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TREES WITH EQUAL ROMAN {2}-DOMINATION NUMBER AND INDEPENDENT ROMAN {2}-DOMINATION NUMBER

Pu Wu[1], Zepeng Li[2], Zehui Shao[1,3,*] and Seyed Mahmoud Sheikholeslami[4]

Abstract. A Roman {2}-dominating function (R{2}DF) on a graph $G = (V, E)$ is a function $f : V \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to either at least one vertex v with $f(v) = 2$ or two vertices v_1, v_2 with $f(v_1) = f(v_2) = 1$. The weight of an R{2}DF f is the value $w(f) = \sum_{u \in V} f(u)$. The minimum weight of an R{2}DF on a graph G is called the Roman {2}-domination number $\gamma_{\{R2\}}(G)$ of G. An R{2}DF f is called an independent Roman {2}-dominating function (IR{2}DF) if the set of vertices with positive weight under f is independent. The minimum weight of an IR{2}DF on a graph G is called the independent Roman {2}-domination number $i_{\{R2\}}(G)$ of G. In this paper, we answer two questions posed by Rahmouni and Chellali.

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1. Introduction

In this paper, we consider only graphs without multiple edges or loops. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a subset $S \subseteq V(G)$ and a vertex $v \in V(G)$, the *open neighborhood* of v in S is the set $N_S(v) = \{u | uv \in E(G) \text{ and } u \in S\}$. The *closed neighborhood* of v in S is the set $N_S[v] = \{v\} \cup N_S(v)$. If $S = V(G)$, then $N_S(v)$ and $N_S[v]$ are denoted by $N(v)$ and $N[v]$, respectively. Let $S \subseteq V(G)$, we write $N_G(S) = \cup_{x \in S} N_G(x)$. The degree of v is $d(v) = |N(v)|$. We will omit the subscript G, that is to say, $N_G(T)$ is denoted by $N(T)$. The *distance* between two vertices u and v in a connected graph G is the length of a shortest uv-path in G. The *diameter* of G, denoted by diam(G), is the maximum value among minimum distances between all pairs of vertices of G. For a vertex v in a rooted tree T, let $C(v)$ and $D(v)$ denote the set of children and descendants of v, respectively and let $D[v] = D(v) \cup \{v\}$. Also, the depth of v, *depth*(v), is the largest distance from v to a vertex in $D(v)$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. We write P_n for the path of order n. A *double star* $DS_{p,q}$ is a tree containing exactly two non-pendant vertices which one is adjacent to p leaves and the other is adjacent to q leaves.

Keywords. Roman {2}-domination, independent Roman {2}-domination, tree, algorithm.

[1] Institute of Computing Science and Technology, Guangzhou University, 510006 Guangzhou, PR China.

[2] School of Information Science and Engineering, Lanzhou University, 730000 Lanzhou, PR China.

[3] School of Information Science and Technology, Chengdu University, 610106 Chengdu, China

[4] Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, I.R. Iran.

*Corresponding author: zshao@gzhu.edu.cn

A function $f : V(G) \to \{0,1,2\}$ is a *Roman dominating function* (RDF) on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an Roman dominating function f is the value $f(V(G)) = \sum_{u\in V(G)} f(u)$. Roman domination was introduced and studied in [7] and later it was extensively studied in the literature [1–5, 8, 14, 15].

A *Roman* $\{2\}$*-dominating function* (R{2}DF) on a graph $G = (V,E)$ is a function $f : V \to \{0,1,2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to either at least one vertex v for which $f(v) = 2$ or two neighbors v_1, v_2 having $f(v_1) = f(v_2) = 1$. The *weight* of an R{2}DF f is the value $w(f) = \sum_{u\in V} f(u)$. An R{2}DF f is called an *independent Roman* $\{2\}$*-dominating function* (IR{2}DF) if the set of vertices with positive weight is independent. The minimum weight of an R{2}DF (resp. IR{2}DF) on a graph G is called the *Roman* $\{2\}$*-domination number* $\gamma_{\{R2\}}(G)$ (resp. *independent Roman* $\{2\}$*-domination number* $i_{\{R2\}}(G)$) of G. An R{2}DF (resp. IR{2}DF) f is called a $\gamma_{\{R2\}}(G)$-function (an $i_{\{R2\}}(G)$-function) if $w(f) = \gamma_{\{R2\}}(G)$ (resp. $w(f) = i_{\{R2\}}(G)$). By the definition of independent Roman {2}-domination, we have

$$\gamma_{\{R2\}}(G) \le i_{\{R2\}}(G). \tag{1}$$

The concept of Roman {2}-domination was introduced in [6] and investigated in [11] and independent Roman {2}-domination was studied in [13], in which bounds involving independent 2-rainbow domination and independent Roman domination numbers are investigated, and the decision version of the independent Roman {2}-domination problem was proved to be NP-complete. Moreover, the following open questions are posed.

Question 1.1. Characterize the graphs (or at least the trees) G for which $\gamma_{\{R2\}}(G) = i_{\{R2\}}(G)$.

Question 1.2. Can you design a linear algorithm for computing the value of $i_{\{R2\}}(T)$ for any tree T?

In this paper, we first settle the Question 1.1 partially and characterize all trees with equal Roman {2}-domination number and independent Roman {2}-domination number, and then we answer to Question 1.2 and give a linear algorithm for computing the value of $i_{\{R2\}}(T)$ for any tree T.

2. Preliminary results

In this section, we present some basic results.

Proposition 2.1. *Let H be a subgraph of a graph G. If $\gamma_{\{R2\}}(H) = i_{\{R2\}}(H)$, $i_{\{R2\}}(G) \le i_{\{R2\}}(H) + s$ and $\gamma_{\{R2\}}(G) \ge \gamma_{\{R2\}}(H) + s$ for some non-negative integer s, then $i_{\{R2\}}(G) = \gamma_{\{R2\}}(G)$.*

Proof. We deduce from the assumptions and (1) that

$$i_{\{R2\}}(G) \ge \gamma_{\{R2\}}(G) \ge \gamma_{\{R2\}}(H) + s = i_{\{R2\}}(H) + s \ge i_{\{R2\}}(G)$$

that this leads to the desired result. □

Proposition 2.2. *Let H be a subgraph of a graph G. If $\gamma_{\{R2\}}(G) = i_{\{R2\}}(G)$, $i_{\{R2\}}(G) \ge i_{\{R2\}}(H) + s$ and $\gamma_{\{R2\}}(G) \le \gamma_{\{R2\}}(H) + s$ for some non-negative integer s, then $\gamma_{\{R2\}}(H) = i_{\{R2\}}(H)$.*

Proof. By (1) and the assumptions, we obtain

$$i_{\{R2\}}(G) = \gamma_{\{R2\}}(G) \le \gamma_{\{R2\}}(H) + s \le i_{\{R2\}}(H) + s \le i_{\{R2\}}(G).$$

Thus all inequalities in the above chain become equalities and so $\gamma_{\{R2\}}(H) = i_{\{R2\}}(H)$. □

Proposition 2.3. *Let G be a graph with $\gamma_{\{R2\}}(G) = i_{\{R2\}}(G)$. If G has a support vertex v with $|L_v| \ge 3$ and $u \in L_v$, then $\gamma_{\{R2\}}(G-u) = i_{\{R2\}}(G-u)$ and there exists a $i_{\{R2\}}(G-u)$-function f such that $f(v) = 2$.*

Proof. Since v is a strong support vertex in $G-u$, there is a $\gamma_{\{R2\}}(G-u)$-function g such that $g(v) = 2$. Clearly, g is an R{2}DF of G and so $\gamma_{\{R2\}}(G) \leq \gamma_{\{R2\}}(G-u)$. Now let f be a $i_{\{R2\}}(G)$-function. Clearly $f(v) \neq 1$. If $f(v) = 2$, then the function f, restricted to $G-u$, is an IR{2}DF of $G-u$ yielding $i_{\{R2\}}(G) \geq i_{\{R2\}}(G-u)$. If $f(v) = 0$, then $f(x) \geq 1$ for each $x \in L_v$ and we may assume that $f(u) = 1$. Now the function f, restricted to $G-u$, is an IR{2}DF of $G-u$ yielding $i_{\{R2\}}(G) \geq i_{\{R2\}}(G-u)+1$. Thus $i_{\{R2\}}(G) \geq i_{\{R2\}}(G-u)$. As in the proof of Observation 2.2, we obtain $\gamma_{\{R2\}}(G-u) = i_{\{R2\}}(G-u)$ and $i_{\{R2\}}(G) = i_{\{R2\}}(G-u)$ and this implies that $f|_{G-u}$ is an $i_{\{R2\}}(G-u)$-function with $f(v) = 2$. This completes the proof. □

Proposition 2.4. *Let G be a graph and $v \in V(G)$. If G' is the graph obtained from G by adding a $K_{1,3}$ centered at c with $V(K_{1,3}) = \{c, c_1, c_2, c_3\}$ and joining v to c_1, then $\gamma_{\{R2\}}(G') = \gamma_{\{R2\}}(G) + 2$ and $i_{\{R2\}}(G') = i_{\{R2\}}(G) + 2$.*

Proof. Clearly, any $\gamma_{\{R2\}}(G)$-function (resp. $i_{\{R2\}}(G)$-function) can be extended to an R{2}DF (resp. IR{2}DF) of G' by assigning a 2 to c and a 0 to c_1, c_2, c_3 and so $\gamma_{\{R2\}}(G') \leq \gamma_{\{R2\}}(G) + 2$ and $i_{\{R2\}}(G') \leq i_{\{R2\}}(G) + 2$. Now let f be a $i_{\{R2\}}(G')$-function. Obviously $f(c) \neq 1$ and $f(c) + f(c_2) + f(c_3) \geq 2$. If $f(c) = 2$, then $f(c_1) = f(c_2) = f(c_3) = 0$ and the function f, restricted to G is an IR{2}DF of G of weight $i_{\{R2\}}(G') - 2$ yielding $i_{\{R2\}}(G') \geq i_{\{R2\}}(G) + 2$. Let $f(c) = 0$. Then we must have $f(c_2) = f(c_3) = 1$. If $f(c_1) = 0$, then as above we have $i_{\{R2\}}(G') \geq i_{\{R2\}}(G) + 2$. Let $f(c_1) \geq 1$. If v has a neighbor w in G with $f(w) \geq 1$, then define $g : V(G) \to \{0, 1, 2\}$ by $g(w) = \min\{2, f(w) + 1\}$ and $g(x) = f(x)$ for $x \in V(G) - \{w\}$, and otherwise define $g : V(G) \to \{0, 1, 2\}$ by $g(v) = 1$ and $g(x) = f(x)$ for $x \in V(G) - \{v\}$. Clearly, g is an IR{2}DF of G of weight at most $i_{\{R2\}}(G') - 2$ and so $i_{\{R2\}}(G') \geq i_{\{R2\}}(G) + 2$. This implies that $i_{\{R2\}}(G') = i_{\{R2\}}(G) + 2$. Similarly, we can see that $\gamma_{\{R2\}}(G') = \gamma_{\{R2\}}(G) + 2$. □

Proposition 2.5. *Let G be a graph and let $v \in V(G)$ be adjacent to two leaves x_1, x_2. If G' is the graph obtained from G by adding a $K_{1,2}$ centered at y with $V(K_{1,2}) = \{y, y_1, y_2\}$ and joining v to y, then $\gamma_{\{R2\}}(G') = \gamma_{\{R2\}}(G) + 2$ and $i_{\{R2\}}(G') = i_{\{R2\}}(G) + 2$.*

Proof. Clearly, any $\gamma_{\{R2\}}(G)$-function (resp. $i_{\{R2\}}(G)$-function) can be extended to an R{2}DF (resp. IR{2}DF) of G' by assigning a 0 to y and a 1 to y_1, y_2 and so $\gamma_{\{R2\}}(G') \leq \gamma_{\{R2\}}(G) + 2$ and $i_{\{R2\}}(G') \leq i_{\{R2\}}(G) + 2$. On the other hand, if f is a $\gamma_{\{R2\}}(G')$-function (resp. $i_{\{R2\}}(G')$-function), then obviously $f(v) + f(x_1) + f(x_2) \geq 2$ and $f(y) + f(y_1) + f(y_2) \geq 2$, and the function f, restricted to G is an R{2}DF (resp. IR{2}DF) of G of weight $\gamma_{\{R2\}}(G') - 2$ (resp. $i_{\{R2\}}(G') - 2$) implying that $\gamma_{\{R2\}}(G') \geq \gamma_{\{R2\}}(G) + 2$ and $i_{\{R2\}}(G') \geq i_{\{R2\}}(G) + 2$. This yields $\gamma_{\{R2\}}(G') = \gamma_{\{R2\}}(G) + 2$ and $i_{\{R2\}}(G') = i_{\{R2\}}(G) + 2$. □

Proposition 2.6. *Let G be a graph and let $v \in V(G)$ be adjacent to the center z of a star $K_{1,s}$ $(s = 1, 2)$. If G' is the graph obtained from G by adding a $K_{1,2}$ centered at y with $V(K_{1,2}) = \{y, y_1, y_2\}$ and joining v to y, then $\gamma_{\{R2\}}(G') = \gamma_{\{R2\}}(G) + 2$ and $i_{\{R2\}}(G') = i_{\{R2\}}(G) + 2$.*

Proof. As above, we can se that $\gamma_{\{R2\}}(G') \leq \gamma_{\{R2\}}(G) + 2$ and $i_{\{R2\}}(G') \leq i_{\{R2\}}(G) + 2$. On the other hand, if f is a $\gamma_{\{R2\}}(G')$-function (resp. $i_{\{R2\}}(G')$-function), then obviously $f(y) + f(y_1) + f(y_2) \geq 2$. If $f(y) = 0$ or $f(v) \geq 1$, the function f, restricted to G is an R{2}DF (resp. IR{2}DF) of G of weight $\gamma_{\{R2\}}(G') - 2$ (resp. $i_{\{R2\}}(G') - 2$) implying that $\gamma_{\{R2\}}(G') \geq \gamma_{\{R2\}}(G) + 2$ and $i_{\{R2\}}(G') \geq i_{\{R2\}}(G) + 2$. Assume that $f(y) \geq 1$ and $f(v) = 0$. To dominate the vertices of $K_{1,s}$, we may assume that $f(z) = 2$. Again, the function f, restricted to G is an R{2}DF (resp. IR{2}DF) of G of weight $\gamma_{\{R2\}}(G') - 2$ (resp. $i_{\{R2\}}(G') - 2$) yielding $\gamma_{\{R2\}}(G') \geq \gamma_{\{R2\}}(G) + 2$ and $i_{\{R2\}}(G') \geq i_{\{R2\}}(G) + 2$. This implies that $\gamma_{\{R2\}}(G') = \gamma_{\{R2\}}(G) + 2$ and $i_{\{R2\}}(G') = i_{\{R2\}}(G) + 2$. □

We now define two classes of graphs H_k and D_k as follows. Let H_k be the tree obtained from $K_{1,k}$ centered at c by adding a pendant path cab (see Fig. 1(a)), and let D_k be the tree obtained from $K_{1,k}$ centered at a by subdividing each edge twice (see Fig. 1(b)). The vertex a in H_k (resp. D_k) is called the *special vertex* of H_k (resp. D_k). The graph obtained from D_k by adding t pendant vertices at a is denoted by $D_{(k,t)}$.

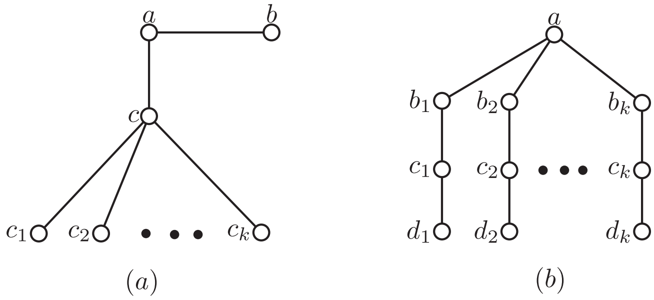

FIGURE 1. (a) the graph H_k; (b) the graph D_k.

Proposition 2.7. *Let G be a graph and let $v \in V(G)$. If G' is the graph obtained from G by adding a H_2 and joining v to a, then $\gamma_{\{R2\}}(G') = \gamma_{\{R2\}}(G) + 3$ and $i_{\{R2\}}(G') = i_{\{R2\}}(G) + 3$.*

Proof. Clearly, any $\gamma_{\{R2\}}(G)$-function (resp. $i_{\{R2\}}(G)$-function) can be extended to an R{2}DF (resp. IR{2}DF) of G' by assigning a 2 to c, a 1 to b and a 0 to the remaining vertices and so $\gamma_{\{R2\}}(G') \leq \gamma_{\{R2\}}(G) + 3$ and $i_{\{R2\}}(G') \leq i_{\{R2\}}(G) + 3$. Now let f be a $i_{\{R2\}}(G')$-function (resp. $\gamma_{\{R2\}}(G')$-function). Obviously $f(c) + f(c_1) + f(c_2) \geq 2$ and $f(a) + f(b) \geq 1$. If $f(a) = 0$ or $f(v) \geq 1$, the function f, restricted to G is an IR{2}DF (resp. R{2}DF) of G of weight $\gamma_{\{R2\}}(G') - 3$ (resp. $i_{\{R2\}}(G') - 3$) and so $\gamma_{\{R2\}}(G') \geq \gamma_{\{R2\}}(G) + 3$ and $i_{\{R2\}}(G') \geq i_{\{R2\}}(G) + 3$. Assume that $f(a) \geq 1$ and $f(v) = 0$. To dominate b, we may assume that $f(a) = 2$. If v has a neighbor w in G with $f(w) \geq 1$, then define $g : V(G) \to \{0, 1, 2\}$ by $g(w) = \min\{2, f(w) + 1\}$ and $g(x) = f(x)$ for $x \in V(G) - \{w\}$, and otherwise define $g : V(G) \to \{0, 1, 2\}$ by $g(v) = 1$ and $g(x) = f(x)$ for $x \in V(G) - \{v\}$. Clearly, g is an IR{2}DF (resp. R{2}DF) of G of weight at most $\gamma_{\{R2\}}(G') - 3$ (resp. $i_{\{R2\}}(G') - 3$) and so $i_{\{R2\}}(G') \geq i_{\{R2\}}(G) + 3$ and $\gamma_{\{R2\}}(G') \geq \gamma_{\{R2\}}(G) + 3$. This implies that $i_{\{R2\}}(G') = i_{\{R2\}}(G) + 3$ and $\gamma_{\{R2\}}(G') = \gamma_{\{R2\}}(G) + 3$. □

Proposition 2.8. *Let G be a graph and let $v \in V(G)$. If G' is the graph obtained from G by adding a $D_{(k,t)}$ with special vertex a and joining v to a where $k + t \geq 2$ and $t \leq 1$, then $\gamma_{\{R2\}}(G') = \gamma_{\{R2\}}(G) + 2k + t$ and $i_{\{R2\}}(G') = i_{\{R2\}}(G) + 2k + t$.*

Proof. Clearly, any $\gamma_{\{R2\}}(G)$-function (resp. $i_{\{R2\}}(G)$-function) can be extended to an R{2}DF (resp. IR{2}DF) of G by assigning a 1 to b_i, d_i $(i = 1, \dots, k)$ and the leaf adjacent to a, if any, and a 0 to the remaining vertices implying that $\gamma_{\{R2\}}(G') \leq \gamma_{\{R2\}}(G) + 2k + t$ and $i_{\{R2\}}(G') \leq i_{\{R2\}}(G) + 2k + t$. Now let f be a $i_{\{R2\}}(G')$-function (resp. $\gamma_{\{R2\}}(G')$-function). Obviously $f(b_i) + f(c_i) + f(d_i) \geq 2$ for each $i \in \{1, \dots, k\}$ and $f(a) + f(L_a) \geq t$. If $f(a) = 0$ or $f(v) \geq 1$, the function f, restricted to G is an IR{2}DF (resp. R{2}DF) of G of weight $i_{\{R2\}}(G') - 2k - t$ (resp. $\gamma_{\{R2\}}(G') - 2k - t$) and so $\gamma_{\{R2\}}(G') \geq \gamma_{\{R2\}}(G) + 2k + t$ and $i_{\{R2\}}(G') \geq i_{\{R2\}}(G) + 2k + t$. Assume that $f(a) \geq 1$ and $f(v) = 0$. Then to dominate the leaf adjacent to a, we may assume have $f(a) \geq 1 + t$. If v has a neighbor w in G with $f(w) \geq 1$, then define $g : V(G) \to \{0, 1, 2\}$ by $g(w) = \min\{2, f(w) + 1\}$ and $g(x) = f(x)$ for $x \in V(G) - \{w\}$, and otherwise define $g : V(G) \to \{0, 1, 2\}$ by $g(v) = 1$ and $g(x) = f(x)$ for $x \in V(G) - \{v\}$. Clearly, g is an IR{2}DF (resp. R{2}DF) of G of weight at most $\gamma_{\{R2\}}(G') - 2k - t$ (resp. $i_{\{R2\}}(G') - 2k - t$) and so $i_{\{R2\}}(G') \geq i_{\{R2\}}(G) + 2k + t$ and $\gamma_{\{R2\}}(G') \geq \gamma_{\{R2\}}(G) + 2k + t$. This implies that $i_{\{R2\}}(G') = i_{\{R2\}}(G) + 2k + t$ and $\gamma_{\{R2\}}(G') = \gamma_{\{R2\}}(G) + 2k + t$. □

Proposition 2.9. *Let G be a graph and let $x \in V(G)$ be a leaf adjacent to a support vertex u of degree 3 and $L_u = \{x, y\}$. If G' is the graph obtained from G by adding a pendant path xa, then $\gamma_{\{R2\}}(G') = \gamma_{\{R2\}}(G) + 1$ and $i_{\{R2\}}(G') = i_{\{R2\}}(G) + 1$.*

Proof. Let f be a $i_{\{R2\}}(G)$-function (resp. $\gamma_{\{R2\}}(G)$-function). If $f(u) \geq 1$, then clearly $f(u) = 2$, $f(x) = 0$ and f can be extended to an IR{2}DF (resp. R{2}DF) of G' by assigning a 1 to a, and if $f(u) = 0$, then clearly

$f(x) = 1$ and f can be extended to an IR{2}DF (resp. R{2}DF) of G' by assigning a 0 to a and reassigning a 2 to x, yielding $\gamma_{\{R2\}}(G') \leq \gamma_{\{R2\}}(G) + 1$ and $i_{\{R2\}}(G') \leq i_{\{R2\}}(G) + 1$.

Now, let g be an $i_{\{R2\}}(G')$-function (resp. $\gamma_{\{R2\}}(G')$-function). If $g(u) \geq 1$, then clearly $g(x) = 0, g(a) = 1$ and the function g, restricted to G is an IR{2}DF (resp. R{2}DF) of G, and if $g(u) = 0$, then $g(y) = 1$ and we may assume that $g(x) = 2$ and the function $h : V(G) \to \{0, 1, 2\}$ defined by $h(x) = 1$ and $h(w) = g(w)$ otherwise, is an IR{2}DF (resp. R{2}DF) of G. Both cases leads to $\gamma_{\{R2\}}(G') \geq \gamma_{\{R2\}}(G) + 1$ and $i_{\{R2\}}(G') \geq i_{\{R2\}}(G) + 1$. Thus $\gamma_{\{R2\}}(G') = \gamma_{\{R2\}}(G) + 1$ and $i_{\{R2\}}(G') = i_{\{R2\}}(G) + 1$ as desired. □

Proposition 2.10. *Let G be a graph and let $v \in V(G)$ be adjacent to two leaves x_1, x_2. If G' is the graph obtained from G by adding a path abc and joining v to a, then $\gamma_{\{R2\}}(G') = \gamma_{\{R2\}}(G) + 2$ and $i_{\{R2\}}(G') = i_{\{R2\}}(G) + 2$.*

Proof. Clearly, any $\gamma_{\{R2\}}(G)$-function (resp. $i_{\{R2\}}(G)$-function) can be extended to an R{2}DF (resp. IR{2}DF) of G' by assigning a 0 to a, c and a 2 to b and so $\gamma_{\{R2\}}(G') \leq \gamma_{\{R2\}}(G) + 2$ and $i_{\{R2\}}(G') \leq i_{\{R2\}}(G) + 2$. On the other hand, if f is a $\gamma_{\{R2\}}(G')$-function (resp. $i_{\{R2\}}(G')$-function), then obviously $f(v) + f(x_1) + f(x_2) \geq 2$ and $f(a) + f(b) + f(c) \geq 2$, and the function f, restricted to G is an R{2}DF (resp. IR{2}DF) of G of weight $\gamma_{\{R2\}}(G') - 2$ (resp. $i_{\{R2\}}(G') - 2$) implying that $\gamma_{\{R2\}}(G') \geq \gamma_{\{R2\}}(G) + 2$ and $i_{\{R2\}}(G') \geq i_{\{R2\}}(G) + 2$. Thus $\gamma_{\{R2\}}(G') = \gamma_{\{R2\}}(G) + 2$ and $i_{\{R2\}}(G') = i_{\{R2\}}(G) + 2$. □

The proof of next result is similar to the proof of Proposition 2.6 and therefore omitted.

Proposition 2.11. *Let G be a graph and let $v \in V(G)$ be adjacent to the center z of a star $K_{1,s}$ $(s = 1, 2)$. If G' is the graph obtained from G by adding a path abc and joining v to a, then $\gamma_{\{R2\}}(G') = \gamma_{\{R2\}}(G) + 2$ and $i_{\{R2\}}(G') = i_{\{R2\}}(G) + 2$.*

3. Trees T with $\gamma_{\{R2\}}(T) = i_{\{R2\}}(T)$

In this section we give a constructive characterization of all trees with equal Roman {2}-domination number and independent Roman {2}-domination number. For a tree T, let

$$W(T) = \{u \in V(T) \mid \text{there exists an } i_{\{R2\}}(T) - \text{function } f \text{ with } f(u) \geq 1\}$$

and

$$W_1(T) = \{u \in V(T) \mid \text{there exists an } i_{\{R2\}}(T) - \text{function } f \text{ with } f(u) = 2\}.$$

In order to present our constructive characterization, we define a family of trees as follows. Let $\mathcal{T}$ be the family of trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ of trees for some $k \geq 1$, where $T_1 \in \{P_1, P_2, P_3, P_4\}$ and $T = T_k$. If $k \geq 2$, T_{i+1} can be obtained from T_i by one of the following operations.

Operation $\mathcal{O}_1$: If $x \in V(T_i)$, then $\mathcal{O}_1$ adds a star $K_{1,3}$ centered at c with $V(K_{1,3}) = \{c, c_1, c_2, c_3\}$ and joins x to c_1 to obtain T_{i+1} (see Fig. 2(a)).

Operation $\mathcal{O}_2$: If $x \in V(T_i)$, then $\mathcal{O}_2$ adds a graph H_2 with a special vertex a and an edge xa to obtain T_{i+1} (see Fig. 2(b)).

Operation $\mathcal{O}_3$: If $x \in V(T_i)$ is adjacent to either two leaves x_1, x_2 or the center x_3 of a star $K_{1,s}$ $(s = 1, 2)$, then $\mathcal{O}_3$ adds a path bac and joins x to a to obtain T_{i+1} (see Fig. 2(c)).

Operation $\mathcal{O}_4$: If $x \in W(T_i)$ and x is adjacent to two leaves d, e or there is a path xyz in T such that $\deg(y) = 2$ and $\deg(z) = 1$, then $\mathcal{O}_4$ adds a pendent path ab and joins x to a to obtain T_{i+1} (see Fig. 2(d)).

Operation $\mathcal{O}_5$: If $x \in V(T_i)$ is adjacent to either two leaves x_1, x_2 or the center x_3 of a star $K_{1,s}$ $(s = 1, 2)$, then $\mathcal{O}_5$ adds a path abc and joins x to a to obtain T_{i+1} (see Fig. 2(e)).

Operation $\mathcal{O}_6$: If $x \in V(T_i)$, then $\mathcal{O}_6$ adds a graph $D_{(k,t)}$ with a special vertex a and an edge xa to obtain T_{i+1}, where $k + t \geq 2$, $t \leq 1$ (see Fig. 2(f)).

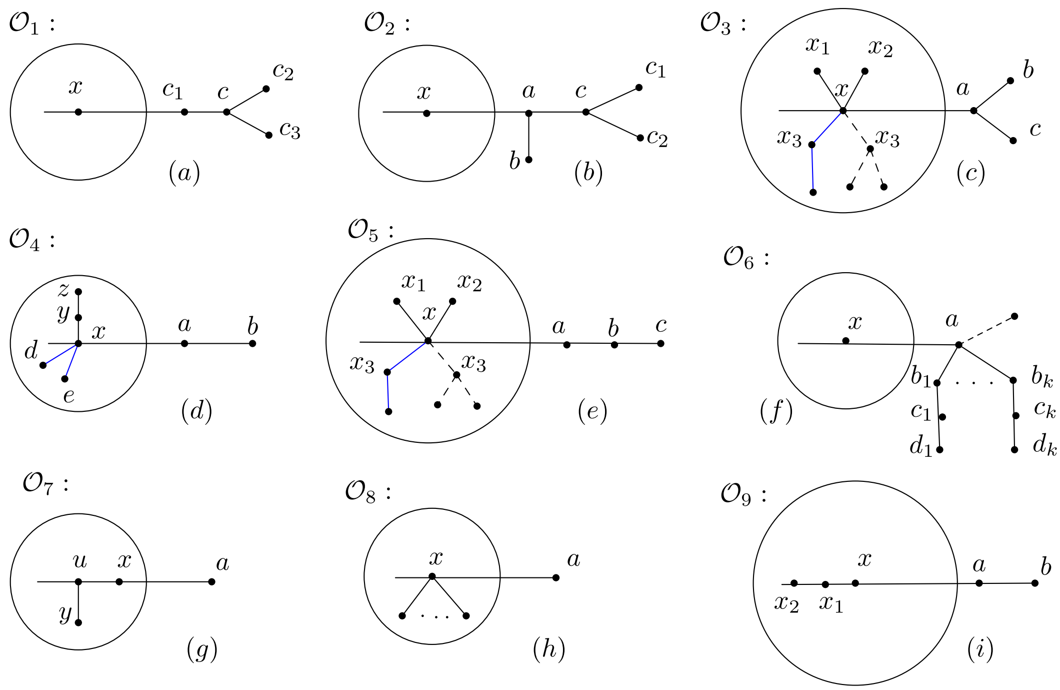

FIGURE 2. The operations $\mathcal{O}_i (i \in \{1, 2, \dots, 9\})$.

Operation $\mathcal{O}_7$: If $x \in V(T_i)$ is a leaf adjacent to a support vertex u of degree 3 and $L_u = \{x, y\}$, then $\mathcal{O}_7$ adds a vertex a and joins a to x to obtain T_{i+1} (see Fig. 2(g)).

Operation $\mathcal{O}_8$: If $x \in W_1(T_i)$ and there are at least two leaves adjacent to x, then $\mathcal{O}_8$ adds a vertex a and an edge ax to obtain T_{i+1} (see Fig. 2(h)).

Operation $\mathcal{O}_9$: If $x \in V(T_i)$ is a leaf and there is a path x_2x_1x in T_i such that $\deg(x_1) = 2$, then $\mathcal{O}_9$ adds a path ab and joins x to a to obtain T_{i+1} (see Fig. 2(i)).

The next result follows immediately from Proposition 2.4.

Lemma 3.1. *If T_i is a tree with $\gamma_{\{R2\}}(T_i) = i_{\{R2\}}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_1$, then $\gamma_{\{R2\}}(T_{i+1}) = i_{\{R2\}}(T_{i+1})$.*

The next result is immediate from Proposition 2.7.

Lemma 3.2. *If T_i is a tree with $\gamma_{\{R2\}}(T_i) = i_{\{R2\}}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_2$, then $\gamma_{\{R2\}}(T_{i+1}) = i_{\{R2\}}(T_{i+1})$.*

The next result follows immediately from Propositions 2.5 ad 2.6.

Lemma 3.3. *If T_i is a tree with $\gamma_{\{R2\}}(T_i) = i_{\{R2\}}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_3$, then $\gamma_{\{R2\}}(T_{i+1}) = i_{\{R2\}}(T_{i+1})$.*

Lemma 3.4. *Let T_i be a tree and T_{i+1} be a graph obtained from T_i by Operation $\mathcal{O}_4$. Then $\gamma_{\{R2\}}(T_{i+1}) = \gamma_{\{R2\}}(T_i) + 1$. Furthermore, if $\gamma_{\{R2\}}(T_i) = i_{\{R2\}}(T_i)$, then $\gamma_{\{R2\}}(T_{i+1}) = i_{\{R2\}}(T_{i+1})$.*

Proof. Let f be a $\gamma_{\{R2\}}(T_i)$-function. Since $|L_x| \geq 2$ or there is a pendant path xyz, we may assume without loss of generality that $f(x) \geq 1$. Then, f can be extended to an R{2}DF of T_{i+1} by assigning a 0 to a and a 1 to b. Hence, $\gamma_{\{R2\}}(T_{i+1}) \leq \gamma_{\{R2\}}(T_i) + 1$. On the other hand, if h is a $\gamma_{\{R2\}}(T_{i+1})$-function, then clearly $h(x) \geq 1$ and hence $h(a) = 0$ and $h(b) = 1$. Then the function h restricted to T_i is an R{2}DF of T_i yielding $\gamma_{\{R2\}}(T_{i+1}) \geq \gamma_{\{R2\}}(T_i) + 1$. Consequently,

$$\gamma_{\{R2\}}(T_{i+1}) = \gamma_{\{R2\}}(T_i) + 1. \tag{2}$$

Now let $\gamma_{\{R2\}}(T_i) = i_{\{R2\}}(T_i)$. Since $x \in W(T_i)$, there exists an $i_{\{R2\}}(T_i)$-function g of T_i such that $g(x) \geq 1$. As above, g can be extended to an IR{2}DF of T_{i+1} by assigning a 0 to a and a 1 to b and this implies that $i_{\{R2\}}(T_{i+1}) \leq i_{\{R2\}}(T_i) + 1$. It follows from $\gamma_{\{R2\}}(T_i) = i_{\{R2\}}(T_i)$ and (2) that $i_{\{R2\}}(T_{i+1}) \leq i_{\{R2\}}(T_i) + 1 = \gamma_{\{R2\}}(T_i) + 1 = \gamma_{\{R2\}}(T_{i+1})$. Now the result follows from (1). □

The next result follows immediately from Propositions 2.10 and 2.11.

Lemma 3.5. *If T_i is a tree with $\gamma_{\{R2\}}(T_i) = i_{\{R2\}}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_5$, then $\gamma_{\{R2\}}(T_{i+1}) = i_{\{R2\}}(T_{i+1})$.*

The next result is immediate by Proposition 2.8.

Lemma 3.6. *If T_i is a tree with $\gamma_{\{R2\}}(T_i) = i_{\{R2\}}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_6$, then $\gamma_{\{R2\}}(T_{i+1}) = i_{\{R2\}}(T_{i+1})$.*

The next result is immediate by Proposition 2.9.

Lemma 3.7. *If T_i is a tree with $\gamma_{\{R2\}}(T_i) = i_{\{R2\}}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_7$, then $\gamma_{\{R2\}}(T_{i+1}) = i_{\{R2\}}(T_{i+1})$.*

Lemma 3.8. *Let T_i be a tree and T_{i+1} be a graph obtained from T_i by Operation $\mathcal{O}_8$, then $\gamma_{\{R2\}}(T_{i+1}) = \gamma_{\{R2\}}(T_i)$. Furthermore, if $\gamma_{\{R2\}}(T_i) = i_{\{R2\}}(T_i)$, then $\gamma_{\{R2\}}(T_{i+1}) = i_{\{R2\}}(T_{i+1})$.*

Proof. Let f be a $\gamma_{\{R2\}}(T_i)$-function. Since x is adjacent to at least two leaves in T_i, w.l.o.g., we can assume that $f(x) = 2$. Hence, f can be extended to an R{2}DF of T_{i+1} by assigning 0 to a, which implies that $\gamma_{\{R2\}}(T_{i+1}) \leq w(f) = \gamma_{\{R2\}}(T_i)$. Conversely, let h be a $\gamma_{\{R2\}}(T_{i+1})$-function. Then we have $h(x) = 2$ and $h|_{T_i}$ is an R{2}DF of T_i. Consequently, we have

$$\gamma_{\{R2\}}(T_{i+1}) = \gamma_{\{R2\}}(T_i). \tag{3}$$

Now we show that $\gamma_{\{R2\}}(T_{i+1}) = i_{\{R2\}}(T_{i+1})$. Since $x \in W_1(T_i)$, there exists an $i_{\{R2\}}(T_i)$-function f' of T_i such that $f'(x) = 2$. Clearly, f' can be extended to an IR{2}DF of T_{i+1} by assigning 0 to a, which implies that $i_{\{R2\}}(T_{i+1}) \leq i_{\{R2\}}(T_i)$. Since $\gamma_{\{R2\}}(T_i) = i_{\{R2\}}(T_i)$, by (3) we have $i_{\{R2\}}(T_{i+1}) \leq i_{\{R2\}}(T_i) = \gamma_{\{R2\}}(T_i) = \gamma_{\{R2\}}(T_{i+1})$. Hence, $\gamma_{\{R2\}}(T_{i+1}) = i_{\{R2\}}(T_{i+1})$. □

Lemma 3.9. *Let T_i be a tree and T_{i+1} be a graph obtained from T_i by Operation $\mathcal{O}_9$, then $\gamma_{\{R2\}}(T_{i+1}) = \gamma_{\{R2\}}(T_i) + 1$. Furthermore, if $\gamma_{\{R2\}}(T_i) = i_{\{R2\}}(T_i)$, then $\gamma_{\{R2\}}(T_{i+1}) = i_{\{R2\}}(T_{i+1})$.*

Proof. Let f be a $\gamma_{\{R2\}}(T_i)$-function. We may assume without loss of generality that $f(x_1) = 0$, $f(x_2) \geq 1$ and $f(x) = 1$. Then f can be extended to an R{2}DF of T_{i+1} by assigning a 0 to a and a 1 to b and so $\gamma_{\{R2\}}(T_{i+1}) \leq \gamma_{\{R2\}}(T_i) + 1$. Conversely, let h be a $\gamma_{\{R2\}}(T_{i+1})$-function. W.l.o.g., we may assume that $h(a) = 0$, $h(x) \geq 1$ and $h(b) = 1$. Then, the function h, restricted to T_i is an R{2}DF of T_i yielding $\gamma_{\{R2\}}(T_{i+1}) \geq \gamma_{\{R2\}}(T_i) + 1$. Consequently, we have

$$\gamma_{\{R2\}}(T_{i+1}) = \gamma_{\{R2\}}(T_i) + 1. \tag{4}$$

Let $\gamma_{\{R2\}}(T_i) = i_{\{R2\}}(T_i)$. If T_i has an $i_{\{R2\}}(T_i)$-function f' such that $f'(x) \neq 0$, then f' can be extended to an IR{2}DF of T_{i+1} by assigning a 1 to b and a 0 to a yielding $i_{\{R2\}}(T_{i+1}) \leq i_{\{R2\}}(T_i) + 1$. Suppose for any $i_{\{R2\}}(T_i)$-function f', we have $f'(x) = 0$. Then $f'(x_1) = 2$ and $f'(x_2) = 0$. Moreover, x_2 has a neighbor x_3 different from x_1 such that $f'(x_3) \neq 0$. Define $g : V(T_{i+1}) \to \{0, 1, 2\}$ by $g(a) = 2$, $g(b) = 0$, $g(x_1) = 1$ and $g(v) = f'(v)$ for $v \in V(T_i) - \{x_1\}$. Clearly, g is an IR{2}DF of T_{i+1} with weight $i_{\{R2\}}(T_i) + 1$ and so $i_{\{R2\}}(T_{i+1}) \leq i_{\{R2\}}(T_i) + 1$. Applying $\gamma_{\{R2\}}(T_i) = i_{\{R2\}}(T_i)$ and (4), we have $i_{\{R2\}}(T_{i+1}) \leq i_{\{R2\}}(T_i) + 1 = \gamma_{\{R2\}}(T_i) + 1 = \gamma_{\{R2\}}(T_{i+1}) \leq i_{\{R2\}}(T_{i+1})$. Hence, $\gamma_{\{R2\}}(T_{i+1}) = i_{\{R2\}}(T_{i+1})$. □

Theorem 3.10. *Let T be a tree of order n. Then $\gamma_{\{R2\}}(T) = i_{\{R2\}}(T)$ if and only if $T \in \mathcal{T}$.*

Proof. We first show that if $T \in \mathcal{T}$ is a tree, then $\gamma_{\{R2\}}(T) = i_{\{R2\}}(T)$. Let $T \in \mathcal{T}$. By the definition of $\mathcal{T}$, we know that there exists a sequence of trees $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ such that $T_1 \in \{P_1, P_2, P_3, P_4\}$, $T_k = T$ and if $k \geq 2$, then T_{i+1} can be obtained from T_i by one of the Operations $\mathcal{O}_i$ $(i \in \{1, 2, \ldots, 9\})$. We proceed by induction on k. If $k = 1$, then the result is trivial. Assume the result holds for each tree $T \in \mathcal{T}$ which can be obtained from a sequence of operations of length $k-1$ and let $T' = T_{k-1}$. By the induction hypothesis, $\gamma_{\{R2\}}(T') = i_{\{R2\}}(T')$. Since $T = T_k$ is obtained by one of the Operations $\mathcal{O}_i (i \in \{1, 2, \ldots, 9\})$ from T', we conclude from the Lemmas 3.1 to 3.9 that $\gamma_{\{R2\}}(T) = i_{\{R2\}}(T)$.

Now, we prove the necessity. The proof is by induction on n. If $n \leq 3$, then $T \in \{P_1, P_2, P_3\}$ and the result is true. Suppose $n \geq 4$ and that the statement holds for all trees of order less than n. Let T be a tree of order n with $\gamma_{\{R2\}}(T) = i_{\{R2\}}(T)$ and let f be an $i_{\{R2\}}$-function of T. If there exists a vertex $v \in V(T)$ with $|L_v| \geq 3$, then let $T' = T - u$ where $u \in L_v$. By Observation 2.3, we have $\gamma_{\{R2\}}(T') = i_{\{R2\}}(T')$ and $v \in W_1(T)$. It follows from the induction hypothesis that $T' \in \mathcal{T}$. Now, T can be obtained from T' by Operation $\mathcal{O}_8$ yielding $T \in \mathcal{T}$. Assume that each vertex of T has at most 2 leaf neighbors. Hence T is not a star and so $\operatorname{diam}(T) \geq 3$. If $\operatorname{diam}(T) = 3$, then T is a double star $DS_{p,q}$ for some $q \geq p \geq 1$, and since each vertex of T has at most 2 leaf neighbors, we conclude that $T \in \{P_4, DS_{1,2}, DS_{2,2}\}$. Obviously $T = P_4 \in \mathcal{T}$. If $T = DS_{1,2}$, the T can be obtained from P_1 by Operation $\mathcal{O}_1$, if $T = DS_{2,2}$, then T can be obtained from P_3 by Operation $\mathcal{O}_3$ and so $T \in \mathcal{T}$. Assume that $\operatorname{diam}(T) \geq 4$.

Let $P = u_1 u_2 \ldots u_k$ be a diametrical path of T such that $d_T(u_2)$ is as large as possible. Among these paths, choose one so that $d_T(u_3)$ is as large as possible. Let $L_{u_2} = \{v_1, \ldots, v_{\deg_T(u_2)-1}\}$ where $u_1 = v_1$. Note that $2 \leq \deg_T(u_2) \leq 3$. We consider the following cases.

Case 1. $\deg_T(u_2) = 3$.

If $\deg_T(u_3) = 2$, then it follows from Proposition 2.4 and the fact $\gamma_{\{R2\}}(T) = i_{\{R2\}}(T)$ that $\gamma_{\{R2\}}(T - T_{u_3}) = i_{\{R2\}}(T - T_{u_3})$. By the induction hypothesis, we have $T - T_{u_3} \in \mathcal{T}$ and T can be obtained from $T - T_{u_3}$ by Operation $\mathcal{O}_1$ implying that $T \in \mathcal{T}$. Assume that $\deg_T(u_3) \geq 3$. If u_3 is a strong support vertex or is adjacent to the center of a star $K_{1,s}$ $(s = 1, 2)$, then we conclude from Propositions 2.5 and 2.6 and the induction hypothesis that $T - T_{u_2} \in \mathcal{T}$. Now, T can be obtained from $T - T_{u_2}$ by Operation $\mathcal{O}_3$ and hence $T \in \mathcal{T}$. Henceforth, we assume u_3 is adjacent to at most one leaf and that u_3 has no child with depth one but u_2. We deduce from these assumption and the fact $d(u_3) \geq 3$ that $d(u_3) = 3$ and u_3 has a child x_1 with depth 0. Let $T' = T - T_{u_3}$. We conclude from Proposition 2.7 and the induction hypothesis that $T' \in \mathcal{T}$. Now, T can be obtained from T' by Operation $\mathcal{O}_2$ and hence $T \in \mathcal{T}$.

Case 2. $\deg_T(u_2) = 2$ and $d(u_3) \geq 4$.

By the choice of diametrical path, we may assume that any child of u_3 with depth one has degree 2. We consider the following subcases.

Subcase 2.1. u_3 has a child y_2 with depth one.

Let $u_3 y_2 y_1$ be a path in T and let $T' = T - T_{u_2}$. If $f(u_3) = 0$, then we have $f(u_2) + f(u_1) \geq 2$ and $f(y_2) + f(y_1) \geq 2$, and the function $g : V(T) \to \{0, 1, 2\}$ defined by $g(u_3) = 1$, $g(u_2) = g(y_2) = 0$, $g(u_1) = g(y_1) = 1$ and $g(x) = f(x)$ otherwise, is an R{2}DF of T of weight less than $\omega(f)$ contradicting the assumption $\gamma_{\{R2\}}(T) = i_{\{R2\}}(T)$. Hence $f(u_3) \geq 1$. It follows that $f(u_2) = 0$ and $f(u_1) = 1$. Thus the function f, restricted to T' is an IR{2}DF of T' of weight $i_{\{R2\}}(T) - 1$ and so $i_{\{R2\}}(T) \geq i_{\{R2\}}(T') + 1$. On the other hand, as in the proof of Lemma 3.4, we have $\gamma_{\{R2\}}(T) = \gamma_{\{R2\}}(T') + 1$. Now the following inequality chain

$$\gamma_{\{R2\}}(T) = i_{\{R2\}}(T) \geq i_{\{R2\}}(T') + 1 \geq \gamma_{\{R2\}}(T') + 1 = \gamma_{\{R2\}}(T)$$

leads to $\gamma_{\{R2\}}(T') = i_{\{R2\}}(T')$ and that the function $f|_{T'}$ is an $i_{\{R2\}}(T')$-function with $f(u_3) \geq 1$ and so $u_3 \in W(T')$. By the induction hypothesis, we have $T' \in \mathcal{T}$. Now T can be obtained from T' by Operation $\mathcal{O}_4$ and so $T \in \mathcal{T}$.

Subcase 2.2. $|L_{u_3}| \geq 2$.

Let $x_1, x_2 \in L_{u_3}$ and let $T' = T - T_{u_2}$. If $f(u_3) = 0$, then we have $f(u_2) + f(u_1) \geq 2$ and $f(x_1) = f(x_2) = 1$, and the function $g : V(T) \to \{0, 1, 2\}$ defined by $g(u_3) = 2$, $g(u_2) = g(x_1) = g(x_2) = 0$, $g(u_1) = 1$ and $g(x) = f(x)$ otherwise, is an R{2}DF of T of weight less than $\omega(f)$ contradicting the assumption $\gamma_{\{R2\}}(T) = i_{\{R2\}}(T)$. Hence $f(u_3) \geq 1$. Now, as in Subcase 2.1, we can see that $T \in \mathcal{T}$.

Case 3. $\deg_T(u_2) = 2$ and $\deg_T(u_3) = 3$.

By the choice of diametrical path, we may assume that any child of u_3 with depth one has degree 2. If u_3 has a child y_2 different from u_2, with depth one, then as in Subcase 2.1, we can see that $T \in \mathcal{T}$. Assume that u_2 is the only child of u_3 with depth one. Since $\deg_T(u_3) = 3$, we deuce that u_3 is adjacent to a leaf, say w. Let $T' = T - u_1$. We conclude from Proposition 2.9 and the induction hypothesis that $T' \in \mathcal{T}$. Now T can be obtained from T' by Operation $\mathcal{O}_7$ and so $T \in \mathcal{T}$.

Case 4. $d(u_2) = d(u_3) = 2$ and $d(u_4) \geq 3$.

By the choice of diametrical path, for any path $u_4 y_3 y_2 y_1$ in T where $y_3 \in C(u_4)$, we have $\deg(y_3) = \deg(y_2) = 2$. If u_4 is a strong support vertex or is adjacent to the center of a star $K_{1,s}$ ($s = 1, 2$), then we conclude from Propositions 2.10 and 2.11 and the induction hypothesis that $T - T_{u_3} \in \mathcal{T}$. Now, T can be obtained from $T - T_{u_3}$ by Operation $\mathcal{O}_5$ and hence $T \in \mathcal{T}$. Henceforth, we assume u_4 is adjacent to at most one leaf and that u_4 has no child with depth one. This implies that $T_{u_4} = D_{(k,t)}$ where $t \leq 1$ and $k + t \geq 2$. Let $T' = T - T_{u_4}$. We conclude from Proposition 2.8 and the induction hypothesis that $T' \in \mathcal{T}$. Now, T can be obtained from T' by Operation $\mathcal{O}_6$ and hence $T \in \mathcal{T}$.

Case 5. $d(u_2) = d(u_3) = d(u_4) = 2$.

Let $T' = T - \{u_1, u_2\}$. By Lemma 3.9, we have $\gamma_{\{R2\}}(T) = \gamma_{\{R2\}}(T') + 1$. We show that $i_{\{R2\}}(T') \leq i_{\{R2\}}(T) - 1$. If $f(u_3) \geq 1$, then $f(u_2) = 0$, $f(u_1) = 1$ and the function $f|_{T'}$ is an IR{2}DF of T' yielding $i_{\{R2\}}(T') \leq i_{\{R2\}}(T) - 1$. Suppose $f(u_3) = 0$. Then $f(u_1) + f(u_2) = 2$ and we may assume that $f(u_1) = 0$ and $f(u_2) = 2$. If $f(u_4) = 2$, then $f(u_5) = 0$ and the function $g : V(T) \to \{0, 1, 2\}$ defined by $g(u_1) = g(u_3) = g(u_5) = 1$, $g(u_2) = g(u_4) = 0$ and $g(v) = f(v)$ for $v \in V(T) - \{u_1, u_2, u_3, u_4, u_5\}$, is an R{2}DF of T with weight $i_{\{R2\}}(T) - 1 = \gamma_{\{R2\}}(T) - 1$ which is a contradiction. Hence, $f(u_4) \leq 1$. If $f(u_4) = 0$, then define $h : V(T') \to \{0, 1, 2\}$ by $h(u_3) = 1$ and $h(v) = f(v)$ for $v\ V(T') - \{u_3\}$, and if $h(u_4) = 1$, then define $h : V(T') \to \{0, 1, 2\}$ by $h(u_4) = 2$ and $h(v) = f(v)$ for $v\ V(T') - \{u_4\}$. Clearly, h is an IR{2}DF of T' of weight $i_{\{R2\}}(T) - 1$, implying that $i_{\{R2\}}(T') \leq i_{\{R2\}}(T) - 1$. We deduce from

$$i_{\{R2\}}(T') \leq i_{\{R2\}}(T) - 1 = \gamma_{\{R2\}}(T) - 1 = \gamma_{\{R2\}}(T') \leq i_{\{R2\}}(T')$$

that $\gamma_{\{R2\}}(T') = i_{\{R2\}}(T')$. It follows from the induction hypothesis on T' that $T' \in \mathcal{T}$. Now, T can be obtained from T' by Operation $\mathcal{O}_9$ and hence $T \in \mathcal{T}$. This completes the proof. □

4. A linear algorithm for computing $i_{\{R2\}}(T)$ for any tree T

To present a linear algorithm, we will use the following notations. For a graph G and a vertex $v \in V(G)$, we denote $G + uw$ the graph obtained from G by adding pendant edge uw. Now, we define

$$i^0_{\{R2\}}(G, u) = \min\{w(f) : f \text{ is an IR\{2\}DF of } G \text{ with } f(u) = 0\},$$
$$i^1_{\{R2\}}(G, u) = \min\{w(f) : f \text{ is an IR\{2\}DF of } G \text{ with } f(u) = 1\},$$
$$i^2_{\{R2\}}(G, u) = \min\{w(f) : f \text{ is an IR\{2\}DF of } G \text{ with } f(u) = 2\},$$
$$i^{00}_{\{R2\}}(G, u) = \min\{w(f) : f \text{ is an IR\{2\}DF of } G - u\},$$
$$i^{01}_{\{R2\}}(G, u) = \min\{w(f) - 1 : f \text{ is an IR\{2\}DF of } G + uw \text{ with } f(u) = 0 \text{ and } f(w) = 1\},$$
$$i^{02}_{\{R2\}}(G, u) = \min\{w(f) - 2 : f \text{ is an IR\{2\}DF of } G + uw \text{ with } f(u) = 0 \text{ and } f(w) = 2\}.$$

The following results are trivial.

Observation 4.1. For any graph G with a specific vertex u, we have

$$i_{\{R2\}}(G) = \min\{i^0_{\{R2\}}(G,u), i^1_{\{R2\}}(G,u), i^2_{\{R2\}}(G,u)\}.$$

Observation 4.2. $i^{00}_{\{R2\}}(G,u) \le i^{01}_{\{R2\}}(G,u)$.

Observation 4.3. $i^{00}_{\{R2\}}(G,u) = i^{02}_{\{R2\}}(G,u)$.

Proof. Let f be an IR{2}DF of $G+uw$ for which $f(u)=0$ and $f(w)=2$ with minimum weight. Then $f|_{G-u}$ is an IR{2}DF of $G-u$ and so $i^{02}_{\{R2\}}(G,u) \ge i^{00}_{\{R2\}}(G,u)$.

On the other hand, any $i_{R2}(G-u)$ can be extended to an IR{2}DF of $G+uw$ by assigning a 2 to w and a 0 to u and hence $i^{00}_{\{R2\}}(G,u) \ge i^{02}_{\{R2\}}(G,u)$. Therefore, we have $i^{00}_{\{R2\}}(G,u) = i^{02}_{\{R2\}}(G,u)$. □

Theorem 4.4. *Suppose G and H are two disjoint graphs with specific vertices u and v, respectively. Let I be the graph obtained by adding the edge uv to $G \cup H$. Consider u as the specific vertex of I. Then the following statements hold.*

(i) $i^0_{\{R2\}}(I,u) = \min\{i^0_{\{R2\}}(G,u) + i^0_{\{R2\}}(H,v), i^{01}_{\{R2\}}(G,u) + i^1_{\{R2\}}(H,v), i^{00}_{\{R2\}}(G,u) + i^2_{\{R2\}}(H,v)\}$;
(ii) $i^1_{\{R2\}}(I,u) = i^1_{\{R2\}}(G,u) + i^{01}_{\{R2\}}(H,v)$;
(iii) $i^2_{\{R2\}}(I,u) = i^2_{\{R2\}}(G,u) + i^{00}_{\{R2\}}(H,v)$;
(iv) $i^{00}_{\{R2\}}(I,u) = i^{00}_{\{R2\}}(G,u) + i_{\{R2\}}(H) = i^{00}_{\{R2\}}(G,u) + \min\{i^0_{\{R2\}}(H,v), i^1_{\{R2\}}(H,v), i^2_{\{R2\}}(H,v)\}$;
(v) $i^{01}_{\{R2\}}(I,u) = \min\{i^{01}_{\{R2\}}(G,u) + i^0_{\{R2\}}(H,v), i^{00}_{\{R2\}}(G,u) + i^1_{\{R2\}}(H,v), i^{00}_{\{R2\}}(G,u) + i^2_{\{R2\}}(H,v)\}$.

Proof. (i) Let f be an $I\{R2\}DF$ of I such that $f(u)=0$ and $w(f) = i^0_{\{R2\}}(I,u)$. If $f(v)=0$, then $f|_G$ is an $I\{R2\}DF$ of G with $f|_G(u)=0$ and $f|_H$ is an $I\{R2\}DF$ of H with $f|_H(v)=0$. Hence, we have $i^0_{\{R2\}}(I,u) \ge i^0_{\{R2\}}(G,u) + i^0_{\{R2\}}(H,v)$. If $f(v)=1$, then $f|_{G+uv}$ is an $I\{R2\}DF$ of $G+uv$ with $f|_{G+uv}(u)=0$ and $f|_{G+uv}(v)=1$, and $f|_H$ is an $I\{R2\}DF$ of H with $f|_H(v)=1$. Hence, we have $i^0_{\{R2\}}(I,u) = w(f|_{G+uv}) - 1 + w(f|_H) \ge i^{01}_{\{R2\}}(G,u) + i^1_{\{R2\}}(H,v)$. If $f(v)=2$, then $f|_{G+uv}$ is an $I\{R2\}DF$ of $G+uv$ with $f|_{G+uv}(u)=0$ and $f|_{G+uv}(v)=2$, and $f|_H$ is an $I\{R2\}DF$ of H with $f|_H(v)=2$. Hence, we have $i^0_{\{R2\}}(I,u) = w(f|_{G+uv}) - 2 + w(f|_H) \ge i^{02}_{\{R2\}}(G,u) + i^2_{\{R2\}}(H,v)$. By Observation 4.3, $i^0_{\{R2\}}(I,u) \ge i^{00}_{\{R2\}}(G,u) + i^2_{\{R2\}}(H,v)$. Thus

$$i^0_{\{R2\}}(I,u) \ge \min\{i^0_{\{R2\}}(G,u) + i^0_{\{R2\}}(H,v), i^{01}_{\{R2\}}(G,u) + i^1_{\{R2\}}(H,v), i^{00}_{\{R2\}}(G,u) + i^2_{\{R2\}}(H,v)\}. \tag{5}$$

Now we prove the inverse inequality. Since any $I\{R2\}DF$ g of G with $g(u)=0$ and any $I\{R2\}DF$ h of H with $h(v)=0$ can form an $I\{R2\}DF$ f' of I with $f'(u)=0$, we have $i^0_{\{R2\}}(G,u) + i^0_{\{R2\}}(H,v) \ge i^0_{\{R2\}}(I,u)$. Also, any I{R2}DF g of $G+uv$ with $g(u)=0$ and $g(v)=1$ and any $I\{R2\}DF$ h of H with $h(v)=1$ can form an I{R2}DF f' of I such that $f'(u)=0$ yielding $i^{01}_{\{R2\}}(G,u) + i^1_{\{R2\}}(H,v) \ge i^0_{\{R2\}}(I,u)$. Similarly, we can see that $i^{00}_{\{R2\}}(G,u) + i^2_{\{R2\}}(H,v) = i^{02}_{\{R2\}}(G,u) + i^2_{\{R2\}}(H,v) \ge i^0_{\{R2\}}(I,u)$. Therefore

$$\begin{aligned} i^0_{\{R2\}}(I,u) \le \min\,\{&i^0_{\{R2\}}(G,u) + i^0_{\{R2\}}(H,v), i^{01}_{\{R2\}}(G,u) + i^1_{\{R2\}}(H,v), \\ &i^{00}_{\{R2\}}(G,u) + i^2_{\{R2\}}(H,v)\}. \end{aligned} \tag{6}$$

Now

(i) follows from (5) and (6).
(ii) It follows from the fact that f is an $IR\{2\}$DF of I with $f(u)=1$ if and only if $f = g \cup h$, where g is an $IR\{2\}$DF of G with $g(u)=1$ and h is an $IR\{2\}$DF of $H+vu$ with $h(v)=0$ and $h(u)=1$.
(iii) Note that f is an $IR\{2\}$DF of I with $f(u)=2$ if and only if $f = g \cup h$, where g is an $IR\{2\}$DF of G with $g(u)=2$ and h is an $IR\{2\}$DF of $H+vu$ with $h(v)=0$ and $h(u)=2$. Using this and Observation 4.3, the result follows.

(iv) It follows from the fact that f is an $IR\{2\}$DF of $I - u$ if and only if $f = g \cup h$, where g is an $IR\{2\}$DF of $G - u$ and h is an $IR\{2\}$DF of H.

(v) Let f be an $I\{R2\}DF$ of $I + uw$ such that $f(u) = 0$, $f(w) = 1$ and $w(f) = i^{01}_{\{R2\}}(I, u)$, where uw is the pendant edge added at u. If $f(v) = 0$, then $f|_{G+uw}$ is an $I\{R2\}DF$ of $G + uw$ with $f|_{G+uw}(u) = 0$ and $f|_{G+uw}(w) = 1$, and $f|_H$ is an $I\{R2\}DF$ of H with $f|_H(v) = 0$. Hence, we have $i^{01}_{\{R2\}}(I, u) = w(f) - 1 = w(f|_{G+uw}) - 1 + w(f|_H) \geq i^{01}_{\{R2\}}(G, u) + i^{0}_{\{R2\}}(H, v)$. If $f(v) = 1$, then $f|_{G-u}$ is an $I\{R2\}DF$ of $G - u$ and $f|_H$ is an $I\{R2\}DF$ of H with $f|_H(v) = 1$. Hence, we have $i^{01}_{\{R2\}}(I, u) = w(f) - 1 = w(f|_{G-u}) + w(f|_H) \geq i^{00}_{\{R2\}}(G, u) + i^{1}_{\{R2\}}(H, v)$. If $f(v) = 2$, then $f|_{G-u}$ is an $I\{R2\}DF$ of $G - u$ and $f|_H$ is an $I\{R2\}DF$ of H with $f|_H(v) = 2$. Hence, we have $i^{0}_{\{R2\}}(I, u) = w(f) - 1 = w(f|_{G-u}) + w(f|_H) \geq i^{00}_{\{R2\}}(G, u) + i^{2}_{\{R2\}}(H, v)$. Hence

$$i^{01}_{\{R2\}}(I, u) \geq \min\{i^{01}_{\{R2\}}(G, u) + i^{0}_{\{R2\}}(H, v), i^{00}_{\{R2\}}(G, u) + i^{1}_{\{R2\}}(H, v), i^{00}_{\{R2\}}(G, u) + i^{2}_{\{R2\}}(H, v)\}. \tag{7}$$

Now we prove the inverse inequality. Since combining any I{R2}DF g of $G + uw$ with $g(u) = 0$ and $g(w) = 1$ and any I{R2}DF h of H with $h(v) = 0$ can form an $I\{R2\}DF$ f' of $I + uw$ with $f'(u) = 0$ and $f'(w) = 1$, we have $i^{01}_{\{R2\}}(G, u) + i^{0}_{\{R2\}}(H, v) \geq i^{01}_{\{R2\}}(I, u)$. Also, any I{R2}DF g of $G - u$ and any I{R2}DF h of H with $h(v) = 1$ can be extended to an I{R2}DF f' of $I + uw$ by setting $f'(u) = 0$ and $f'(w) = 1$, and so $i^{00}_{\{R2\}}(G, u) + i^{1}_{\{R2\}}(H, v) \geq i^{01}_{\{R2\}}(I, u)$ Finally, any $I\{R2\}DF$ g of $G - u$ and any $I\{R2\}DF$ h of H with $h(v) = 2$ can be extended to an $I\{R2\}DF$ f' of I by setting $f'(u) = 0$ and $f'(w) = 1$, and hence $i^{00}_{\{R2\}}(G, u) + i^{2}_{\{R2\}}(H, v) \geq i^{01}_{\{R2\}}(I, u)$. Consequently,

$$i^{01}_{\{R2\}}(I, u) \leq \min\{i^{01}_{\{R2\}}(G, u) + i^{0}_{\{R2\}}(H, v), i^{00}_{\{R2\}}(G, u) + i^{1}_{\{R2\}}(H, v), i^{00}_{\{R2\}}(G, u) + i^{2}_{\{R2\}}(H, v)\}, \tag{8}$$

and (v) follows from (7) and (8).

□

If the vertices of a tree T have an ordering $[v_1, v_2, \ldots, v_n]$ such that v_i is a leaf of $T_i{=}T[\{v_i, v_{i+1}, \ldots, v_n\}]$ for $1 \leq i \leq n - 1$, then $[v_1, v_2, \ldots, v_n]$ is called a tree ordering of T, where the only neighbor v_j of v_i with $j > i$ is called the parent of v_i. Lemma 4.1 and Theorem 4.4 give the following dynamic programming algorithm for computing $i_{\{R2\}}(T)$ for any tree T.

Algorithm $i_{\{R2\}}$ Domination
Input: A tree T with a tree ordering $[v_1, v_2, \ldots, v_n]$.
Output: the independent Roman {2}-domination number $i_{\{R2\}}(T)$ of T.
begin
  for $i = 1$ to n **do**
    $i^{00}_{\{R2\}}(v_i) \leftarrow 0$;
    $i^{0}_{\{R2\}}(v_i) \leftarrow \infty$;
    $i^{01}_{\{R2\}}(v_i) \leftarrow \infty$;
    $i^{1}_{\{R2\}}(v_i) \leftarrow 1$;
    $i^{2}_{\{R2\}}(v_i) \leftarrow 2$;
  end for
  for $i = 1$ to $n - 1$ **do**
    let v_j be the parent of v_i;

$i_{\{R2\}}(v_i) = \min\{i^0_{\{R2\}}(v_i), i^1_{\{R2\}}(v_i), i^2_{\{R2\}}(v_i)\}$
$i^0_{\{R2\}}(v_j) = \min\{i^0_{\{R2\}}(v_j) + i^0_{\{R2\}}(v_i), i^{01}_{\{R2\}}(v_j) + i^1_{\{R2\}}(v_i), i^{00}_{\{R2\}}(v_j) + i^2_{\{R2\}}(v_i)\}\}.$
$i^1_{\{R2\}}(v_j) = i^1_{\{R2\}}(v_j) + i^{01}_{\{R2\}}(v_i)$
$i^2_{\{R2\}}(v_j) = i^2_{\{R2\}}(v_j) + i^{00}_{\{R2\}}(v_i)$
$i^{01}_{\{R2\}}(v_j) = \min\{i^{01}_{\{R2\}}(v_j) + i^0_{\{R2\}}(v_i), i^{00}_{\{R2\}}(v_j) + i^1_{\{R2\}}(v_i), i^{00}_{\{R2\}}(v_j) + i^2_{\{R2\}}(v_i)\}$
$i^{00}_{\{R2\}}(v_i) = i^{00}_{\{R2\}}(v_j) + \min\{i^0_{\{R2\}}(v_i), i^1_{\{R2\}}(v_i), i^2_{\{R2\}}(v_i)\}$
end for
return $i_{\{R2\}}(v_n)$;
end.

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