

COMPLEXITY OF INHERITANCE OF \mathcal{F} -CONVEXITY FOR RESTRICTED GAMES INDUCED BY MINIMUM PARTITIONS

A. SKODA^{1,*}

Abstract. Let $G = (N, E, w)$ be a weighted communication graph. For any subset $A \subseteq N$, we delete all minimum-weight edges in the subgraph induced by A . The connected components of the resultant subgraph constitute the partition $\mathcal{P}_{\min}(A)$ of A . Then, for every cooperative game (N, v) , the \mathcal{P}_{\min} -restricted game (N, \bar{v}) is defined by $\bar{v}(A) = \sum_{F \in \mathcal{P}_{\min}(A)} v(F)$ for all $A \subseteq N$. We prove that we can decide in polynomial time if there is inheritance of \mathcal{F} -convexity, *i.e.*, if for every \mathcal{F} -convex game the \mathcal{P}_{\min} -restricted game is \mathcal{F} -convex, where \mathcal{F} -convexity is obtained by restricting convexity to connected subsets. This implies that we can also decide in polynomial time for any unweighted graph if there is inheritance of convexity for Myerson's graph-restricted game.

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1. INTRODUCTION

Given a finite set N with $|N| = n$, we consider a weighted communication graph $G = (N, E, w)$ where w is a weight function defined on the set E of edges. For a given subset A of N , we denote by $E(A)$ the set of edges in E with both end-vertices in A , and by $\Sigma(A)$ the subset of minimum-weight edges in $E(A)$. Let $G_A = (A, E(A))$ be the graph induced by A , and $\tilde{G}_A = (A, E(A) \setminus \Sigma(A))$ be the graph obtained by deleting the minimum-weight edges in G_A . Skoda [13] introduced the correspondence \mathcal{P}_{\min} on N which associates to every subset $A \subseteq N$, the partition $\mathcal{P}_{\min}(A)$ of A into the connected components of \tilde{G}_A . Then for every cooperative game (N, v) the \mathcal{P}_{\min} -restricted game (N, \bar{v}) is defined by:

$$\bar{v}(A) = \sum_{F \in \mathcal{P}_{\min}(A)} v(F), \text{ for all } A \subseteq N. \quad (1.1)$$

v is the characteristic function of the game, $v : 2^N \rightarrow \mathbb{R}$, $A \mapsto v(A)$ and satisfies $v(\emptyset) = 0$. Compared to the original game (N, v) , the \mathcal{P}_{\min} -restricted game (N, \bar{v}) takes into account the combinatorial structure of the graph and the ability of players to cooperate in a given coalition and therefore different aspects of cooperation restrictions. In particular, assuming that the edge-weights reflect the degrees of relationships between players, $\mathcal{P}_{\min}(A)$ gives a partition of a coalition A into subgroups where players are in privileged relationships (with respect to the minimum relationship degree in A). Many other correspondences have been considered to define restricted games (see, *e.g.*, [3–5, 7, 10–12]).

Keywords. Cooperative game, restricted game, graph partitions, convexity, complexity.

¹ Université de Paris I, Centre d'Economie de la Sorbonne, 106-112 Bd de l'Hôpital, 75013 Paris, France.

*Corresponding author: alexandre.skoda@univ-paris1.fr

For a given correspondence a classical problem is to study what conditions guarantee that some desirable properties of the original game, as superadditivity or convexity, are transmitted to the restricted game. We say that there is inheritance of a given property if for every game satisfying this property, the restricted-game still satisfies it. For correspondences associated with a graph as \mathcal{P}_{\min} , a characterization of inheritance of a given property relies on necessary and sufficient conditions on the underlying graph. Grabisch [11] investigated the inheritance of superadditivity and convexity for the correspondence \mathcal{P}_{\min} . They proved that there is always inheritance of superadditivity for \mathcal{P}_{\min} . On the contrary they observed that inheritance of convexity requires heavy restrictions on the underlying graph. Of course, inheritance of convexity is of particular interest as it implies that good properties are inherited, for instance the non-emptiness of the core, and that the Shapley value is in the core. But inheritance of convexity is also a strong requirement. For instance, inheritance of convexity for \mathcal{P}_{\min} implies that there are at most three different edge-weights [14]. Hence weaker properties than convexity can be usefully considered. Following alternative definitions of convexity in combinatorial optimization and game theory when restricted families of subsets (not necessarily closed under union and intersection) are considered (see, e.g., [6,7,9]), Grabisch [11] introduced the \mathcal{F} -convexity by restricting convexity to the family \mathcal{F} of connected subsets of G . \mathcal{F} -convexity is particularly relevant in the context of communication graphs as connected subsets correspond to coalitions whose members can communicate. Skoda [13] characterized inheritance of \mathcal{F} -convexity for \mathcal{P}_{\min} by five necessary and sufficient conditions on the edge-weights. Of course the study of inheritance of \mathcal{F} -convexity is also a first key step to characterize inheritance of convexity in the general case.

Skoda [13] also highlighted a relation between Myerson's restricted game introduced by Myerson [12] and the \mathcal{P}_{\min} -restricted game linking also convexity to \mathcal{F} -convexity. Given an unweighted communication graph G on N , Myerson's restricted game (N, v^M) is defined by $v^M(A) = \sum_{F \in \mathcal{P}_M(A)} v(F)$ for all $A \subseteq N$, where $\mathcal{P}_M(A)$ is the set of connected components of G_A . Any Myerson's restricted game can be obtained as a restriction of the \mathcal{P}_{\min} -restricted game associated with a specific weighted graph. The latter is obtained by the following procedure. Let G be the unweighted graph associated with Myerson's game. A weighted graph G' is obtained by adding to G a dominating vertex s (s is linked to all vertices of G) and by setting edge-weights with only two different values. A complete description of the procedure is given by Skoda ([13], Sect. 4). Moreover, Skoda [13] proved that inheritance of convexity for Myerson's game associated with G is equivalent to inheritance of \mathcal{F} -convexity for the \mathcal{P}_{\min} -restricted game associated with G' .

In the present paper, we investigate the following decision problem: "Given a weighted graph, is there inheritance of \mathcal{F} -convexity for the correspondence \mathcal{P}_{\min} ?" Note that the "No" answer implies the existence of an \mathcal{F} -convex game such that the associated \mathcal{P}_{\min} -restricted game is not \mathcal{F} -convex. Although this problem looks at first sight non-polynomial, a deeper analysis of the graph conditions established by Skoda [13] leads to its tractability. We prove that this decision problem can be solved in polynomial time if we restrict the original games to the family of superadditive games and this constitutes the main result.

Theorem 1.1. *Let $G = (N, E, w)$ be an arbitrary weighted graph. Let us consider the correspondence \mathcal{P}_{\min} . We can verify inheritance of \mathcal{F} -convexity from (N, v) to (N, \bar{v}) for all \mathcal{F} -convex and superadditive game (N, v) in $O(n^6)$ time.*

This result implies that we can also solve in polynomial time the following decision problem associated with Myerson's restricted game: "Given an unweighted graph, is there inheritance of convexity for the correspondence \mathcal{P}_M ?" Of course, directly testing inheritance of convexity (resp. \mathcal{F} -convexity) for the correspondence \mathcal{P}_{\min} is highly non-polynomial. We would have to consider all convex games (N, v) and for each of them, to compute the \mathcal{P}_{\min} -restricted game (N, \bar{v}) and to verify the convexity (resp. \mathcal{F} -convexity) of (N, \bar{v}) , i.e., for all subsets $A, B \in 2^N$ (resp. for all subsets A, B such that A, B , and $A \cap B$ are connected), to verify that $\bar{v}(A \cup B) + \bar{v}(A \cap B) \geq \bar{v}(A) + \bar{v}(B)$. The characterization given by Skoda [13] implies that we can verify inheritance of \mathcal{F} -convexity by checking that some specific distributions of edge-weights are satisfied on all stars, paths, cycles, pans, and adjacent cycles of the graph G . These conditions on edge-weights will be referred to as Star, Path, Cycle, Pan, and Adjacent cycles conditions respectively. A pan graph corresponds to the union of a simple cycle C and an edge $e = \{i, j\}$ satisfying $|V(C) \cap \{i, j\}| = 1$. A tadpole graph corresponds to the union of a simple cycle C and

an elementary path P satisfying $|V(C) \cap V(P)| = 1$ and such that $V(C) \cap V(P)$ corresponds to one of the two end-vertices of P . Pan and tadpole graphs are usually considered in graph theory to extend results obtained on paths and cycles. Skoda [13] highlighted the importance of these graphs for inheritance of \mathcal{F} -convexity as they give simple counter-examples even if the Star, Path and Cycles conditions are satisfied. These counter-examples led to the definition of the Pan Condition². Let us recall that inheritance of convexity for the correspondence \mathcal{P}_M requires the cycle-completeness of the underlying graph G [16]. A graph G is cycle-complete if for any cycle C in G , the subset of vertices of C induces a complete subgraph of G . In other words, all possible chords of a cycle have to exist in G . Cycle-completeness is not a necessary condition for inheritance of convexity for \mathcal{P}_{\min} , but the Pan condition can be seen in some sense as a counterpart of the cycle-completeness condition as it requires the existence of at least one chord in some specific cycles. More surprisingly, the cycle-completeness can be directly related to the Adjacent cycles condition. Specifically, Skoda [13] proved that, if we consider the specific weighted graph G' built from G as previously described, then the Adjacent cycles condition on G' is equivalent to the cycle-completeness of the graph G . Hence, it is sufficient to check the Adjacent cycles condition to verify inheritance of convexity for the Myerson correspondence \mathcal{P}_M . Let us note that the Star, Path, Cycle, and Pan conditions are always satisfied on G' . This last point is only due to the specificity of G' , in particular the restriction of edge-weights to only two values, and checking these conditions is mandatory to decide inheritance of \mathcal{F} -convexity for \mathcal{P}_{\min} with arbitrary graphs. Of course directly checking all these conditions would also lead to a non-polynomial algorithm as the number of paths and cycles can be exponential. We prove in this paper that we only have to take into account a polynomial number of specific paths and cycles and therefore we can decide in $O(n^6)$ time whether there is inheritance of \mathcal{F} -convexity.

Moreover, to establish this result we have to more deeply analyze the relations between the Star, Path, Cycle, Pan and Adjacent cycles conditions characterizing inheritance of \mathcal{F} -convexity. It is of independent interest as we prove that some of them are not completely independent of the others and we highlight their relations which for the sake of simplicity were only implicitly contained in [13]. In particular we observe that the Cycle condition splits into an intermediary cycle condition which is a consequence of the Star and Path conditions and a condition on chords of the cycles. Also a part of the Pan condition is a consequence of Star and Path conditions. We prove that the Cycle condition is a consequence of the Star, Path and Adjacent cycles conditions. Hence inheritance of \mathcal{F} -convexity can be characterized by only four of the previous conditions: the Star, Path, Pan, and Adjacent cycles conditions.

Star condition can easily be checked in polynomial time. Checking the other conditions in polynomial time requires a deeper and careful analysis. In a first fundamental step, we prove that to verify Path condition in polynomial time we only have to consider one minimum-weight spanning tree T in G and to verify Path condition on T , and to test intermediary cycle condition for the fundamental cycles associated with T . Then the main step, which is also the most difficult one, deals with the Adjacent cycles condition which *a priori* involves any pair of adjacent cycles (C, C') . A first contradiction to the Adjacent cycle condition is the existence of two cycles C and C' with two common adjacent edges e_1 and e_2 of non-maximum weight. We prove that if there exist two such cycles then there also exist two specific cycles \tilde{C} and \tilde{C}' with two common edges of non-maximum weight which are associated with shortest paths in some particular subgraphs depending only on the pair (e_1, e_2) . As a result of their features such cycles \tilde{C} and \tilde{C}' can be detected in polynomial time. A similar result holds for the second possible contradiction to the Adjacent cycle condition which corresponds to the existence of two adjacent cycles C and C' with only one common non-maximum weight edge e_1 . Hence, looking for \tilde{C} and \tilde{C}' for any pair of adjacent edges (resp. for any edge) in G , we can detect any contradiction to the Adjacent cycles condition. Therefore the Adjacent cycles condition can be checked in polynomial time. Assuming that the Star, Path and Adjacent cycles condition are satisfied, the Cycle condition is necessarily satisfied. Finally, using the Cycle condition, we establish a new condition equivalent to the Pan condition that we can easily verify in polynomial time using an appropriate shortest path.

² These counter-examples were at first observed with pan graphs and then generalized to tadpole graphs. We have chosen the term Pan condition to make reference to the most simple counter-examples but the Pan condition is actually stated in the more general setting of tadpole graphs.

The article is organized as follows. In Section 2, we give preliminary definitions and results. In particular, we recall the definitions of convexity and \mathcal{F} -convexity. In Section 3, we recall necessary and sufficient conditions established by Skoda [13] on graph edge-weights to have inheritance of \mathcal{F} -convexity for the correspondence \mathcal{P}_{\min} . In Section 4, we present the proof of Theorem 1.1 which is the main result of the present paper.

2. PRELIMINARY DEFINITIONS AND RESULTS

Let N be a given set. We denote by 2^N the set of all subsets of N . A game (N, v) is *zero-normalized* if $v(i) = 0$ for all $i \in N$. A game (N, v) is *superadditive* if, for all $A, B \in 2^N$ such that $A \cap B = \emptyset$, $v(A \cup B) \geq v(A) + v(B)$. For any given subset $\emptyset \neq S \subseteq N$, the unanimity game (N, u_S) is defined by:

$$u_S(A) = \begin{cases} 1 & \text{if } A \supseteq S, \\ 0 & \text{otherwise.} \end{cases} \quad (2.1)$$

We note that (N, u_S) is superadditive for all $S \neq \emptyset$.

Let us consider a game (N, v) . For arbitrary subsets A and B of N , we define the value:

$$\Delta v(A, B) := v(A \cup B) + v(A \cap B) - v(A) - v(B).$$

A game (N, v) is *convex* if its characteristic function v is supermodular, *i.e.*, $\Delta v(A, B) \geq 0$ for all $A, B \in 2^N$. We note that u_S is supermodular for all $S \neq \emptyset$. Let \mathcal{F} be a *weakly union-closed family*³ of subsets of N such that $\emptyset \notin \mathcal{F}$. A game v on 2^N is said to be *\mathcal{F} -convex* if $\Delta v(A, B) \geq 0$, for all $A, B \in \mathcal{F}$ such that $A \cap B \in \mathcal{F}$. If $\mathcal{F} = 2^N \setminus \{\emptyset\}$ and if the game is superadditive, then \mathcal{F} -convexity corresponds to convexity.

For a given graph $G = (N, E)$, we say that a subset $A \subseteq N$ is connected if the induced graph $G_A = (A, E(A))$ is connected. In this paper, \mathcal{F} will be the family of connected subsets of N .

A correspondence \mathcal{P} on N associates to every subset $A \subseteq N$ a partition $\mathcal{P}(A)$ of A and to every game (N, v) , the \mathcal{P} -restricted game (N, \bar{v}) defined by $\bar{v}(A) = \sum_{F \in \mathcal{P}(A)} v(F)$ for all $A \subseteq N$.

We recall the following theorem established by Skoda [13].

Theorem 2.1. *Let $G = (N, E, w)$ be an arbitrary weighted graph, and \mathcal{P} an arbitrary correspondence on N . If for each non-empty subset $S \subseteq N$, (N, \bar{u}_S) is superadditive, then the following claims are equivalent.*

- (1) *For all non-empty subset $S \subseteq N$, the game (N, \bar{u}_S) is \mathcal{F} -convex.*
- (2) *For all $i \in N$ and for all $A, B \in \mathcal{F}$, $A \subseteq B \subseteq N \setminus \{i\}$ such that $A \cup \{i\} \in \mathcal{F}$, we have for all $A' \in \mathcal{P}(A \cup \{i\})$, $\mathcal{P}(A)|_{A'} = \mathcal{P}(B)|_{A'}$.*

We will use the previous theorem with the correspondence \mathcal{P}_{\min} . Let us recall that there is always inheritance of superadditivity for \mathcal{P}_{\min} [11], therefore the condition of Theorem 2.1 on inheritance of superadditivity for unanimity games is trivially satisfied.

Let $G = (N, E, w)$ be a weighted graph. We set $|N| = n$ and $|E| = m$. We assume that all weights are strictly positive and denote the weight of an edge $e_k = \{i, j\}$ in E by w_k or w_{ij} . The adjacency matrix A of G is an $n \times n$ matrix defined by:

$$a_{ij} = \begin{cases} w_{ij} & \text{if } \{i, j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

³ \mathcal{F} is weakly union-closed if $A \cup B \in \mathcal{F}$ for all $A, B \in \mathcal{F}$ such that $A \cap B \neq \emptyset$ [8]. Weakly union-closed families were introduced and analysed by Algaba [1] (see also [2]) and called union stable systems.

3. A CHARACTERIZATION OF INHERITANCE OF \mathcal{F} -CONVEXITY

In this section we recall necessary and sufficient conditions on the weight vector w established by Skoda [13] for the inheritance of \mathcal{F} -convexity from the original communication game (N, v) to the \mathcal{P}_{\min} -restricted game (N, \bar{v}) .

A star S_k corresponds to a tree with one internal vertex and k leaves. We consider a star S_3 with vertices $1, 2, 3, 4$ and edges $e_1 = \{1, 2\}$, $e_2 = \{1, 3\}$ and $e_3 = \{1, 4\}$.

Star condition. For every star of type S_3 of G , the edge-weights w_1, w_2, w_3 satisfy, after renumbering the edges if necessary:

$$w_1 \leq w_2 = w_3.$$

Path condition. For every elementary path $\gamma = (1, e_1, 2, e_2, 3, \dots, m, e_m, m+1)$ in G and for all i, j, k such that $1 \leq i < j < k \leq m$, the edge-weights satisfy:

$$w_j \leq \max(w_i, w_k).$$

Proposition 3.1. Let $G = (N, E, w)$ be a weighted graph. If for every unanimity game (N, u_S) , the \mathcal{P}_{\min} -restricted game (N, \bar{u}_S) is \mathcal{F} -convex, then the Star and Path conditions are satisfied.

For a given cycle $C = \{1, e_1, 2, e_2, \dots, m, e_m, 1\}$ in G with $m \geq 3$, we denote by $E(C)$ the set of edges $\{e_1, e_2, \dots, e_m\}$ of C and by $\hat{E}(C)$ the set composed of $E(C)$ and of the chords of C in G .

Weak cycle condition. For every simple cycle of G , $C = \{1, e_1, 2, e_2, \dots, m, e_m, 1\}$ with $m \geq 3$, the edge-weights satisfy, after renumbering the edges if necessary:

$$w_1 \leq w_2 \leq w_3 = \dots = w_m = M \tag{3.1}$$

where $M = \max_{e \in E(C)} w(e)$.

Intermediary cycle condition.

(1) Weak Cycle condition.

(2) Moreover $w(e) \leq w_2$ for all chord incident to 2, and $w(e) \leq \hat{M} = \max_{e \in \hat{E}(C)} w(e)$ for all chord non incident to 2. Moreover:

- If $w_1 \leq w_2 < \hat{M}$ then $w(e) = \hat{M}$ for all $e \in \hat{E}(C)$ non-incident to 2. If e is a chord incident to 2 then $w_1 \leq w_2 = w(e) < \hat{M}$ or $w(e) < w_1 = w_2 < \hat{M}$.
- If $w_1 < w_2 = \hat{M}$, then $w(e) = \hat{M}$ for all $e \in \hat{E}(C) \setminus \{e_1\}$.

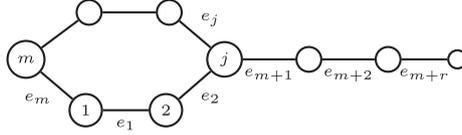
Cycle condition.

(1) Weak Cycle condition.

(2) Moreover $w(e) = w_2$ for all chord incident to 2, and $w(e) = \hat{M}$ for all $e \in \hat{E}(C)$ non incident to 2.

Proposition 3.2. Let $G = (N, E, w)$ be a weighted graph.

1. If G satisfies the Path condition then the Weak Cycle condition is satisfied.
2. If G satisfies the Star and Path conditions then the Intermediary Cycle condition is satisfied.
3. If for every unanimity game (N, u_S) , the \mathcal{P}_{\min} -restricted game (N, \bar{u}_S) is \mathcal{F} -convex, then the Cycle condition is satisfied.

FIGURE 1. Pan formed by the union of C and P .

Weak Pan condition. For all connected subgraphs corresponding to the union of a simple cycle $C = \{e_1, e_2, \dots, e_m\}$ with $m \geq 3$, and an elementary path P such that there is an edge e in P with $w(e) \leq \min_{1 \leq k \leq m} w_k$ and $|V(C) \cap V(P)| = 1$, the edge-weights satisfy:

- (a) either $w_1 = w_2 = w_3 = \dots = w_m = \hat{M}$,
 (b) or $w_1 = w_2 < w_3 = \dots = w_m = \hat{M}$,

where $\hat{M} = \max_{e \in \hat{E}(C)} w(e)$. In this last case $V(C) \cap V(P) = \{2\}$.

Pan condition.

- (1) Weak Pan condition.
 (2) If Claim (b) of the Weak Pan condition is satisfied and if moreover $w(e) < w_1$ then $\{1, 3\}$ is a maximum weight chord of C .

Proposition 3.3. Let $G = (N, E, w)$ be a weighted graph.

- (1) If G satisfies the Star, and Path conditions then the Weak Pan condition is satisfied.
 (2) If for every unanimity game (N, u_S) , the \mathcal{P}_{\min} -restricted game (N, \bar{u}_S) is \mathcal{F} -convex, then the Pan condition is satisfied.

We recall the proof of Proposition 3.3 as we will need it to prove a new formulation of the Pan condition in Section 4.

Proof. (1) Let us consider $C = \{1, e_1, 2, e_2, 3, \dots, m, e_m, 1\}$, and $P = \{j, e_{m+1}, m+1, e_{m+2}, m+2, \dots, e_{m+r}, m+r\}$ with $j \in \{1, \dots, m\}$, as represented in Figure 1. We can assume w.l.o.g. that $e = e_{m+r}$ (restricting P if necessary) and that $w_{m+j} > w_{m+r} = w(e)$ for all $1 \leq j \leq r-1$.

Applying Claim 2 of Proposition 3.2 to the cycle C , we have after renumbering the edges if necessary:

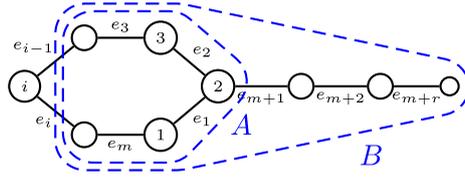
$$w_1 \leq w_2 \leq w_3 = \dots = w_m = \hat{M}. \quad (3.2)$$

Let us first assume $3 \leq j \leq m$. Applying Proposition 3.1 to the path $\{2, e_1, 1, e_m, m, \dots, j+1, e_j, j, e_{m+1}, m+1, \dots, m+r-1, e_{m+r}, m+r\}$, we have $w_j \leq \max(w_1, w(e)) = w_1$. Then (3.2) implies (a).

Let us now assume $j \in \{1, 2\}$. If $r = 1$ then $w_{m+1} = w(e) \leq w_1$. Otherwise, applying Proposition 3.1 to the path $\{e_1, e_{m+1}, \dots, e_{m+r}\}$, we have $w_{m+1} \leq \max(w_1, w(e)) = w_1$. If $j = 1$, Proposition 3.1 applied to the star defined by $\{e_1, e_m, e_{m+1}\}$, implies $w_{m+1} \leq w_1 = w_m$. Hence (3.2) still implies (a). If $j = 2$, Proposition 3.1 applied to the star defined by $\{e_1, e_2, e_{m+1}\}$, implies $w_{m+1} \leq w_1 = w_2$. If $w_1 = w_2 = \hat{M}$ then (a) is satisfied. Otherwise we have $w_1 = w_2 < \hat{M}$ and (b) is satisfied. Hence the Weak Pan condition is satisfied.

(2) Now if Claim (b) of the Weak Pan condition is satisfied, let us assume by contradiction that $\{1, 3\} \notin E(C)$. Claim 3 of Proposition 3.2 implies $w(e) = \hat{M}$ (resp. $w(e) = w_2$) for any chord e of C non incident (resp. incident) to 2. Therefore we can assume w.l.o.g. that C has no maximum weight chord (otherwise we can replace C by a smaller cycle which still contains the vertices 1, 2, 3). Let us consider $i \in V(C) \setminus \{1, 2, 3\}$, $A = V(C) \setminus \{i\}$ and $B = A \cup V(P)$ as represented in Figure 2.

Then $\mathcal{P}_{\min}(A) = \{\{2\}, \{3, 4, \dots, i-1\}, \{i+1, \dots, m, 1\}\}$, $\mathcal{P}_{\min}(A \cup \{i\}) = \{A \cup \{i\} \setminus \{2\}, \{2\}\}$, and $\mathcal{P}_{\min}(B) = \{B \setminus \{m+r\}, \{m+r\}\}$ or $\mathcal{P}_{\min}(B) = \{B\}$ (this last case can occur if there exists an edge e' in G_B with


 FIGURE 2. $w_{m+r} < w_1 = w_2 < w_3 = \dots = w_m = M$.

$w(e') < w(e)$ or with $m+r$ as an end-vertex and $w(e') > w(e)$). Therefore $A' := A \cup \{i\} \setminus \{2\} \in \mathcal{P}_{\min}(A \cup \{i\})$, but $\mathcal{P}_{\min}(B)_{|A'} = \{A \setminus \{2\}\} \neq \mathcal{P}_{\min}(A)_{|A'}$ and it contradicts Theorem 2.1. \square

Lemma 3.4. *Let $G = (N, E, w)$ be a weighted graph satisfying the Star and Path conditions. Then for all pairs (C, C') of adjacent simple cycles in G , we have:*

$$\hat{M} = \max_{e \in \hat{E}(C)} w(e) = \max_{e \in E(C)} w(e) = \max_{e \in E(C')} w(e) = \max_{e \in \hat{E}(C')} w(e) = \hat{M}'. \quad (3.3)$$

Proof. We first consider $M = \max_{e \in E(C)} w(e)$ and $M' = \max_{e \in E(C')} w(e)$. Let us consider two adjacent cycles C and C' with $M < M'$. There is at least one edge e_1 common to C and C' . Then we have $w_1 \leq M < M'$ and therefore e_1 is a non-maximum weight edge in C' . By Claim 1 of Proposition 3.2 the Weak Cycle condition is satisfied. It implies that there are at most two non-maximum weight edges in C' . Therefore there exists an edge e'_2 in C' adjacent to e_1 with $w'_2 = M'$. As $M' > M$, e'_2 is not an edge of C . Let e_2 be the edge of C adjacent to e_1 and e'_2 . Then we have $w_2 \leq M < M'$ but it contradicts the Star condition applied to $\{e_1, e_2, e'_2\}$ (two edge-weights are strictly smaller than w'_2). Therefore $M = M'$. Finally by Claim 2 of Proposition 3.2 the Intermediary Cycle condition is satisfied and we have $\hat{M} = M = M' = \hat{M}'$. \square

Adjacent cycles condition. *For all pairs (C, C') of adjacent simple cycles in G such that:*

- (a) $V(C) \setminus V(C') \neq \emptyset$ and $V(C') \setminus V(C) \neq \emptyset$,
- (b) C has at most one non-maximum weight chord,
- (c) C and C' have no maximum weight chord,
- (d) C and C' have no common chord,

then C and C' cannot have two common non-maximum weight edges. Moreover C and C' have a unique common non-maximum weight edge e_1 if and only if there are non-maximum weight edges $e_2 \in E(C) \setminus E(C')$ and $e'_2 \in E(C') \setminus E(C)$ such that e_1, e_2, e'_2 are adjacent and:

- $w_1 = w_2 = w'_2$ if $|E(C)| \geq 4$ and $|E(C')| \geq 4$.
- $w_1 = w_2 \geq w'_2$ or $w_1 = w'_2 \geq w_2$ if $|E(C)| = 3$ or $|E(C')| = 3$.

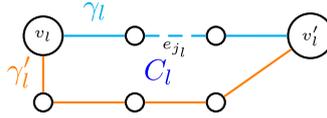
Proposition 3.5. *Let $G = (N, E, w)$ be a weighted graph. If for every unanimity game (N, u_S) the \mathcal{P}_{\min} -restricted game (N, \bar{u}_S) is \mathcal{F} -convex, then the Adjacent Cycles condition is satisfied.*

Finally the following characterization of inheritance of \mathcal{F} -convexity was established by Skoda [13].

Theorem 3.6. *Let $G = (N, E, w)$ be a weighted graph. For every superadditive and \mathcal{F} -convex game (N, v) , the \mathcal{P}_{\min} -restricted game (N, \bar{v}) is \mathcal{F} -convex if and only if the Path, Star, Cycle, Pan, and Adjacent cycles conditions are satisfied.*

4. PROOF OF THEOREM 1.1

We now use the characterization given in Theorem 3.6 to prove Theorem 1.1. We will prove that to verify Star, Path, Cycle, Pan, and Adjacent cycles conditions we only have to consider a polynomial number of paths, stars,

FIGURE 3. Paths γ_l and γ'_l in C_l .

pans and cycles. Therefore, given a weighted graph, we can decide in polynomial time if there is inheritance of \mathcal{F} -convexity from underlying games to \mathcal{P}_{\min} -restricted games. We assume that graphs are represented by their adjacency matrices.

Proposition 4.1. *Let $G = (N, E, w)$ be a weighted graph. Star condition can be verified in $O(n^2)$ time.*

Proof. Given a node i in N , we denote by E_i the set of edges incident to i . If $|E_i| \geq 3$, then we select two edges e_1, e_2 in E_i and replace E_i by $E_i \setminus \{e_1, e_2\}$. If $w_1 < w_2$ we set $\min = w_1$, $\max = w_2$, otherwise $\min = w_2$, $\max = w_1$. Then we apply the following procedure. While $E_i \neq \emptyset$ we select an edge $e \in E_i$. If $\min = \max$ then if $w(e) \leq \max$ we set $\min = w(e)$, otherwise we stop. Otherwise we have $\min < \max$ and if $w(e) \neq \max$ we stop. Otherwise we replace E_i by $E_i \setminus \{e\}$. If the procedure stops with $E_i \neq \emptyset$, then there is a contradiction to the Star condition. Otherwise the Star condition is satisfied for the star centered in i . Hence Star condition can be verified for the star centered in i in $O(n)$ time. As we have to repeat this procedure for all nodes in N we can verify Star condition in $O(n^2)$ time. \square

We now assume that the Star condition is satisfied by G .

Adding an edge to a spanning tree creates a unique cycle, called a *fundamental cycle*. Let T be a minimum-weight spanning tree of G and $\gamma = \{1, e_1, 2, e_2, \dots, m, e_m, m+1\}$ be an elementary path of G . Let $e_{j_1}, e_{j_2}, \dots, e_{j_k}$, with $1 \leq j_1 < j_2 < \dots < j_k \leq m$, be the edges in $E(\gamma) \setminus E(T)$. For every $e_{j_l} \in E(\gamma) \setminus E(T)$, we denote by C_l the fundamental cycle in G associated with T and e_{j_l} .

Remark 4.2. Of course two fundamental cycles associated with a spanning tree can be adjacent.

We suppose that G satisfies the Star condition and that every fundamental cycle associated with T satisfies the Weak Cycle condition. (Note that we only have to check for each fundamental cycle that there are at most two edges of non-maximum weight and that these edges are adjacent. And there are exactly $m - n + 1$ fundamental cycles associated with T .) For each fundamental cycle C_l , we denote by M_l its maximum edge-weight. As T is a minimum-weight spanning tree we have $w_{j_l} = M_l \geq w(e)$ for all $e \in E(C_l)$. Let γ_l be the restriction of γ to C_l and v_l, v'_l its end-vertices. Let $\gamma'_l := C_l \setminus \gamma_l$ be the other path in C_l linking v_l and v'_l as represented in Figure 3.

Note that γ'_l contains at least one edge otherwise $v_l = v'_l$ and γ is not an elementary path. We denote by $\gamma_{l,l+1}$ the part of γ linking v'_l to v_{l+1} . $\gamma_{l,l+1}$ can be reduced to one vertex (if $v'_l = v_{l+1}$). We denote by $\gamma_{0,1}$ (resp. $\gamma_{k,m+1}$) the part of γ linking 1 to v_1 (resp. v'_k to $m+1$). $\gamma_{0,1}$ (resp. $\gamma_{k,m+1}$) can also be reduced to one vertex. Then $\tilde{\gamma} = \gamma_{0,1} \cup \gamma'_1 \cup \gamma_{1,2} \cup \gamma'_2 \cup \dots \cup \gamma_{k-1,k} \cup \gamma'_k \cup \gamma_{k,m+1}$ corresponds to a path in T linking 1 to $m+1$. Let us observe that if two successive fundamental cycles C_l, C_{l+1} are adjacent then $\gamma'_l \cup \gamma_{l,l+1} \cup \gamma'_{l+1}$ is not a simple path ($\gamma_{l,l+1}$ is reduced to one vertex but γ'_l and γ'_{l+1} have common edges). In this case we delete the common edges to get a simple path (in fact we get a unique elementary path in T) as represented in Figures 4a and 4b. Keeping the same notations for simplicity, we now consider that $\tilde{\gamma}$ is an elementary path in T linking 1 to $m+1$.

Lemma 4.3. *If C_l and C_{l+1} are adjacent then $M_l = M_{l+1}$.*

Proof. As the Star condition is satisfied and as the fundamental cycles associated with T satisfy the Weak Cycle condition, the result is an immediate consequence of Lemma 3.4. \square

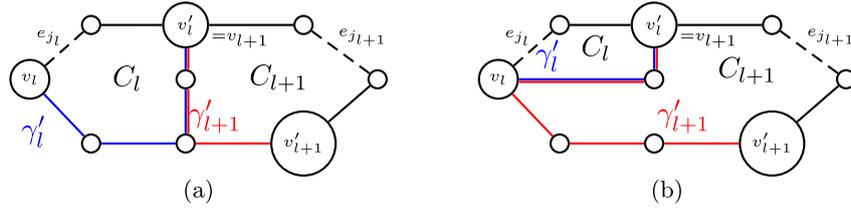


FIGURE 4. We replace $\gamma'_l \cup \gamma_{l,l+1} \cup \gamma'_{l+1}$ by $(\gamma'_l \cup \gamma'_{l+1}) \setminus (\gamma'_l \cap \gamma'_{l+1})$. We can have $\gamma'_l \subset \gamma'_{l+1}$.

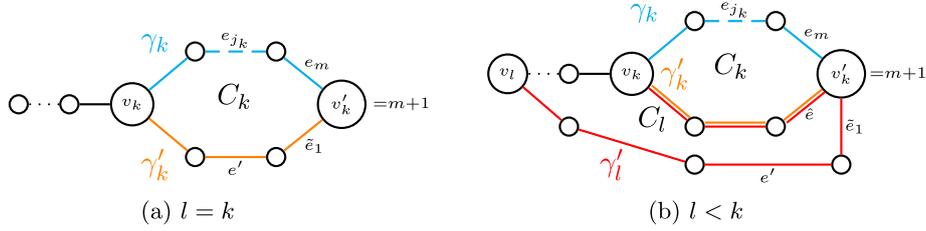


FIGURE 5. \tilde{e}_1 in γ'_l .

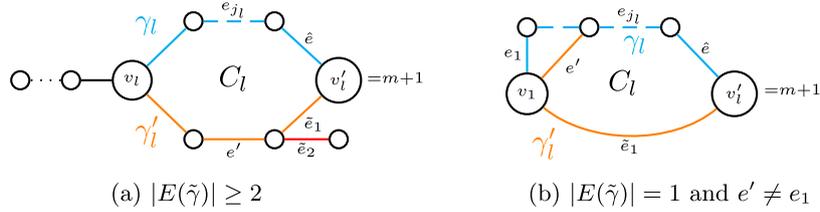


FIGURE 6. e' in $E(C_l)$ adjacent to \tilde{e}_1 .

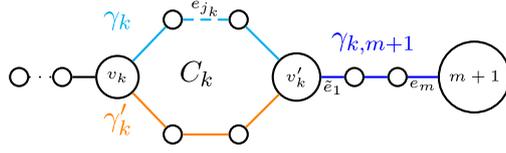
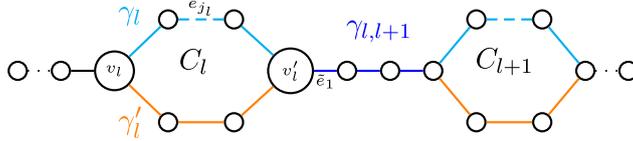
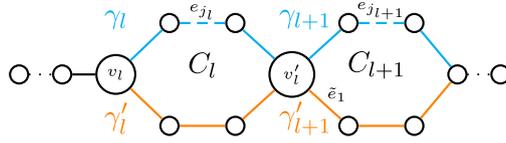
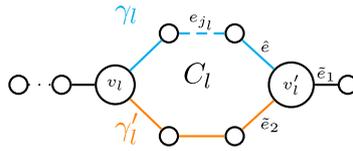
Lemma 4.4. *Let $G = (N, E, w)$ be a weighted graph satisfying the Star condition. Let us consider a minimum-weight spanning tree T and an elementary path $\gamma = \{1, e_1, 2, e_2, \dots, m, e_m, m+1\}$ in G . Let us assume that every fundamental cycle associated with T satisfies the Weak Cycle condition. Then for every fundamental cycle C_l associated with T and e_{j_l} in $E(\gamma) \setminus E(T)$, with $1 \leq l \leq k$, either $M_l = w_1$ or w_m or there exists an edge $e \in E(\tilde{\gamma})$ such that $M_l = w(e)$.*

Proof. Let us recall that $w_{j_l} = M_l$.

Case 1: Let us consider C_k and let us assume $v'_k = m+1$, i.e., C_k incident to $m+1$. Then $e_m \in E(\gamma_k)$. If $e_{j_k} = e_m$ or if $w_m = M_k$ the result is satisfied. Hence we can assume $e_{j_k} \neq e_m$ and $w_m < M_k$. Let \tilde{e}_1 be the (last) edge of $\tilde{\gamma}$ incident to $m+1$.

By definition of $\tilde{\gamma}$, \tilde{e}_1 belongs to a path γ'_l in C_l with $1 \leq l \leq k$ as represented in Figure 5a for $l = k$ and Figure 5b for $l < k$. Note that if $l < k$ then $C_l, C_{l+1}, \dots, C_{k-1}, C_k$ are adjacent and by Lemma 4.3 we have $M_l = M_k$. If $w(\tilde{e}_1) = M_k$ we take $e = \tilde{e}_1$. Otherwise we have $w(\tilde{e}_1) < M_k$. Let \hat{e} be the edge in $E(C_l) \setminus \{\tilde{e}_1\}$ incident to v'_k . If $l = k$ then $\hat{e} = e_m$ and we have $w(\hat{e}) < M_k$. Otherwise we apply Star condition to $\{e_m, \hat{e}, \tilde{e}_1\}$ and $w(\hat{e}) < M_k$ is still satisfied. Then the Weak Cycle condition applied to C_l implies that there is an edge e' in $E(C_l)$ adjacent to \tilde{e}_1 with $w(e') = M_k$. If e' belongs to $\tilde{\gamma}$ we take $e = e'$. Otherwise, if $|E(\tilde{\gamma})| \geq 2$ there is an edge \tilde{e}_2 in $\tilde{\gamma}$ adjacent to e' and \tilde{e}_1 as represented in Figure 6a.

Star condition applied to $\{e', \tilde{e}_1, \tilde{e}_2\}$ implies $w(\tilde{e}_2) = M_k$ and we take $e = \tilde{e}_2$. Finally if $|E(\tilde{\gamma})| = 1$ then $v_l = v_1 = 1$. Hence either $e' = e_1$ and the result is satisfied or e' is adjacent to e_1 as represented in Figure 6b.

FIGURE 7. \tilde{e}_1 in $\gamma_{k,m+1}$.FIGURE 8. $v'_l \neq v_{l+1}$, \tilde{e}_1 in $\gamma_{l,l+1}$.FIGURE 9. $v'_l = v_{l+1}$, \tilde{e}_1 in γ'_{l+1} .FIGURE 10. \hat{e} in γ_l , \tilde{e}_2 in γ'_l .

Then Star condition applied to $\{e_1, e', \tilde{e}_1\}$ implies $w(e') = w_1$. Let us observe that we can have $e' = e_{j_l}$, and hence Star condition has to be satisfied for G (and not only for T) in the assumptions.

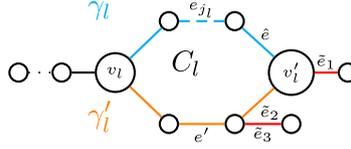
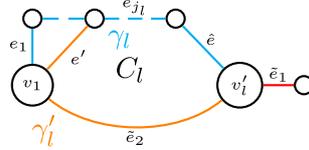
Case 2: Let us now consider C_l with $l \leq k$ and $v'_l \neq m+1$. Then there exists an edge $\tilde{e}_1 \in E(\tilde{\gamma}) \setminus E(C_l)$ incident to v'_l . If $l = k$, then \tilde{e}_1 is the edge of $\gamma_{k,m+1}$ incident to v'_k as represented in Figure 7.

If $l < k$, we can assume that C_l and C_{l+1} are not adjacent. Otherwise Lemma 4.3 implies $M_l = M_{l+1}$ and we can replace C_l by C_{l+1} . If $v'_l \neq v_{l+1}$, \tilde{e}_1 is the edge of $\gamma_{l,l+1}$ incident to v'_l as represented in Figure 8.

If $v'_l = v_{l+1}$, \tilde{e}_1 is the first edge of γ'_{l+1} incident to v'_l as represented in Figure 9.

Let us denote by $\tilde{\gamma}_{1,l}$ the part of $\tilde{\gamma}$ linking 1 to v'_l . Let \tilde{e}_2 be the edge of $\tilde{\gamma}_{1,l}$ incident to v'_l as represented in Figure 10 with \tilde{e}_2 in γ'_l . If $\tilde{e}_2 \notin \gamma'_l$ then there exists a fundamental cycle C_r for some r , with $1 \leq r \leq l$, such that $\tilde{e}_2 \in \gamma'_r$ and $C_r, C_{r+1}, \dots, C_{l-1}, C_l$ are adjacent. By Lemma 4.3 we have $M_r = M_l$. Therefore we can suppose $\tilde{e}_2 \in \gamma'_l$ replacing C_l by C_r if necessary. Note that in this last case C_l may be adjacent to C_{l+1} but $\tilde{e}_1 \notin E(C_l)$. Let \hat{e} be the edge of γ_l incident to v'_l as represented in Figure 10.

If $w(\tilde{e}_2) = M_l$, we take $e = \tilde{e}_2$. Otherwise we have $w(\tilde{e}_2) < M_l$. If $w(\hat{e}) = M_l$, Star condition applied to $\{\hat{e}, \tilde{e}_1, \tilde{e}_2\}$ implies $w(\tilde{e}_1) = M_l$ and we can take $e = \tilde{e}_1$. If now $w(\hat{e})$ and $w(\tilde{e}_2) < M_l$, then the Weak Cycle condition applied to C_l implies that there is an edge e' in $E(C_l)$ adjacent to \tilde{e}_2 with $w(e') = M_l$. If e' belongs to $\tilde{\gamma}$ we take $e = e'$. Otherwise, if $|E(\tilde{\gamma}_{1,l})| \geq 2$ there is an edge \tilde{e}_3 in $\tilde{\gamma}$ adjacent to e' and \tilde{e}_2 as represented in Figure 11.


 FIGURE 11. $|E(\tilde{\gamma}_{1,l})| \geq 2$, \tilde{e}_3 in $\tilde{\gamma}$ adjacent to e' and \tilde{e}_2 .

 FIGURE 12. $|E(\tilde{\gamma}_{1,l})| = 1$, $v_l = v_1 = 1$ and $e' \neq e_1$.

Star condition applied to $\{e', \tilde{e}_2, \tilde{e}_3\}$ implies $w(\tilde{e}_3) = M_l$ and we take $e = \tilde{e}_3$. Finally if $|E(\tilde{\gamma}_{1,l})| = 1$ then $v_l = v_1 = 1$. Then either $e' = e_1$ and the result is satisfied or e' is adjacent to e_1 as represented in Figure 12.

Then as $w(\tilde{e}_2) < w(e')$, Star condition applied to $\{e', e_1, \tilde{e}_2\}$ implies $w(e') = w_1$. Note that the situation is similar to Case 1 with $l < k$ replacing \tilde{e}_1 by \tilde{e}_2 . \square

Using for instance Prim's algorithm, we can find a minimum-weight spanning tree T in $O(n^2)$ time. The following proposition shows that the Path condition is satisfied for G if it is satisfied for T and if all fundamental cycles associated with T satisfy the Weak Cycle condition, and therefore can be verified in polynomial time.

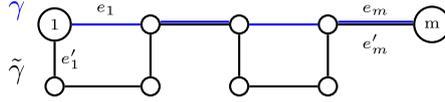
Proposition 4.5. *Let $G = (N, E, w)$ be a weighted connected graph satisfying the Star condition. Let T be a minimum-weight spanning tree of G . If T satisfies the Path condition and if each fundamental cycle associated with T satisfies the Weak Cycle condition, then G also satisfies the Path condition. Therefore the Path condition can be verified in $O(n^3)$ time.*

Proof. Let $\gamma = \{1, e_1, 2, e_2, \dots, m, e_m, m+1\}$ be a path of G . We have to prove that for every j , $2 \leq j \leq m-1$, we have $w_j \leq \max(w_1, w_m)$.

Let us consider $e \in E(\gamma) \setminus E(T)$ and C the fundamental cycle associated with T and e . Let M be the maximum edge-weight in C . We necessarily have $w(e) = M$ otherwise T is not a minimum-weight spanning tree. We have to prove:

$$M \leq \max(w_1, w_m). \quad (4.1)$$

Let $\tilde{\gamma}$ be the path in T related to γ linking 1 to $m+1$ defined previous pages. Lemma 4.4 implies that either $M = w_1$ or w_m and then (4.1) is obviously satisfied or there exists $\tilde{e} \in E(\tilde{\gamma})$ such that $w(\tilde{e}) = M$. Let e'_1 (resp. e'_m) be the edge of $\tilde{\gamma}$ incident to 1 (resp. $m+1$). Path condition applied to $\tilde{\gamma}$ implies $w(\tilde{e}) \leq \max(w'_1, w'_m)$. If e_1 (resp. e_m) is not an edge of T , then it forms a fundamental cycle containing e'_1 (resp. e'_m) as represented in Figure 13 and as T is a minimum-weight spanning tree we have $w'_1 \leq w_1$ (resp. $w'_m \leq w_m$). If e_1 (resp. e_m) is an edge of T , then Path condition applied to $e_1 \cup \tilde{\gamma}$ (resp. $e_m \cup \tilde{\gamma}$) implies $w(\tilde{e}) \leq \max(w_1, w'_m)$ (resp. $w(\tilde{e}) \leq \max(w'_1, w_m)$). If both e_1 and e_m are edges of T , then Path condition applied to $e_1 \cup \tilde{\gamma} \cup e_m$ implies $w(\tilde{e}) \leq \max(w_1, w_m)$. Hence in every case, $w(\tilde{e}) \leq \max(w_1, w_m)$ and therefore (4.1) is satisfied. As M is the maximum edge-weight in C , we get $w(e) \leq \max(w_1, w_m)$ for all $e \in E(C)$. Therefore we also have $w(e) \leq \max(w_1, w_m)$ for all e in a fundamental cycle associated with T . Let e be a remaining edge in $E(\gamma) \cap E(T)$. e is necessarily in $\tilde{\gamma}$ (more precisely in one of the subpaths $\gamma_{0,1}, \gamma_{1,2}, \dots, \gamma_{k,m+1}$) and therefore satisfies $w(e) \leq \max(w_1, w_m)$.

FIGURE 13. γ and $\tilde{\gamma}$, $e_1 \notin E(T)$, $e_m \in E(T)$.

Let us now analyze the complexity. Prim's algorithm provides a minimum-weight spanning tree in time $O(n^2)$.

Then we verify Path condition for T . For each vertex $v \in V$, we build the arborescence in T rooted in v . It can be done using a breadth first search in $O(n^2)$. During the search we can verify the Path condition. Indeed we only have to compare the weight of an edge added to the arborescence to the weight of its predecessor in the arborescence. If there is an increase of edge-weights followed by a decrease then it contradicts the Path condition (We can assign a label $\{+, -\}$ to each vertex entering the arborescence, and therefore associated with the entering edge, corresponding to an increase or decrease, and we compare it to the label of the preceding node). Hence we can verify the Path condition on T in $O(n^3)$ time.

For a given fundamental cycle associated with T , the Cycle condition can be verified in $O(n)$ time. Indeed, as T contains at most $n - 1$ edges, a fundamental cycle contains at most n edges. Then a minimum-weight edge can be found after at most $n - 1$ comparisons. A minimum-weight edge among the $n - 1$ remaining edges can be found using at most $n - 2$ comparisons. Actually we only have to consider the two edges adjacent to the minimum weight edge previously found. Finally we only have to verify the equality of edge-weights for the remaining edges. It requires at most $n - 3$ comparisons. There exists $m - (n - 1)$ edges in $E \setminus E(T)$ and therefore $m - (n - 1)$ fundamental cycles associated with T . Hence we can verify the Cycle condition for all fundamental cycles associated with T in $O(nm)$.

Finally Path condition for G can be verified in $O(n^2 + n^3 + nm)$ time and therefore in $O(n^3)$ time (as $m \leq \frac{n(n-1)}{2}$). \square

We now prove that the Adjacent cycles condition can be verified in $O(n^6)$ time. We first establish that we only need to verify the condition on specific cycles built with shortest paths.

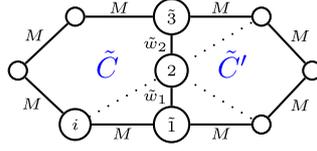
Lemma 4.6. *Let $G = (N, E, w)$ be a weighted graph satisfying the Star and Path (and therefore Weak and Intermediary Cycle) conditions. Let $e_1 = \{1, 2\}$ and $e_2 = \{2, 3\}$ be two adjacent edges of G . Let us define $\tilde{E} := \{e \in E; w(e) > \max(w_1, w_2)\}$ and $\tilde{G} := (N \setminus \{2\}, \tilde{E}, w|_{\tilde{E}})$. Let γ be a shortest path in \tilde{G} , if it exists, linking 1 and 3. Let us consider the following claims:*

- (1) *There exist two simple cycles C and C' in G satisfying conditions a, b, c, d, of the Adjacent Cycles condition, and such that:*
 - (e) *e_1 and e_2 are two common non-maximum weight edges of C and C' .*
- (2) *There exist two simple cycles \tilde{C} and \tilde{C}' satisfying conditions a, b, c, d, of the Adjacent Cycles condition, and such that:*
 - (f) *\tilde{C} and \tilde{C}' have two common non-maximum weight edges $\tilde{e}_1 = \{\tilde{1}, 2\}$, $\tilde{e}_2 = \{2, \tilde{3}\}$.*
 - (g) *$\tilde{C} \setminus \{2\}$ corresponds to a shortest path $\tilde{\gamma}$ in \tilde{G} linking $\tilde{1}$ and $\tilde{3}$. $\tilde{\gamma}$ is a subpath of γ and for some $i \in V(\tilde{\gamma}) \setminus \{\tilde{1}, \tilde{3}\}$, $\tilde{C}' \setminus \{2\}$ corresponds to a shortest path γ' in $\tilde{G}_{N \setminus \{2, i\}}$ linking $\tilde{1}$ and $\tilde{3}$.*

Then Claim 1 implies Claim 2. Moreover for given adjacent edges e_1 and e_2 of G , Claim 2 can be verified in $O(n^3)$ time.

Remark 4.7. It can happen that $\tilde{\gamma} = \gamma$ and then $1 = \tilde{1}$ and $3 = \tilde{3}$.

Remark 4.8. Let us observe that $|V(C)| \geq 4$ and $|V(C')| \geq 4$, otherwise by the Weak Cycle condition C or C' would have a maximum weight chord. Let us also observe that by the Weak Cycle condition and Lemma 3.4, all edges of γ and γ' have the same weight M .


 FIGURE 14. $i \in \tilde{\gamma} = \tilde{C} \setminus \{2\}$ and $i \notin \gamma' = \tilde{C}' \setminus \{2\}$.

Proof of Lemma 4.6. We assume w.l.o.g. $w_1 \leq w_2$. Let us assume that Claim 1 is satisfied. Lemma 3.4 implies $\max_{e \in \hat{E}(C)} w(e) = \max_{e \in \hat{E}(C')} w(e) = M$. Let γ be a shortest path in \tilde{G} linking 1 to 3. Then $\hat{C} = \{2, e_1, 1\} \cup \gamma \cup \{3, e_2, 2\}$ is a simple cycle in G adjacent to C or $\hat{C} = C$. The Weak Cycle condition and Lemma 3.4 imply that e_1 and e_2 are edges of non-maximum weight in \hat{C} and all edges of γ have weight $M = \max_{e \in \hat{E}(C)} w(e)$. As $M > w_2 \geq w_1$, the Intermediary Cycle condition implies $w(e) = M$ for all $e \in \hat{E}(\hat{C})$ non incident to 2 and $w(e) = w_2$ or $w(e) < w_1 = w_2$ for any chord incident to 2. Hence \hat{C} has no chord of maximum weight M otherwise γ is not a shortest path.

Case 1: Let us assume that \hat{C} has no chord. Then we take $\tilde{C} = \hat{C}$. If $V(\tilde{C}) \setminus V(C') \neq \emptyset$ we select a node i in $V(\tilde{C}) \setminus V(C')$. Otherwise we have $V(\tilde{C}) \subseteq V(C')$. $V(\tilde{C}) \subset V(C')$ would imply that C' has a chord of maximum weight M , a contradiction. Hence $V(\tilde{C}) = V(C')$ and in this case we have $V(\tilde{C}) \setminus V(C) = V(C') \setminus V(C) \neq \emptyset$. Then we select a node i in $V(\tilde{C}) \setminus V(C)$ (we can interchange C and C'). By construction of i there exists a path in $\tilde{G}_{N \setminus \{2, i\}}$ linking 1 to 3 (i.e., a path in C' or C). Therefore there exists a shortest path γ' in $\tilde{G}_{N \setminus \{2, i\}}$ linking 1 to 3. Then the cycle $\tilde{C}' = \{2, e_1, 1\} \cup \gamma' \cup \{3, e_2, 2\}$ is convenient. We have $V(\tilde{C}) \setminus V(\tilde{C}') \neq \emptyset$ as $i \in V(\tilde{C}) \setminus V(\tilde{C}')$. If $V(\tilde{C}') \setminus V(\tilde{C}) = \emptyset$ then $V(\tilde{C}') \subseteq V(\tilde{C})$. $V(\tilde{C}') = V(\tilde{C})$ is not possible as $i \in V(\tilde{C}) \setminus V(\tilde{C}')$. Then $V(\tilde{C}') \subset V(\tilde{C})$ and γ' is a path strictly shorter than γ linking 1 and 3 in \tilde{G} , a contradiction. \tilde{C} and \tilde{C}' have no maximum weight chord because γ and γ' are shortest path linking 1 and 3 in respectively \tilde{G} and $\tilde{G}_{N \setminus \{2, i\}}$. \tilde{C} has no chord. \tilde{C} and \tilde{C}' satisfy conditions a to d, 2f and 2g.

If e_1 and e_2 are given, we can obtain a shortest path γ , if it exists, using a Breadth First Search algorithm in $O(n^2)$ time. Then to decide if \hat{C} has a chord or not, take $O(n)$ time. Indeed \hat{C} cannot have a maximum weight chord otherwise γ is not a shortest path, and any non-maximum weight chord is incident to 2. Therefore we only have to check in G if there exists a neighbor of 2 in $\gamma \setminus \{1, 3\}$. In the worst case, we have to consider $n - 3$ vertices. For a given node $i \in \gamma \setminus \{1, 3\}$ checking the existence of $\{2, i\}$ in the adjacency matrix is in $O(1)$ time. Therefore we can check the existence of a chord in $O(n)$ time. If \hat{C} is chordless then for a given $i \in V(\gamma)$ we look for a shortest path γ' in $\tilde{G}_{N \setminus \{2, i\}}$ linking 1 and 3 using the BFS algorithm in $O(n^2)$ time. In the worst case we repeat this for all $i \in V(\gamma)$. Hence in this case (\hat{C} without chord) we need $O(n^3)$ time to decide if Claim 2 is satisfied.

Case 2: Let us now assume that \hat{C} has a chord \hat{e} . Then it is necessarily a non-maximum weight chord incident to 2 with $w(\hat{e}) \leq w_2$ (by the Intermediary Cycle condition). We choose arbitrarily the end-vertex i of any chord $\hat{e}_0 = \{2, i\}$ of \hat{C} . We select two other edges in $\hat{E}(\hat{C})$, $\hat{e}_1 = \{2, k\}$ and $\hat{e}_2 = \{2, j\}$, with $3 \leq j < i < k$, which are as close as possible from \hat{e}_0 , i.e., j is the maximum possible index for such an edge \hat{e}_2 and k is the minimum possible index for such an edge \hat{e}_1 . Hence if $j = 3$ (resp. $k = 1$) then $\hat{e}_2 = e_2$ (resp. $\hat{e}_1 = e_1$). We consider the cycle $\tilde{C} = \{k, \hat{e}_1, 2, \hat{e}_2, j, e_j, j + 1, \dots, e_{i-1}, i, e_i, \dots, e_{k-1}, k\}$ as represented in Figure 15.

By renumbering as follows, $\tilde{1} = k$, $\tilde{e}_1 = \hat{e}_1$, $\tilde{2} = 2$, $\tilde{e}_2 = \hat{e}_2$, $\tilde{3} = j$, $\tilde{e}_{\tilde{i}-1} = e_{i-1}$, $\tilde{i} = i$, $\tilde{e}_{\tilde{i}} = e_i, \dots, \tilde{e}_m = e_{k-1}$, $k = \tilde{1}$ we get $\tilde{C} = \{\tilde{1}, \tilde{e}_1, \tilde{2}, \tilde{e}_2, \tilde{3}, \dots, \tilde{e}_{\tilde{i}-1}, \tilde{i}, \tilde{e}_{\tilde{i}}, \dots, \tilde{e}_m, \tilde{1}\}$. \tilde{C} has only one non-maximum weight chord $\tilde{e}_0 = \{2, \tilde{i}\}$ corresponding to $\hat{e}_0 = \{2, i\}$ and \tilde{e}_1, \tilde{e}_2 are the two non-maximum weight edges of \tilde{C} . The restricted path $\tilde{\gamma} = \{\tilde{3}, \tilde{e}_3, \dots, \tilde{e}_{\tilde{i}-1}, \tilde{i}, \tilde{e}_{\tilde{i}}, \dots, \tilde{e}_m, \tilde{1}\}$ is a shortest path linking $\tilde{3}$ to $\tilde{1}$ in \tilde{G} as γ is a shortest path linking 1 to 3 in \tilde{G} . As C and C' have no common chord, $\tilde{e}_0 = \{2, \tilde{i}\}$ cannot be a common chord of C and C' . We can assume w.l.o.g. that $\tilde{e}_0 = \{2, \tilde{i}\}$ is not a chord of C' . Let γ_1 (resp. γ_2) be the part of γ linking 1 and k

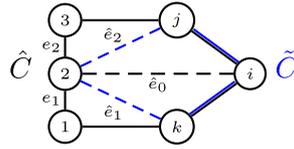


FIGURE 15. \hat{C} and \tilde{C} .

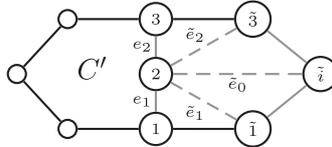


FIGURE 16. Path $\{\tilde{1}, 1\} \cup (C' \setminus \{2\}) \cup \{3, \tilde{3}\}$.

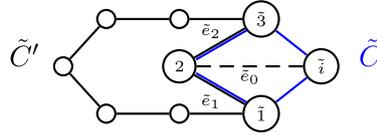


FIGURE 17. Adjacent cycles \tilde{C} and \tilde{C}' .

(resp. 3 and j), $\gamma_1 = \{k, e_k, k + 1, \dots, p, e_p, 1\}$ (resp. $\gamma_2 = \{3, e_3, 4, \dots, j - 1, e_{j-1}, j\}$) then $\gamma_1 \cup (C' \setminus \{2\}) \cup \gamma_2$ corresponds to a path linking k to j and therefore $\tilde{1}$ to $\tilde{3}$ in $\tilde{G} \setminus \{2, \tilde{i}\}$ as represented in Figure 16.

Therefore there exists a shortest path γ' in $\tilde{G} \setminus \{2, \tilde{i}\}$ linking $\tilde{1}$ to $\tilde{3}$. Then we can consider the cycles \tilde{C} and $\tilde{C}' = \{2, \tilde{e}_1, \tilde{1}\} \cup \gamma' \cup \{\tilde{3}, \tilde{e}_2, 2\}$ as represented in Figure 17.

We can achieve the proof as in the first case. By construction \tilde{C} has only one non-maximum weight chord $\tilde{e}_0 = \{2, \tilde{i}\}$ which is not a chord of \tilde{C}' ($\tilde{i} \notin \tilde{C}'$) and \tilde{C} and \tilde{C}' have no maximum weight chords. \tilde{C} and \tilde{C}' satisfy conditions a to d, 2f and 2g (replacing e_1 (resp. e_2) by \tilde{e}_1 (resp. \tilde{e}_2)).

As in Case 1, we can obtain \hat{C} , if it exists, using a Breadth First Search algorithm in $O(n^2)$ time. Then to find a chord $\tilde{e}_0 = \{2, \tilde{i}\}$ of \hat{C} , we need at most $O(n)$ time. To select the edges $\tilde{e}_1 = \{2, k\}$ and $\tilde{e}_2 = \{2, j\}$, we also need at most $O(n)$ comparisons. To build the shortest path γ' in $\tilde{G} \setminus \{2, \tilde{i}\}$ using a Breadth First Search algorithm, we need $O(n^2)$ time. Hence to verify Claim 2 when \hat{C} has a chord, we need $O(n^2)$ time. Note that if there exists a chord in \tilde{C} we do not need to check Claim g for all $i \in V(\tilde{\gamma}) \setminus \{\tilde{1}, \tilde{3}\}$. We check the condition only for the node i corresponding to the chord we have found. \square

We consider that the Adjacent cycles condition can be divided in two parts. The first part is the condition of non-existence of cycles with two common non-maximum weight edges. Then the second part is the condition on edge-weights for cycles with a unique common non-maximum weight edge.

Proposition 4.9. *The first part of the Adjacent cycles condition can be verified in $O(n^6)$ time.*

Proof. We have to verify that for any pair of adjacent edges $e_1 = \{1, 2\}$ and $e_2 = \{2, 3\}$, e_1 and e_2 are not common non-maximum weight edges of two adjacent cycles (*i.e.*, Claim 1 of Lem. 4.6 is not satisfied). By Lemma 4.6 it is sufficient to prove that Claim 2 of Lemma 4.6 is not satisfied and it can be done in $O(n^3)$ time. There are at most $\sum_{i \in V} C_n^2 = n^2 \binom{n-1}{2}$ pairs of adjacent edges. Hence we need at most n^3 iterations of the procedure considered in Lemma 4.6 and therefore the first part of the Adjacent cycles condition can be verified in $O(n^6)$ time. \square

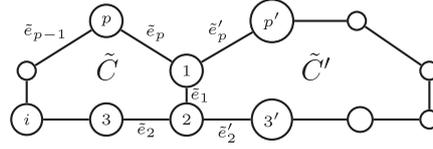


FIGURE 18. $\tilde{C} = \{1, \tilde{e}_1, 2, \tilde{e}_2, \dots, p, \tilde{e}_p, 1\}$ and $\tilde{C}' = \{1, \tilde{e}_1, 2, \tilde{e}'_2, \dots, p', \tilde{e}'_{p'}, 1\}$.

Lemma 4.10. *Let $G = (N, E, w)$ be a weighted graph satisfying the Star and Path (and therefore Intermediary Cycle) conditions. Let us assume that the first part of the Adjacent Cycles condition is satisfied. Then the following claims are equivalent:*

- (1) *There exist two simple cycles C and C' in G satisfying conditions a, b, c, d, of the Adjacent Cycles condition, and such that:*
 - (e) *C and C' have a unique common non-maximum weight edge $e_1 = \{1, 2\}$ and $w(e) > w_1$ for all $e \in (E(C) \cup E(C')) \setminus \{e_1\}$.*
- (2) *There exist two simple cycles \tilde{C} and \tilde{C}' in G satisfying conditions a, b, c, d, of the Adjacent Cycles condition, and such that:*
 - (f) *\tilde{C} and \tilde{C}' have a unique common non-maximum weight edge $\tilde{e}_1 = \{1, 2\}$ and $w(e) > w_1$ for all $e \in (E(\tilde{C}) \cup E(\tilde{C}')) \setminus \{\tilde{e}_1\}$.*
 - (g) *$\tilde{C} \setminus \{\tilde{e}_1\}$ corresponds to a shortest path $\tilde{\gamma}$ in $\tilde{G} = (V, E \setminus \{\tilde{e}_1\}, w|_{E \setminus \{\tilde{e}_1\}})$ linking 1 and 2 and for some $i \in V(\tilde{\gamma}) \setminus \{1, 2\}$, $\tilde{C}' \setminus \{\tilde{e}_1\}$ corresponds to a shortest path $\tilde{\gamma}'$ in $\tilde{G}_{N \setminus \{i\}}$ linking 1 and 2.*

Let us observe that using the Intermediary Cycle condition we only need to verify that the edges in $E(C)$ adjacent to e_1 have a weight strictly greater than w_1 . Indeed if it is satisfied then $w(e) > w_1$ for all $e \in E(C) \setminus \{e_1\}$.

Remark 4.11. Let us note that by the Intermediary Cycle condition $C, C', \tilde{C}, \tilde{C}'$ can have at most two non-maximum weight edges. If they all have two non-maximum weight edges then e_1 and all these edges are adjacent (incident to 1 or 2).

Proof. Let us consider $C = \{1, e_1, 2, e_2, \dots, m, e_m, 1\}$. Then $\gamma = C \setminus \{e_1\} = \{2, e_2, \dots, m, e_m, 1\}$ is a path linking 1 and 2 in \tilde{G} . Hence there is a shortest path $\tilde{\gamma} = \{2, \tilde{e}_2, \tilde{3}, \tilde{e}_3, \dots, p, \tilde{e}_p, 1\}$ linking 1 and 2 in \tilde{G} . Then we can build the cycle $\tilde{C} := \{1, e_1, 2, \tilde{e}_2, \tilde{3}, \tilde{e}_3, \dots, p, \tilde{e}_p, 1\}$ adding the edge e_1 to the path $\tilde{\gamma}$. Either $\tilde{C} = C$, or \tilde{C} and C are adjacent. In this last case Lemma 3.4 implies $\max_{e \in E(C)} w(e) = \max_{e \in E(\tilde{C})} w(e)$. Hence e_1 is a non-maximum weight edge in \tilde{C} . Claim e implies $w_2 > w_1$ (resp. $w_m > w_1$). If $\tilde{e}_2 = e_2$ (resp. $\tilde{e}_p = e_m$) then $w(\tilde{e}_2) = w_2$ (resp. $w(\tilde{e}_p) = w_m$). Otherwise Star condition applied to $\{e_1, e_2, \tilde{e}_2\}$ (resp. e_1, e_m, \tilde{e}_p) implies $w(\tilde{e}_2) = w_2$ (resp. $w(\tilde{e}_p) = w_m$). Then the Intermediary Cycle condition implies $w_1 < w(\tilde{e}_2) \leq w(\tilde{e}_3) = \dots = w(\tilde{e}_p) = M$ with $M = \max_{e \in E(\tilde{C})} w(e)$, exchanging \tilde{e}_2 and \tilde{e}_p if necessary. The Intermediary Cycle condition also implies that for any chord e in \tilde{C} , we have $w(e) \leq w_2$ if e is incident to 2 (in fact, as $w_1 < w_2$, the Star condition applied to $\{e_1, \tilde{e}_2, e\}$ implies $w(e) = w_2$) and $w(e) = M$ otherwise. In any case, e would contradict the optimality of $\tilde{\gamma}$. Hence \tilde{C} has no chord.

Let us assume $V(\tilde{C}) \subseteq V(C) \cap V(C')$. As $V(C) \setminus V(C') \neq \emptyset$ (resp. $V(C') \setminus V(C) \neq \emptyset$), we have $V(\tilde{C}) \neq V(C)$ (resp. $V(\tilde{C}) \neq V(C')$). Then at least one edge \tilde{e} of \tilde{C} is a chord of C . As C has no maximum weight chord, \tilde{e} is a chord of non-maximum weight in C and then \tilde{e} is also an edge in $E(\tilde{C}) \setminus \{e_1\}$ of non-maximum weight (by Lemma 3.4, as C and \tilde{C} are adjacent the maximum weight in C and \tilde{C} is M). Following the Intermediary Cycle condition, \tilde{e} is the unique edge of \tilde{C} of non-maximum weight adjacent to e_1 as represented in Figure 19.

By the same reasoning (interchanging C and C'), there exists an edge \tilde{e}' of \tilde{C} which is a chord of C' and \tilde{e}' is an edge of \tilde{C} of non-maximum weight. \tilde{e}' necessarily corresponds to the unique edge of \tilde{C} of non-maximum weight adjacent to e_1 . Therefore $\tilde{e} = \tilde{e}'$. Hence C and C' have a common chord, a contradiction.

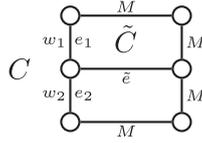


FIGURE 19. \tilde{e} non-maximum weight chord (resp. edge) in C (resp. \tilde{C}).

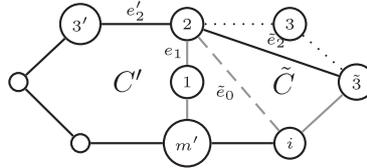


FIGURE 20. C' and \tilde{C} , with $i \notin C'$.

Therefore we have either $V(\tilde{C}) \setminus V(C) \neq \emptyset$ or $V(\tilde{C}) \setminus V(C') \neq \emptyset$. In every case we can choose $i \in V(\tilde{C}) \setminus \{1, 2\}$ such that there exists a path $\hat{\gamma}$ in $\tilde{G}_{N \setminus \{i\}}$ linking 1 and 2. $\hat{\gamma} = \gamma$ if $i \notin V(C)$ or $\hat{\gamma} = C' \setminus \{e_1\} = \{2, e'_2, 3', e'_3, \dots, m', e'_{m'}, 1\}$ if $i \notin V(C')$ as represented in Figure 20.

Therefore we can find a shortest path $\tilde{\gamma}'$ in $\tilde{G}_{N \setminus \{i\}}$ linking 1 and 2 as represented in Figure 18. Adding to $\tilde{\gamma}'$ the edge e_1 we get the cycle $\tilde{C}' = \{1, e_1, 2, \tilde{e}'_2, \tilde{3}', \tilde{e}'_{\tilde{3}'}, \dots, \tilde{p}', \tilde{e}'_{\tilde{p}'}, 1\}$. \tilde{C}' has no chord otherwise $\tilde{\gamma}'$ is not a shortest path in $\tilde{G}_{N \setminus \{i\}}$ (by the same reasoning as for \tilde{C}). As $i \in V(\tilde{C}) \setminus V(\tilde{C}')$ we have $V(\tilde{C}) \setminus V(\tilde{C}') \neq \emptyset$. We have already seen that $\max_{e \in E(C)} w(e) = \max_{e \in E(\tilde{C})} w(e) = \max_{e \in E(\tilde{C}')} w(e)$, and that if C contains a second non-maximum weight edge then \tilde{C} and \tilde{C}' also contain a second non-maximum weight edge of same weight. Therefore if $V(\tilde{C}') \subset V(\tilde{C})$ then $\tilde{\gamma}'$ is a path linking 1 and 2 in \tilde{G} shorter than $\tilde{\gamma}$, a contradiction. Therefore we have $V(\tilde{C}') \setminus V(\tilde{C}) \neq \emptyset$. \square

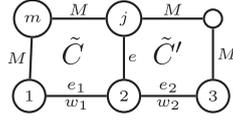
We first verify a weaker condition than the second part of the Adjacent Cycles condition.

Weak second part of the Adjacent Cycles condition. If e_1 is a unique non-maximum weight edge common to two simple cycles (C, C') in G satisfying the conditions a, b, c, d, of the Adjacent Cycles condition, then there are non-maximum weight edges $e_2 \in E(C) \setminus \{e_1\}$ and $e'_2 \in E(C') \setminus \{e_1\}$ such that e_1, e_2, e'_2 are adjacent and $w_1 = w_2 \geq w'_2$ or $w_1 = w'_2 \geq w_2$.

Note that we cannot have $e_2 = e'_2$ otherwise e_1 and e_2 are two non-maximum weight edges common to C and C' as $w_1 = w_2 = w'_2$. Therefore we have $e_2 \in E(C) \setminus E(C')$ and $e'_2 \in E(C') \setminus E(C)$.

Proposition 4.12. *Let $G = (N, E, w)$ be a weighted graph satisfying the Star and Path (and therefore Weak and Intermediary Cycle) conditions, and the first part of the Adjacent Cycles condition. Then the Weak Second Part of the Adjacent Cycles condition can be verified in $O(n^3m)$ time.*

Proof. Let e_1 be a common non-maximum weight edge of two adjacent cycles C and C' satisfying the conditions a, b, c, d, of the Adjacent Cycles condition. Let $e_2 \in E(C) \setminus e_1$ and $e'_2 \in E(C') \setminus e_1$ be edges incident to the same end-vertex of e_1 (at this step of the proof e_2 may be equal to e'_2 and e_2 or e'_2 may have maximum weight in C or C'). Let us assume that we have neither $w_1 = w_2 \geq w'_2$ nor $w_1 = w'_2 \geq w_2$. If $e_2 = e'_2$ then $w_2 = w'_2$ and therefore we have $w_1 < w_2 = w'_2$ or $w_1 > w_2 = w'_2$. This last case is not possible as e_1, e_2 would be two non-maximum weight edges common to C and C' . If $e_2 \neq e'_2$ then Star condition applied to e_1, e_2, e'_2 also implies $w_1 < w_2 = w'_2$. (By the same reasoning we get $w_1 < w_m = w'_{m'}$.) Then the Intermediary Cycle condition implies $w(e) > w_1$ for all $e \in (E(C) \cup E(C')) \setminus \{e_1\}$. Then by Lemma 4.10, e_1 is also a non-maximum weight


 FIGURE 21. $w(e) < w_1 = w_2$.

edge common to two cycles \tilde{C} and \tilde{C}' such that $\tilde{C} \setminus \{e_1\}$ corresponds to a shortest path $\tilde{\gamma}$ linking 1 and 2 in $\tilde{G} = (N, E \setminus \{e_1\}, w|_{E \setminus \{e_1\}})$ and for some $i \in V(\tilde{\gamma}) \setminus \{1, 2\}$, $\tilde{C}' \setminus \{e_1\}$ corresponds to a shortest path in $\tilde{G}_{N \setminus \{i\}}$ linking 1 and 2. Moreover we still have $w(e) > w_1$ for all $e \in (E(\tilde{C}) \cup E(\tilde{C}')) \setminus \{e_1\}$. Therefore for every edge $e_1 \in E$, we only have to verify that such a couple of adjacent cycles (\tilde{C}, \tilde{C}') cannot exist. Therefore we look for a shortest path $\tilde{\gamma}$ linking 1 and 2 in \tilde{G} . If it exists then we look for a shortest path $\tilde{\gamma}'$ in $\tilde{G}_{N \setminus \{i\}}$ with $i \in V(\tilde{\gamma}) \setminus \{1, 2\}$, linking 1 and 2. If such a path $\tilde{\gamma}'$ exists, we only have to verify that $w(e) = w_1$ for some edge e of $\tilde{\gamma}$ or $\tilde{\gamma}'$ (in fact the Weak Cycle condition implies that such an edge is necessarily adjacent to e_1).

Note that by the Star and Weak Cycle conditions and Lemma 3.4, all elementary paths linking 1 and 2 in \tilde{G} have either all their edges of same weight $M > w_1$ or one edge incident to 1 or 2 with weight $w_2 > w_1$ and their remaining edges with weight $M > w_2$. Therefore we can find a shortest path $\tilde{\gamma}$ linking 1 and 2 in \tilde{G} using a BFS algorithm in $O(n^2)$. Then we can also use a BFS algorithm to find a shortest path in $\tilde{G}_{N \setminus \{i\}}$ with $i \in \tilde{\gamma}$. In the worst case we have to consider all nodes $i \in \tilde{\gamma}$. Therefore we can verify in $O(n^3)$ time if e_1 satisfies the Weak Second Part of the Adjacent Cycles condition. Therefore the Weak Second Part of the Adjacent cycles condition can be verified in $O(n^3m)$ time. \square

Proposition 4.13. *Let $G = (N, E, w)$ be a weighted graph satisfying the Star and Path (and therefore Intermediary Cycle) conditions, and the Weak Second Part of the Adjacent Cycles condition. Then the Cycle condition is satisfied.*

Proof. Let us consider a cycle $C = \{1, e_1, 2, e_2, \dots, m, e_m, 1\}$. The Intermediary Cycle condition implies that $w_1 \leq w_2 \leq w_3 = \dots = w_m = M$ after renumbering if necessary and that $w(e) \leq w_2$ for all chord incident to 2 and $w(e) \leq M$ for all other chords. Moreover:

- If $w_1 \leq w_2 < M$ then $w(e) = M$ for all $e \in E(C)$ non-incident to 2. If e is a chord incident to 2 then $w_1 \leq w_2 = w(e) < M$ or $w(e) < w_1 = w_2 < M$.
- If $w_1 < w_2 = M$, then $w(e) = M$ for all $e \in E(C) \setminus \{e_1\}$.

Therefore if $w_1 < w_2 = M$ the Cycle condition is satisfied. If $w_1 \leq w_2 < M$, let us assume by contradiction that there exists a chord incident to 2 with $w(e) < w_2$. Star condition applied to $\{e, e_1, e_2\}$ implies $w_1 = w_2$ and that for any other chord e' incident to 2 we have $w(e') = w_1 = w_2$. Hence we can assume that C has only one chord e incident to 2 replacing if necessary C by a smaller cycle using the chords of C incident to 2 which are as close as possible to e . We have $e = \{2, j\}$ with $4 \leq j \leq m$. We can consider two adjacent cycles $\tilde{C} := \{2, e, j, e_j, \dots, m, e_m, 1, e_1, 2\}$ and $\tilde{C}' := \{2, e_2, 3, e_3, \dots, e_{j-1}, j, e, 2\}$ as represented in Figure 21. e is a unique common edge to \tilde{C} and \tilde{C}' such that $w(e) < w_1 = w_2 \leq M$. If \tilde{C} (resp. \tilde{C}') has a chord then we can replace \tilde{C} (resp. \tilde{C}') by a smaller cycle. Therefore we can suppose that \tilde{C} and \tilde{C}' have no chords. Then $w(e) < w_1 = w_2$ contradicts the Weak Second Part of the Adjacent Cycles condition. Therefore any edge incident to 2 has weight w_2 .

Finally if $w_1 = w_2 = M$, let us also assume by contradiction that there exists a chord e in C with $w(e) < w_2 = M$. As $w_1 = w_2 = M$, we can always assume e incident to 2 after renumbering if necessary and then we can establish the same contradiction as before. \square

Assuming now that the Star, Path, and Cycle conditions are satisfied we give a new formulation of the Pan condition. Then we will analyze its complexity.

Lemma 4.14. *Let $G = (N, E, w)$ be a connected weighted graph satisfying the Star, Path, and Cycle conditions. The Pan condition is satisfied if and only if for all simple cycle $C = \{1, e_1, 2, e_2, \dots, m, e_m, 1\}$, with $w_1 = w_2 \leq w_3 = \dots = w_m = \hat{M} = \max_{e \in \hat{E}(C)} w(e)$, one of the following claim is satisfied:*

- 1) $w_1 = w_2 = \hat{M}$.
- 2) $w_1 = w_2 < \hat{M}$ and $\{1, 3\}$ is a chord of C .
- 3) $\sigma(E) = w_1 = w_2 < \hat{M}$, where $\sigma(E) = \min_{e \in E} w(e)$.

Proof. Let us first assume that the Pan condition is satisfied. Let us consider a simple cycle $C = \{1, e_1, 2, e_2, \dots, m, e_m, 1\}$, with $w_1 = w_2 \leq w_3 = \dots = w_m = \hat{M}$. If $w_1 = w_2 = \hat{M}$ then Claim 1 is satisfied. Otherwise we have $w_1 = w_2 < \hat{M}$. If $\{1, 3\}$ is a chord of C then Claim 2 is satisfied. Otherwise $\{1, 3\}$ is not a chord of C and let us assume $\sigma(E) < w_1 = w_2$. Then there exists $e \in E \setminus E(C)$ such that $w(e) < w_1 = w_2$. Note that the Cycle condition implies that a chord of C incident (resp. non incident) to 2 has weight w_2 (resp. \hat{M}). Therefore we actually have $e \in E \setminus \hat{E}(C)$. As G is connected there is a path P containing e (P can be restricted to e) such that $|V(C) \cap V(P)| = 1$. Then as $w(e) < w_1 = w_2 < \hat{M}$, Pan condition implies that $V(C) \cap V(P) = \{2\}$ and that $\{1, 3\}$ is a maximum weight chord of C , a contradiction. Therefore Claim 3 is satisfied.

Let us now consider a simple cycle $C = \{1, e_1, 2, e_2, \dots, m, e_m, 1\}$, and an elementary path P such that there is an edge $e \in P$ with $w(e) \leq \min_{1 \leq k \leq m} w_k$ and $|V(C) \cap V(P)| = 1$. The Cycle condition implies $w_1 \leq w_2 \leq w_3 = \dots = w_m = \hat{M}$ after renumbering if necessary. Moreover the Path and Star conditions imply $w_1 = w_2 \leq \hat{M}$ and if $w_1 = w_2 < \hat{M}$ then $V(C) \cap V(P) = \{2\}$ (as the Weak Pan condition is satisfied by Proposition 3.3). If C satisfies Claim 1 then the Pan condition is trivially satisfied. If C satisfies Claim 2 then $\{1, 3\}$ is a chord of C and as $w_1 = w_2 < \hat{M}$ we have $V(C) \cap V(P) = \{2\}$. Moreover the (Intermediary) Cycle condition implies that $\{1, 3\}$ is a maximum weight chord. Therefore the Pan condition is satisfied. Finally if C satisfies Claim 3, we also have $V(C) \cap V(P) = \{2\}$ and as $\sigma(E) = w_1$, we necessarily have $w(e) = w_1$. Therefore the Pan condition is still satisfied. \square

Proposition 4.15. *Let $G = (N, E, w)$ be a connected weighted graph satisfying the Star, Path conditions and the Cycle condition (or the Weak Second Part of the Adjacent Cycles condition). Then Pan condition can be verified in $O(n^5)$ time.*

Proof. Let $C = \{1, e_1, 2, e_2, \dots, m, e_m, 1\}$ be a simple cycle of G . The Cycle condition implies $w_1 \leq w_2 \leq w_3 = \dots = w_m = M = \max_{e \in E(C)} w(e)$, after renumbering if necessary. Let $\tilde{\gamma}$ be a shortest path linking 1 and 3 in $G_{N \setminus \{2\}}$ and let \tilde{C} be the simple cycle formed by e_1 , e_2 and $\tilde{\gamma}$. If $\tilde{C} \neq C$ then Lemma 3.4 implies that $\max_{e \in E(\tilde{C})} w(e) = M$. Lemma 4.14 implies that Pan condition for C is equivalent to Pan condition for \tilde{C} . Then if $w_1 = w_2 < M$ either the chord $\{1, 3\}$ exists (i.e. $\tilde{\gamma} = \{1, \{1, 3\}, 3\}$ and $\tilde{C} = \{1, e_1, 2, e_2, 3, \{1, 3\}, 1\}$) otherwise $w_1 = w_2 = \sigma(E)$. Hence to verify the Pan condition we just need to consider any pair of adjacent edges $e_1 = \{1, 2\}$ and $e_2 = \{2, 3\}$ with $w_1 = w_2$. Let us consider the graph $\tilde{G} = (N, \tilde{E}, w|_{\tilde{E}})$ with $\tilde{E} = \{e \in E; w(e) \geq w_1\}$. Then we look for a shortest path γ linking 1 to 3 in the graph $\tilde{G}_{N \setminus \{2\}}$ using a BFS algorithm in $O(n^2)$ time. If γ exists and if γ has at least two edges, we consider the simple cycle $C = \{1, e_1, 2, e_2, 3\} \cup \gamma$. Note that as γ is a shortest path linking 1 and 3, $\{1, 3\}$ cannot be a chord of \tilde{C} . Then applying Lemma 4.14 to \tilde{C} we only need to verify that either $w_1 = w_2 = w_3$ or $w_1 = w_2 = \sigma(E)$. $\sigma(E)$ can be computed in $O(m)$ time. We have at most $\sum_{i \in V} C_n^2 = n^2 \binom{n-1}{2}$ pairs e_1, e_2 to take into account. Therefore the Pan condition can be verified in $O(n^5)$. \square

Proposition 4.16. *Let $G = (N, E, w)$ be a weighted graph satisfying the Weak Second Part of the Adjacent Cycles and the Pan conditions. Then the Second part of the Adjacent Cycles condition is satisfied.*

Proof. Let e_1 be a unique non-maximum weight edge common to two simple cycles (C, C') in G satisfying the conditions a, b, c, d, of the Adjacent Cycles condition. The Weak Second Part of the Adjacent Cycles condition

implies that there are non-maximum weight edges $e_2 \in E(C) \setminus \{e_1\}$ and $e'_2 \in E(C') \setminus \{e_1\}$ such that e_1, e_2, e'_2 are adjacent and $w_1 = w_2 \geq w'_2$ or $w_1 = w'_2 \geq w_2$. If $|E(C)| = 3$ or $|E(C')| = 3$ then the second part of the Adjacent Cycles condition is satisfied. If $|E(C)| \geq 4$ and $|E(C')| \geq 4$, let us assume $w_1 = w_2 > w'_2$ (resp. $w_1 = w'_2 > w_2$). By assumption C (resp. C') have no maximum weight chord therefore the Pan condition applied to C and e'_2 (resp. C' and e_2) implies $w_1 = w_2 = \max_{e \in E(C)} w(e)$ (resp. $w_1 = w'_2 = \max_{e \in E(C')} w(e)$), a contradiction. Therefore $w_1 = w_2 = w'_2$ and the second part of the Adjacent Cycles condition is satisfied. \square

Finally, Propositions 4.1, 4.5, 4.9, 4.12, 4.13, 4.15, 4.16, and Theorem 3.6 imply Theorem 1.1.

5. CONCLUSION

We think that the method we have presented to prove that inheritance of \mathcal{F} -convexity is a polynomial problem, especially, the use of minimum spanning trees and shortest paths to select specific paths and cycles, can be useful for further developments. In a forthcoming paper, we prove that the characterization of inheritance of convexity for the correspondence \mathcal{P}_{\min} given by Skoda [14] can also be checked in polynomial time (see [15] for a preliminary version). But the previous method could also be used to obtain polynomial algorithms to check inheritance of convexity or \mathcal{F} -convexity for other correspondences as for example the correspondence \mathcal{P}_G presented by Grabisch [11] which associates to a subset its partition into relatively strongest components. Of course it is not certain that such algorithms do exist.

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