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ON THE PRECONDITIONED PROJECTIVE ITERATIVE METHODS FOR THE LINEAR COMPLEMENTARITY PROBLEMS *

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Abstract. This paper aims to propose the new preconditioning approaches for solving linear complementarity problem (LCP). Some years ago, the preconditioned projected iterative methods were presented for the solution of the LCP, and the convergence of these methods has been analyzed. However, most of these methods are not correct, and this is because the complementarity condition of the preconditioned LCP is not always equivalent to that of the un-preconditioned original LCP. To overcome this shortcoming, we present a new strategy with a new preconditioner that ensures the solution of the same problem is correct. We also study the convergence properties of the new preconditioned iterative method for solving LCP. Finally, the new approach is illustrated with the help of a numerical example.

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1. Introduction

For a given real vector $q \in R^n$ and a given matrix $A \in R^{n\times n}$ the linear complementarity problem abbreviated as LCP (A, q), consists in finding vectors $w, z \in R^n$ such that,

$$w = Az + q, z^T w = 0, z \geq 0, w \geq 0, \tag{1.1}$$

where, z^T denotes the transpose of the vector z.

Many problems in various scientific computing, economics and engineering areas can lead to the solution of the linear complementarity problem (LCP) and its generalizations. For example, quadratic programming, Nash equilibrium point of a bimatrix game, nonlinear obstacle problems, invariant capital stock, optimal stopping, contact, and structural mechanics, free boundary value problem for journal bearings, traffic equilibriums, manufacturing systems, etc. For more details, see [1–3] and the references therein. Because of the wide applications, the research on the numerical methods for solving the LCP has attracted much attention. Generally, there are some iterative methods for the solution of the LCP, including the projected methods [4–16] and the modulus-type algorithms [17–23]; see [24] for a survey of the solvers for LCPs.

Keywords. Linear complementarity problems, preconditioning, Projected model, M-matrix, GAOR.

* *Dedicated to Professor Apostolos Hadjidimos.*

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Some scholars also suggested the preconditioning techniques for LCPs the so-called Preconditioned Projected Iterative (PPI) methods [25–29]. Unfortunately, most of these models do not always work. This is because the complementarity condition of the preconditioned LCP is not always equivalent to that of the un-preconditioned original LCP. In other words, the complementarity condition is not transitive. *i.e.*, if P is the preconditioner of a PPI method, then from $z^T Pw = 0$ we cannot conclude that $z^T w = 0$. To overcome this shortcoming, we establish a new strategy with a new preconditioned that ensures the solution of the same problem is correct. Numerical experiments are also given to illustrate our assertions.

The outline of this paper is as follows. In Section 2 we review some preliminaries about matrices and LCPs. In Section 3, we study the class of projective iterative methods for solving LCPs. In Section 4, the existing PPI approach for solving the LCPs problem is presented and we point out the error of these models with a numerical example. In Section 5, we establish a new model of preconditioning to solve an LCP. The numerical results about the new method are shown and discussed in Section 6. Finally, in Section 7, we end the paper by our conclusions.

2. Prerequisite

We begin with some basic notation and preliminary results which we refer to later. First of all, the matrix $A = (a_{ij}) \in R^{n\times n}$ $(0 \leq i, j \leq n)$ is nonnegative (positive) if for any $i, j; a_{ij} \geq 0$ $(a_{ij} > 0)$. In this case we write $A \geq 0(A > 0)$. Similarly, for n-dimensional vectors x, by identifying them with $n \times 1$ matrices, we can also define $x \geq 0(x > 0)$. Moreover, we denote the spectral radius of A by $\rho(A)$.

Definition 2.1 ([30–33])**.**

(a) a matrix $A = (a_{ij})$ is called a Z-matrix if for any $i \neq j, a_{ij} \leq 0$;
(b) a matrix $A = (a_{ij})$ is called an M-matrix, if $A = \alpha I - B$; $B \geq 0$ and $\alpha > \rho(B)$;
(c) a Z-matrix is an M-matrix, if A is nonsingular, and $A^{-1} \geq 0$;
(d) a Z-matrix is an M-matrix, if $Ax \geq 0$ implies $x \geq 0$;
(e) a matrix $A = (a_{ij})$ is called H-matrix if and only if $< A >= (m_{ij}) \in R^{n\times n}$ is an M-matrix, where;

$$m_{i,i} = |a_{i,i}|;\ m_{i,j} = -|a_{i,j}|,\ i \neq j;$$

(f) a real square matrix $A = (a_{ij})$ is called *monomial matrix* if and only if it's each row and column contains only one nonzero entry and the remaining entries are zero.

Definition 2.2 ([4])**.** For any vector $x \in R^n$, vector x_+ is defined such that $[x_i]_+ = \max\{0, x_i\}$. Then, for any $x, y \in R^n$, the following facts hold:

(i) $(x + y)_+ \leq x_+ + y_+$.
(ii) $x_+ - y_+ \leq (x - y)_+$.
(iii) $|x| = x_+ - (-x)_+$.
(iv) $x \leq y$ implies $x_+ \leq y_+$.

Lemma 2.3 ([4])**.** *Let $A \in R^{n\times n}$ be an H-matrix with positive diagonal elements. Then the* LCP(A, q) *has a unique solution $Z^* \in R^n$.*

Lemma 2.4 ([4])**.** LCP(A, q) *can be equivalently transformed to a fixed-point system of equations;*

$$z = (z - \alpha E(Az + q))_+,$$

where α is some positive constant and E is a diagonal matrix with positive diagonal elements.

3. Projected iterative methods

Let us consider LCP (1.1). The LCP(A, q) is equivalent to the following zero-finding formulation;

$$\min(z, (Az + q)) = 0. \tag{3.1}$$

And the zero-finding formulation is equivalent to the following fixed-point formulation;

$$\max(0, z - (Az + q)) = z. \tag{3.2}$$

This fixed point model used in some iterative methods for LCPs called the projected iterative methods. We describe shortly this class of iterative methods. For any iteration, we have the following splitting;

$$A = M - N.$$

Then from equation (3.2):

$$\max(0, z - (Mz - Nz + q)) = z.$$

Let us split the matrix A into $A = D - L - U$ in which $D = \text{diag}(A)$, L and U are strictly lower and strictly upper triangular matrices, respectively. Then the Generalized AOR (GAOR) algorithm [7, 14–17] is as follows:

$$z^{(k+1)} = (z^{(k)} - D^{-1}[-\alpha\Omega L Z^{(k+1)} + (\Omega A - \alpha\Omega L)z^{(k)} + \Omega q])_+, \tag{3.3}$$

where, α is a real parameter and $\Omega = (w_1, \ldots, w_n)$ is a real diagonal relaxation matrix.

In next lemma, we have the convergence theorem, proposed in [7] for the GAOR methods.

Lemma 3.1. *Let $A \in R^{n\times n}$ be an H-matrix with positive diagonal elements. Moreover, let*

$$G = I - \alpha\Omega D^{-1}|L|, \quad F = |I - D^{-1}(\Omega A - \alpha L)|,$$

then, for any initial vector $z^{(0)} \in R^n$, the iterative sequence $z^{(k)}$ generated by the GAOR method converges to the unique solution z^* of the LCP (A, q) and;

$$\rho(G^{-1}F) \leq \max\{|1 - w_i| + w_i\rho(|J|)\} < 1,$$

whenever,

$$0 < w_i < 2/(1 + \rho(|J|)), \quad 0 \leq \alpha \leq 1,$$

where $\rho(|J|)$ is the spectral radius of the Jacobi iteration matrix $(J = D^{-1}(L + U))$.

4. The existing PPI models and their shortcomings

In the existing PPI models [25–29], first we choose a preconditioner P and then using $\bar{A} = PA$, $\bar{q} = Pq$, equation (1.1) is transformed into the following problem:

$$\bar{w} = \bar{A}z + \bar{q}, z^T\bar{w} = 0, z \geq 0, \bar{w} \geq 0. \tag{4.1}$$

Then, by any iterative methods and under certain conditions, we can solve equation (4.1). For example, in [25], a model of $(I + S) - type$ preconditioner, presented as follows:

$$P(\gamma\beta) = I + S(\gamma\beta) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -\gamma_2 a_{21} - \beta_2 & 1 & \cdots & 0 \\ -\gamma_3 a_{31} - \beta_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ -\gamma_n a_{n1} - \beta_n & 0 & \cdots & 1 \end{pmatrix}, \tag{4.2}$$

where, $\gamma = [0, \gamma_2, \ldots, \gamma_i, \ldots, \gamma_n] \in R^{n\times n}$, $0 \le \gamma_i \le 1$ and $(1-\gamma_i)a_{i1} \le \beta_i \le -\gamma_i a_{i1}$, $i = 2, \ldots, n$.

Then, they used the following preconditioned GAOR method (PGAOR) to solve the linear complementarity problem:

$$z^{(k+1)} = (z^{(k)} - \hat{D}^{-1}[-\alpha\Omega\hat{L}Z^{(k+1)} + (\Omega\hat{A} - \alpha\Omega\hat{L})z^{(k)} + \Omega\hat{q}])_+, \tag{4.3}$$

where $\hat{D}$, $\hat{L}$ are diagonal and strictly lower triangular parts of $\hat{A} = P(\gamma\beta)A$, respectively.

Unfortunately, these models does not always work and this is because the complementarity condition of the preconditioned LCP is not always equivalent to that of the un-preconditioned original LCP. In the following, by a counterexample, we demonstrate the solutions of equation (4.1) is not always the solution of original LCP (1.1).

Example 4.1. Consider LCP(A, q) with following system:

$$A = \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/3 & 1 & -1/3 \\ -1/2 & -1/2 & 1 \end{pmatrix}, q = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

It is easy to see that A is a nonsingular M-matrix and the solution of this LCP is $z = [0, 1, 0]^T$, $w^T = [1/2, 0, 1/2]$. So, we consider two preconditioners and test them:

Case one. Consider the diagonal preconditioner $P_1 = \text{diag}(\gamma_1, \gamma_2, \gamma_3)$, where $\gamma_i > 0$. In this case the initial LCP problem becomes:

$$\bar{w} := (P_1 w) = (P_1 A)z + P_1 q, \quad z^T(P_1 w) = 0.$$

And the solution is $z = [0, 1, 0]^T$, $\bar{w}^T = [\gamma_1/2, 0, \gamma_3/2]$. So, the complementarity condition is satisfied $z^T\bar{w} = 0$ and in addition, the actual w can be recovered dividing by γ_1 and γ_3 (or simply $w = {P_1}^{-1}\bar{w}$). So we can see that in this case, the complementarity condition of the preconditioned LCP is equivalent to that of the original LCP.

In the next case, we can see that the complementarity condition will not satisfied.

Case two. Consider the preconditioner (4.2) for the Example 4.1. So we have:

$$P(\gamma\beta) = \begin{pmatrix} 1 & 0 & 0 \\ -\gamma_2/3 - \beta_2 & 1 & 0 \\ -\gamma_3/2 - \beta_3 & 0 & 1 \end{pmatrix},$$

where, $0 \le \gamma_i \le 1$ and $-1/3(1-\gamma_2) \le \beta_2 \le 1/3\gamma_2$, $-1/2(1-\gamma_3) \le \beta_3 \le 1/2\gamma_3$. By choosing $\gamma_i = 1$, $\beta_i = 0$ we have:

$$\hat{A} = \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & 5/6 & -1/2 \\ 0 & -3/4 & 3/4 \end{pmatrix}, q = \begin{pmatrix} 1 \\ -2/3 \\ 3/2 \end{pmatrix}.$$

Now, if we use the model (4.3) or any projective iterative methods to solve the LCP $(\hat{A},\ \hat{q}\ )$, we obtain:

$$z = [0, 4/5, 0]^T.$$

On the other hand, if we insert the true solution $z = [0, 1, 0]^T$ for the LCP $(\hat{A},\ \hat{q}\ )$ we get:

$$\hat{w} = \hat{A}z + \hat{q} = \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & 5/6 & -1/2 \\ 0 & -3/4 & 3/4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2/3 \\ 3/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/6 \\ 3/4 \end{pmatrix}.$$

And,

$$z^T\hat{w} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/6 \\ 3/4 \end{pmatrix} = 1/6 \neq 0.$$

Therefore, since the solution of the problems (4.3) is not the solution of equation (1.1) and also do not satisfy the complementarity condition of the original LCP, the obtained results of this case are incorrect. In sum, we have seen that by the existing PPI models, the solution of equation (4.1) is not always the solution of the original LCP (1.1). In the next section, we present a new preconditioned strategy to solve the LCP (1.1).

5. Proposed model

In this section, we present our preconditioned GAOR algorithm to solve equation (1.1).

Algorithm 5.1. Preconditioned GAOR method for LCP

Step 1. Choose a symmetric preconditioner P, an initial vector $z^{(0)} \in R^n$, and the parameters $\alpha, \Omega, \varepsilon$.
Step 2. Set $\tilde{z}^{(0)} = P^{-1}z^{(0)}$, $\tilde{q} = Pq$, $\tilde{A} = \mathrm{PAP} = \tilde{D} - \tilde{L} - \tilde{U}$, where $\tilde{D}$, $\tilde{L}$, and $\tilde{U}$ are diagonal, strictly lower and strictly upper triangular parts of $\tilde{A}$, respectively.

Step 3. For $k = 0, 1, 2,$ do

$$\tilde{z}^{(k+1)} = (\tilde{z}^{(k)} - \tilde{D}^{-1}[-\alpha\Omega\tilde{L}\tilde{Z}^{(k+1)} + (\Omega\tilde{A} - \alpha\Omega\tilde{L})\tilde{z}^{(k)} + \Omega\tilde{q}])_+. \tag{5.1}$$

Step 4. If $\|P(\tilde{z}^{(k+1)} - \tilde{z}^{(k)})\| \leq \varepsilon$, then stop and set $z = P\tilde{z}^{(k+1)}$; otherwise, set $k = k + 1$ and go to Step 3.

It is worth mentioning that in Step 4, instead of $\|(\tilde{z}^{(k+1)} - \tilde{z}^{(k)})\| \leq \varepsilon$, we use $\|P(\tilde{z}^{(k+1)} - \tilde{z}^{(k)})\| \leq \varepsilon$. This is because the vectors $\tilde{z}^{(k+1)}$ and $\tilde{z}^{(k)}$ may be very close to each other and make the Algorithm 5.1 end, while the actual vectors $z^{(k+1)}$ and $z^{(k)}$ may be far apart. Now, if we test the complementarity condition of this algorithm, we have:

$$\tilde{z}^T\tilde{w} = (P^{-1}z)^T(\mathrm{PAP}P^{-1}z + Pq) = z^TP^{-T}PAz + z^TP^{-T}Pq.$$

Since P is symmetric, then $P^{-T} = P^{-1}$. Then we get:

$$\tilde{z}^T\tilde{w} = z^TAz + z^Tq = z^T(Az + q) = 0.$$

Therefore, the algorithm do not destroy the complementarity condition required for both the original problem and the preconditioned one. The other constraint is $\tilde{z} = p^{-1}z \geq 0$. We must choose a symmetric preconditioner with nonnegative inverse. We suggest to use *monomial matrix* (Def. 2.1(f)) as a preconditioner. Monomial matrices form a group and have a suitable properties. The product of monomial matrices is a monomial matrix. The inverse of a monomial matrix is again a monomial matrix and is equal to the transpose matrix in which every nonzero entry is replaced by its inverse [31–33]. So if we choose the preconditioner P with nonnegative elements, then $P^{-1} \geq 0$ and $\tilde{z} \geq 0$.

Next, we propose a new preconditioner as in the following algorithm:

Algorithm 5.2. A new preconditioner

Step 1. Choose $\gamma, d \in R - \{0\}$ and set $P_{n\times n}$ as a zeros matrix.
Step 2. For $r = 1, 2, \ldots, n$, do:
$k_r = \min\{j|a_{rj} = a_{jr}\}, j = r + 1, \ldots, n.$
If $k_r = \{\}$; ser $k_r = n + 1$.
Step 3. Set $k_v = \min\{k_r\}$.

Step 4. Set $p_{vk_v} = p_{k_v v} = -\gamma a_{vk_v}$,
Step 5. For $i = \{1, 2, \ldots, n\} - \{v, k_v\}$, set $p_{ii} = d$.

As a matter of fact, the new preconditioner is as follows:

$$P = T + S, \tag{5.2}$$

where,

$$T = (t_{ij})_{i=j} = \begin{cases} d, & \text{for } \; i = j \neq (v, k_v), \\ 0, & \text{otherwise}. \end{cases} \tag{5.3}$$

$$S = (s_{ij})_{i \neq j} = \begin{cases} -\gamma a_{vk_v}, & \text{for } \; i = v, j = k_v, \\ -\gamma a_{rk_i}, & \text{for } \; i = k_v, j = v, \\ 0, & \text{otherwise}. \end{cases} \tag{5.4}$$

It is easy to see that the new preconditioner P in equation (5.2) is a monomial matrix. For example, consider the following matrix:

$$A = \begin{pmatrix} 1 & -0.02 & -0.3 & -0.4 \\ -0.02 & 1 & -0.07 & -0.08 \\ -0.03 & -0.07 & 1 & -0.1 \\ -0.1 & -0.3 & -0.2 & 1 \end{pmatrix}.$$

Then by Algorithm 5.2, we have:

$$k_1 = 2, k_2 = 3, k_3 = 5, K_4 = 5.$$

And,

$$k_v = k_1 = 2, v = 1.$$

So, $p_{12} = p_{21} = -\gamma a_{12}, p_{33} = p_{44} = d$. Therefore, the preconditioner is as follows:

$$A = \begin{pmatrix} 0 & 0.02\gamma & 0 & 0 \\ 0.02\gamma & 0 & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & d \end{pmatrix}.$$

Next, we prove the convergence of Algorithm 5.2.

Theorem 5.3. *Let A be an M-matrix. Then $\tilde{A} = \mathrm{PAP}$ with the preconditioner (5.2) is also an M-matrix if $\gamma, d > 0$.*

Proof. Let P be the preconditioner (5.2). Then after some algebra, for $\tilde{A} = \mathrm{PAP} = (\tilde{a}_{ij})$ we have:

$$\tilde{a}_{ii} = \begin{cases} (\gamma a_{vk_v})^2, & \text{for } \; i = v, k_v, \\ d^2, & \text{otherwise}. \end{cases}$$

$$(\tilde{a}_{ij})_{i \neq j} = \begin{cases} \gamma^2 (a_{vk_v})^3, & \text{for } \; i = v, j = k_v, \\ -d\gamma a_{k_v j} a_{vk_v}, & \text{for } \; i = v, j \neq k_v, \\ -d\gamma a_{vj} a_{rk_v}, & \text{for } \; i = k_v, j \neq v, \\ \gamma^2 (a_{vk_v})^3, & \text{for } \; i = k_v, j = v, \\ -d\gamma a_{ik_v} a_{vk_v}, & \text{for } \; i \neq k_v, j = v, \\ -d\gamma a_{iv} a_{vk_v}, & \text{for } \; i \neq v, j = k_v, \\ d^2 a_{ij}, & \text{otherwise}. \end{cases}$$

Therefore, if $\gamma, d > 0$ we have $\tilde{a}_{ii} > 0$ and $(\tilde{a}_{ij})_{i \neq j} < 0$, *i.e.*, $\tilde{A}$ is a Z-matrix with positive diagonal elements. Now, by using Definition 2.1(d), we prove that $\tilde{A}$ is an M-matrix. Suppose that $\tilde{A}x = y \geq 0$. Then $x = \tilde{A}^{-1}y$. Since A is an M-matrix, then by Definition 2.1(c), $A^{-1} \geq 0$. Also since $P^{-1} \geq 0$, we have $P^{-1}A^{-1}P^{-1} \geq 0$. So, $x = \tilde{A}^{-1}y \geq 0$ which completes the proof. □

Theorem 5.4. *Suppose that A is an M-matrix. Then, for any initial vector $z^{(0)} \in R^n$, the iterative sequence $\{z^{(k)}\}$ generated by the Algorithm 5.1, converges to the unique solution z^* of the* LCP *(A, q).*

Proof. From Theorem 5.3, $\tilde{A} = \mathrm{PAP}$ is an M-matrix. Thus from Lemma 3.1, Algorithm 5.1 converges to the unique solution of LCP(A, q). □

In the end of this section, we solve Example 4.1 by our model.

Consider LCP (A, q) of Example 4.1. By choosing $d = 1, \gamma = 1$, the preconditioner (5.2) is as follows:

$$P = \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}.$$

Then by Algorithm 5.1,

$$\tilde{A} = \begin{pmatrix} 1/4 & -1/4 & -1/8 \\ -1/6 & 1 & -1/6 \\ -1/8 & -1/4 & 1/4 \end{pmatrix}, \tilde{q} = \begin{pmatrix} 1/2 \\ -1 \\ 1/2 \end{pmatrix}, \tilde{z} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Therefore,

$$z = P\tilde{z} = [0, 1, 0]^T.$$

6. Numerical results

In this section, we give an example to illustrate the results obtained in previous sections. These examples computed with "MATLAB R 2014 a" on a personal computer. Here, for GAOR method we set $\alpha = 1, \Omega = I$. In addition, for preconditioner (5.2), we take $\gamma = 0.5$, $d = 0.2$. The initial approximation of z is $z^{(0)} = (1, 1, \ldots, 1)^T \in R^n, n = m^2$. The stopping criterion imposed in all the runs is as follows:

$$\|(z^{(k+1)}) - z^{(k)})\|_\infty \leq 10^{-16},$$
$$\|P(\tilde{z}^{(k+1)}) - \tilde{z}^{(k)})\|_\infty \leq 10^{-16}.$$

Example 6.1 ([14])**.** Consider the sparse LCP (A, q), where:

$$A = \bar{A} + 4I_n \in R^{n\times n}, \bar{A} = \mathrm{tridiag}(-I_m, S, -I_m) \in R^{n\times n},$$
$$S = \mathrm{tridiag}(-1, 4, -1) \in R^{m\times m},$$
$$z^* = [1, 2, 1, 2, \ldots]^T \in R^n, q = -Az^*.$$

Evidently, A is an M-matrix. So, the LCP (A, q) has a unique solution. Then, we solved $n \times n$ the M-matrix yielded by the GAOR iterative methods, and the preconditioned GAOR (Algorithm 5.1). In Table 1, for different problem sizes of n, we list the iteration steps and the CPU times with respect to the GAOR and PGAOR methods. We also report the spectral radius of iteration matrices of GAOR and PGAOR by ρ and $\tilde{\rho}$, respectively.

From the table, we observe that, all tested methods can quickly compute a satisfactory approximation to the solution of the LCP(A, q). Moreover, we can see that the spectral radius of the preconditioned iterative method is superior to the basic iterative method. Also, the PGAOR method require less iteration steps than the GAOR method. Furthermore, the PGAOR method costs less computing times than the GAOR method. Therefore, in terms of computing efficiency, the PGAOR method outperform the classic GAOR method.

TABLE 1. Result of Example 6.1.

Method	GAOR			PGAOR		
n	Iter	CPU	ρ	Iter	CPU	$\tilde{\rho}$
25	24	0.001524	0.18750	13	0.000720	0.00728
100	30	0.015129	0.23015	15	0.009381	0.00919
400	33	0.167703	0.24444	15	0.076553	0.00977
900	33	1.077159	0.24744	15	0.515363	0.00989
1600	33	4.530280	0.24853	15	2.016781	0.00994
2500	34	9.778764	0.24905	15	4.312249	0.00996

7. Conclusions

It is shown that the existing preconditioned projective iterative methods proposed do not always provide the solution original for linear complementarity problem and hence the obtained results are not always correct. Therefore, in this paper, we have proposed a new strategy of the preconditioned GAOR method for linear complementarity problem. We have also studied how the iterative method for LCP is affected if the system is preconditioned by the new model.

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