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GLOBAL DISTRIBUTION CENTER NUMBER OF SOME GRAPHS AND AN ALGORITHM

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Abstract. The global center is a newly proposed graph concept. For a graph $G = (V(G), E(G))$, a set $S \subseteq V(G)$ is a *global distribution center* if every vertex $v \in V(G) \backslash S$ is adjacent to a vertex $u \in S$ with $|N[u] \cap S| \geq |N[v] \cap (V(G) \backslash S)|$, where $N(v) = \{u \in V(G) | uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$. The global distribution center number of a graph G is the minimum cardinality of a global distribution center of G. In this paper, we investigate the global distribution center number for special families of graphs. Furthermore, we develop a polynomial time heuristic algorithm to find the set of the global distribution center for general graphs.

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1. Introduction

The study of networks has become an important area of multidisciplinary research involving computer science, mathematics, chemistry, social sciences, informatics and other theoretical and applied sciences [12]. Computer networks can be modeled on the grounds of graphs where hosts, servers or hubs can be considered as vertices and edges as connecting medium between them. The effectiveness and robustness of a network for link or node failures are very important concepts in the design of communication networks. In the literature, various measures have been defined to measure the robustness of networks, and a variety of graph theoretic parameters have been used to derive formulas to calculate network vulnerability. Recently, several interesting works have been studied on graph theoretic parameters in [1,2,4]. The vertex is actually a possible location to find fault or some damaged devices in a computer network [8]. A distribution center for a set of products is a structure or a group of units used to store goods that are to be distributed to retailers, to wholesalers, or directly to consumers. Distribution centers are usually thought of as being demand driven. Very recently, Desormeaux *et al.* have defined a new concept for distribution as namely distribution center number and global distribution center number in [7]. Next we give some basic terminology for graphs.

Keywords. Network design and communication, complex networks, distribution centers, global distribution center number, trees.

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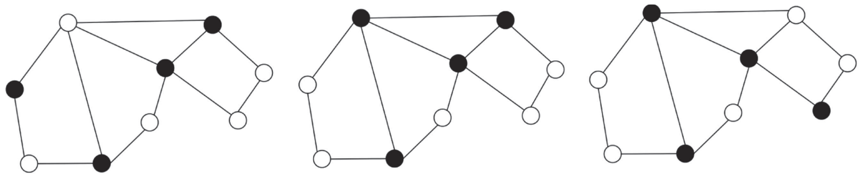

FIGURE 1. The alternatives gdc-sets of G.

A network is usually described by an undirected simple graph. Let $G = (V(G), E(G))$ be a simple undirected graph of order n and size m. We begin by recalling some standard definitions that we need throughout this paper. For any vertex $v \in V(G)$, the *open neighborhood* of v is $N(v) = \{u \in V(G)|uv \in E(G)\}$ and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V(G)$, its open neighborhood is the set $N(S) = \bigcup_{v \in S} N(v)$, and its closed neighborhood is the set $N[S] = N(S) \cup S$. The *boundary* of S, denoted $\partial(S)$ is $\partial(S) = N(S) \cap (V(G)\backslash S)$, that is, the boundary is the set of vertices in $V(G)\backslash S$ that are adjacent to at least one vertex in S. The *degree of vertex* v in G denoted by $\deg(v)$, that is, the size of its open neighborhood. The *distance* $d(u, v)$ between two vertices u and v in G is the length of a shortest path between them. The *diameter* of G, denoted by $\text{diam}(G)$ is the largest distance between two vertices in $V(G)$. The *maximum degree* of G is $\max\{\deg(v)|v \in V(G)\}$ and is denoted by $\Delta(G)$. The *minimum degree* of G is $\min\{\deg(v)|v \in V(G)\}$ and is denoted by $\delta(G)$. A vertex v is said to be *pendant* vertex if $\deg(v) = 1$. A vertex u is called *support* if u is adjacent to a pendant vertex [11]. A set $S \subseteq V(G)$ is a *dominating set* if every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The minimum cardinality taken over all dominating sets of G is called the *domination number* of G and is denoted by $\gamma(G)$ [8]. A set $S \subseteq V(G)$ is a *total dominating set* if every vertex in $V(G)$ is adjacent to at least one vertex in S. The minimum cardinality taken over all total dominating sets of G is called the *total domination number* of G and is denoted by $\gamma_t(G)$ [11].

A non-empty set of vertices $S \subseteq V(G)$ is a *distribution center* if every vertex $v \in \partial(S)$ is adjacent to a vertex $u \in S$ with $|N[u] \cap S| \geq |N[v] \cap (V(G)\backslash S)|$. The minimum cardinality of a distribution center of a graph G is the *distribution center number* $\text{dc}(G)$, and a distribution center of G with cardinality $\text{dc}(G)$ is called a dc-*set* of G [7]. Similarly, a set $S \subseteq V(G)$ is a *global distribution center*, if every vertex $v \in V(G)\backslash S$ is adjacent to a vertex $u \in S$ with $|N[u] \cap S| \geq |N[v] \cap (V(G)\backslash S)|$. The *global distribution center number* $\text{gdc}(G)$ is the minimum cardinality of a global distribution center of G. A global distribution center with cardinality $\text{gdc}(G)$ is called a gdc-*set* of G [7]. For example, we consider the graph G of order 9 and size 12 in Figure 1, where some alternative gdc-sets of G are illustrated by the set of darkened vertices. Clearly, $\text{gdc}(G) = 4$ is obtained.

Our aim in this paper is to consider the computing the global distribution center number of some trees, grid graphs and bipartite graphs. In Section 2, well-known basic results are given for the global distribution center number. Then, the global distribution center numbers of some trees, grids and bipartite graphs are computed in Section 3. We give in Section 4, a polynomial time heuristic algorithm to find $\text{gdc}(G)$ for an arbitrary graph G and discuss the algorithm. Finally, in Section 5, we present our conclusions.

2. Basic results

In this section, well-known basic results are given with regard to global distribution center numbers of graphs.

Theorem 2.1 ([7]). *For any graph G, $\gamma(G) \leq \text{gdc}(G)$ and $\text{dc}(G) \leq \text{gdc}(G) \leq |V(G)|$.*

Theorem 2.2 ([7]). *For the non-trivial path P_n of order n, $\text{gdc}(P_n) = \lfloor \frac{n}{2} \rfloor$.*

Theorem 2.3 ([7]). *For the cycle C_n of order n, $\text{gdc}(C_n) = \lceil \frac{n}{2} \rceil$.*

Theorem 2.4 ([7]). *For the complete graph K_n of order n, $\text{gdc}(K_n) = \lceil \frac{n}{2} \rceil$.*

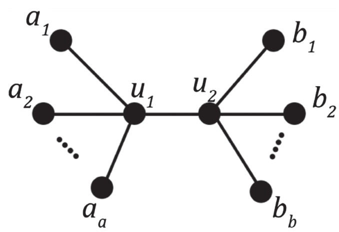

FIGURE 2. The graph $S(a,b)$.

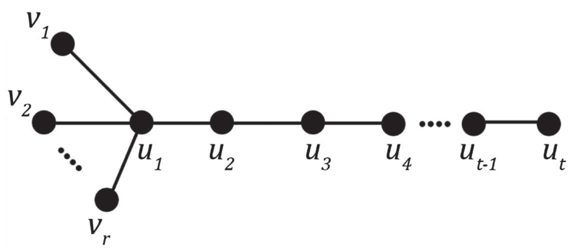

FIGURE 3. The graph $C(t,r)$.

Theorem 2.5 ([7])**.** *For the wheel graph W_n of order $n \geq 4$,* $\mathrm{gdc}(W_n) = 3$*.*

Theorem 2.6 ([7])**.** *For the complete bipartite graph $K_{p,q}$,where $p \leq q$,* $\mathrm{gdc}(K_{p,q}) = p$*.*

Theorem 2.7 ([7])**.** *For any non-trivial tree T,* $\mathrm{gdc}(T) \leq \lfloor \frac{5\gamma_t(T)-2}{4} \rfloor$*.*

Theorem 2.8 ([7])**.** *If G is a bipartite graph with no isolated vertices, then* $\mathrm{gdc}(G) \leq \lfloor \frac{n}{2} \rfloor$*.*

3. Global distribution center numbers of some trees, grids and bipartite graphs

Definition 3.1 ([9])**.** The double star graph $S(a,b)$, where $a,b \geq 0$, is the graph consisting of the union of two star graphs $K_{1,a}$ and $K_{1,b}$ together with an edge joining their centers. We show the graph $S(a,b)$ in Figure 2.

Theorem 3.2. *Let $S(a,b)$ be a double star graph of order $a+b$, where $a,b \geq 0$. Then,* $\mathrm{gdc}(S(a,b)) = 2$*.*

Proof. Let the vertices of $S(a,b)$ be $V(S(a,b)) = V(K_{1,a}) \cup V(K_{1,b})$, and let the center vertices of $K_{1,a}$ and $K_{1,b}$ be u_1 and u_2, respectively. We have $\deg(u_1) = a+1$, $\deg(u_2) = b+1$ and $\deg(a_i) = \deg(b_i) = 1$ for every vertices a_i and b_i, where vertices a_i are the vertices of $K_{1,a}$ and vertices b_i are the vertices of $K_{1,b}$. Let S be a gdc-set of $S(a,b)$, also let $|S| = 1$. If S includes a vertex a_i, then we get $|N[a_i] \cap \{a_i\}| < |N[u_1] \cap (V(S(a,b)\backslash\{a_i\})|$. Similarly, if S includes a vertex b_i, then we get $|N[b_i] \cap \{b_i\}| < |N[u_1] \cap (V(S(a,b)\backslash\{b_i\})|$. Moreover, if S includes the vertex u_1, then we get $|N[u_1] \cap \{u_1\}| < |N[u_2] \cap (V(S(a,b)\backslash\{u_1\})|$. Similarly, if S includes the vertex u_2, then we get $|N[u_2] \cap \{u_2\}| < |N[u_1] \cap (V(S(a,b)\backslash\{u_2\})|$. Thus, we obtain $|S| \geq 2$, that is, $\mathrm{gdc}(S(a,b)) \geq 2$. Clearly, $\gamma_t(S(a,b)) = 2$. So, we get $\mathrm{gdc}(S(a,b)) \leq 2$ by the Theorem 2.7. As a result, $\mathrm{gdc}(S(a,b)) = 2$ is obtained. □

Definition 3.3 ([3])**.** The comet graph $C(t,r)$ is the graph obtained by identifying one end of the path P_t with the center of the star graph $K_{1,r}$. We show the graph of $C(t,r)$ in Figure 3.

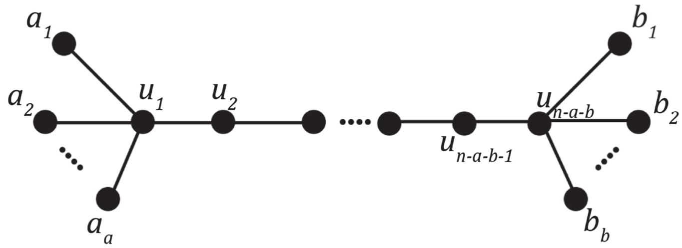

FIGURE 4. The graph $\mathrm{DC}(n, a, b)$.

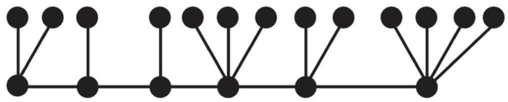

FIGURE 5. The graph $P_6^*(2, 1, 1, 3, 2, 4)$.

Theorem 3.4. *If* $C(t, r)$ *is a comet graph of order* $t + r$*, then* $\mathrm{gdc}(C(t, r)) = 1 + \left\lfloor \frac{t-1}{2} \right\rfloor$.

Proof. Let $V(C(t, r)) = V(P_{t-1}) \cup V(K_{1,r})$, where $V(P_{t-1}) = \{u_2, u_3, \ldots, u_t\}$ and $V(K_{1,r}) = \{u_1, v_1, \ldots, v_r\}$. Let S be a gdc-set of $C(t, r)$. By the definition of gdc-set, the set S must include the vertex u_1. In the continuation of the proof, the vertices that must be added to S have to be as in the proof of Theorem 2.2. Thus, we obtain $\mathrm{gdc}(C(t, r)) = 1 + \left\lfloor \frac{t-1}{2} \right\rfloor$. □

Definition 3.5 ([6]). For $a, b \geq 1$ and $n \geq a + b + 2$ the double comet $\mathrm{DC}(n, a, b)$ is a tree which is composed of a path containing $n - a - b$ vertices with a pendant vertices attached to one of ends of the path and b pendant vertices attached to the other end of the path. Thus, $\mathrm{DC}(n, a, b)$ has n vertices and $a + b$ leaves. We show the graph $\mathrm{DC}(n, a, b)$ in Figure 4.

Theorem 3.6. *If* $\mathrm{DC}(n, a, b)$ *is a double comet graph with* $a, b \geq 2$ *and* $n \geq a + b + 3$*, then* $\mathrm{gdc}(\mathrm{DC}(n, a, b)) = 2 + \left\lfloor \frac{n-a-b-2}{2} \right\rfloor$.

Proof. Let $V(\mathrm{DC}(n, a, b)) = V(P_{n-a-b-2}) \cup V(K_{1,a}) \cup V(K_{1,b})$, where $V(P_{n-a-b-2}) = \{u_2, u_3, \ldots, u_{n-a-b-1}\}$, $V(K_{1,a}) = \{u_1, a_1, a_2, \ldots, a_a\}$ and $V(K_{1,b}) = \{u_{n-a-b}, b_1, b_2, \ldots, b_b\}$. Let S be a gdc-set of $\mathrm{DC}(n, a, b)$. Since the size of S must be minimum, the vertices u_1 and u_{n-a-b} must be in S. Hence, the remaining vertices form a path graph of order $(n - a - b - 2)$. Clearly, no three consecutive vertices on the path are in $V - S$. So, the set S is formed similarly as in the proof of the Theorem 2.1. Thus, we obtain $\mathrm{gdc}(\mathrm{DC}(n, a, b)) = 2 + \left\lfloor \frac{n-a-b-2}{2} \right\rfloor$. □

Definition 3.7 ([10]). Let $p_1, p_2, \ldots, p_n$ be non-negative integers and the graph G be such a graph, where $|V(G)| = n$. The *thorn graph* of the graph G with parameters $p_1, p_2, \ldots, p_n$ is obtained by attaching p_i new vertices of degree one to the vertex u_i of the graph G, where $i = \overline{1, n}$. The thorn graph of the graph G will be denoted by G^* or if the respective parameters need to be specified, by $G^*(p_1, p_2, \ldots, p_n)$. We display the graph $P_6^*(2, 1, 1, 3, 2, 4)$ in Figure 5.

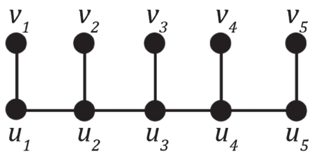

FIGURE 6. The graph $P_5 \odot K_1$.

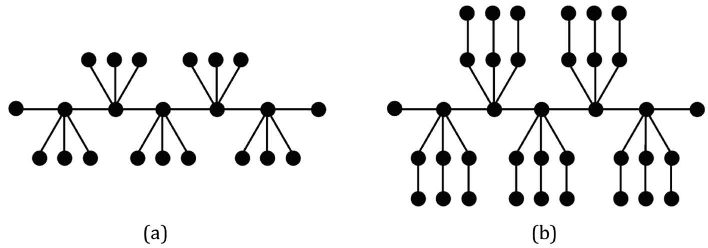

FIGURE 7. *Panel a*: the graph $C_{(3,0)}P_7$, and *Panel b*: the graph $C_{(3,1)}P_7$.

Theorem 3.8. *If G^* is a thorn graph of any graph G of order n, then* $\text{gdc}(G^*) = n$.

Proof. By the definition of the thorn graph, we have a lot of vertices with degree 1. Clearly, the number of these vertices is greater than n. Let p_i be a vertex with degree 1. Clearly, $|N[p_i]| = 2$ and let S be a gdc-set of G^*. If S includes all vertices of G, then S is a gdc-set of G^*. Because, we have $|N[u_i] \cap S| \geq |N[p_i] \cap (V(G^*) \backslash S)|$ for every vertex $u_i \in V(G^*)$. So, $\text{gdc}(G^*) \leq n$. If S includes vertices p_i with degree 1, then S can not include the vertices of G. But, since we have $|N[p_i]| = 2$ for every vertex p_i, $|N[u_i] \cap S| = 1$ is obtained. Furthermore, we get $|N[p_i] \cap (V(G^*) \backslash S)| = 1$ if S does not include the vertex p_i. It is easy to see that the set S must not include the vertices p_i by the definition of the gdc-set. Clearly, we have $\text{gdc}(G^*) \geq n$. Thus, $\text{gdc}(G^*) = n$ is obtained. □

Definition 3.9 ([13])**.** The graph obtained by joining a pendant edge at each vertex of a path P_n is called a comb graph and is denoted by $P_n \odot K_1$. We show the graph $P_5 \odot K_1$ in Figure 6.

Corollary 3.10. *If $P_n \odot K_1$ is a comb graph of order $2n$, then* $\text{gdc}(P_n \odot K_1) = n$.

Proof. The proof is very similar to the proof of Theorem 3.8. □

Definition 3.11 ([14])**.** $C_{(m,0)}P_n$ is a generalized Caterpillar obtained from the path graph P_n by attaching m vertices of degree one to each vertex of degree two of P_n. The tree $C_{(m,1)}P_n$ is a generalized Caterpillar obtained from the path graph P_n by attaching m vertices of degree two to each vertex of degree two of P_n. We show the graph $C_{(3,0)}P_7$ and $C_{(3,1)}P_7$ in Figure 7.

Corollary 3.12. *If $C_{(m,0)}P_n$ is a generalized Caterpillar graph of order $n + m(n-2)$, where $m \geq 2$, then* $\text{gdc}(C_{(m,0)}P_n) = n - 2$.

Proof. The proof is very similar to the proof of Theorem 3.8. □

Theorem 3.13. *If $C_{(m,1)}P_n$ is a generalized Caterpillar graph of order $n + m(n-2)$, where $m \geq 2$, then* $\text{gdc}(C_{(m,1)}P_n) = m(n-2) + \left\lceil \frac{n-2}{2} \right\rceil$.

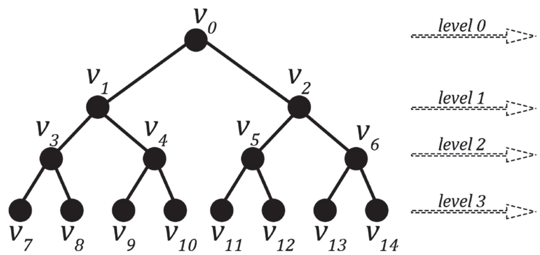

FIGURE 8. The tree H_3^2.

Proof. The graph $C_{(m,1)}P_n$ has $n + m(n-2)$ vertices. Let $V(C_{(m,1)}P_n) = V_1 \cup V_2 \cup V_3$, where $V_1 = \{u_i \in V(P_n) : 1 \leq i \leq n\}$, $V_2 = \{v_i \in V(C_{(m,1)}P_n) - V(P_n) : \deg(v_i) = 2 \text{ and } 1 \leq i \leq m(n-2)\}$ and $V_3 = \{w_i \in V(C_{(m,1)}P_n) - V(P_n) : \deg(w_i) = 1 \text{ and } 1 \leq i \leq m(n-2)\}$.

It is clear that $\deg(u_1) = \deg(u_n) = 1$ and $\deg(u_i) = 2$, where $i \in \{2, \ldots, n-1\}$ for the vertices of V_1. Let S be a gdc-set of $C_{(m,1)}P_n$. Since $\deg(w_i) = 1$ and $\deg(v_i) = 2$ for each vertex $w_i \in V_3$ and $v_i \in V_2$, and also to reduce the size S, the set S must include the all vertices of V_2. Furthermore, S must include the vertices $\{u_2, u_4, \ldots, u_{n-3}, u_{n-1}\}$. Therefore, $|S| = m(n-2) + \left\lceil \frac{n-2}{2} \right\rceil$. It is also clear that this set S for graph $C_{(m,1)}P_n$ is unique. No other set with gdc-set is found. Hence, we obtain $\text{gdc}(C_{(m,1)}P_n) = m(n-2) + \left\lceil \frac{n-2}{2} \right\rceil$. □

Definition 3.14 ([5])**.** A complete k-ary tree H_n^k with depth n is all leaves have the same depth and all internal vertices have degree k. A complete k-ary tree has $\frac{k^{n+1}-1}{k-1}$ vertices and $\frac{k^{n+1}-1}{k-1} - 1$ edges. We show the tree H_3^2 in Figure 8.

Theorem 3.15. *If H_n^k is a complete k-ary tree of order $\frac{k^{n+1}-1}{k-1}$, where $n \geq 2$ and $k \geq 2$, then*

$$\text{gdc}(H_n^k) = \begin{cases} \left(\sum_{i=0}^{(n/3)-1} k^{3i+2} \right) + 1 & , n \equiv 0 (\text{mod}\, 3); \\ \left(\sum_{i=0}^{\lfloor n/3 \rfloor} k^{3i} \right) & , n \equiv 1 (\text{mod}\, 3); \\ \left(\sum_{i=0}^{\lfloor n/3 \rfloor} k^{3i+1} \right) & , n \equiv 2 (\text{mod}\, 3). \end{cases}$$

Proof. Let S be a gdc-set of H_n^k. It is easy to see that the set S is a minimum dominating set of H_n^k, that is, we have three cases depending on n.

Case 1: $n \equiv 0 (\text{mod}\, 3)$.

The minimum gdc-set of H_n^k contains both the vertices on the levels $(n-1-3i)$ for $0 \leq i \leq \lfloor n/3 \rfloor$ and the vertex on the level 0. Thus, $|S| = \text{gcd}(H_n^k) = \left(\sum_{i=0}^{(n/3)-1} k^{3i+2} \right) + 1.$

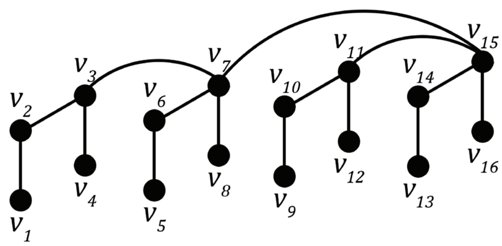

FIGURE 9. The binomial tree B_4.

Case 2: $n \equiv 1 (\mathrm{mod}\, 3)$.

The minimum gdc-set of H_n^k contains the vertices on the levels $(n-1-3i)$ for $0 \leq i \leq \lfloor n/3 \rfloor$. Thus, $|S| = \mathrm{gcd}(H_n^k) = \sum_{i=0}^{\lfloor n/3 \rfloor} k^{3i}$.

Case 3: $n \equiv 2 (\mathrm{mod}\, 3)$.

The minimum gdc-set of H_n^k contains the vertices on the levels $(n-1-3i)$ for $0 \leq i \leq \lfloor n/3 \rfloor$. Thus, $|S| = \mathrm{gcd}(H_n^k) = \sum_{i=0}^{\lfloor n/3 \rfloor} k^{3i+1}$.

□

Definition 3.16 ([5])**.** The binomial tree of order $n \geq 0$ with root R is the tree B_n defines as follows:

(i) If $n = 0$, then $B_n = B_0 = R$, *i.e.*, the binomial tree of order zero consists of a single root R.
(ii) If $n > 0$, then $B_n = R, B_0, B_1, \ldots, B_{n-1}$, *i.e.*, the binomial tree of order $n > 0$ comprises the root R and n binomial subtrees $B_0, B_1, \ldots, B_{n-1}$. We show the binomial trees B_4 and B_5 in Figures 9 and 10, respectively.

Theorem 3.17. *If B_n is a binomial tree of order 2^n, then* $\mathrm{gdc}(B_n) = 2^{n-1}$ *for* $n \geq 1$.

Proof. The binomial tree B_n has 2^n vertices and B_n consists of two copies of B_{n-1}, B_{n-1} consists of two copies of B_{n-2} and so on. For $n \leq 3$, $\mathrm{gdc}(B_1) = 1, \mathrm{gdc}(B_2) = 2$ and $\mathrm{gdc}(B_3) = 4$. Let $n = 4$, and let S_1 be a gdc-set of B_4. It is easy to see that we have $\mathrm{gdc}(B_4) = 8$ for the set $S_1 = \{v_1, v_3, v_5, v_7, v_{10}, v_{12}, v_{13}, v_{15}\}$ (see Fig. 9).

Let S be a gdc-set of B_5. Since B_5 consists of two copies of B_4, we must also take vertices which are comprises the vertices $v_1, v_3, v_5, v_7, v_{10}, v_{12}, v_{13}, v_{15}$ for the second copies of B_4, that is, the vertices $v'_1, v'_3, v'_5, v'_7, v'_{10}, v'_{12}, v'_{13}, v'_{15}$ must be taken into S as in Figure 10. Thus, we get $S = \{v_1, v_3, v_5, v_7, v_{10}, v_{12}, v_{13}, v_{15}, v'_1, v'_3, v'_5, v'_7, v'_{10}, v'_{12}, v'_{13}, v'_{15}\}$. Clearly, we obtain $\mathrm{gdc}(B_5) = 2 \cdot \mathrm{gdc}(B_4) = 16$.

By the similar way, $\mathrm{gdc}(B_6) = 2 \cdot \mathrm{gdc}(B_5) = 32$ and $\mathrm{gdc}(B_7) = 2 \cdot \mathrm{gdc}(B_6) = 64$ are obtained. For $n \geq 5$, we get the following recurrence formula:

$$\mathrm{gdc}(B_n) = 2 \cdot \mathrm{gdc}(B_{n-1}) \text{ for } n \geq 5. \quad (3.1)$$

By the formula (3.1), we have

$$\begin{aligned}\operatorname{gdc}(B_n) &= 2 \cdot \operatorname{gdc}(B_{n-1})\\ &= 2(2 \cdot \operatorname{gdc}(B_{n-2})) = 2^2 \cdot \operatorname{gdc}(B_{n-2})\\ &\vdots\\ &= 2^{n-4} \cdot \operatorname{gdc}(B_4).\end{aligned}$$

Furthermore, we obtain

$$\operatorname{gdc}(B_n) = 2^i \cdot \operatorname{gdc}(B_{n-i}), \text{ where } 1 \le i \le n-1. \tag{3.2}$$

The formula (3.2) can be proved by mathematical induction. When $i = 1$, we have $\operatorname{gdc}(B_n) = 2^1 \cdot \operatorname{gdc}(B_{n-1})$ and it is true by the formula (3.1). We assume that the result is true for $i = k$ and we will prove that (3.2) is true for $i = k + 1$. By induction hypothesis and (3.1), we get:

$$\begin{aligned}\operatorname{gdc}(B_n) &= 2^k \cdot \operatorname{gdc}(B_{n-k})\\ &= 2^k(2 \cdot \operatorname{gdc}(B_{n-k-1}))\\ &= 2^{k+1} \cdot \operatorname{gdc}(B_{n-k-1}).\end{aligned}$$

This implies that the statement is true for $i = k + 1$. So, we obtain

$$\operatorname{gdc}(B_n) = 2^i \cdot \operatorname{gdc}(B_{n-i}), \text{ where } 1 \le i \le n-4.$$

Since the initial condition is $n = 4$, which is achieved for $i = n - 4$, we have the following formula for solution (3.2):

$$\begin{aligned}\operatorname{gdc}(B_n) &= 2^{n-4} \cdot \operatorname{gdc}(B_{n-(n-4)})\\ &= 2^{n-4} \cdot \operatorname{gdc}(B_4)\\ &= 8(2^{n-4})\\ &= 2^{n-1}.\end{aligned}$$

□

Theorem 3.18. *For an $m \times n$ grid graph $P_{m,n}$, $\operatorname{gdc}(P_{m,n}) = m \left\lfloor \frac{n}{2} \right\rfloor$, if $m \ge 2$ and $n \ge 3$.*

Proof. A two-dimensional grid graph is an $m \times n$ lattice graph $P_{m,n}$ that can be represented as a Cartesian product $P_m \times P_n$ of path graphs on m and n vertices. By the Theorem 2.2, we have $\operatorname{gdc}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$ for a nontrivial path P_n of order n. This implies $\operatorname{gdc}(P_{m,n}) = m \left\lfloor \frac{n}{2} \right\rfloor$. □

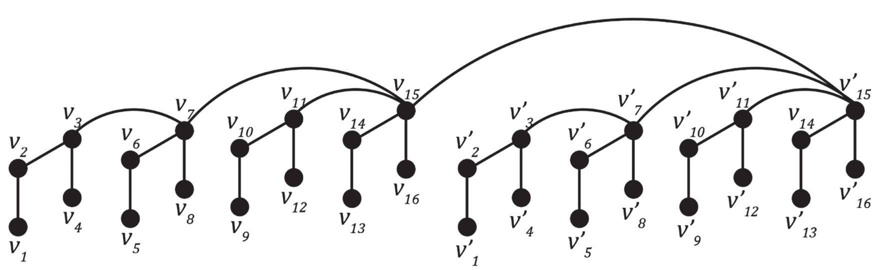

FIGURE 10. The binomial tree B_5.

Theorem 3.19. *Let G be a connected bipartite graph with bipartition V_1 and V_2, where $|V_i| = n_j$, $\delta_j = \min\{\deg(v) : v \in V_j\}$ and $\Delta_j = \max\{\deg(v) : v \in V_j\}$, for $j = 1, 2$. Let s_j and b_j be the numbers of the minimum and maximum degree's of V_j, for $j = 1, 2$.*

(i) *Let $\Delta_1 = n_2$, $\Delta_2 = n_1$ and $\delta_j = 1$, for $j = 1, 2$. If $b_j = 1$ and $s_j = n_j - 1$, for $j = 1, 2$, then $\mathrm{gdc}(G) = 2$.*
(ii) *Let $\Delta_1 = n_2$, $\Delta_2 = n_1$. If $b_j = 1$ for $j = 1, 2$ and minimum degree is less than two and the degree's of the remaining vertices are not maximum, then $\mathrm{gdc}(G) = 2$.*
(iii) *Let $\Delta_1 = n_2$, and $\Delta_2 = n_1$. If k is the number of vertices whose degree are greater than three, then $\mathrm{gdc}(G) \leq \min\left\{2 + k, \left\lfloor \frac{n_1 + n_2}{2} \right\rfloor\right\}$.*

Proof. Let $v_x \in V_1$, $v_y \in V_2$, $\deg(v_x) = n_2$, $\deg(v_y) = n_1$ and let S be a gdc-set of G. Since the size of S must be minimum, the vertices v_x and v_y must be in S.

(i) Clearly, each vertex of $V(G) - \{v_x, v_y\}$ is adjacent to the vertices v_x and v_y. Since $\delta_j = 1$ and $s_j = n_j - 1$ for $j = 1, 2$, we obtain $|N[v_w] \cap (V(G) - S)| = 1$ for each vertex $v_w \in V(G) - \{v_x, v_y\}$. So, $|N[v_x] \cap S| = |N[v_y] \cap S| = 2$. Thus, we get $\mathrm{gdc}(G) = 2$.
(ii) The proof is very similar to the previous case (i). In this case, $|N[v_w] \cap (V(G) - S)| \leq 2$ is obtained for each vertex $v_w \in V(G) - \{v_x, v_y\}$. Furthermore, we know $|N[v_x] \cap S| = |N[v_y] \cap S| = 2$. Thus, we get $\mathrm{gdc}(G) = 2$.
(iii) Let v_t be a vertex whose degree is greater than three in $V(G) - \{v_x, v_y\}$. If $v_t \in V_1$, then the vertex is adjacent to $\deg(v_t)$-vertices in V_2. Similarly, if $v_t \in V_2$, then the vertex is adjacent to $\deg(v_t)$-vertices in V_1. Thus, the vertex v_t must be added to gdc-set of G. Because, we get $|N[v_t] \cap (V(G) - S)| \geq 3$. Furthermore, the vertices with degree three can be added to gdc-set of G. Since the number of these vertices is k, we get $\mathrm{gdc}(G) \leq \lfloor \frac{n}{2} \rfloor$. Furthermore, we have by the Theorem 2.8 for any connected bipartite graph G of order n. As a result, we get $\mathrm{gdc}(G) \leq \min\left\{2 + k, \left\lfloor \frac{n_1 + n_2}{2} \right\rfloor\right\}$.

□

4. A heuristic algorithm for computing global distribution center number

Desormeaux *et al.* who proposed distribution centers on graphs ask the following question in [7]: What is the complexity of determining the value of $\mathrm{gdc}(G)$ for an arbitrary graph G? In particular, can these two values be determined in polynomial time for any tree T? In this section, we develop a polynomial time heuristic algorithm given as a pseudocode in Algorithm 1 to find $\mathrm{gdc}(G)$ for an arbitrary graph G.

In line 3 of Algorithm 1, the matrix M is obtained using the Floyd–Warshall algorithm. It is clear that a global distribution center does not contain any vertex with degree one. So, $Possible\_S$ contains possible vertices which may be in the global distribution center. We sort the vertices in $Possible\_S$ as S in decreasing order by the $rowSum(M)$. The first vertex in S is the farthest vertex from the center of the graph, that is, it is the most likely to be deleted in S. The for loop of lines 6–8 computes separately the number of neighbors in S and in $V - S$ of each vertex in V. Within the for loop of lines 9–34, we determine whether a vertex in S is deleted. This loop invariant works as follows:

If any vertex $i \in S$ has no neighbor in S, then it is not deleted in S and the loop continues with the next vertex in S to be examined. Otherwise (lines 17–20), we check the constraint in the definition of global distribution center (GDC) is satisfied for the neighbors of i in S after i is deleted. In line 22, if any neighbors of i in $V - S$ has no a neighbor in S other than i, then i is not deleted. In lines 25–28, we check the constraint in the definition of GDC is satisfied for the neighbors of i in $V - S$ after i is deleted. If a vertex i is deleted in S, then the number of neighbors in S and in $V - S$ of vertex i is updated as in lines 31–33. In the worst case of Algorithm 1, the for loop in line 9 runs at most $|V|$ times. The union of $neighbor\_S$ and $neighbor\_V$ is the list of adjacents of a vertex in the graph. The inner loops in line 16 and 24 run totally as the degree of a vertex in the graph. By the Handshaking Lemma, the GDC algorithm runs in $O|V|$ time.

Algorithm 1: Global Distribution Center Algorithm

```
Data: An undirected graph G = (V, E)
Result: A non-empty set S ⊆ V which is a general distribution center
begin
    n = |V|;
    M = A matrix of dimension nxn with all-pairs shortest path lenghts in G;
    Possible_S = The set of vertices whose degrees are greater than one;
    S = sort the vertices in Possible_S in decreasing order by the rowSum(M);
    for i = 1 to n do
        neighbor_S_number[i]=the number of neighbors in S of vertex i ∈ V ;
        neighbor_V_number[i]=the number of neighbors in V−S of vertex i ∈ V ;
    for each vertex i ∈ S do
        v_erasable=True;
        u_erasable=True;
        neighbor_S = a list of neighbor vertices of i in S;
        neighbor_V = a list of neighbor vertices of i in V−S;
        if neighbor_S is empty then
            continue; //next iteration of i
        else
            for each vertex u ∈ neighbor_S do
                if neighbor_S_number[u] ≤ neighbor_V_number[i] then
                    u_erasable=False;
                    break; //terminates the current loop
        for each vertex v ∈ neighbor_V do
            if neighbor_S_number[v] ≤ 1 then
                v_erasable=False;
                break //terminates the current loop
            for each vertex w ∈ Adj[v] in S do
                if neighbor_S_number[w] ≤ neighbor_V_number[v] then
                    v_erasable=False;
                    break //terminates the current loop
        if v_erasable = True and u_erasable=True then
            Delete vertex i in S;
            for each vertex u ∈ Adj[i] do
                neighbor_S_number[u] − −
                neighbor_V_number[u] + +;
    Return S;
end
```

We tested our algorithm on several different classes of graphs such as complete graph, complete bipartite graph, cycle and path graphs. We obtained the same global distribution center number as given in the theorems in Section 2. The results of the tests satisfied the upper bound in Theorem 2.8 for bipartite graphs with no isolated vertices. Furthermore, we tested the proposed algorithm on double star graph, comet graph, thorn graph, Caterpillar graph, complete k-ary tree, binomial tree and grid graph which are presented in Section 3. We found the same global distribution center number as given in the theorems in this section. Besides, we also tested the algorithm on any graph and verified the results manually.

5. Conclusion

In this paper, we considered global distribution centers in graphs that are usually thought of as being demand driven. We investigate the global distribution center number for selected families of graphs. Finally, we proposed a polynomial time heuristic algorithm to find the set of the global distribution center for arbitrary graphs. Proving the complexity class of finding is a global distribution center set is the subject of future work.

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