and there are games where *Z*-equilibrium is preferable to the Nash equilibrium for the two players (in the case where Nash equilibrium is not Pareto optimal); see the following example.

Example 2.9. Consider the bi-matrix game

$$(A, B) = \begin{pmatrix} (-2,-2) & (1,-2) \\ (2,1) & (-2,2) \\ (0,1) & (0,1) \end{pmatrix}$$

where $X = \{1,2,3\}$ and $Y = \{1,2\}$ are the set of pure strategies for row player and column player, respectively. It is easy to see that the pair of strategies $(1,2) \in X \times Y$ is a Nash equilibrium and the pair of strategies $(2,1) \in X \times Y$ is a *Z*-equilibrium. As $a_{21} > a_{12}$ and $b_{12} > b_{21}$, this example shows that *Z*-equilibrium insures better payoffs for the two players than their payoffs in Nash-equilibrium.

Remark 2.10. – Condition (1) of Definition 2.5 is equivalent to the following

$$\begin{cases} l_1 = \max\limits_{i\in X} \min\limits_{j\in Y} a_{ij} \le a_{k^*l^*} \\ l_2 = \max\limits_{j\in Y} \min\limits_{i\in X} b_{ij} \le b_{k^*l^*} \end{cases}$$

which means that for player I (resp. II), the strategy profile (k^*, l^*) yields a payoff that is greater than or equal to her/his security level l_1 (resp. l_2). In other words, Z-equilibrium is individually rational for the players.

– We have the following equivalence (k^*, l^*) is a *Z*-equilibrium for G1 $\Longleftrightarrow$ (k^*, l^*) is a Pareto optimal solution of the (BMP) $\langle S, (F_1, F_2)\rangle$, where $F_1(i,j) = a_{ij}, F_2(i,j) = b_{ij}, S = \{(i,j) | l_1 \le a_{ij},\ l_2 \le b_{ij}\}$. We remark that the set of *Z*-equilibria is equal to the core of the game.

Remark 2.11. If Nash equilibrium is Pareto optimal then it is a Z-equilibrium.

Remark 2.12. *Z*-equilibrium (*in pure strategies*) always exists in bi-matrix games. Indeed, Bouchama *et al.* [4] proved that a finite game always has a *Z*-equilibrium in pure strategies and they established the equivalence between the concept of solution of a constraint satisfaction problem (CSP) and the *Z*-equilibrium of its associated game. Furthermore, they proposed a backtrack search based procedure for computing *Z*-equilibrium of the associated game.

In the following, we investigate *Z*-equilibrium in *mixed strategies*. Let us consider the sets of mixed strategies of players I and II, which represent weights assigned to their pure strategies

$$P = \left\{ p^T = (p_1, \dots, p_m), \sum_{i=1}^{m} p_i = 1,\ p_i \in [0,1] \right\} \quad \text{and}$$

$$Q = \left\{ q^T = (q_1, \dots, q_n), \sum_{j=1}^{n} q_j = 1,\ q_j \in [0,1] \right\},$$

respectively, where T represents the transpose operator.

The mixed strategies specify probabilities that players choose their particular pure strategies. Then, a mixed strategy game $G2$ is given as follows

$$G2 = \langle \{I, II\}, P \times Q, (A, B)\rangle.$$

When players I and II choose one mixed strategy from their own strategy set, say p and q, respectively, the expected payoff of player I and player II are respectively $E_1(p,q) = p^T A q$ and $E_2(p,q) = p^T B q$.

Definition 2.13. A pair $(p^*, q^*) \in P \times Q$ is said to be a mixed strategy *Z*-equilibrium of the game $G2$ if it satisfies the following two conditions:

(1) $\begin{cases} \forall p \in P, \quad \exists q \in Q, \quad p^T Aq \leq p^{*T} Aq^*; \\ \forall q \in Q, \quad \exists p \in P, \ p^T Bq \leq p^{*T} Bq^*; \end{cases}$

(2) the pair (p^*, q^*) is a Pareto optimal solution to the (BMP) $\langle P \times Q, \ (p^T Aq, \ p^T Bq)\rangle$.

Remark 2.14. – Condition (1) of Definition 2.13 is equivalent to

$$\begin{cases} \alpha_1 = \max_{p \in P} \min_{q \in Q} (p^T Aq) \leq p^{*T} Aq^* \\ \alpha_2 = \max_{q \in Q} \min_{p \in P} (p^T Bq) \leq p^{*T} Bq^*. \end{cases}$$

– Definition 2.13 means that (p^*, q^*) is a Pareto solution to the (BMP) $\langle D, (p^T Aq, p^T Bq)\rangle$, where $D = \{(p, q) \in P \times Q, \quad \alpha_1 \leq p^T Aq \quad and \quad \alpha_2 \leq p^T Bq\}$.

Proposition 2.15. *Let $(k^*, l^*) \in X \times Y$ be a Z-equilibrium of the game $G1$, then the pair $(p^*, q^*) \in P \times Q$ such that $p^* = (p_1, p_2, \ldots, p_m)^T, \ \ p_{k^*} = 1, \ p_k = 0, \ \forall k \neq k^*$ and $q^* = (q_1, q_2, \ldots, q_n)^T, \ \ q_{l^*} = 1, \ p_l = 0, \ \forall l \neq l^*$, is not necessary a mixed strategy Z-equilibrium for the game G2.*

Proof. Indeed, let us consider the following bi-matrix game. □

Example 2.16.

$$(A, B) = \begin{pmatrix} (3,4) & (6,2) \\ (5,3) & (2,5) \end{pmatrix}$$

where $X = \{1, \ 2\}$ and $Y = \{1, \ 2\}$ are the sets of pure strategies for row player and column player, respectively. The pair of strategies $(1, 1) \in X \times Y$ is a *Z*-equilibrium. Let $p^* = (1, 0)$ and $q^* = (1, 0)$, we have $p^{*T} Aq^* = 3$ and there exists $\overline{p} = (\frac{3}{4}, \frac{1}{4}) \in P$, such that $\forall q = (q_1, q_2) \in Q$, $p^T Aq = q_1(\frac{14}{4}) + q_2(5) > 3$, which means that the pair (p^*, q^*) doesn't satisfy condition (1) of Definition 2.13. Consequently, (p^*, q^*) is not a mixed strategy *Z*-equilibrium for the game $G2$.

3. Liu uncertainty theory

In the following, we present some definitions and properties of the Liu uncertainty theory that we will use in this paper. For more details we refer the reader to Liu [20–22].

Let Γ be a nonempty set, and L a σ-algebra over Γ. Each element E in L is called an event. A set function $\mathcal{M}$ from L to $[0, 1]$ is called an uncertain measure if it satisfies the normality, duality and subadditivity axioms.
Axiom 1 (Normality). $\mathcal{M}(\Gamma) = 1$ for the universal set Γ.
Axiom 2 (Duality). $\mathcal{M}(E) + \mathcal{M}(E^c) = 1$ for any event E.
Axiom 3 (Subadditivity). For every countable sequence of events $E_1, \ E_2, \ \ldots,$ we have

$$\mathcal{M}\left(\bigcup_{i \geq 1} E_i\right) \leq \sum_{i \geq 1} \mathcal{M}(E_i).$$

$(\Gamma, L, \mathcal{M})$ is called an uncertainty space.

Let $(\Gamma_k, L_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots n$. Let $\Gamma = \Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_n$ and $L = L_1 \times L_2 \times \ldots \times L_n$. Then the product uncertain measure $\mathcal{M}$ on the product σ-algebra L is defined by the following axiom:
Axiom 4 (Product Axiom). Let $(\Gamma_k, L_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots, n$. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left(\prod_{k=1}^{n} \Delta_k\right) = \min_{1 \leq k \leq n} \mathcal{M}_k(\Delta_k), \quad \Delta_k \in L_k, \quad k = 1, 2, \ldots, n.$$

Remark 3.1. A major difference between probability theory and Liu uncertainty theory is in the product axiom. Because of this difference, Liu uncertainty theory and probability theory operational laws are different.

In the following, we enumerate the basic results used in this paper.

Definition 3.2. A function $\Phi : \mathbb{R} \longrightarrow [0,1]$ is an uncertainty distribution if and only if it is an increasing function except $\Phi(x) \equiv 0$ and $\Phi(x) \equiv 1$ (see [33]).

Definition 3.3 ([20])**.** An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, L, \mathcal{M})$ to the set of real numbers.

Let ξ be an uncertain variable, the concept of uncertainty distribution is defined by

$$\Phi(x) = \mathcal{M}(\xi \leq x), \forall x \in \mathbb{R}$$

For example an uncertain variable ξ is called normal if it has a normal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}, x \in \mathbb{R}$$

denoted by $\mathcal{N}(e, \sigma)$, where e and σ are real numbers with $\sigma > 0$.

Definition 3.4 ([22])**.** An uncertainty distribution $\phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x satisfying $0 < \phi(x) < 1$, and

$$\lim_{x \longrightarrow -\infty} \phi(x) = 0, \quad \lim_{x \longrightarrow +\infty} \phi(x) = 1.$$

Remark 3.5. A regular uncertainty distribution ϕ has a unique inverse function $\alpha \longrightarrow \phi^{-1}(\alpha)$, for each $\alpha \in]0,1[$. The inverse ϕ^{-1} is called the inverse uncertainty distribution of ξ. For example, a normal uncertainty distribution, is regular.

The inverse uncertainty distribution of normal uncertain variable $N(e, \sigma)$ is

$$\phi^{-1}(\alpha) = e + \frac{\sigma\sqrt{3}}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right).$$

Definition 3.6 ([21])**.** The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \min_{1 \leq i \leq n} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots B_n$.

Definition 3.7 ([20])**.** Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite.

If ξ is a regular variable with an uncertainty distribution Φ, then the expected value may be calculated as follows

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^{0} \Phi(x)\mathrm{d}x$$

Definition 3.8 ([20])**.** Let ξ be a regular uncertain variable, and $\alpha \in (0,1)$. Then

$$\xi_{\sup}(\alpha) = \sup\{r|\mathcal{M}\{\xi \geq r\} \geq \alpha\} = \Phi^{-1}(1-\alpha)$$

is called the α-optimistic value of ξ.

Let ξ be an uncertain variable. Then $\xi_{\sup}(\alpha)$ is a decreasing function of α.

Lemma 3.9 ([22])**.** *Let ξ and η be independent uncertain variables with finite expected values. Then for any real numbers a and b, we have*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta].$$

Lemma 3.10 ([22])**.** *Let ξ and η be independent regular uncertain variables, and $\alpha \in (0,1)$. Then for any non-negative real numbers a and b, we have*

$$(a\xi + b\eta)_{\sup}(\alpha) = a\xi_{\sup}(\alpha) + b\eta_{\sup}(\alpha).$$

Ranking criteria

Let ξ and η be two independent regular uncertain variables and α and r be two given real numbers with $\alpha \in]0,1]$. Then we can compare the given two variables as follows.

- $W(C_1)$ (Expected value criterion): $\xi \geq \eta$ if and only if $E[\xi] \geq E[\eta]$.
- $W(C_2)$ (Optimistic value criterion): $\xi \geq \eta$ if and only if $\xi_{\sup}(\alpha) \geq \eta_{\sup}(\alpha)$ for some predetermined confidence level $\alpha \in]0,1]$.
- $W(C_3)$ (Uncertain measure criterion or Chance Criterion): $\xi \geq \eta$ if and only if $\mathcal{M}\{\xi \geq r\} \geq \mathcal{M}\{\eta \geq r\}$ for some predetermined level r.

Uncertain bi-criteria programming problem

Let us consider the uncertain bi-criteria maximization problem

$$\langle C, (\tilde{g}_1, \tilde{g}_2)\rangle,$$

where the decision vectors $z \in C$ are crisp, $C \subset \mathbb{R}^N$, $N \in \mathbb{N}^*$. $\tilde{g}_1(z) = g_1(z,\zeta)$ and $\tilde{g}_2(z) = g_2(z,\zeta)$ are the objective functions and ζ is an uncertain vector.

Based on the previous ranking criteria, we propose three extensions of the notion of Pareto optimal solution (Def. 2.1) to the uncertain maximization problem $\langle C, (\tilde{g}_1, \tilde{g}_2)\rangle$.

Definition 3.11. For predetermined real numbers α and r with $\alpha \in]0,1]$, $z^* \in C$ is said Pareto efficient under criterion $W(\theta)$ to the uncertain bi-criteria maximization problem (UBMP) $\langle C, (\tilde{g}_1, \tilde{g}_2)\rangle$ if there is no feasible solution z such that

$$(G_{1,W}(z^*,\theta),\ G_{2,W}(z^*,\theta)) \leq (G_{1,W}(z,\theta),\ G_{2,W}(z,\theta)),$$

where $\forall l = 1,2,\quad G_{l,W}(z,\theta) = \begin{cases} E[\tilde{g}_l(z)], & \text{if} \quad \theta = C_1; \\ \sup\{r|\mathcal{M}\{\tilde{g}_l(z) \geq r\} \geq \alpha\}, & \text{if} \quad \theta = C_2; \\ \mathcal{M}\{\tilde{g}_l(z) \geq r\}, & \text{if} \quad \theta = C_3. \end{cases}$

4. Uncertain bi-matrix game

In many real game situations, players may not know precisely the outcomes or payoffs of their strategies and no samples are available to estimate a probability distribution of the involved uncertainty. Therefore, the players have to rely on their experiences and subjective judgements of experts to evaluate their belief degree that each event will occur. Liu [23] showed that fuzzy theory and probability theory are not appropriate to model belief degree and we have to deal with it by Liu uncertainty theory.

In this section, Liu uncertainty theory is used to represent the indeterminacy of the payoffs.

Consider the uncertain payoffs bi-matrix games $G3$ and $G4$:

$$G3 = \langle \{I, II\}, X \times Y, (\tilde{A}, \tilde{B}) \rangle$$
$$G4 = \langle \{I, II\}, P \times Q, (\tilde{A}, \tilde{B}) \rangle$$

$\{I, II\}$ represents the two players. $\tilde{A} = (\xi_{ij})$ is the payoffs matrix of player I and $\tilde{B} = (\eta_{ij})$ is the payoffs matrix of player II, where ξ_{ij} and η_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$ are uncertain variables.

4.1. Solutions of the games $G3$ and $G4$

For a predetermined confidence level α (resp. β) $\in]0, 1]$ and predetermined level u (resp. v) of the row (resp. the column) player, we have the following definitions.

Definition 4.1. Let $\theta \in \{C_1,\ C_2,\ C_3\}$. A pair (k^*, l^*) is called a Z-equilibrium (in pure strategies) under criterion $W(\theta)$ for $G3$, if the following two conditions hold:

(1) $\begin{cases} \forall x \in \{1, \ldots m\}, \exists y \in \{1, \ldots n\},\ f_{1,W}(x, y, \theta) \le f_{1,W}(k^*, l^*, \theta); \\ \forall y \in \{1, \ldots n\},\ \exists x \in \{1, \ldots m\}, f_{2,W}(x, y, \theta) \le f_{2,W}(k^*, l^*, \theta); \end{cases}$

(2) (k^*, l^*) is Pareto efficient under criterion $W(\theta)$ to the UBMP $\langle P \times Q,\ (f_{1,W}(x, y, \theta), f_{2,W}(x, y, \theta)) \rangle$,

where $f_{1,W}(x, y, \theta) = \begin{cases} E[\xi_{xy}], & \text{if} \quad \theta = C_1; \\ \sup\{u | \mathcal{M}\{\xi_{xy} \ge u\} \ge \alpha\}, & \text{if} \quad \theta = C_2; \\ \mathcal{M}\{\xi_{xy} \ge u\}, & \text{if} \quad \theta = C_3; \end{cases}$ and

$$f_{2,W}(x, y, \theta) = \begin{cases} E[\eta_{xy}], & \text{if} \quad \theta = C_1; \\ \sup\{v | \mathcal{M}\{\eta_{xy} \ge v\} \ge \beta\}, & \text{if} \quad \theta = C_2; \\ \mathcal{M}\{\eta_{xy} \ge v\}, & \text{if} \quad \theta = C_3. \end{cases}$$

Remark 4.2

- In this approach, under the criterion $W(\theta)$, we assume that the aim of the row player is to maximize $f_{1,W}(x, y, \theta)$ and the aim of the column player is to maximize $f_{2,W}(x, y, \theta)$, $\theta \in \{C_1, C_2, C_3\}$.
- For $\theta \in \{C_1, C_2, C_3\}$, under criterion $W(\theta)$, Condition (1) of Definition 4.1 means that for any deviation x (resp. y) of the row player (resp. column player), the column player (resp. row player) punishes him by choosing a strategy y (resp. x) so that the resulting payoff $f_{1,W}(x, y, \theta)$ (resp. $f_{2,W}(x, y, \theta)$ is less or equal to $f_{1,W}(k^*, l^*, \theta)$ (resp. $f_{2,W}(k^*, l^*, \theta)$).

Definition 4.3. A pair (p^*, q^*) is called a mixed strategy Z-equilibrium under criterion $W(\theta)$ for $G4$, if the following two conditions hold:

(1) $\begin{cases} \forall p \in P,\ \exists q \in Q,\ F_{1,W}(p, q, \theta) \le F_{1,W}(p^*, q^*, \theta); \\ \forall q \in Q,\ \exists p \in P,\ F_{2,W}(p, q, \theta) \le F_{2,W}(p^*, q^*, \theta); \end{cases}$

(2) (p^*, q^*) is Pareto efficient under criterion $W(\theta)$ to the uncertain bi-criteria maximization problem (UBMP) $\langle P \times Q,\ (F_{1,W}(p, q, \theta), F_{2,W}(p, q, \theta)) \rangle$,

where $F_{1,W}(p, q, \theta) = \begin{cases} E[p^T \tilde{A} q], & \text{if} \quad \theta = C_1; \\ \sup\{u | \mathcal{M}\{p^T \tilde{A} q \ge u\} \ge \alpha\}, & \text{if} \quad \theta = C_2; \\ \mathcal{M}\{p^T \tilde{A} q \ge u\}, & \text{if} \quad \theta = C_3; \end{cases}$

and $F_{2,W}(p, q, \theta) = \begin{cases} E[p^T \tilde{B} q], & \text{if} \quad \theta = C_1; \\ \sup\{v | \mathcal{M}\{p^T \tilde{B} q \ge v\} \ge \beta\}, & \text{if} \quad \theta = C_2; \\ \mathcal{M}\{p^T \tilde{B} q \ge v\}, & \text{if} \quad \theta = C_3. \end{cases}$

Remark 4.4

- Under the criterion $W(\theta)$, we assume that the aim of the row player is to maximize $F_{1,W}(p, q, \theta)$ and the aim of the column player is to maximize $F_{2,W}(p, q, \theta)$, $\theta \in \{C_1, C_2, C_3\}$.

– Condition (1) of Definition 4.3 means that for any deviation p (resp. q) of the row player (resp. column player), the column player (resp. row player) punishes him by choosing a strategy q (resp. p) so that the resulting $F_1, W(p,q,\theta$ (resp. $F_{2,W}(p,q,\theta)$ is less or equal than $F_{1,W}(p^*,q^*,\theta)$ (resp. $F_{2,W}(p^*,q^*,\theta)$), $\theta \in \{C_1,C_2,C_3\}$.

Remark 4.5

– The approach considered in Perchet [34] to deal with uncertainties is different from the approach that we consider in this paper. Perchet [34] studied the notion of robust Nash equilibria which unify different notions of Nash equilibria with uncertainties for N-player games where $A_k \subset \mathbb{R}^{n_k}$, the action set of player $k \in \mathcal{N} = \{1,\dots,N\}$, is a compact and convex set and his payoff function $u_k : \prod_{k\in\mathcal{N}} A_k \longrightarrow \mathbb{R}$ is multilinear.

In Perchet [34], no assumptions are made on the mathematical structure or theory of uncertainty that supports his approach to uncertainty. The author proceeds under the assumption that the uncertainties are represented, for every player k, by a given mapping $\Upsilon_k : A_{-k} \longrightarrow \mathbb{R}^{n_k}$, where $A_{-k} = \prod_{t\neq k} A_t$. Which is different from the framework of this present work. Indeed, Liu uncertainty theory [20] models belief degrees and the uncertainty on the payoff is given as an uncertain variable (a measurable function on the uncertainty space). To solve the bimatrix game $G3$, we used three different ranking criteria of uncertain variables to define new equilibria as it is explained in Remarks 4.2 and 4.4.

– Different from the present study, the uncertainty in the payoff functions $f_i(x,y),\ i=1,\dots,n$ considered in Larbani and Lebbah [18] and in Nessah et al [31] is due to the lack of information about the realizations of an unknown parameter y and x is the strategy profile which is crisp. The authors suppose that the players know the domain where these parameters can take their values but completely ignore their behavior. The proposed solutions are based on the notion of Z-equilibrium Zhukovskii [45] and the max-min principle for decision making theory.

4.2. Existence conditions

Our focus in this section is the existence of the equilibria introduced in section 4.1 when the functions $F_{i,w}$ and $f_{i,w}$, $i=1,2$ are real valued.

For a predetermined confidence level α (resp. β) $\in]0,1]$ and predetermined level u (resp. v) of the row (resp. the column) player, we have the following results.

The existence of a Z-equilibrium for $G3$ (in *pure strategies*) under criterion $W(\theta)$ is given as follows.

Theorem 4.6. (1) *If the uncertain variables ξ_{ij} and η_{ij}, $i=1,\dots m$ and $j=1,\dots,n$ have finite expected values, there exist at least one Z-equilibrium under criterion $W(C_1)$.*

(2) *If the uncertain variables ξ_{ij} and η_{ij}, $i=1,\dots m$ and $j=1,\dots,n$ are regular, then there exist at least one Z-equilibrium under criterion $W(C_2)$.*

(3) *The game $G3$ always has a Z-equilibrium under criterion $W(C_3)$.*

Proof. A Z-equilibrium under criterion $W(\theta)$ is a Z-equilibrium for the bimatrix game $\langle X \times Y, (f_{1,W}(i,j,\theta), f_{2,W}(i,j,\theta)\rangle$. The existence of this concept is guaranteed by the existence theorem in Bouchama *et al.* [4]. □

Example 4.7. The condition of regular uncertain variables in Theorem 4.6 can not be dropped. For example, let us consider the non regular uncertain variable ξ with the uncertain function $\phi(x) = \frac{1}{2}(1+\exp(1000-x))^{-1},\ \ x \in \mathbb{R}$. Then, it is easy to verify that $\sup\{r|\mathcal{M}\{\xi \geq r\} \geq \alpha\} = \sup\{r|1-\mathcal{M}\{\xi \leq r\} \geq \alpha\} = \sup\{r|\phi(r) \leq 1-\alpha\} = \sup\{r|\phi(r) \leq 0.5\} = +\infty$, for $\alpha = 0.5$, which means that without regularity of ξ, the existence of the quantity $\sup\{r|\mathcal{M}\{\xi \geq r\} \geq \alpha\}$ is not guaranteed.

Theorem 4.8. *Assume that the uncertain variables ξ_{ij} and η_{ij} in the payoff matrices $\tilde{A}$ and $\tilde{B}$, $i=1,\dots m$ and $j=1,\dots,n$, are*

(1) *independent with finite expected values, then there exist at least one mixed strategy Z-equilibrium under criterion* $W(C_1)$*;*
(2) *independent and regular, then there exist at least one mixed strategy Z-equilibrium under criterion* $W(C_2)$*;*
(3) *independent and regular with finite expected values, then there exist at least one mixed strategy Z-equilibrium under criterion* $W(C_3)$*.*

Proof. (1) Existence of a mixed strategy Z-equilibrium under criterion $W(C_1)$.
Let $\tilde{\alpha}_1 = \max_{p\in P}\min_{q\in Q} E[p^T\tilde{A}q]$ and $\tilde{\alpha}_2 = \max_{q\in Q}\min_{p\in P} E[p^T\tilde{B}q]$. Due to the linearity of the expected value operator (Lem. 3.9), we have, $E[p^T\tilde{A}q] = p^T\tilde{A}_E q$ and $E[p^T\tilde{B}q] = p^T\tilde{B}_E q$. Where $\tilde{A}_E = (E(\xi_{ij}))_{i,j}$ and $\tilde{B}_E = (E(\eta_{i,j})_{i,j})$.
Since the functions $F_{1,W}(p,q,C_1) = E[p^T\tilde{A}q] = p^T E[\tilde{A}]q$ and $F_{2,W}(p,q,C_1) = E[p^T\tilde{B}q] = p^T E[\tilde{B}]q$ are continuous on the compact set $P\times Q$, then $\tilde{\alpha}_1$ exists. We conclude the existence of at least one mixed strategy Z-equilibrium under criterion $W(C_1)$.

(2) Existence of a mixed strategy Z-equilibrium under criterion $W(C_2)$.
For $(i,j)\in\{1,\dots,m\}\times\{1,\dots,n\}$, we denote by Φ_{ij} and Ψ_{ij} the distributions of ξ_{ij} and η_{ij} respectively. Then, using the regularity and independence of the variables (Lem. 3.10), we obtain

$$\begin{cases} F_{1,W}(p,q,C_2) = \max\{u|\mathcal{M}\{p^T\tilde{A}q \geq u\} \geq \alpha\} = p^T\tilde{A}^{\alpha}_{\text{sup}}q \\ F_{2,W}(p,q,C_2) = \max\{v|\mathcal{M}\{p^T\tilde{B}q \geq v\} \geq \beta\} = p^T\tilde{B}^{\beta}_{\text{sup}}q \end{cases}$$

where $\begin{cases} \tilde{A}^{\alpha}_{\text{sup}} = (\Phi^{-1}_{ij}(1-\alpha))_{i,j}, \\ \tilde{B}^{\beta}_{\text{sup}} = (\Psi^{-1}_{ij}(1-\beta))_{i,j}. \end{cases}$

$F_{1,W}(p,q,C_2)$ and $F_{2,W}(p,q,C_2)$ are continuous on the compact set $P\times Q$, then we have the existence of at least one mixed strategy Z-equilibrium under criterion $W(C_2)$.

(3) Existence of a mixed strategy Z-equilibrium under criterion $W(C_3)$.
First, we prove that the function $F_{1,W}(p,q,C_3) = \mathcal{M}\{p^T\tilde{A}q \geq u\}$ is continuous on $P\times Q$.
Let $(p^k,q^k)_{k\in\mathbb{N}}, p^k = (p^k_1,\dots,p^k_m)^T \in P$ and $q^k = (q^k_1,\dots,q^k_n)^T \in Q$ be a sequence which converges to (p^L,q^L), $p^L = (p^L_1,\dots,p^L_m)^T \in P$ and $q^L = (q^L_1,\dots,q^L_n)^T \in Q$.
We shall prove that

$$F_{1,W}(p^k,q^k,C_3) \longrightarrow F_{1,W}(p^L,q^L,C_3), \quad k\longrightarrow\infty.$$

For this purpose, let $\Phi_{(p^k)^T\tilde{A}q^k}$ be the uncertainty distribution of $(p^k)^T\tilde{A}q^k$ and $\Phi_{(p^L)^T\tilde{A}q^L}$ be the uncertainty distribution of $(p^L)^T\tilde{A}q^L$.
Due to the independence and the regularity of the variable, we have

$$F_{1,W}(p^k,q^k,C_3) = \mathcal{M}\{(p^k)^T\tilde{A}q^k \geq u\} = 1 - \Phi_{(p^k)^T\tilde{A}q^k}(u). \tag{4.1}$$

On the other hand, using the Markov inequality (see [20]), we have for any given real number $\epsilon > 0$,

$$\mathcal{M}\{|(p^k)^T\tilde{A}q^k - (p^L)^T\tilde{A}q^L| \geq \epsilon\} \leq \frac{\sum_{i,j}(p^k_i q^k_j - p^L_i q^L_j)E(\xi_{i,j})}{\epsilon}.$$

Since the expected values of the variables $\xi_{i,j}$ are finite, we obtain

$$\lim_{k\longrightarrow\infty}\mathcal{M}\{|(p^k)^T\tilde{A}q^k - (p^L)^T\tilde{A}q^L| \geq \epsilon\} \leq \lim_{k\longrightarrow\infty}\frac{\sum_{i,j}(p^k_i q^k_j - p^l_i q^L_j)E(\xi_{i,j})}{\epsilon} = 0.$$

Hence

$$\lim_{k\longrightarrow\infty}\mathcal{M}\{|(p^k)^T\tilde{A}q^k - (p^L)^T\tilde{A}q^L| \geq \epsilon\} = 0$$

which means that the sequence of uncertain variables $((p^k)^T \tilde{A} q^k)$ converges in distribution to the uncertain variable $((p^L)^T \tilde{A} q^L)$. Thus, by the definition of convergence with respect to distribution of uncertain variables, we have

$$\lim_{k \longrightarrow \infty} \Phi_{(p^k)^T \tilde{A} q^k}(u) = \Phi_{(p^L)^T \tilde{A} q^L}(u). \tag{4.2}$$

From equations (4.1) and (4.2), we obtain

$$\lim_{k \longrightarrow \infty} F_{1,W}(p^k, q^k, C_3) = F_{1,W}(p^L, q^L, C_3).$$

We deduce that the function $(p, q) \longrightarrow F_{1,W}(p, q, C_1)$ is continuous on $P \times Q$. By a similar process, we can prove that $F_{2,W}(p, q, C_3) = \mathcal{M}\{p^{\mathrm{T}} \tilde{B} q \geq v\}$ is continuous on $P \times Q$. Consequently, we have the existence of at least one mixed strategy Z-equilibrium under criterion $W(C_3)$.

□

Remark 4.9. A crisp number c is a special uncertain variable. It is the constant function $\xi(\gamma) \equiv c$ on the uncertainty space $(\Gamma, L, \mathcal{M})$. Therefore, a bi-matrix game without uncertainties can be considered as an uncertain bi-matrix game on the form $G3$ where the uncertain variables ξ_{ij} and η_{ij} in the payoff matrices $\tilde{A}$ and $\tilde{B}$, $i = 1, \ldots, m$ and $j = 1, \ldots, n$, are constants. Consequently, an uncertain bi-matrix game is a generalization of the bi-matrix game without uncertainties. Since the expected values of constants variables are finite, we obtain the existence of a Z-equilibrium for deterministic bi-matrix game.

– If we assume that in game $G4$, uncertain variables $\xi_{ij},\ \eta_{ij},\ i = 1, \ldots m;\ j = 1, \ldots, n$ are constant, Definition 4.3 coincides with Definition 2.13. This means that a mixed strategy Z-equilibrium is a Z-equilibrium under criterion $W(C_1)$.

Corollary 4.10. *$G2$ has at least one mixed strategy Z-equilibrium.*

Proof. In game $G2$, the real numbers a_{ij} and $b_{ij}, i = 1, \ldots, m;\ j = 1, \ldots, n$ are regular and independent uncertain variables, then conditions of Theorem 4.8 are satisfied by the matrices A and B. We conclude that G2 has a mixed strategy Z-equilibrium. □

Remark 4.11. This corollary is also a particular case of the existence theorem in Ferhat and Radjef [11]. The authors proved an existence result of the Z-equilibria for a mixed strategic multicriteria game.

The condition of regular uncertain variables can not be dropped in Theorem 4.8. See the following examples.

Example 4.12. Assume that in game $G3$ in the matrix $\tilde{A}$, $\tilde{\xi}_{11}$ is the non regular uncertain variables with the uncertain distribution $\phi_{11}(x) = \frac{1}{2}(1 + \exp(1000 - x))^{-1}, \quad x \in \mathbb{R}$. Then, for the vectors $p = (1, 0, \ldots, 0)^T$ and $q = (1, 0, \ldots, 0)^T$ we have $p^T \tilde{A} q = \tilde{a_{11}}$. By the same way as in Example 4.7, we prove that the quantity $\sup\{r | \mathcal{M}\{p^T \tilde{A} q \geq r\} \geq \alpha\}$ is not real valued. Which means that without regularity of $\xi_{1,1}$ the existence of the quantity $\sup\{r | \mathcal{M}\{p^T \tilde{A} q \geq r\} \geq \alpha\}$ is not guaranteed.

Example 4.13. Take an uncertainty space $(\Gamma, L, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with power set and $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = 0.5$. we consider in game $G4$, $\tilde{A} = (\xi_{i,j})$ with $\xi_{1,2} \equiv 1$; $\xi_{i,j} \equiv 0$ for $(i,j) \neq (1,1)$, $(i,j) \neq (1,2)$ and $\xi_{1,1}(\gamma) = \begin{cases} 1 & \gamma = \gamma_1; \\ -1 & \gamma = \gamma_2. \end{cases}$ with the uncertainty distribution

$$\phi_{1,1}(x) = \begin{cases} 0 & x < -1; \\ 0.5 & -1 \leq x < 1; \\ 1 & x \geq 1. \end{cases}$$

$\xi_{1,1}$ is not regular.

Let us compute the function $(p,q) \longmapsto F_{1,w}(p,q,C_3) = \mathcal{M}\{p^t\tilde{A}q \geq u\}$ where u is a predetermined level which we take equal to $\frac{1}{2}$.

For all $(p,q) \in P\times Q, \mathcal{M}\{p^t\tilde{A}q \geq u\} = \mathcal{M}\{p_1q_1\xi_{1,1} \geq \frac{1}{2} - p_1q_2\}$

$$= \begin{cases} \mathcal{M}\{\xi_{1,1} \geq \frac{1}{p_1q_1}(\frac{1}{2} - p_1q_2)\} & p_1q_1 \neq 0; \\ \mathcal{M}\{0 \geq \frac{1}{2} - p_1q_2\} & p_1q_1 = 0. \end{cases}$$

$$= \begin{cases} 0 & p_1q_1 \neq 0 \quad \text{and} \quad p_1(q_1+q_2) < \frac{1}{2}; \\ 0.5 & p_1q_1 \neq 0; \quad \frac{1}{2} \leq p_1(q_1+q_2) \quad \text{and} \quad p_1(q_2-q_1) < \frac{1}{2}; \\ 1 & \frac{1}{2} \leq p_1(q_2-q_1); \\ 0 & \frac{1}{2}p_1q_1 = 0 \quad \text{and} \quad \frac{1}{2} - p_1q_2 > 0; \\ 1 & p_1q_1 = 0 \quad \text{and} \quad \frac{1}{2} - p_1q_2 \leq 0. \end{cases}$$

Let $(\overline{p},\overline{q}) \in P\times Q$, such that $\overline{p} = (1,0,\ldots,0)^t$ and $\overline{q} = (\frac{1}{4},\frac{1}{4},\frac{1}{2},0,\ldots,0)^t$. We have $F_{1,w}(\overline{p},\overline{q},C_3) = 0.5$ and $\lim_{p_1\longrightarrow 1^+; q_1\longrightarrow \frac{1}{2}^+; q_2\longrightarrow \frac{1}{2}^+} F_{1,w}(\overline{p},\overline{q},C_3) = 0$.

$F_{1,w}$ is not continuous on the compact $P\times Q$. Which is an assumption to prove the existence of the mixed strategy Z-equilibrium under criterion $W(C_3)$.

4.3. Relation with the concepts of equilibria introduced in Gao [12]

In this section, we recall the concepts introduced in Gao [12] which are based on Nash equilibrium. Then, we compare our results with those of Gao.

Definition 4.14 ([12])**.** A pair (p^*,q^*) is called an expected Nash equilibrium, if it satisfies

$$\begin{cases} \forall p \in P, & E[p^T\tilde{A}q^*] \leq E[p^{*T}\tilde{A}q^*]; \\ \forall q \in Q, & E[p^{*T}\tilde{B}q] \leq E[p^{*T}\tilde{B}q^*]. \end{cases}$$

Definition 4.15 ([12])**.** A pair (p^*,q^*) is called an (α,β)-optimistic Nash-equilibrium if it satisfies

$$\begin{cases} \forall p \in P, & \max\{u|\mathcal{M}\{p^{*T}\tilde{A}q^* \geq u\} \geq \alpha\} \geq \max\{u|\mathcal{M}\{p^T\tilde{A}q^* \geq u\} \geq \alpha\}; \\ \forall q \in Q, & \max\{v|\mathcal{M}\{p^{*T}\tilde{B}q^* \geq v\} \geq \beta\} \geq \max\{v|\mathcal{M}\{p^{*T}\tilde{B}q \geq v\} \geq \beta\}. \end{cases}$$

Definition 4.16 ([12])**.** A pair (p^*,q^*) is called an (u,v)-most uncertainty Nash equilibrium if it satisfies

$$\begin{cases} \forall p \in P, & \mathcal{M}\{p^{*T}\tilde{A}q^* \geq u\} \geq \mathcal{M}\{p^T\tilde{A}q^* \geq u\}; \\ \forall q \in Q, & \mathcal{M}\{p^{*T}\tilde{B}q^* \geq v\} \geq \mathcal{M}\{p^{*T}\tilde{B}q \geq v\}. \end{cases}$$

Remark 4.17. We have the following relations.

- If the expected Nash equilibrium is Pareto efficient under criterion $W(C_1)$ to the (UBMP) $\langle P\times Q,\ (p^T\tilde{A}q,\ p^T\tilde{B}q)\rangle$, then it is a mixed strategy Z-equilibrium under criterion $W(C_1)$.
- If an (α,β)-optimistic Nash equilibrium is Pareto efficient under criterion $W(C_2)$ to the (UBMP) $\langle P\times Q,\ (p^T\tilde{A}q,\ p^T\tilde{B}q)\rangle$, then it is a mixed strategy Z-equilibrium under criterion $W(C_2)$.
- If an (u,v)-most uncertain Nash equilibrium is Pareto efficient under criterion $W(C_3)$ to the (UBMP) $\langle P\times Q,\ (p^T\tilde{A}q,\ p^T\tilde{B}q)\rangle$, then it is a mixed strategy Z-equilibrium under criterion $W(C_3)$.

5. Computation of the solutions

The computation of the equilibria introduced in Section 4.1 is considered in this here. We show how the problem of computation of the introduced concepts is transformed into mathematical programming problems. An algorithm is developed for computing the Z-equilibria.

Theorem 5.1. *Let ξ_{ij} and η_{ij} in payoff matrices $\widetilde{A}$ and $\widetilde{B}$ respectively be independent regular uncertain variables with finite expected values, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Then*

(1) *Let $\theta \in \{C_1,\ C_2\}$, a sufficient condition that the strategy profile (p^*, q^*) be a Z-equilibrium under criterion $W(\theta)$ of the game G3 is that there exists a pair (β_1, β_2), $\beta_i > 0$, $i = 1, 2$, such that (p^*, q^*) is a solution of the following mathematical programming problem*

$$(\tilde{P}_\theta)\begin{cases}\max\left(\beta_1 F_{1,W}(p,q,\theta) + \beta_2 F_{2,W}(p,q,\theta)\right)\\ \text{subject to}\\ F_{1,W}(p,q,\theta) \geq \tilde{\alpha}_1^\theta\\ F_{2,W}(p,q,\theta) \geq \tilde{\alpha}_2^\theta\\ \displaystyle\sum_{j=1}^{n} q_j - 1 = 0\\ \displaystyle\sum_{i=1}^{m} p_i - 1 = 0\\ 0 \leq q_j,\ j = 1, \ldots, n\\ 0 \leq p_i,\ i = 1, \ldots, m\end{cases}$$

$$\text{where}\quad\begin{cases}\tilde{\alpha}_1^\theta = \max V\\ \text{subject to}\\ V \leq \displaystyle\sum_{i=1}^{m} \tilde{a}_{ij}^\theta p_i,\ j = 1, \ldots, n\\ \displaystyle\sum_{i=1}^{m} p_i - 1 = 0\\ 0 \leq p_i,\ i = 1, \ldots, m\end{cases}\quad\text{and}\quad\begin{cases}\tilde{\alpha}_2^\theta = \max W\\ \text{subject to}\\ W \leq \displaystyle\sum_{j=1}^{n} \tilde{b}_{ij}^\theta q_j,\ i = 1, \ldots, m\\ \displaystyle\sum_{j=1}^{n} q_j - 1 = 0\\ 0 \leq q_j,\ j = 1, \ldots, n\end{cases}$$

where $\tilde{a}_{ij}^\theta = \begin{cases} E(\xi_{ij}) & if \theta = C_1\\ \Phi_{ij}^{-1}(1-\alpha) & if \theta = C_2\end{cases}$ *and* $\tilde{b}_{ij}^\theta = \begin{cases} E(\eta_{ij}) & if \theta = C_1\\ \Psi_{ij}^{-1}(1-\beta) & if \theta = C_2\end{cases}$

α and β are given in Section 4.1.

(2) *a sufficient condition for the strategy profile (p^*, q^*) to be a Z-equilibrium under criterion $W(C_3)$ of the game G3 is that there exists a pair (β_1, β_2), $\beta_i > 0$, $i = 1, 2$, such that (p^*, q^*) is a solution of the following mathematical programming problem*

$$(\tilde{P}_{C_3})\begin{cases}\max\left(\beta_1 \mathcal{M}\{p^T\tilde{A}q \geq u\} + \beta_2 \mathcal{M}\{p^T\tilde{B}q \geq v\}\right)\\ \text{subject to}\\ -\displaystyle\sum_{i,j} p_i q_j \phi_{ij}^{-1}(1-\tilde{\lambda}_1) \leq u\\ -\displaystyle\sum_{i,j} p_i q_j \phi_{ij}^{-1}(1-\tilde{\lambda}_2) \leq v\\ \displaystyle\sum_{j=1}^{n} q_j - 1 = 0\\ \displaystyle\sum_{i=1}^{m} p_i - 1 = 0\\ 0 \leq q_j,\ j = 1, \ldots, n\\ 0 \leq p_i,\ i = 1, \ldots, m\end{cases}$$

where $\tilde{\lambda}_1 = \tilde{\alpha}_1^{C_3} = \max_{p\in P}\min_{q\in Q} \mathcal{M}\{p^T\tilde{A}q \geq u\}$ *and* $\tilde{\lambda}_2 = \tilde{\alpha}_2^{C_3} = \max_{q\in Q}\min_{p\in P} \mathcal{M}\{p^T\tilde{B}q \geq v\}$.

Proof. (1) The set of the Z-equilibria under criterion $W(C_1)$ is given by all the Pareto solution of the bi-criteria maximization problem:

$$\left\langle \tilde{S}(\tilde{\alpha}_1^{C_1}, \tilde{\alpha}_2^{C_1}), (F_{1,W}(p,q,C_1), F_{2,W}(p,q,C_1)) \right\rangle \tag{5.1}$$

where $\tilde{\alpha}_1^{C_1} = \max_{p\in P} \min_{q\in Q} E[p^T \tilde{A} q]$, $\tilde{\alpha}_2^{C_1} = \max_{q\in Q} \min_{p\in P} E[p^T \tilde{B} q]$
$F_{1,W}(p,q,C1) = p^T \widetilde{A}_E q \quad , F_{2,W}(p,q,C_1) = p^T \widetilde{B}_E q$ and
$\tilde{S}(\tilde{\alpha}_1, \tilde{\alpha}_2) = \Big\{ (p,q) \quad such \quad that \quad p^T \tilde{A}_E q \geq \tilde{\alpha}_1^{C_1}, \quad p^T \tilde{B}_E q \geq \tilde{\alpha}_2^{C_1} \Big\}$.

Solutions of problem (5.1) can be found by solving the following one [42]

$$\max_{(p,q)\in \tilde{S}(\tilde{\alpha}_1^{C_1}, \tilde{\alpha}_2^{C_1})} (p^T(\beta_1 \tilde{A}_E + \beta_2 \tilde{B}_E) q),$$

where $\beta_1, \ \beta_2 \geq 0$ and $\beta_1 + \beta_2 = 1$.

On the other hand it is well known that the problem $\tilde{\alpha}_1^{C_1} = \max_{p\in P} \min_{q\in Q} (p^T \tilde{A}_E q)$ is equivalent to

$$\begin{cases} \tilde{\alpha}_1^{C_1} = \max V \\ \text{subject to} \\ V \leq \displaystyle\sum_{i=1}^{m} \tilde{a}_{ij}^{C_1} p_i, \ j = 1, \ldots, n \\ \displaystyle\sum_{i=1}^{m} p_i - 1 = 0 \\ 0 \leq p_i, \ i = 1, \ldots, m \end{cases}$$

which concludes the proof. The proof is similar under criterion $W(C_2)$.

In part 2, we proceed in the same way as in 1) and we take into account the equivalences from Liu [20]:

$$\mathcal{M}\{p^T \tilde{A} q \geq u\} \geq \tilde{\lambda}_1 \Longleftrightarrow \left(- \sum_{i,j} p_i q_j \phi_{ij}^{-1}(1 - \tilde{\lambda}_1) \leq u \right) \quad \text{and}$$

$$\mathcal{M}\{p^T \tilde{B} q \geq v\} \geq \tilde{\lambda}_2 \Longleftrightarrow \left(- \sum_{i,j} p_i q_j \psi_{ij}^{-1}(1 - \tilde{\lambda}_2) \leq v \right)$$

□

In Corollary 5.2, we present a way for computing mixed strategy Z-equilibria of the bi-matrix game $G2$ (Def. 2.13).

Corollary 5.2. *A sufficient condition that the strategy profile (p^*, q^*) be a mixed Z-equilibrium of the bi-matrix game $G2$ is that there exists a pair (β_1, β_2), $\beta_i > 0$, $i = 1, 2$, such that (p^*, q^*) is a solution of the following mathematical programming problem $(P4)$.*

$$(P4)\begin{cases}\max\left(p^T(\beta_1 A+\beta_2 B)q\right)\\ \text{subject to}\\ p^TAq\geq\alpha_1\\ p^TBq\geq\alpha_2\\ \displaystyle\sum_{j=1}^{n}q_j-1=0\\ \displaystyle\sum_{i=1}^{m}p_i-1=0\\ 0\leq q_j,\ \ j=1,\dots,n\\ 0\leq p_i,\ \ i=1,\dots,m\end{cases}$$

$$\text{where}\quad\begin{cases}\alpha_1=\max V\\ \text{subject to}\\ V\leq\displaystyle\sum_{i=1}^{m}a_{ij}p_i,\ \ j=1,\dots n\\ \displaystyle\sum_{i=1}^{m}p_i-1=0\\ 0\leq p_i,\ \ i=1,\dots,m\end{cases}\quad\text{and}\quad\begin{cases}\alpha_2=\max W\\ \text{subject to}\\ W\leq\displaystyle\sum_{j=1}^{n}b_{ij}q_j,\ \ i=1\dots m\\ \displaystyle\sum_{j=1}^{n}q_j-1=0\\ 0\leq q_j,\ \ j=1,\dots,n\end{cases}.$$

Proof. According to Remark 4.9, in order to obtain the result of corollary 5.2 it is sufficient to set $\tilde{A}_E = A$, in problem $(\tilde{P}_{C_1})$. □

5.1. Algorithm

The following algorithm can be used to compute Z-equilibrium under criterion $W(C_k)$, $k = 1, 2, 3$ are described as follows.

1: Initialization: Let $\tilde{A}$, $\tilde{B}$, $\beta_1 \geq 0$, $\beta_2 \geq 0$ $(\beta_1 + \beta_2 = 1)$ be given.
2: Compute $\tilde{A}_E$, $\tilde{B}_E$, $\tilde{A}_{\text{sup}}$, $\tilde{B}_{\text{sup}}$.
3: if $k = 1$ or 2, use simplex code for computing $\alpha_1^{C_k}$ and $\alpha_2^{C_k}$;
if $k = 3$, use the maillage of the sets $[0, 1]^{m-1}$ and $[0, 1]^{n-1}$ for computing $\alpha_1^{C_k}$ and $\alpha_2^{C_k}$ of maxmin problem.
4: Use IBBA code for solving the problem $(\tilde{P}_{C_k})$.

In state 3 of the algorithm, if $k = 3$, the sets of constraints P and Q are reduced to $[0, 1]^{m-1}$ and $[0, 1]^{n-1}$ respectively by substituting the constraints $\sum_{i=1}^{m} p_i = 1$ and $\sum_{j=1}^{m} q_j = 1$ in the objective function.

This algorithm is based on the IBBA code. It is useful to note that the problems $(\tilde{P}_{C_1})$, $(\tilde{P}_{C_2})$ and $(\tilde{P}_{C_3})$ are non-concave because of matrices $\tilde{A}_E$, $\tilde{B}_E$, $\tilde{A}_{\text{sup}}$ and $\tilde{B}_{\text{sup}}$ being non definte.

To obtain the global optimal solutions of $(\tilde{P}_{C_1})$, $(\tilde{P}_{C_2})$ and $(\tilde{P}_{C_3})$, we use the IBBA solver thanks to Frédéric Messine from LAPLACE-N7; IBBA is a deterministic global optimization code based on interval analysis and affine relaxations [26–28].

6. Numerical example

Let f_1 and f_2 be 2 firms in competition for a given market. Assume that the two firms want to market the same product and they lack of observed data on the consumer's demands of their products to estimate a probability distribution of the involved uncertainty. For example, in the case of a new product as considered in Gao [12]. Then, the two firms have to invite some domain experts to evaluate their belief degree that each event will occur. Using Liu uncertainty theory the demands of the product can be understood as uncertain variables and the outcomes resulting of the actions of the two firms are uncertain variables. Assume that, the payoff matrix of the first firm (player I) is

$$\tilde{A} = \begin{pmatrix} \mathcal{N}(110,14) & \mathcal{N}(60,9) \\ \mathcal{N}(70,10) & \mathcal{N}(30,6) \end{pmatrix}$$

and the payoff matrix of the second firm (player II) is

$$\tilde{B} = \begin{pmatrix} \mathcal{N}(40,4) & \mathcal{N}(55,8) \\ \mathcal{N}(45,6) & \mathcal{N}(70,11) \end{pmatrix}.$$

In this section, the IBBA algorithm is used for solving the Mathematical programming problems.

In the following, we consider the notations $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ 1-p_1 \end{pmatrix}$ and $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ 1-q_1 \end{pmatrix}$.

(1) The firms adopt the expected value criterion. We have

$$\tilde{A}_E = \begin{pmatrix} 110 & 60 \\ 70 & 30 \end{pmatrix} \quad \textit{and} \quad \tilde{B}_E = \begin{pmatrix} 40 & 55 \\ 45 & 70 \end{pmatrix}$$

and the problem $(\tilde{P}1)$, with $\beta_1 = \beta_2 = \frac{1}{2}$ is

$$\begin{cases} \max((p_1, 1-p_1) \begin{pmatrix} 75 & \frac{115}{2} \\ \frac{115}{2} & 50 \end{pmatrix} \begin{pmatrix} q_1 \\ 1-q_1 \end{pmatrix}) \\ \text{subject to} \\ (p_1, 1-p_1) \begin{pmatrix} 110 & 60 \\ 70 & 30 \end{pmatrix} \begin{pmatrix} q_1 \\ 1-q_1 \end{pmatrix} \geq \tilde{\alpha}_1(C_1) \\ (p_1, 1-p_1) \begin{pmatrix} 40 & 55 \\ 45 & 70 \end{pmatrix} \begin{pmatrix} q_1 \\ 1-q_1 \end{pmatrix} \geq \tilde{\alpha}_2(C_1) \\ \\ 0 \leq q_1 \leq 1 \\ 0 \leq p_1 \leq 1 \end{cases}$$

$$\text{where} \quad \begin{cases} \tilde{\alpha}_1 = \max V \\ \text{subject to} \\ V \leq 40p_1 + 70, \\ V \leq 30p_1 + 30 \\ 0 \leq p_1 \leq 1 \end{cases} \quad \textit{and} \quad \begin{cases} \tilde{\alpha}_2 = \max W \\ \text{subject to} \\ W \leq -15q_1 + 55 \\ W \leq -25q_1 + 70 \\ 0 \leq q \leq 1 \end{cases}.$$

Solving these three mathematical programming problems, we obtain the vector $((0.6972, 0.3028), (0.2519, 0.7481))$ which is an EZE. The equilibrium strategy of the player I is $p^* = (0.6972, 0.3028)$ and the equilibrium strategy of player II is $q^* = (0.2519, 0.7481)$, with the expected payoff $p^{*T} A_E q^* = 62.7482$ and $p^{*T} B_E q^* = 55.0007$ of the two firms, respectively.

(2) The firms adopt the Optimistic value criterion.
When $\alpha = 0,85$ and $\beta = 0,90$, we obtain

$$\tilde{A}^{\alpha}_{\text{sup}} = (\phi^{-1}_{ij}(1-\alpha))_{ij} = \begin{pmatrix} 96.6113 & 51.3930 \\ 59.4803 & 24.2620 \end{pmatrix}$$

$$\tilde{B}^{\beta}_{\text{sup}} = (\psi^{-1}_{ij}(1-\beta))_{ij} = \begin{pmatrix} 36.3658 & 45.3089 \\ 37.7316 & 56.6747 \end{pmatrix}$$

The problem $(\tilde{P}_{C_2})$, with $\beta_1 = \beta_2 = \frac{1}{2}$ becomes:

$$\begin{cases} \max((p_1, 1-p_1)\begin{pmatrix} 66.4886 & 48.3509 \\ 48.6060 & 40.4683 \end{pmatrix}\begin{pmatrix} q_1 \\ 1-q_1 \end{pmatrix}) \\ \text{subject to} \\ (p_1, 1-p_1)\begin{pmatrix} 96.6113 & 51.3930 \\ 59.4803 & 24.2620 \end{pmatrix}\begin{pmatrix} q_1 \\ 1-q_1 \end{pmatrix} \geq \tilde{\alpha}_1^{C_2} \\ (p_1, 1-p_1)\begin{pmatrix} 36.3658 & 45.3089 \\ 37.7316 & 56.6747 \end{pmatrix}\begin{pmatrix} q_1 \\ 1-q_1 \end{pmatrix} \geq \tilde{\alpha}_2^{C_2} \\ 0 \leq q_1 \leq 1 \\ 0 \leq p_1 \leq 1 \end{cases}$$

$$\text{where} \quad \begin{cases} \tilde{\alpha}_1^{C_2} = \max V \\ \text{subject to} \\ V \leq 37.1310p_1 + 59.4803 \\ V \leq 27.1310p_1 + 24.2620 \\ 0 \leq p_1 \leq 1 \end{cases} \quad \text{and} \quad \begin{cases} \tilde{\alpha}_2^{C_2} = \max W \\ \text{subject to} \\ W \leq -8.9430q_1 + 45.3089 \\ W \leq -18.943q_1 + 56.6747 \\ 0 \leq q \leq 1 \end{cases}$$

Solving the previous three mathematical programming problems, we obtain an $(0.85, 0.90)$-optimistic equilibrium strategies $p^* = (0.6984, 0.3016)$ for the first player and $q^* = (0.2866, 0.7134)$ for the second player. The value of the uncertain bi-matrix game is $F_{1,W}(p^*, q^*, C_2) = p^{*T} A^{\alpha}_{\text{sup}} q^* = 55.3055$ for the first player and $F_{2,W}(p^*, q^*, C_2) = p^{*T} B^{\beta}_{\text{sup}} q^* = 45.3093$ for the second player.
$((0.6984, 0.3016), (0.2866, 0.7134))$ is an $(0.85, 0.9)$-OZE.

(3) Suppose that the payoff levels of the players I and II are $u = 50$ and $v = 40$, respectively. In the problem $(\widetilde{P}_{C_3})$, $p^T \widetilde{A} q = N(e_1, \sigma_1)$ and $p^T \widetilde{B} q = N(e_2, \sigma_2)$ are normal uncertain variables. Consequently,

- $\mathcal{M}\{p^T \tilde{A} q \leq u\} = \left[1 + \exp\left(\frac{\pi(e_1 - u)}{\sqrt{3}\sigma_1}\right)\right]^{-1} = \left[1 + \exp\left(\frac{\pi(30p_1 + 40q_1 + 30 + 10p_1q_1 - u)}{\sqrt{3}(3p_1 + 4q_1 + p_1q_1 + 6)}\right)\right]^{-1}$, with
  $e_1 = p^T \begin{pmatrix} 110 & 60 \\ 70 & 30 \end{pmatrix} q = 30p_1 + 40q_1 + 10p_1q_1 + 30$
  $\sigma_1 = p^T \begin{pmatrix} 14 & 9 \\ 10 & 6 \end{pmatrix} q = 3p_1 + 4q_1 + 6 + p_1q_1.$
- $\mathcal{M}\{p^T \tilde{B} q \leq v\} = \left[1 + \exp\left(\frac{\pi(e_2 - v)}{\sqrt{3}\sigma_2}\right)\right]^{-1} = \left[1 + \exp\left(\frac{\pi(-15p_1 - 25q_1 + 70 + 10p_1q_1 - v)}{\sqrt{3}(-3p_1 - 5q_1 + 11 + p_1q_1)}\right)\right]^{-1}$, with
  $e_2 = p^T \begin{pmatrix} 40 & 55 \\ 45 & 70 \end{pmatrix} q = -15p_1 - 25q_1 + 70 + 10p_1q_1.$
  $\sigma_2 = p^T \begin{pmatrix} 4 & 8 \\ 6 & 11 \end{pmatrix} q = -3p_1 - 5q_1 + 11 + p_1q_1.$

Problem $(\tilde{P}_{C_3})$ becomes

$$\begin{cases} \max\left(1 - \frac{1}{2}\left[1 + \exp\left(\frac{\pi(30p_1 + 40q_1 + 30 + 10p_1q_1 - u)}{\sqrt{3}(3p_1 + 4q_1 + 6 + p_1q_1)}\right)\right]^{-1} - \frac{1}{2}\left[1 + \exp\left(\frac{\pi(-15p_1 - 25q_1 + 70 + 10p_1q_1 - v)}{\sqrt{3}\,(-3p_1 - 5q_1 + 11 + p_1q_1)}\right)\right]^{-1}\right) \\ \text{subject to} \\ -\left(30p_1 + 40q_1 + 30 + 10p_1q_1 + \frac{(3p_1 + 4q_1 + 6 + p_1q_1)\sqrt{3}}{\pi}\ln\left(\frac{1-\tilde{\lambda}_1}{\tilde{\lambda}_1}\right)\right) \leq u \\ -\left(-15p_1 - 25q_1 + 70 + 10p_1q_1 + \frac{(-3p_1 - 5q_1 + 11 + p_1q_1)\sqrt{3}}{\pi}\ln\left(\frac{1-\tilde{\lambda}_2}{\tilde{\lambda}_2}\right)\right) \leq v \\ 0 \leq q_1 \leq 1 \\ 0 \leq p_1 \leq 1 \end{cases}$$

where

$$\tilde{\lambda}_1 = \max_{p_1 \in [0,1]} \min_{q_1 \in [0,1]} \left(1 - \left[1 + \exp\left(\frac{\pi(30p_1 + 40q_1 + 30 + 10p_1q_1 - u)}{\sqrt{3}(3p_1 + 4q_1 + 6 + p_1q_1)}\right)\right]^{-1}\right) \quad \text{and}$$

$$\tilde{\lambda_2} = \max_{q_1 \in [0,1]} \min_{p_1 \in [0,1]} \left(1 - \left[1 + \exp\left(\frac{\pi(-15p_1 - 25q_1 + 70 + 10p_1q_1 - v)}{\sqrt{3}(-3p_1 - 5q_1 + 11 + p_1q_1)}\right)\right]^{-1}\right).$$

The solution is given by $\overline{p}_1 = 0.8186$, $\overline{q}_1 = 0.3724$.

– The equilibrium strategies of player I: $(0.8186, 0.1814)$ and the payoff for player I is $F_{1,W}(p^*, q^*, C_3) = 0.9817$.
– The equilibrium strategies of player II: $(0.3724, 0.6276)$ with the payoff for player II is $F_{2,W}(p^*, q^*, C_3) = 0.9514$.

$((0.8186, 0.1814), (0.3724, 0.6276))$ is a $(50, 40)$-MUZE. At this equilibrium point, the first firm chooses the mixed strategy $(0.8186, 0.1814)$ yielding a payoff level of 50 with uncertain measure 98% and the second firm chooses the mixed strategy $q = (0.3724, 0.6276)$ yielding a payoff level of 40 with uncertain measure 95%.

7. Conclusion

In this paper, we introduced definitions of Z-equilibria in bi-matrix game with uncertain payoffs according to different ranking criteria of Liu uncertainty theory. Moreover, an existence theorem for each proposed concept is provided. Using mathematical programming, a method for computing each proposed equilibrium is presented and illustrated with a numerical example. In the near future, we intend to extend the results of this paper to continuous games and multiple criteria games.

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