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A COMPREHENSIVE STUDY OF A BACKLOGGING EOQ MODEL WITH NONLINEAR HEPTAGONAL DENSE FUZZY ENVIRONMENT

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Abstract. This paper deals with an adaptation of an application of nonlinear heptagonal dense fuzzy number. The concept of linear and as well as non-linear for both symmetric and asymmetric heptagonal dense fuzzy number is introduced here. We develop a new ranking method for non-linear heptagonal dense fuzzy number also. Considering a backorder inventory model with non-linear heptagonal dense fuzzy demand rate we have utilized a modified centroid method for defuzzification. For decision maker's aspects, numerical examples, comparative study with other dense fuzzy numbers and a sensitivity analysis show the superiority of the nonlinear heptagonal dense fuzzy number. Finally, graphical illustrations are made to justify the model followed by a conclusion.

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1. Introduction

1.1. Review of literature on fuzzy numbers

Zadeh [13] developed the concept on Fuzzy Set theory which strongly deals with the facts of nonrandom uncertainty. This theory has been applying with great success in various fields recent time. Chang and Zadeh [47] introduced the concept of fuzzy mapping and control. Fuzzy numbers like L-R fuzzy number, triangular fuzzy number, trapezoidal fuzzy number, pentagonal fuzzy number, hexagonal fuzzy number, heptagonal fuzzy number, half circle fuzzy number, exponential fuzzy number, Gaussian fuzzy number etc. are quite well-known general fuzzy numbers. But due to insertion of learning experiences this fuzzy numbers themselves can be modified into some other fuzzy numbers. Researchers like Felix and Devadoss [1] (Decagonal fuzzy number), De and Mahata [33] (cloudy fuzzy number), De and Beg [34, 35] (dense fuzzy number), De [42] (triangular dense fuzzy lock set) etc. have worked along these directions.

Also, the ranking of fuzzy number is an important component of the decision making process. Several attempts have been made [3, 24–26, 28, 29, 45, 50, 51, 54, 55, 57] to defuzzify the fuzzy numbers. At the early stage, Chen and Lu [14, 15], Liu and Han [53] etc. have used cut set and decision-maker's preference to construct ranking

Keywords. Heptagonal dense fuzzy number, centroid method, inventory problem, optimization.

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function. On the other hand, for ranking several fuzzy numbers the centroid method has been studied by Cheng [4], Chu and Tsao [48], and Wang and Lee [56] extensively. Also, some adaptive defuzzification methods has been developed to explore the process of human thinking in the present decade. To defuzzify the fuzzy objective Halgamuge *et al.* [27] used neural networks, Song and Leland [21] and Yager [23] etc. proposed an adaptive learning technique for solving the behavioral management problems. A parameterized defuzzification method with Gaussian based distribution transformation and polynomial transformation has been discussed by Filev and Yager [5], Jiang and Li [49] etc. But in fact, no method gives a right effective defuzzification output based on learning experiences yet. Therefore, in this paper we have studied several types of fuzzy numbers under dense fuzzy rule which is strongly associated with learning experiences over any kind of decision making process.

1.2. Motivation on economic order quantity (EOQ) model under fuzzy environment

Nowadays, modern researchers have been enriching traditional backorder EOQ model using different approximation and methodologies of uncertain parameters. In the literature, numerous research articles have been found to characterize the inventory problems. Researchers like De and Sana [38,39] have developed a backlogging EOQ model under intuitionistic fuzzy set (IFS) using the score function of the objective value. De *et al.* [32] have applied the IFS technique via interpolating by pass to develop a backorder EOQ model. However, in IFS environment De [41] investigated a special type of EOQ model where the natural idle time (general closing time duration per day) has been considered. Das *et al.* [20] considered a step order fuzzy model for time dependent backlogging over idle time. Also, at the same time De and Sana [37] developed an alternative fuzzy EOQ model with backlogging for selling price and promotional effort sensitive demand. Recently, De and Sana [40] developed the (p, q, r, l) model for stochastic demand under intuitionistic fuzzy aggregation with Bonferroni mean. Using the learning effect on fuzzy parameters Kazemi *et al.* [17–19] developed an EOQ model for imperfect quality items and they incorporated the human forgetting effect also. Accumulating several fuzzy inventory models Shekarian *et al.* [9] discussed a comprehensive review of the inventory models. Items of imperfect quality were analyzed by Shekarian *et al.* [8] and Patro *et al.* [22]. The fuzzy inventory model has also been enriched with the hands of some other contemporary researchers [6,7,43,52]. Karmakar *et al.* [30] studied a pollution sensitive production inventory model in which the dense fuzzy number and its new defuzzyfication method have been applied intelligently. Karmakar *et al.* [31] developed another pollution sensitive remanufacturing model using triangular dense fuzzy lock set which deserve a right milestone in the related field itself. As per latest literature survey we have seen that articles on learning experiences studied by Maity *et al.* [44] which includes an EOQ model for two decision makers' single decision using triangular dense fuzzy lock set very well.

1.3. Literature review over heptagonal fuzzy number

The use of heptagonal fuzzy number in different fields has been enlisted chronologically in the following Table 1.

From the above study we see that linear membership function with symmetry is only taken most of the cases but the concept of non-linear membership function was absent. In this article, we have studied all types of fuzzy number for linear and non-linear membership functions. In addition, we see that not a single article has been developed yet using the dense fuzzy rule which is associated with learning experiences of human being over several fuzzy numbers. Therefore, we have incorporated dense fuzzy rule to each of the fuzzy numbers (triangular fuzzy, trapezoidal fuzzy, pentagonal fuzzy, hexagonal fuzzy and heptagonal fuzzy) for linear and non-linear cases into a classical backorder EOQ model. In this study, we derive different types of heptagonal dense fuzzy number namely symmetric linear heptagonal dense fuzzy number, asymmetric linear heptagonal dense fuzzy number, symmetric nonlinear heptagonal dense fuzzy number and asymmetric nonlinear heptagonal dense fuzzy numbers also. We consider the classical backorder EOQ model for the applicability of new fuzzy approach in the related fields. Utilizing centroid method, a numerical analysis along with graphical illustration has been made. We have shown that nonlinear heptagonal dense fuzzy model gives optimum result with respect to the other models stated as earlier.

Table 1. Use of heptagonal fuzzy number in the literature.

Authors information	Types of membership function	Main contribution	Application area
Selvakumari and Lavanya [46]	Linear membership function with symmetry	Define Heptagonal fuzzy number and alpha cut	Game problem
Sudha and Karunambigai [2]	Linear membership function with symmetry	Propose a ranking method	Transportation problem
Rathi *et al.* [12]	Linear membership function with symmetry	Define arithmetic operation and alpha cut	Assignment problem
Namarta *et al.* [16]	Generalized Linear membership function with symmetry	Propose a ranking method	Ranking problem
Rathi and Balamohan [11]	Linear membership function	Define Heptagonal fuzzy number, alpha cut and ranking	Assignment problem
Rathi and Balamohan [10]	Symmetric membership function	Propose a ranking method	Multi criterion decision making problem
Maity *et al.* this paper	Linear and non-linear membership function	Heptagonal dense fuzzy number	Application in inventory management problem

2. Preliminaries

Here we introduce some useful definitions of fuzzy set and the corresponding ranking methods.

Definition 2.1. A fuzzy set $\widetilde{A}$ defined on the universal set of real numbers R, is said to be a fuzzy number if it satisfies the following properties:

(i) $\widetilde{A}$ is convex.
(ii) $\widetilde{A}$ is normal *i.e.*, $\exists x_0 \in R$ such that $\mu_{\widetilde{A}}(x_0) = 1$.
(iii) $\mu_{\widetilde{A}}(x)$ is piecewise continuous.
(iv) A_α must be closed interval for every $\alpha \in [0, 1]$.
(v) The support of $\widetilde{A}$, *i.e.*, support($\widetilde{A}$) must be bounded.

2.1. Heptagonal fuzzy number (HFN)

In this section we develop different types of heptagonal fuzzy number and their corresponding membership functions with graphical illustrations.

Definition 2.2. A fuzzy number $\widetilde{A} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ is said to be HFN if it satisfies the following conditions.

(i) $\mu_{\widetilde{A}}(x)$ is a continuous function in the interval [0, 1]
(ii) $\mu_{\widetilde{A}}(x)$ is strictly increasing and continuous function on $[a_1, a_2]$ and $[a_3, a_4]$
(iii) $\mu_{\widetilde{A}}(x)$ takes value k in the interval $[a_2, a_3]$ and $[a_5, a_6]$ where $0 < k < 1$
(iv) $\mu_{\widetilde{A}}(x)$ is strictly decreasing and continuous function on $[a_4, a_5]$ and $[a_6, a_7]$
(v) $\mu_{\widetilde{A}}(x)$ takes value 1 at the point a_4.

Definition 2.3. Two heptagonal fuzzy numbers $\widetilde{A} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ and $\widetilde{B} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7)$ are said to be equal if $a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3,\ a_4 = b_4, a_5 = b_5, a_6 = b_6, a_7 = b_7$.

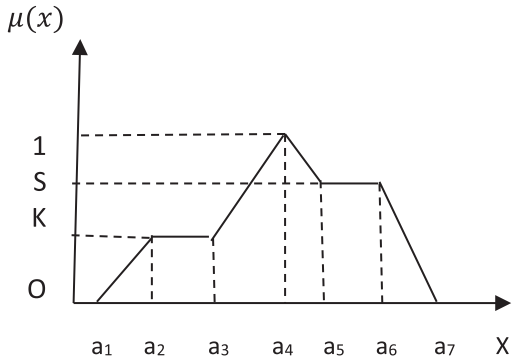

FIGURE 1. Membership function of LHFNA.

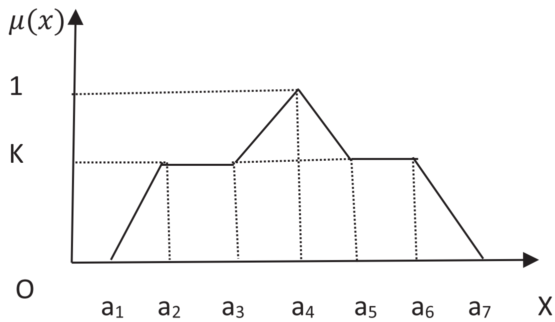

FIGURE 2. Membership function of LHFNS.

Now, we try to define some new types of heptagonal fuzzy number in their different form.

2.2. Linear Heptagonal fuzzy number with asymmetry (LHFNA)

Definition 2.4. A linear heptagonal fuzzy number is defined as $\widetilde{A}_{\text{LAS}} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7; k, s)$ whose membership function is given by

$$\mu_{\widetilde{A}_{\text{LAS}}}(x) = \begin{cases} 0 & \text{if } x < a_1 \\ k\left(\frac{x-a_1}{a_2-a_1}\right) & \text{if } a_1 \leq x \leq a_2 \\ k & \text{if } a_2 \leq x \leq a_3 \\ k + (1-k)\frac{x-a_3}{a_4-a_3} & \text{if } a_3 \leq x \leq a_4 \\ 1 & \text{if } x = a_4 \\ s + (1-s)\frac{a_5-x}{a_5-a_4} & \text{if } a_4 \leq x \leq a_5 \\ s & \text{if } a_5 \leq x \leq a_6 \\ s\left(\frac{a_7-x}{a_7-a_6}\right) & \text{if } a_6 \leq x \leq a_7 \\ 0 & \text{if } x > a_7. \end{cases} \quad (1)$$

The graphical representation of LHFNA is shown in Figure 1.

Note. (i) If $k = s$ the asymmetry heptagonal fuzzy number becomes symmetry heptagonal fuzzy number shown in Figure 2.

(ii) For asymmetry heptagonal fuzzy number may be $k < s$ or $k > s$.

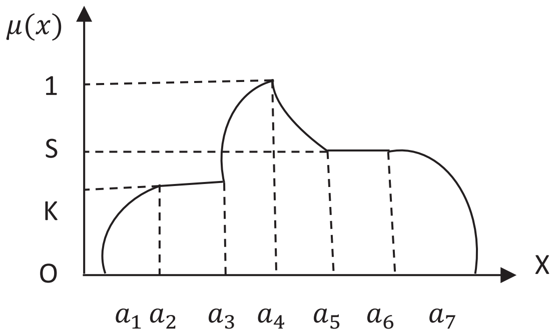

FIGURE 3. Membership function of NHFNA.

(iii) When $k = s = 1$ the heptagonal fuzzy number will be converted to trapezoidal fuzzy number.
(iv) When $a_3 = a_4$ and $a_4 = a_5$ then the heptagonal fuzzy number is also converted to trapezoidal fuzzy number. In this case if $k, s \neq 1$ then it transformed to generalized trapezoidal fuzzy number.

3. Nonlinear heptagonal fuzzy number with asymmetry (NHFNA)

Definition 3.1. A nonlinear heptagonal fuzzy number with asymmetry is defined as $\widetilde{A}_{\mathrm{NA}} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7; k, s)_{(n_1, n_2; m_1, m_2)}$ whose membership function is stated as

$$\mu_{\widetilde{A}_{\mathrm{NA}}}(x) = \begin{cases} 0 & \text{if } x < a_1 \\ k\left(\frac{x-a_1}{a_2-a_1}\right)^{n_1} & \text{if } a_1 \leq x \leq a_2 \\ k & \text{if } a_2 \leq x \leq a_3 \\ k + (1-k)\left(\frac{x-a_3}{a_4-a_3}\right)^{n_2} & \text{if } a_3 \leq x \leq a_4 \\ 1 & \text{if } x = a_4 \\ s + (1-s)\left(\frac{a_5-x}{a_5-a_4}\right)^{m_1} & \text{if } a_4 \leq x \leq a_5 \\ s & \text{if } a_5 \leq x \leq a_6 \\ s\left(\frac{a_7-x}{a_7-a_6}\right)^{m_2} & \text{if } a_6 \leq x \leq a_7 \\ 0 & \text{if } x > a_7. \end{cases} \tag{2}$$

The graphical interpretation of NHFNA is shown in Figure 3.

Note. (i) If $k = s$ the asymmetry non-linear heptagonal fuzzy number becomes symmetry non-linear heptagonal fuzzy number shown in Figure 4.

(ii) When $k = s = 1$ the non-linear heptagonal fuzzy number will be converted to non-linear trapezoidal fuzzy number.
(iii) if $n_1 = n_2 = m_1 = m_2 = 1$ then non-linear asymmetric heptagonal fuzzy number becomes linear heptagonal asymmetric fuzzy number.
(iv) If $k = 1, s = 1, n_1 = n_2 = m_1 = m_2 = 1$ then non-linear heptagonal fuzzy number becomes linear Trapezoidal fuzzy number.

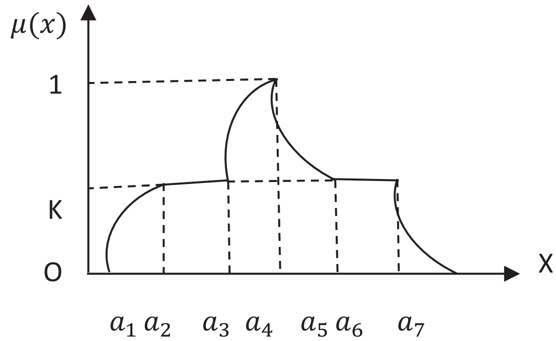

FIGURE 4. Membership function of NHFNS.

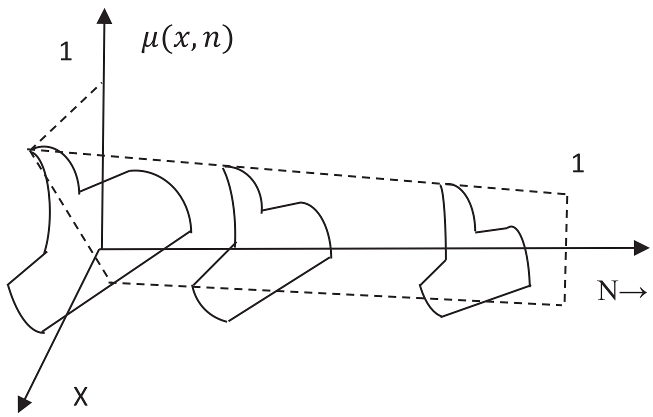

FIGURE 5. Membership function of NHDFNA.

4. Nonlinear heptagonal dense fuzzy number with asymmetry (NHDFNA)

Definition 4.1. A nonlinear heptagonal dense fuzzy number with symmetry is defined as $\widetilde{A}_{\mathrm{NDA}} = \left(d_1\left(1-\frac{\rho_1}{1+n}\right), d_1\left(1-\frac{\rho_2}{1+n}\right), d_1\left(1-\frac{\rho_3}{1+n}\right), d_1, d_1\left(1+\frac{\sigma_1}{1+n}\right), d_1\left(1+\frac{\sigma_2}{1+n}\right), d_1\left(1+\frac{\sigma_3}{1+n}\right); k, s\right)_{(n_1,n_2;m_1,m_2)}$ whose membership function is written as

$$\mu_{\mathrm{NDA}}(x) = \begin{cases} k\left\{\frac{x-d_1\left(1-\frac{\rho_1}{1+n}\right)}{\frac{d_1(\rho_1-\rho_2)}{1+n}}\right\}^{n_1} & \text{if } d_1\left(1-\frac{\rho_1}{1+n}\right) \le x \le d_1\left(1-\frac{\rho_2}{1+n}\right) \\ k & \text{if } d_1\left(1-\frac{\rho_2}{1+n}\right) \le x \le d_1\left(1-\frac{\rho_3}{1+n}\right) \\ k+(1-k)\left\{\frac{x-d_1\left(1-\frac{\rho_3}{1+n}\right)}{\frac{d_1\rho_3}{1+n}}\right\}^{n_2} & \text{if } d_1\left(1-\frac{\rho_3}{1+n}\right) \le x \le d_1 \\ s+(1-s)\left\{\frac{d_1\left(1+\frac{\sigma_1}{1+n}\right)-x}{\frac{d_1\sigma_1}{1+n}}\right\}^{m_1} & \text{if } d_1 \le x \le d_1\left(1+\frac{\sigma_1}{1+n}\right) \\ s & \text{if } d_1\left(1+\frac{\sigma_1}{1+n}\right) \le x \le d_1\left(1+\frac{\sigma_2}{1+n}\right) \\ s\left\{\frac{d_1\left(1+\frac{\sigma_3}{1+n}\right)-x}{\frac{d_1(\sigma_3-\sigma_2)}{1+n}}\right\}^{m_2} & \text{if } d_1\left(1+\frac{\sigma_2}{1+n}\right) \le x \le d_1\left(1+\frac{\sigma_3}{1+n}\right) \end{cases} \tag{3}$$

The graphical interpretation is shown in Figure 5.

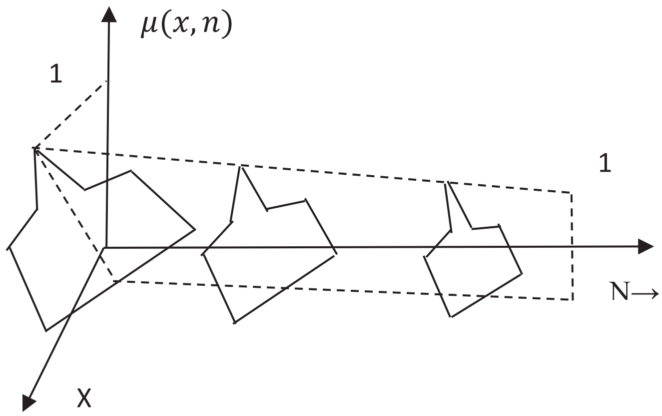

FIGURE 6. Membership function of LHDFNS.

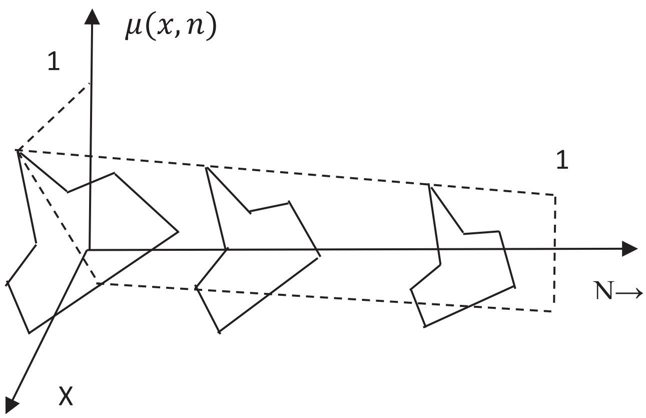

FIGURE 7. Membership function of LHDFNA.

Note. Case I: If $n_1 = n_2 = m_1 = m_2 = 1$ and $k = s = 1$ then non-linear asymmetric heptagonal dense fuzzy number becomes linear heptagonal symmetric dense fuzzy number shown in Figure 6.

Case II: If $n_1 = n_2 = m_1 = m_2 = 1$then non-linear asymmetric heptagonal dense fuzzy number becomes linear heptagonal asymmetric dense fuzzy number shown in Figure 7.

Case III: If $k = s = 1$ then non-linear asymmetric heptagonal dense fuzzy number becomes non-linear heptagonal symmetric dense fuzzy number shown in Figure 8.

5. Ranking of heptagonal fuzzy number with symmetry (HFNS)

Let, A is a set of fuzzy numbers defined on the set of real numbers and the ranking of fuzzy numbers is actually a function $F : A \to R$ which maps each fuzzy number into a real line. We propose a new method to find out the ranking of fuzzy numbers. Let us consider the linear heptagonal symmetric fuzzy number with height w is $A = (a_1, a_2, a_3, a_4, a_5, a_6, a_7; w/2, w/2)$ (shown in Fig. 9). The heptagon is divided into three triangular and three rectangular regions. Using centroid formula, we can find the ranking values as follows:

Let, T_1 denotes the centroid of the triangle whose vertices are $(a_1, 0), (a_2, 0), \left(a_2, \frac{w}{2}\right)$
T_2 denotes the centroid of the rectangle whose vertices are $(a_2, 0), (a_3, 0), \left(a_2, \frac{w}{2}\right), \left(a_3, \frac{w}{2}\right)$

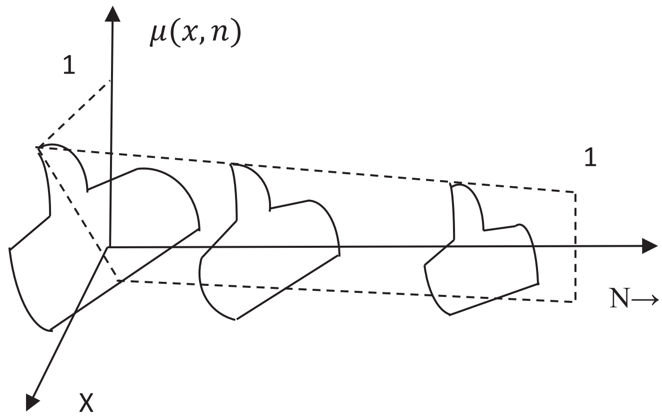

FIGURE 8. Membership function of NHDFNS.

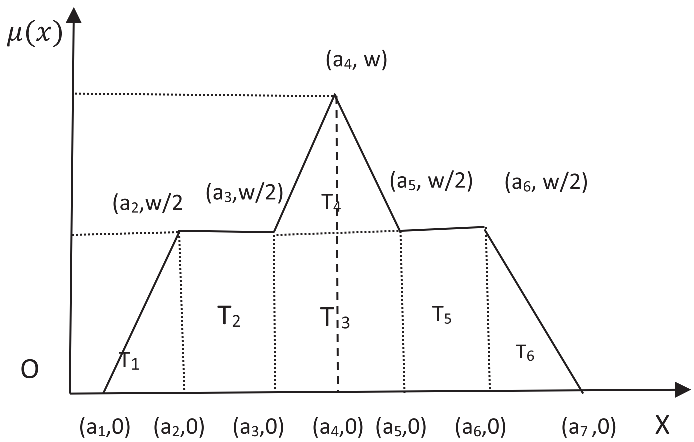

FIGURE 9. HFNS splits into different regions.

T_3 denotes the centroid of the rectangle whose vertices are $(a_3,0),(a_5,0),\left(a_3,\frac{w}{2}\right),\left(a_5,\frac{w}{2}\right)$
T_4 denotes the centroid of the triangle whose vertices are $(a_4,w),\left(a_3,\frac{w}{2}\right),\left(a_5,\frac{w}{2}\right)$
T_5 denotes the centroid of the rectangle whose vertices are $(a_5,0),(a_6,0),\left(a_5,\frac{w}{2}\right),\left(a_6,\frac{w}{2}\right)$
T_6 denotes the centroid of the triangle whose vertices are $(a_6,0),(a_7,0),\left(a_6,\frac{w}{2}\right)$ then

$$T_1=\left(\frac{a_1+2a_2}{3},\frac{w}{6}\right),T_2=\left(\frac{a_1+a_3}{2},\frac{w}{4}\right),T_3=\left(\frac{a_3+a_5}{2},\frac{w}{4}\right),$$
$$T_4=\left(\frac{a_4+a_5+a_3}{3},\frac{2w}{3}\right),T_5=\left(\frac{a_5+a_6}{2},\frac{w}{4}\right),T_6=\left(\frac{2a_6+a_7}{3},\frac{w}{6}\right).$$

By adding these all we can get the new ranking as

$$\text{Rank(A)}=\left(\frac{2a_1+7a_2+8a_3+2a_4+8a_5+7a_6+2a_7}{6},\frac{21w}{12}\right). \tag{4}$$

Now if A and B are two different heptagonal fuzzy numbers then Rank(A) and Rank(B) will satisfy the following:

(i) $R(A)>R(B)$ then $A>B$, (ii) $R(A)<R(B)$ then $A<B$, (iii) $R(A)=R(B)$ then $A\approx B$.

6. Defuzzification based on centroid method

We developed the defuzzification of non-linear symmetric heptagonal fuzzy number as,

$$R = \frac{\int_{a_1}^{a_2} x.\mu(x)\,dx + \int_{a_2}^{a_3} x.\mu(x)\,dx + \int_{a_3}^{a_4} x.\mu(x)\,dx + \int_{a_4}^{a_5} x.\mu(x)\,dx + \int_{a_5}^{a_6} x.\mu(x)\,dx + \int_{a_6}^{a_7} x.\mu(x)\,dx}{\int_{a_1}^{a_2} \mu(x)\,dx + \int_{a_2}^{a_3} \mu(x)\,dx + \int_{a_3}^{a_4} \mu(x)\,dx + \int_{a_4}^{a_5} \mu(x)\,dx + \int_{a_5}^{a_6} \mu(x)\,dx + \int_{a_6}^{a_7} \mu(x)\,dx}$$

$$= \frac{\int_{a_1}^{a_2} x.\left\{k\left(\frac{x-a_1}{a_2-a_1}\right)^{n_1}\right\}dx + \int_{a_2}^{a_3} x.k\,dx + \int_{a_3}^{a_4} x.\left\{k+(1-k)\left(\frac{x-a_3}{a_4-a_3}\right)^{n_2}\right\}dx + \int_{a_4}^{a_5} x.\left\{k+(1-k)\left(\frac{a_5-x}{a_5-a_4}\right)^{m_1}\right\}dx + \int_{a_5}^{a_6} x.k\,dx + \int_{a_6}^{a_7} x.\left\{k\left(\frac{a_7-x}{a_7-a_6}\right)^{m_2}\right\}dx}{\int_{a_1}^{a_2} \left\{k\left(\frac{x-a_1}{a_2-a_1}\right)^{n_1}\right\}dx + \int_{a_2}^{a_3} k.dx + \int_{a_3}^{a_4} \left\{k+(1-k)\left(\frac{x-a_3}{a_4-a_3}\right)^{n_2}\right\}dx + \int_{a_4}^{a_5} \left\{k+(1-k)\left(\frac{a_5-x}{a_5-a_4}\right)^{m_1}\right\}dx + \int_{a_5}^{a_6} k.dx + \int_{a_6}^{a_7} \left\{k\left(\frac{a_7-x}{a_7-a_6}\right)^{m_2}\right\}dx}$$

$$= \frac{\frac{k(a_2-a_1)^2}{n_1+2} + \frac{ka_1(a_2-a_1)}{n_1+1} + \frac{k\left(a_6^2-a_2^2\right)}{2} + \frac{(1-k)(a_4-a_3)^2}{n_2+2} + \frac{a_3(1-k)(a_4-a_3)}{n_2+1} - \frac{(1-k)(a_5-a_4)^2}{m_1+2} + \frac{a_5(1-k)(a_5-a_4)}{m_1+1} - \frac{k(a_7-a_6)^2}{m_2+2} + \frac{ka_7(a_7-a_6)}{m_2+1}}{\frac{k(a_2-a_1)}{n_1+1} + k(a_6-a_2) + \frac{(1-k)(a_4-a_3)}{n_2+1} + \frac{(1-k)(a_5-a_4)}{m_1+1} + \frac{k(a_7-a_6)}{m_2+1}}. \tag{5}$$

(i) For linear heptagonal fuzzy number, we put $n_1 = 1, n_2 = 1, m_1 = 1, m_2 = 1$

$$R_{\text{NEW}} = \frac{\frac{k(a_2-a_1)^2}{3} + \frac{ka_1(a_2-a_1)}{2} + \frac{k\left(a_6^2-a_2^2\right)}{2} + \frac{(1-k)(a_4-a_3)^2}{3} + \frac{a_3(1-k)(a_4-a_3)}{2} - \frac{(1-k)(a_5-a_4)^2}{3} + \frac{a_5(1-k)(a_5-a_4)}{2} - \frac{k(a_7-a_6)^2}{3} + \frac{ka_7(a_7-a_6)}{2}}{\frac{k(a_2-a_1)}{2} + k(a_6-a_2) + \frac{(1-k)(a_4-a_3)}{2} + \frac{(1-k)(a_5-a_4)}{2} + \frac{k(a_7-a_6)}{2}}. \tag{6}$$

(ii) For $k = 1, n_1 = 1, n_2 = 1, m_1 = 1, m_2 = 1$ we have

$$P' = \frac{a_7^2 + a_6^2 + a_7 a_6 - a_1 a_2 - a_2^2 - a_1^2}{6}, Q' = \frac{a_7 + a_6 - a_2 - a_1}{2}, R' = \frac{\left(a_7^2 + a_6^2 + a_7 a_6 - a_1 a_2 - a_2^2 - a_1^2\right)}{3.\left(a_7 + a_6 - a_2 - a_1\right)}. \tag{7}$$

7. Implication of HFN in inventory management problems

If we compare heptagonal fuzzy number with the traditional triangular or trapezoidal fuzzy number then it looks like complex in construction and as well as in definition, but heptagonal fuzzy number gives additional possibility to represent imperfect knowledge that leads to model in real life problems. Heptagonal fuzzy numbers represent linguistic information like {worst, very bad, bad, good, very good, excellent} or the purity of a metal be of {50%, 60%, 70%, 80%, 90%, 99%}. The information is retained throughout the model by the decision maker for further analysis. HFN can find its applications in many optimization problems, decision making problems like inventory management (or supply chain) problems, and economic problems etc. which need seven parameters. For example, if we consider the demand parameter of an inventory management problem then it is more practical to view this demand as {negligible, poor, low, average, high, very high} in our day to day life. Beyond this, in particular case of dynamics of tumor growth, the growth rate consists of seven points and is difficult to represent by using Triangular or Trapezoidal Fuzzy numbers. Therefore, Heptagonal fuzzy number can find its vital applications in solving the problem.

8. Assumption and notation of the inventory model

Notations

C_1 : Holding cost per quantity per unit time.
C_2 : Shortage cost per unit quantity per unit time.
C_3 : Set up cost per unit time period per cycle.
D_2 : Demand in Shortage period ($D_2 = D_1 e^{-t_2}$)
Q_1 : Inventory level at time t_1.
Q_2 : Shortage during the time t_2.

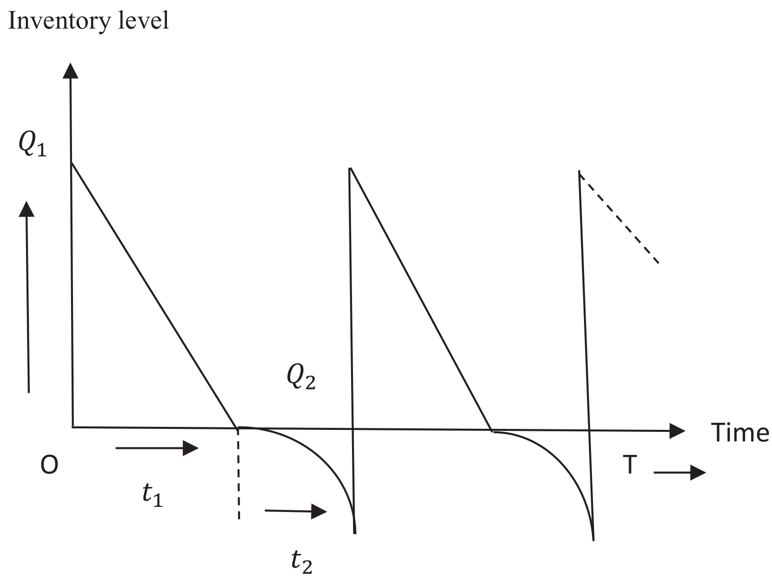

FIGURE 10. Backorder Inventory model.

Assumptions

We have the following assumptions

(i) Demand rate is uniform and known.
(ii) Rate of replenishment is finite.
(iii) Lead time is zero/negligible.
(iv) Shortage are allowed and fully backlogged.

8.1. Formulation of crisp mathematical model

Let the inventory starts at time $t = 0$ with order quantity Q_1 and demand rate D_1. After time $t = t_1$ the inventory reaches zero level and the shortage starts and it continues up to time $t = t_1+t_2$. Let Q_2 be the shortage quantity during that time period t_2. Also, we assume, that the shortage time demand rate is depending on the duration of shortage time t_2. Therefore, the mathematical problem associated to the proposed model is shown in Figure 10 and the necessary calculations are given below.

From Figure 10, using triangle law and area under curvature we get the following relations:

$$T = t_1 + t_2 \tag{8}$$

$$Q_1 = D_1 t_1. \tag{9}$$

$$\text{And } Q_2 = \int_0^{t_2} D_2 \mathrm{d}t_2 = \int_0^{t_2} D_1 \mathrm{e}^{-t_2} \mathrm{d}t_2 = D_1 \int_0^{t_2} \mathrm{e}^{-t_2} \mathrm{d}t_2 = D_1[1 - \mathrm{e}^{-t_2}]. \tag{10}$$

$$\text{Holding cost} = \frac{1}{2} C_1 Q_1 t_1. \tag{11}$$

$$\text{Shortage cost} = \frac{1}{2} C_2 Q_2 t_2 = \frac{1}{2} C_2 D_1 \left(1 - \mathrm{e}^{-t_2}\right) t_2. \tag{12}$$

$$\text{Set up cost} = C_3. \tag{13}$$

$$\text{Therefore, total inventory cost} = \frac{1}{2} C_1 Q_1 t_1 + \frac{1}{2} C_2 Q_2 t_2 + C_3. \tag{14}$$

And hence total average inventory cost $= \dfrac{1}{t_1+t_2}\left(\dfrac{1}{2}C_1Q_1t_1+\dfrac{1}{2}C_2Q_2t_2+C_3\right)$

$$= \frac{1}{2}C_1D_1\frac{t_1^2}{t_1+t_2}+\frac{1}{2}C_2D_1\frac{\left(1-\mathrm{e}^{-t_2}\right)t_2}{t_1+t_2}+\frac{C_3}{t_1+t_2}=D_1\left\{\frac{1}{2}C_1\frac{t_1^2}{t_1+t_2}\;+\;\frac{1}{2}C_2\frac{\left(1-\mathrm{e}^{-t_2}\right)t_2}{t_1+t_2}\right\}+\frac{C_3}{t_1+t_2}\cdot \quad (15)$$

Thus, our problem is given by

$$\text{Minimize } Z\left(t_1,t_2\right)=D_1\psi\left(t_1,t_2\right)+\varphi\left(t_1,t_2\right) \quad (16)$$

$$\text{Where}\begin{cases}\psi\left(t_1,t_2\right)=\frac{1}{2}C_1\frac{t_1^2}{t_1+t_2}+\frac{1}{2}C_2\frac{\left(1-\mathrm{e}^{-t_2}\right)t_2}{t_1+t_2}\\ \varphi\left(t_1,t_2\right)=\frac{C_3}{t_1+t_2}\cdot\end{cases} \quad (17)$$

Subject to the conditions (8)–(10).

9. Fuzzy mathematical model

Let the demand rate of our propose model assumes flexible values following non-random uncertainty. Also let such kind of uncertainty corresponds to a dense fuzzy set which is strongly associated with nonlinear asymmetric heptagonal dense fuzzy set. Therefore, the objective function of the crisp model (16) can be transformed to

$$\widetilde{Z}=\widetilde{D}_1\psi\left(t_1,t_2\right)+\varphi\left(t_1,t_2\right) \quad (18)$$

where ψ and φ are given by (17). Now, (18) can also be written as

$$\widetilde{D}_1=\frac{(\widetilde{Z}-\varphi)}{\psi}\cdot \quad (19)$$

The membership function of the nonlinear asymmetric heptagonal dense fuzzy set is defined by

$$\mu\left(\widetilde{D}_1\right)=\begin{cases}k\left\{\frac{D_1-d_1\left(1-\frac{\rho_1}{1+n}\right)}{\frac{d_1(\rho_1-\rho_2)}{1+n}}\right\}^{n_1} & \text{if } d_1\left(1-\frac{\rho_1}{1+n}\right)\le D_1\le d_1\left(1-\frac{\rho_2}{1+n}\right)\\ k & \text{if } d_1\left(1-\frac{\rho_2}{1+n}\right)\le D_1\le d_1\left(1-\frac{\rho_3}{1+n}\right)\\ k+(1-k)\left\{\frac{D_1-d_1\left(1-\frac{\rho_3}{1+n}\right)}{\frac{d_1\rho_3}{1+n}}\right\}^{n_2} & \text{if } d_1\left(1-\frac{\rho_3}{1+n}\right)\le D_1\le d_1\\ s+(1-s)\left\{\frac{d_1\left(1+\frac{\sigma_1}{1+n}\right)-D_1}{\frac{d_1\sigma_1}{1+n}}\right\}^{m_1} & \text{if } d_1\le D_1\le d_1\left(1+\frac{\sigma_1}{1+n}\right)\\ s & \text{if } d_1\left(1+\frac{\sigma_1}{1+n}\right)\le D_1\le d_1\left(1+\frac{\sigma_2}{1+n}\right)\\ s\left\{\frac{d_1\left(1+\frac{\sigma_3}{1+n}\right)-D_1}{\frac{d_1(\sigma_3-\sigma_2)}{1+n}}\right\}^{m_2} & \text{if } d_1\left(1+\frac{\sigma_2}{1+n}\right)\le D_1\le d_1\left(1+\frac{\sigma_3}{1+n}\right)\end{cases} \quad (20)$$

and corresponding index value is given by

$$I\left(\widetilde{D}_1\right)=\frac{\sum_{n=0}^{N}\left\{\int_{a_1}^{a_2}D_1.\mu\left(D_1\right)\mathrm{d}D_1+\int_{a_2}^{a_3}D_1.\mu\left(D_1\right)\mathrm{d}D_1+\int_{a_3}^{a_4}D_1.\mu\left(D_1\right)\mathrm{d}D_1+\int_{a_4}^{a_5}D_1.\mu\left(D_1\right)\mathrm{d}D_1+\int_{a_5}^{a_6}D_1.\mu\left(D_1\right)\mathrm{d}D_1+\int_{a_6}^{a_7}D_1.\mu\left(D_1\right)\mathrm{d}D_1\right\}}{\sum_{n=0}^{N}\left\{\int_{a_1}^{a_2}\mu\left(D_1\right)\mathrm{d}D_1+\int_{a_2}^{a_3}\mu\left(D_1\right)\mathrm{d}D_1+\int_{a_3}^{a_4}\mu\left(D_1\right)\mathrm{d}D_1+\int_{a_4}^{a_5}\mu\left(D_1\right)\mathrm{d}D_1+\int_{a_5}^{a_6}\mu\left(D_1\right)\mathrm{d}D_1+\int_{a_6}^{a_7}\mu\left(D_1\right)\mathrm{d}D_1\right\}}$$
$$=\frac{P}{Q}$$

where

$$P=\sum_{n=0}^{N}\left\{\int_{a_1}^{a_2}D_1.k\left\{\frac{D_1-d_1\left(1-\frac{\rho_1}{1+n}\right)}{\frac{d_1(\rho_1-\rho_2)}{1+n}}\right\}^{n_1}\mathrm{d}D_1+\int_{a_2}^{a_3}D_1.k\mathrm{d}D_1+\int_{a_3}^{a_4}D_1.\left\{k+(1-k)\left\{\frac{D_1-d_1\left(1-\frac{\rho_3}{1+n}\right)}{\frac{d_1\rho_3}{1+n}}\right\}^{n_2}\right\}\mathrm{d}D_1\right.$$

$$+\int_{a_4}^{a_5} D_1.\left\{s+(1-s)\left\{\frac{d_1\left(1+\frac{\sigma_1}{1+n}\right)-D_1}{\frac{d_1\sigma_1}{1+n}}\right\}^{m_1}\right\}\mathrm{d}D_1+\int_{a_5}^{a_6} D_1.s\mathrm{d}D_1+\int_{a_6}^{a_7} D_1.s\left\{\frac{d_1\left(1+\frac{\sigma_3}{1+n}\right)-D_1}{\frac{d_1(\sigma_3-\sigma_2)}{1+n}}\right\}^{m_2}\mathrm{d}D_1\Bigg\}$$

$$Q=\sum_{n=0}^{N}\left\{\int_{a_1}^{a_2} k\left\{\frac{D_1-d_1\left(1-\frac{\rho_1}{1+n}\right)}{\frac{d_1(\rho_1-\rho_2)}{1+n}}\right\}^{n_1}\mathrm{d}D_1+\int_{a_2}^{a_3} k\mathrm{d}D_1+\int_{a_3}^{a_4}\left\{k+(1-k)\left\{\frac{D_1-d_1\left(1-\frac{\rho_3}{1+n}\right)}{\frac{d_1\rho_3}{1+n}}\right\}^{n_2}\right\}\mathrm{d}D_1\right.$$

$$\left.+\int_{a_4}^{a_5}\left\{s+(1-s)\left\{\frac{d_1\left(1+\frac{\sigma_1}{1+n}\right)-D_1}{\frac{d_1\sigma_1}{1+n}}\right\}^{m_1}\right\}\mathrm{d}D_1+\int_{a_5}^{a_6} s\mathrm{d}D_1+\int_{a_6}^{a_7} s\left\{\frac{d_1\left(1+\frac{\sigma_3}{1+n}\right)-D_1}{\frac{d_1(\sigma_3-\sigma_2)}{1+n}}\right\}^{m_2}\mathrm{d}D_1\right\}. \tag{21}$$

Aft simplification we have, $I\left(\widetilde{D}_1\right)=\frac{P}{Q}$ where (21)

$$P=\sum_{n=0}^{N}\left\{\frac{kd_1^2(\rho_1-\rho_2)^2}{(n_1+2)(1+n)^2}+\frac{kd_1^2(\rho_1-\rho_2)\left(1-\frac{\rho_1}{1+n}\right)}{(n_1+1)(1+n)}+\frac{k}{2}\left\{d_1^2-d_1^2\left(1-\frac{\rho_2}{1+n}\right)^2\right\}+\frac{(1-k)d_1^2\rho_3^2}{(n_1+2)(1+n)^2}\right.$$
$$+\frac{(1-k)d_1^2\rho_3\left(1-\frac{\rho_3}{1+n}\right)}{(n_2+1)(1+n)}+\frac{s}{2}\left\{d_1^2\left(1+\frac{\sigma_2}{1+n}\right)^2-d_1^2\right\}-\frac{(1-s)d_1^2\sigma_1^2}{(m_1+2)(1+n)^2}+\frac{(1-s)d_1^2\sigma_1\left(1+\frac{\sigma_1}{1+n}\right)}{(m_1+1)(1+n)}$$
$$\left.-\frac{sd_1^2(\sigma_3-\sigma_2)^2}{(m_2+2)(1+n)^2}+\frac{sd_1^2(\sigma_3-\sigma_2)\left(1+\frac{\sigma_3}{1+n}\right)}{(m_2+1)(1+n)}\right\} \tag{22}$$

and

$$Q=\sum_{n=0}^{N}\left\{\frac{kd_1(\rho_1-\rho_2)}{(1+n)(n_1+1)}+\frac{kd_1\rho_2}{1+n}+\frac{(1-k)d_1\rho_3}{(1+n)(n_2+1)}+\frac{sd_1\sigma_2}{1+n}+\frac{(1-s)d_1\sigma_1}{(1+n)(m_1+1)}+\frac{sd_1(\sigma_3-\sigma_2)}{(1+n)(m_2+1)}\right\}. \tag{23}$$

Now we obtain the membership function of the fuzzy objective by using (19)

$$\mu\left(\widetilde{Z}\right)=\begin{cases} k\left\{\frac{\frac{Z-\varphi}{\psi}-d_1\left(1-\frac{\rho_1}{1+n}\right)}{\frac{d_1(\rho_1-\rho_2)}{1+n}}\right\}^{n_1} & \text{if } \varphi+d_1\psi\left(1-\frac{\rho_1}{1+n}\right)\le Z\le\varphi+d_1\psi\left(1-\frac{\rho_2}{1+n}\right)\\ k & \text{if } \varphi+d_1\psi\left(1-\frac{\rho_2}{1+n}\right)\le Z\le\varphi+d_1\psi\left(1-\frac{\rho_3}{1+n}\right)\\ k+(1-k)\left\{\frac{\frac{Z-\varphi}{\psi}-d_1\left(1-\frac{\rho_3}{1+n}\right)}{\frac{d_1\rho_3}{1+n}}\right\}^{n_2} & \text{if } \varphi+d_1\psi\left(1-\frac{\rho_3}{1+n}\right)\le Z\le\varphi+d_1\psi\\ s+(1-s)\left\{\frac{d_1\left(1+\frac{\sigma_1}{1+n}\right)-\frac{Z-\varphi}{\psi}}{\frac{d_1\sigma_1}{1+n}}\right\}^{m_1} & \text{if } \varphi+d_1\psi\le Z\le\varphi+d_1\psi\left(1+\frac{\sigma_1}{1+n}\right)\\ s & \text{if } \varphi+d_1\psi\left(1+\frac{\sigma_1}{1+n}\right)\le Z\le\varphi+d_1\psi\left(1+\frac{\sigma_2}{1+n}\right)\\ s\left\{\frac{d_1\left(1+\frac{\sigma_3}{1+n}\right)-\frac{Z-\varphi}{\psi}}{\frac{d_1(\sigma_3-\sigma_2)}{1+n}}\right\}^{m_2} & \text{if } \varphi+d_1\psi\left(1+\frac{\sigma_2}{1+n}\right)\le Z\le\varphi+d_1\psi\left(1+\frac{\sigma_3}{1+n}\right). \end{cases} \tag{24}$$

After defuzzification by centroid method we have index value of Z as follows

$$I\left(\widetilde{Z}\right)=\frac{\sum_{n=0}^{N}\left\{\int_{a_1}^{a_2}Z.\mu(Z)\,\mathrm{d}Z+\int_{a_2}^{a_3}Z.\mu(Z)\,\mathrm{d}Z+\int_{a_3}^{a_4}Z.\mu(Z)\,\mathrm{d}Z+\int_{a_4}^{a_5}Z.\mu(Z)\,\mathrm{d}Z+\int_{a_5}^{a_6}Z.\mu(Z)\,\mathrm{d}Z+\int_{a_6}^{a_7}Z.\mu(Z)\,\mathrm{d}Z\right\}}{\sum_{n=0}^{N}\left\{\int_{a_1}^{a_2}\mu(Z)\,\mathrm{d}Z+\int_{a_2}^{a_3}\mu(Z)\,\mathrm{d}Z+\int_{a_3}^{a_4}\mu(Z)\,\mathrm{d}Z+\int_{a_4}^{a_5}\mu(Z)\,\mathrm{d}Z+\int_{a_5}^{a_6}\mu(Z)\,\mathrm{d}Z+\int_{a_6}^{a_7}\mu(Z)\,\mathrm{d}Z\right\}}$$
$$=\frac{P}{Q} \tag{25}$$

where

$$P=\sum_{n=0}^{N}\left\{\int_{a_1}^{a_2}Z.\mu(Z)\,\mathrm{d}Z+\int_{a_2}^{a_3}Z.\mu(Z)\,\mathrm{d}Z+\int_{a_3}^{a_4}Z.\mu(Z)\,\mathrm{d}Z+\int_{a_4}^{a_5}Z.\mu(Z)\,\mathrm{d}Z+\int_{a_5}^{a_6}Z.\mu(Z)\,\mathrm{d}Z+\int_{a_6}^{a_7}Z.\mu(Z)\,\mathrm{d}Z\right\}$$

$$= \sum_{n=0}^{N} \left\{ \frac{kd_1^2\psi^2 (\rho_1 - \rho_2)^2}{(n_1+2)(1+n)^2} + \frac{kd_1\psi(\rho_1-\rho_2)\left(\varphi + d_1\psi\left(1 - \frac{\rho_1}{1+n}\right)\right)}{(n_1+1)(1+n)} + \frac{k}{2}\left[(\varphi + d_1\psi)^2 - \left\{\varphi + d_1\psi\left(1 - \frac{\rho_3}{1+n}\right)\right\}^2\right] \right.$$
$$+ \frac{(1-k)\, d_1^2\psi^2\rho_3^2}{(n_1+2)(1+n)^2} + \frac{(1-k)\, d_1\psi\rho_3\left(\varphi + d_1\psi\left(1 - \frac{\rho_3}{1+n}\right)\right)}{(n_2+1)(1+n)} + \frac{s}{2}\left[\left\{\varphi + d_1\psi\left(1 + \frac{\sigma_2}{1+n}\right)\right\}^2 - \{\varphi + d_1\psi\}^2\right]$$
$$\left. - \frac{(1-s)\, d_1^2\psi^2\sigma_1^2}{(m_1+2)(1+n)^2} + \frac{(1-s)\, d_1\psi\sigma_1\left\{\varphi + d_1\psi\left(1 + \frac{\sigma_1}{1+n}\right)\right\}}{(m_1+1)(1+n)} - \frac{sd_1^2\psi^2(\sigma_3-\sigma_2)^2}{(m_2+2)(1+n)^2} + \frac{sd_1\psi\left\{\varphi + d_1\psi\left(1 + \frac{\sigma_3}{1+n}\right)\right\}}{(m_2+1)(1+n)} \right\} \tag{26}$$

and

$$Q = \sum_{n=0}^{N} \left\{ \frac{kd_1\psi(\rho_1-\rho_2)}{(1+n)(n_1+1)} + \frac{kd_1\psi\rho_2}{1+n} + \frac{(1-k)\, d_1\psi\rho_3}{(1+n)(n_2+1)} + \frac{sd_1\psi\sigma_2}{1+n} + \frac{(1-s)\, d_1\psi\sigma_1}{(1+n)(m_1+1)} + \frac{sd_1\psi(\sigma_3-\sigma_2)}{(1+n)(m_2+1)} \right\}. \tag{27}$$

9.1. Particular cases

(i) If we take $\rho_2 = \rho_1, \rho_3 = \rho_1$ and $\sigma_1 = \sigma_3, \sigma_2 = \sigma_3$ and $k = s = 0$ then we get

$$I\left(\widetilde{D}_1\right) = \frac{\sum_{n=0}^{N}\left\{\frac{d_1^2\rho_1^2}{(n_1+2)(1+n)^2} + \frac{d_1^2\rho_1\left(1-\frac{\rho_1}{1+n}\right)}{(n_2+1)(1+n)} - \frac{d_1^2\sigma_3^2}{(m_1+2)(1+n)^2} + \frac{d_1^2\sigma_3\left(1+\frac{\sigma_3}{1+n}\right)}{(m_1+1)(1+n)}\right\}}{\sum_{n=0}^{N}\left\{\frac{d_1\rho_1}{(1+n)(n_2+1)} + \frac{d_1\sigma_3}{(1+n)(m_1+1)}\right\}} \quad \text{and}$$

$$I\left(\widetilde{Z}\right) = \frac{\sum_{n=0}^{N}\left\{\frac{d_1^2\psi^2\rho_1^2}{(n_1+2)(1+n)^2} + \frac{d_1\psi\rho_1\left(\varphi + d_1\psi\left(1-\frac{\rho_1}{1+n}\right)\right)}{(n_2+1)(1+n)} - \frac{d_1^2\psi^2\sigma_3^2}{(m_1+2)(1+n)^2} + \frac{d_1\psi\sigma_3\left\{\varphi + d_1\psi\left(1+\frac{\sigma_3}{1+n}\right)\right\}}{(m_1+1)(1+n)}\right\}}{\sum_{n=0}^{N}\left\{\frac{d_1\psi\rho_1}{(1+n)(n_2+1)} + \frac{d_1\psi\sigma_3}{(1+n)(m_1+1)}\right\}}.$$

Gives the triangular dense fuzzy model [30, 44].

(ii) If we take $\rho_2 = \rho_3 = \rho'$ where $\rho_2 < \rho' < \rho_3, \sigma_1 = \sigma_2 = \sigma'$ where $\sigma_1 < \sigma' < \sigma_2$ and $k = s = 1$ then we get

$$I\left(\widetilde{D}_1\right) = \frac{\sum_{n=0}^{N}\left\{\frac{d_1^2(\rho_1-\rho')^2}{(n_1+2)(1+n)^2} + \frac{d_1^2(\rho_1-\rho')\left(1-\frac{\rho_1}{1+n}\right)}{(n_1+1)(1+n)} + \frac{1}{2}\left\{d_1^2 - d_1^2\left(1 - \frac{\rho'}{1+n}\right)^2\right\} + \frac{1}{2}\left\{d_1^2\left(1 + \frac{\sigma'}{1+n}\right)^2 - d_1^2\right\} - \frac{d_1^2(\sigma_3-\sigma')^2}{(m_2+2)(1+n)^2} + \frac{d_1^2(\sigma_3-\sigma')\left(1+\frac{\sigma_3}{1+n}\right)}{(m_2+1)(1+n)}\right\}}{\sum_{n=0}^{N}\left\{\frac{d_1(\rho_1-\rho')}{(1+n)(n_1+1)} + \frac{d_1\rho'}{1+n} + \frac{d_1\sigma'}{1+n} + \frac{d_1(\sigma_3-\sigma')}{(1+n)(m_2+1)}\right\}}$$

and $I\left(\widetilde{Z}\right) = \frac{P}{Q}$ where

$$P = \sum_{n=0}^{N} \left\{ \frac{d_1^2\psi^2(\rho_1-\rho')^2}{(n_1+2)(1+n)^2} + \frac{d_1\psi(\rho_1-\rho')\left(\varphi + d_1\psi\left(1-\frac{\rho_1}{1+n}\right)\right)}{(n_1+1)(1+n)} + \frac{1}{2}\left[(\varphi + d_1\psi)^2 - \left\{\varphi + d_1\psi\left(1 - \frac{\rho'}{1+n}\right)\right\}^2\right] \right.$$
$$\left. + \frac{1}{2}\left[\left\{\varphi + d_1\psi\left(1 + \frac{\sigma'}{1+n}\right)\right\}^2 - \{\varphi + d_1\psi\}^2\right] - \frac{d_1^2\psi^2(\sigma_3-\sigma')^2}{(m_2+2)(1+n)^2} + \frac{d_1\psi\left\{\varphi + d_1\psi\left(1+\frac{\sigma_3}{1+n}\right)\right\}}{(m_2+1)(1+n)} \right\}$$

and

$$Q = \sum_{n=0}^{N}\left\{\frac{d_1\psi(\rho_1-\rho')}{(1+n)(n_1+1)} + \frac{d_1\psi\rho'}{1+n} + \frac{d_1\psi\sigma'}{1+n} + \frac{d_1\psi(\sigma_3-\sigma')}{(1+n)(m_2+1)}\right\}$$

gives the trapezoidal dense fuzzy model.

(iii) If we take $\rho_3=\rho_2$ and $\sigma_1=\sigma_2$ then we get $I\left(\widetilde{D}_1\right)=\frac{P}{Q}$ where

$$P=\sum_{n=0}^{N}\left\{\frac{kd_1^2\left(\rho_1-\rho_2\right)^2}{\left(n_1+2\right)\left(1+n\right)^2}+\frac{kd_1^2\left(\rho_1-\rho_2\right)\left(1-\frac{\rho_1}{1+n}\right)}{\left(n_1+1\right)\left(1+n\right)}+\frac{k}{2}\left\{d_1^2-d_1^2\left(1-\frac{\rho_2}{1+n}\right)^2\right\}+\frac{\left(1-k\right)d_1^2\rho_2^2}{\left(n_1+2\right)\left(1+n\right)^2}\right.$$
$$+\frac{\left(1-k\right)d_1^2\rho_2\left(1-\frac{\rho_2}{1+n}\right)}{\left(n_2+1\right)\left(1+n\right)}+\frac{s}{2}\left\{d_1^2\left(1+\frac{\sigma_2}{1+n}\right)^2-d_1^2\right\}-\frac{\left(1-s\right)d_1^2\sigma_2^2}{\left(m_1+2\right)\left(1+n\right)^2}+\frac{\left(1-s\right)d_1^2\sigma_2\left(1+\frac{\sigma_2}{1+n}\right)}{\left(m_1+1\right)\left(1+n\right)}$$
$$\left.-\frac{sd_1^2\left(\sigma_3-\sigma_2\right)^2}{\left(m_2+2\right)\left(1+n\right)^2}+\frac{sd_1^2\left(\sigma_3-\sigma_2\right)\left(1+\frac{\sigma_3}{1+n}\right)}{\left(m_2+1\right)\left(1+n\right)}\right\}$$

and

$$Q=\sum_{n=0}^{N}\left\{\frac{kd_1\left(\rho_1-\rho_2\right)}{\left(1+n\right)\left(n_1+1\right)}+\frac{kd_1\rho_2}{1+n}+\frac{\left(1-k\right)d_1\rho_2}{\left(1+n\right)\left(n_2+1\right)}+\frac{sd_1\sigma_2}{1+n}+\frac{\left(1-s\right)d_1\sigma_2}{\left(1+n\right)\left(m_1+1\right)}+\frac{sd_1\left(\sigma_3-\sigma_2\right)}{\left(1+n\right)\left(m_2+1\right)}\right\}$$

also $I\left(\widetilde{Z}\right)=\frac{P_1}{Q_1}$ where

$$P_1=\sum_{n=0}^{N}\left\{\frac{kd_1^2\psi^2\left(\rho_1-\rho_2\right)^2}{\left(n_1+2\right)\left(1+n\right)^2}+\frac{kd_1\psi\left(\rho_1-\rho_2\right)\left(\varphi+d_1\psi\left(1-\frac{\rho_1}{1+n}\right)\right)}{\left(n_1+1\right)\left(1+n\right)}+\frac{k}{2}\left[\left(\varphi+d_1\psi\right)^2-\left\{\varphi+d_1\psi\left(1-\frac{\rho_2}{1+n}\right)\right\}^2\right]\right.$$
$$+\frac{\left(1-k\right)d_1^2\psi^2\rho_2^2}{\left(n_1+2\right)\left(1+n\right)^2}+\frac{\left(1-k\right)d_1\psi\rho_2\left(\varphi+d_1\psi\left(1-\frac{\rho_2}{1+n}\right)\right)}{\left(n_2+1\right)\left(1+n\right)}+\frac{s}{2}\left[\left\{\varphi+d_1\psi\left(1+\frac{\sigma_2}{1+n}\right)\right\}^2-\left\{\varphi+d_1\psi\right\}^2\right]$$
$$\left.-\frac{\left(1-s\right)d_1^2\psi^2\sigma_2^2}{\left(m_1+2\right)\left(1+n\right)^2}+\frac{\left(1-s\right)d_1\psi\sigma_2\left\{\varphi+d_1\psi\left(1+\frac{\sigma_2}{1+n}\right)\right\}}{\left(m_1+1\right)\left(1+n\right)}-\frac{sd_1^2\psi^2\left(\sigma_3-\sigma_2\right)^2}{\left(m_2+2\right)\left(1+n\right)^2}+\frac{sd_1\psi\left\{\varphi+d_1\psi\left(1+\frac{\sigma_3}{1+n}\right)\right\}}{\left(m_2+1\right)\left(1+n\right)}\right\}$$

and

$$Q_1=\sum_{n=0}^{N}\left\{\frac{kd_1\psi\left(\rho_1-\rho_2\right)}{\left(1+n\right)\left(n_1+1\right)}+\frac{kd_1\psi\rho_2}{1+n}+\frac{\left(1-k\right)d_1\psi\rho_2}{\left(1+n\right)\left(n_2+1\right)}+\frac{sd_1\psi\sigma_2}{1+n}+\frac{\left(1-s\right)d_1\psi\sigma_2}{\left(1+n\right)\left(m_1+1\right)}+\frac{sd_1\psi\left(\sigma_3-\sigma_2\right)}{\left(1+n\right)\left(m_2+1\right)}\right\}$$

gives the pentagonal dense fuzzy model.

(iv) If we take $\sigma_1=\sigma_2$ then we get $I\left(\widetilde{D}_1\right)=\frac{P}{Q}$ where

$$P=\sum_{n=0}^{N}\left\{\frac{kd_1^2\left(\rho_1-\rho_2\right)^2}{\left(n_1+2\right)\left(1+n\right)^2}+\frac{kd_1^2\left(\rho_1-\rho_2\right)\left(1-\frac{\rho_1}{1+n}\right)}{\left(n_1+1\right)\left(1+n\right)}+\frac{k}{2}\left\{d_1^2-d_1^2\left(1-\frac{\rho_2}{1+n}\right)^2\right\}+\frac{\left(1-k\right)d_1^2\rho_3^2}{\left(n_1+2\right)\left(1+n\right)^2}\right.$$
$$+\frac{\left(1-k\right)d_1^2\rho_3\left(1-\frac{\rho_3}{1+n}\right)}{\left(n_2+1\right)\left(1+n\right)}+\frac{s}{2}\left\{d_1^2\left(1+\frac{\sigma_2}{1+n}\right)^2-d_1^2\right\}$$
$$\left.-\frac{\left(1-s\right)d_1^2\sigma_2^2}{\left(m_1+2\right)\left(1+n\right)^2}+\frac{\left(1-s\right)d_1^2\sigma_2\left(1+\frac{\sigma_2}{1+n}\right)}{\left(m_1+1\right)\left(1+n\right)}-\frac{sd_1^2\left(\sigma_3-\sigma_2\right)^2}{\left(m_2+2\right)\left(1+n\right)^2}+\frac{sd_1^2\left(\sigma_3-\sigma_2\right)\left(1+\frac{\sigma_3}{1+n}\right)}{\left(m_2+1\right)\left(1+n\right)}\right\}$$

and

$$Q=\sum_{n=0}^{N}\left\{\frac{kd_1\left(\rho_1-\rho_2\right)}{\left(1+n\right)\left(n_1+1\right)}+\frac{kd_1\rho_2}{1+n}+\frac{\left(1-k\right)d_1\rho_3}{\left(1+n\right)\left(n_2+1\right)}+\frac{sd_1\sigma_2}{1+n}+\frac{\left(1-s\right)d_1\sigma_2}{\left(1+n\right)\left(m_1+1\right)}+\frac{sd_1\left(\sigma_3-\sigma_2\right)}{\left(1+n\right)\left(m_2+1\right)}\right\}$$

also $I\left(\widetilde{Z}\right)=\frac{P_1}{Q_1}$, where

TABLE 2. Optimal solution of EOQ model.

Model	Time (t_1^*)	Time (t_2^*)	Q_1^*	Q_2^*	Minimum cost (Z^*)
Crisp	3.04	3.04	304.67	95.24	430.21
Triangular Dense fuzzy	3.0639	3.0639	303.13	94.31	427.72
Trapezoidal Dense fuzzy	3.0139	3.0139	290.17	91.55	501.96
Pentagonal Dense fuzzy	3.1034	3.1034	288.81	88.88	472.76
Hexagonal Dense fuzzy	3.0413	3.0413	283.12	88.64	413.39
Heptagonal Dense fuzzy	3.0427	3.0427	282.62	88.45	412.89

$$P_1 = \sum_{n=0}^{N} \left\{ \frac{kd_1^2\psi^2(\rho_1-\rho_2)^2}{(n_1+2)(1+n)^2} + \frac{kd_1\psi(\rho_1-\rho_2)\left(\varphi+d_1\psi\left(1-\frac{\rho_1}{1+n}\right)\right)}{(n_1+1)(1+n)} + \frac{k}{2}\left[(\varphi+d_1\psi)^2 - \left\{\varphi+d_1\psi\left(1-\frac{\rho_3}{1+n}\right)\right\}^2\right] \right.$$
$$+ \frac{(1-k)d_1^2\psi^2\rho_3^2}{(n_1+2)(1+n)^2} + \frac{(1-k)d_1\psi\rho_3\left(\varphi+d_1\psi\left(1-\frac{\rho_3}{1+n}\right)\right)}{(n_2+1)(1+n)} + \frac{s}{2}\left[\left\{\varphi+d_1\psi\left(1+\frac{\sigma_2}{1+n}\right)\right\}^2 - \{\varphi+d_1\psi\}^2\right]$$
$$\left. - \frac{(1-s)d_1^2\psi^2\sigma_2^2}{(m_1+2)(1+n)^2} + \frac{(1-s)d_1\psi\sigma_2\left\{\varphi+d_1\psi\left(1+\frac{\sigma_2}{1+n}\right)\right\}}{(m_1+1)(1+n)} - \frac{sd_1^2\psi^2(\sigma_3-\sigma_2)^2}{(m_2+2)(1+n)^2} + \frac{sd_1\psi\left\{\varphi+d_1\psi\left(1+\frac{\sigma_3}{1+n}\right)\right\}}{(m_2+1)(1+n)} \right\}$$

and

$$Q_1 = \sum_{n=0}^{N} \left\{ \frac{kd_1\psi(\rho_1-\rho_2)}{(1+n)(n_1+1)} + \frac{kd_1\psi\rho_2}{1+n} + \frac{(1-k)d_1\psi\rho_3}{(1+n)(n_2+1)} + \frac{sd_1\psi\sigma_2}{1+n} + \frac{(1-s)d_1\psi\sigma_2}{(1+n)(m_1+1)} + \frac{sd_1\psi(\sigma_3-\sigma_2)}{(1+n)(m_2+1)} \right\}$$

gives the hexagonal dense fuzzy model.

9.2. Numerical example

Let us consider $C_1 = 2.5, C_2 = 1.8, C_3 = 1200, d_1 = 100, k = 0.6, s = 0.3, \rho_1 = 0.5, \rho_2 = 0.45, \rho_3 = 0.33, \sigma_1 = 0.25, \sigma_2 = 0.3, \sigma_3 = 0.4$ then we get the following results.

From the Table 2, we see that the minimum objective value came from the heptagonal dense fuzzy model having average inventory cost \$412.89 for 6.08 days cycle time with 282.62 units of order quantity with respect to 88.45 units of backorder quantity. The objective values for the other cases assume higher values.

9.3. Sensitivity analysis

Based on the numerical example (Case of nonlinear heptagonal dense fuzzy sets) considered above for the simple EOQ model, we now calculate the corresponding outputs for changing inputs parameter one by one. The sensitivity analysis is performed by changing of each parameter $\rho_1, \rho_2, \rho_3, \sigma_1, \sigma_2, \sigma_3, C_1, C_2$ and C_3 by +30%, +20%, −20% and −30% considering one at a time and keeping the remaining parameters unchanged.

Table 3 shows that the shortage cost per unit item c_2 and the fuzzy system parameters $\rho_1, \sigma_1, \sigma_2, \sigma_3$ are slightly sensitive towards model minimum with refence to all changes from −30% to +30%. For these parameters the objectives values (average inventory cost) getting ranges from \$400.54 to \$425.24 with respect to the range of the order quantity 280.59–284.84 units only. From the Table 4 we see that the system parameters ρ_2, ρ_3 and holding cost c_1 per order quantity per unit time and set up cost c_3. per unit time period per cycle are highly sensitive.

TABLE 3. Optimal solution of Heptagonal dense fuzzy model.

Learning experience (N)	Time (t_1^*)	Time (t_2^*)	Q_1^*	Q_2^*	Minimum cost (Z^*)
0	3.0406	3.0406	270.31	84.65	413.20
1	3.0416	3.0416	276.03	86.41	413.06
2	3.0421	3.0421	279.15	87.38	412.98
3	3.0425	3.0425	281.18	88.00	412.93
4	3.0427	3.0427	282.62	88.45	412.89

TABLE 4. Sensitivity analysis with parametric changes from (−30% to +30%).

Parameter	% Change	t_1^* days	t_2^* days	Q_1^*	Q_2^*	Minimum cost Z^*($)	$\frac{Z^*-Z_*}{Z_*}100\%$
ρ_1	+30	3.064	3.064	281.62	87.62	410.65	−4.54
	+20	3.0568	3.0568	281.97	87.9	411.41	−4.36
	−20	...	...	...	...	...	...
	−	...	...	...	...	...	...
ρ_2	+30	...	...	...	...	...	...
	+20	...	...	...	...	...	...
	−20	3.0518	3.0518	290.17	90.59	469.1	9.04
	−30	3.0547	3.0547	293.75	91.63	502.39	16.78
ρ_3	+30	3.0945	3.0945	287.37	88.66	463.08	7.64
	+20	3.0771	3.0771	285.79	88.59	446.8	3.86
	−20	3.0088	3.0088	279.45	88.29	377.15	−12.33
	−30	2.992	2.992	277.84	88.2	358.57	−16.65
σ_1	+30	3.0404	3.0404	283.37	88.74	413.64	−3.85
	+20	3.0413	3.0413	283.12	88.64	413.39	−3.9
	−20	3.044	3.044	282.14	88.27	412.43	−4.13
	−30	3.0446	3.0446	281.9	88.18	412.2	−4.18
σ_2	+30	3.0205	3.0205	284.84	89.7	422.03	−1.9
	+20	3.028	3.028	284.08	89.27	419.03	−2.59
	−20	3.0572	3.0572	281.25	87.67	406.59	−5.49
	−30	3.0643	3.0643	280.59	87.29	403	−0.03
σ_3	+30	3.0298	3.0298	282.86	88.85	407.93	0.18
	+20	3.0337	3.0337	282.74	88.71	409.62	−4.78
	−20	3.0534	3.0534	282.68	88.2	416.01	−3.3
	−30	...	...	...	...	...	...
c_1	+30	2.6632	2.6632	247.37	86.41	464.06	87
	+20	2.7735	2.7735	257.61	87.08	447.75	4.08
	−20	3.4095	3.4095	316.68	89.81	374.27	−13
	−30	...	...	...	...	...	...
c_2	+30	3.0267	3.0267	281.13	88.38	425.24	−1.15
	+20	3.032	3.032	281.63	88.4	421.13	−2.11
	+20	3.0533	3.0533	283.61	88.5	404.65	−5.94
	−30	3.0586	3.0586	284.09	88.52	400	−6.89
c_3	+30	3.4902	3.4902	324.18	90.05	465.71	8.25
	+20	3.3479	3.3479	310.96	89.62	448.89	4.34
	−20	2.7035	2.7035	251.11	86.66	372.86	−13.33
	−30	2.517	2.517	233.79	85.39	350	−18.45

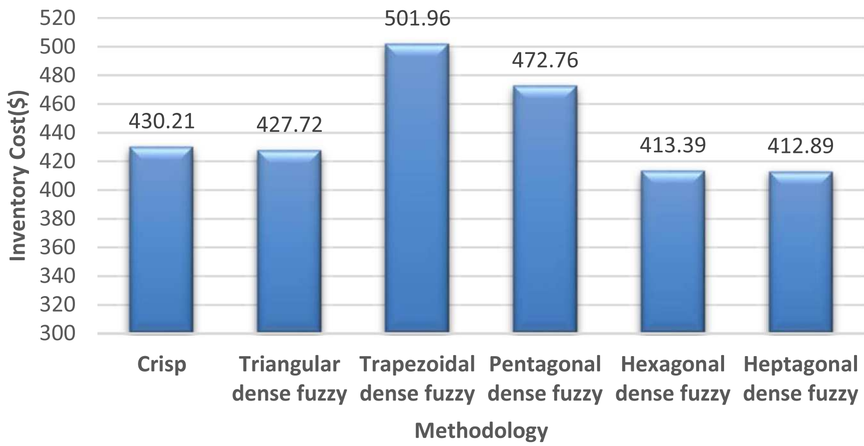

FIGURE 11. Inventory cost *vs.* Methodology.

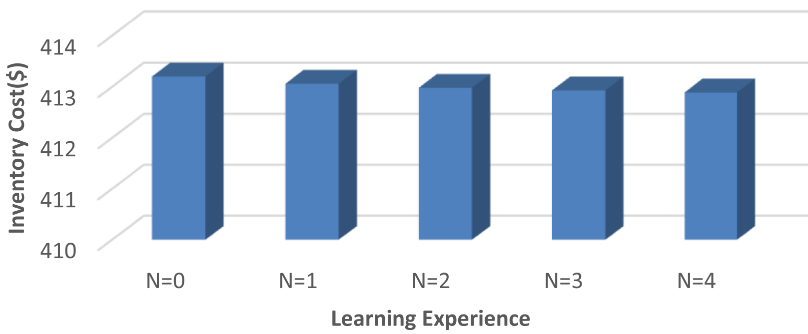

FIGURE 12. Inventory cost for heptagonal dense fuzzy model.

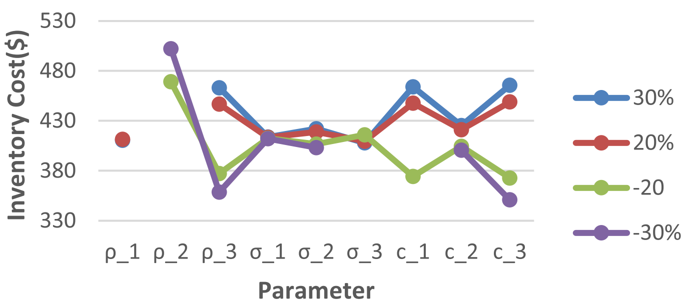

FIGURE 13. Graphical illustration of sensitivity analysis.

10. GRAPHICAL ILLUSTRATION

We have studied graphically over the numerical outputs (Tab. 1) of the model. Figure 11 shows the average cost values of the model gets minimum in nonlinear heptagonal dense fuzzy environment. Figure 12 shows that whenever we are going to enhance learning experience (for details see Appendix 11) the average inventory cost gradually decreases. Figure 13 shows that the variations of the cost functions are negligible for the fuzzy system parameters $\sigma_1, \sigma_2, \sigma_3$. and shortage cost c_2. also. The unit holding cost, the set up cost and the parameter $\rho_1,\ \rho_2,\ \rho_3$ are most fluctuating parameters keeping objectives values around \$350.83–\$469.1 alone.

11. Conclusion

Here we have discussed a simple backorder EOQ model under nonlinear heptagonal dense fuzzy environment. In fact, we have studied the model with the effect of learning experiences over the demand rate by means of heptagonal dense fuzzy number. However, for comparative study we assume the demand rate may follows triangular dense fuzzy number, trapezoidal dense fuzzy number, pentagonal dense fuzzy number and hexagonal dense fuzzy number also. Throughout the whole study, because of numerical illustration and system cost minimization of the problem, we observe that the nonlinear heptagonal dense fuzzy environment is much suitable for any decision maker of an inventory management problem with respect to other existing fuzzy environments. For better justification of the model a sensitivity analysis and graphical illustrations have been analyzed broadly.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this article.

Appendix

(De and Mahata [32], De and Beg [33,34]): Suppose x is a crisp number then under non-randomly uncertainties it will be "around x".

Our focus of attentions is that such uncertainty basically depends upon the frequency of interaction or time variable. This kind of fuzzy variation is generally called the variation due to learning effect in decision making problems. As for example, if we consider the demand rate of an inventory process then the ambiguity is very high at the beginning because the decision maker has no information over how many people will accept their items. As the time (it may be cycle time) or frequency of interaction progresses the DM will begin to gather more information over the nature of expected demand through the run of inventory and learn whether it is below expected or over expected (usually mean below crisp or above crisp value). Such kind of phenomena is viewed as "Inventory system is under learning experience".

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