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TRANSIENT AND STEADY-STATE ANALYSIS OF A QUEUING SYSTEM HAVING CUSTOMERS' IMPATIENCE WITH THRESHOLD

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Abstract. In many practical queuing situations reneging and balking can only occur if the number of customers in the system is greater than a certain threshold value. Therefore, in this paper we study a single server Markovian queuing model having customers' impatience (balking and reneging) with threshold, and retention of reneging customers. The transient analysis of the model is performed by using probability generating function technique. The expressions for the mean and variance of the number of customers in the system are obtained and a numerical example is also provided. Further the steady-state solution of the model is obtained. Finally, some important queuing models are derived as the special cases of this model.

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1. Introduction

Queuing systems possessing customers' impatience with threshold find their applications in computer networks with time-out mechanisms, call centers with impatient phone call, and hospital emergency rooms handling critical patients. A customer is said to be impatient if he tends to join the line only when a short wait is required and has a tendency to stay in the line if the wait has been adequately small. At the point when this impatience becomes adequately strong and customers leave before being served, the administrator of the company must take action to reduce the congestion level for better results. The models we consider here find applications in helping the administration to give satisfactory benefit for its customers with tolerable waiting.

The pioneer work on queuing models with customers' impatience is studied in [4, 5, 7, 8, 19, 20]. The transient behavior of the $M/M/1/N$ queue for a general N has been discussed in [21]. The expressions obtained there are so complex that these cannot be used to obtain parameters of the queue length such as the mean in the explicit form. In [17], authors obtain the closed form solution of the problem discussed in [21] by using a very simple and elegant algebraic method. The time-dependent solution of an $M/M/1$ queuing model with balking is obtained in [9] using probability generating function technique. In [6], the authors analyze $M/M/1$ priority queues with two classes of customers and constant impatience time. Using truncation method in [18], the steady-state

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probabilities of $M/M/s$ queuing system with customers' impatience and retrials is obtained. In [10], author derives the transient solution of a correlated input queuing system with catastrophic and restorative effects facing customers' impatience. The busy period of a single server Markovian queuing system is obtained in [1] by employing probability generating function technique. In [2], the transient solution of a two-heterogenous servers queue with impatient customers is obtained. An $M/M/1$ queue with working vacations, Bernoulli schedule vacation interruption, balking and reneging is considered in [23] where the authors obtain its steady-state probabilities. An $M/M/1$ queue with customers' impatience and multiple vacations is studied in [3].

From a business point of view, it is remarked that customer impatience could lead to loss of potential customers, which affects the business of the company. Thus, in order to retain impatient customers, more and more companies adopt various strategies. In a pioneering paper [11] on queuing systems with customer retention, an $M/M/1/N$ queueing model with retention of reneging customers is studied. Later in [12], authors extend their own work by adding balking to it. The steady-state solution of a single-server Markovian feedback queueing model with retention of reneging customers is obtained in [22]. In [15], authors incorporate retention to a discrete-time queuing system and derive the steady-state probabilities of $Geo/Geo/1/N$ queuing system. The transient solution of a two-heterogeneous servers Markovian queuing model with retention of reneging customers is obtained in [13]. In [14], a multi-server Markovian queuing system with balking, reneging and retention of reneging customers is considered in which its transient solution is obtained.

The applicability of this model can be seen in communication network systems. If there are numerous packets lined up in the system, local packets are always accepted. If the remote packets are less than a threshold value packets waiting in the node for process, then a new arrival either decides not to join the system or departs after joining the system. Further, if the number of packets in the waiting line exceeds the threshold value packets, these may time-out and get dropped before being transmitted. Thus, these systems can be represented as queuing models with impatience behavior. Therefore, these queuing models have wide applications to determine the impact of the threshold level in communication networks.

Above mentioned application motivates us to analyze the behavior of a queuing system having customers' impatience and retention of reneging customers with threshold. Since there is no work on transient solution of this model. Therefore, in this paper, we study a single server Markovian queuing system having customers' impatience and retention of reneging customers with threshold and obtain its transient as well as steady-state solution.

Rest of the paper is arranged as follows: in Section 2 queuing model is described. The transient analysis of the model is provided in Section 3. In Section 4 steady-state solution of the model is derived. Section 5 deals with the special cases of the model. Finally the paper is concluded in Section 6.

2. Queuing model description

We consider a single server queuing model with balking and retention of reneging customers in which the customers arrive according to a Poisson process with mean rate λ. On arrival a customer either decides to join the queue with probability one if the number of customers in the system is less than a threshold value k. If there are k customers or more ahead of him, then he joins the queue with probability β and may balk with probability $1-\beta$. The service time distribution is negative exponential with parameter μ. The queue discipline is first-come-first-served (FCFS). The capacity of the system is infinite. After joining the queue each customer will wait for a certain length of time T for his service to begin. If it has not begun by then he may get renege with probability p and may remain in the queue for his service with probability $q(=1-p)$ if certain customer retention strategy is used. This time T is a random variable which follows negative exponential distribution with parameter ξ. It is assumed that the reneging can only occur if the number of customers in the system are greater than a certain threshold value k. Therefore, the average reneging rate is given by the following function:

$$\xi_n = \begin{cases} 0, & 0 < n \leq k \\ (n-k)\xi, & n \geq k+1 \end{cases}.$$

Let $\{X(t), t \geq 0\}$ be the number of customers present in the system at time t. Let $P_n(t) = P\{X(t) = n\}, n = 0, 1, \ldots$ be the probability that there are n customers in the system at time t, and $P(z,t)$ the corresponding probability generating function. We assume that there is no customer in the system at $t = 0$.

The queuing model under investigation is governed by the following differential-difference equations:

$$\frac{\mathrm{d}P_0(t)}{\mathrm{d}t} = -\lambda P_0(t) + \mu P_1(t), n = 0 \tag{2.1}$$

$$\frac{\mathrm{d}P_n(t)}{\mathrm{d}t} = -\left(\lambda + \mu\right) P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t), n = 1, 2, \ldots, k-1 \tag{2.2}$$

$$\frac{\mathrm{d}P_k(t)}{\mathrm{d}t} = -\left(\beta\lambda + \mu\right) P_k(t) + \lambda P_{k-1}(t) + (\mu + \xi p) P_{k+1}(t), n = k \tag{2.3}$$

$$\frac{\mathrm{d}P_n(t)}{\mathrm{d}t} = -\left(\beta\lambda + \mu + (n-k)\xi p\right) P_n(t) + \beta\lambda P_{n-1}(t) + (\mu + (n-k+1)\xi p) P_{n+1}(t), n = k+1, \ldots. \tag{2.4}$$

3. Transient solution of the model

In this section, we present the transient solution of the model. The probability generating function technique is used to obtain the time-dependent system size probabilities.

Theorem 3.1. *The time-dependent probabilities of the system size of a Markovian queuing model with single server, balking and retention of reneging customers which is governed by the differential-difference equations (2.1)–(2.4) are given by:*

$$P_i(t) = b_{i,0}(t) + \mu \int_0^t b_{i,k-2}(u) P_{k-1}(t-u)\mathrm{d}u, i = 0, 1, \ldots, k-2$$

$$P_{k-1}(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(-1)^m}{\Psi} \left(\frac{\mu}{\Psi}\right)^m \left(\frac{2\Psi}{\alpha}\right)^{n+1} (n+1) \binom{n}{m} \left[\int_0^t M(t-u) \int_0^u N^{C(m)}(u-v) \exp\{-(\beta\lambda + \mu - \xi p)v\} \frac{I_{n+1}(\alpha v)}{v} \mathrm{d}u\mathrm{d}v\right.$$

and, for $n = 1, 2, \ldots$

$$P_{n+k-1}(t) = n\gamma^n \int_0^t \exp\{-(\beta\lambda + \mu - \xi p)(t-u)\} \frac{I_n(\alpha(t-u))}{(t-u)} P_{k-1}(u)\mathrm{d}u$$

where $b_{i,j}(t)$ denotes the Laplace inverse of $b^*_{i,j}(s)$ (defined in the proof), $N^{C(m)}(t)$ is m-fold convolution of $N(t)$ with itself with $N^{C(0)} = \delta(t)$, the Dirac delta function, $\Psi = \mu - \xi p$, $\alpha = 2\sqrt{\beta\lambda(\mu - \xi p)}$.

Proof. Define the probability generating function $P(z,t)$ for the transient state probabilities $P_n(t)$ by

$$P(z,t) = \sum_{n=0}^{k-1} P_n(t) + \sum_{n=0}^{\infty} P_{n+k}(t) z^{n+1};\ P(z,0) = 1 \tag{3.1}$$

with

$$\sum_{n=0}^{k-1} P_n(t) = R_{k-1}(t). \tag{3.2}$$

Adding the equations (2.1) and (2.2), we get

$$\frac{\mathrm{d}(R_{k-1}(t))}{\mathrm{d}t} = -\lambda P_{k-1}(t) + \mu P_k(t). \tag{3.3}$$

Now, multiplying the equation (2.3) and (2.4) by z^n, summing over the respective range of n, we obtain

$$\begin{aligned}\frac{\mathrm{d}\left[\sum_{n=0}^{\infty} P_{n+k}(t) z^{n+1}\right]}{\mathrm{d}t} &= [(\mu-\xi p)(z^{-1}-1)+\beta\lambda(z-1)]\sum_{n=0}^{\infty} P_{n+k}(t) z^{n+1} \\ &\quad + \lambda z P_{k-1}(t) - \mu P_k(t) + \xi p(1-z)\frac{\partial P(z,t)}{\partial z}\cdot \end{aligned} \tag{3.4}$$

Adding the equations (3.3) and (3.4), we obtain the following partial differential equation

$$\begin{aligned}\frac{\partial P(z,t)}{\partial t} - \xi p(1-z)\frac{\partial P(z,t)}{\partial z} &= [(\mu-\xi p)(z^{-1}-1)+\beta\lambda(z-1)][P(z,t) \\ &\quad - R_{k-1}(t)] + \lambda(z-1)P_{k-1}(t). \end{aligned} \tag{3.5}$$

Solving the equation (3.5) we get

$$\begin{aligned} P(z,t) &= \exp\left\{[(\mu-\xi p)(z^{-1}-1)+\beta\lambda(z-1)]t\right\} + \int_0^t [\lambda(z-1)P_{k-1}(u) \\ &\quad - ((\mu-\xi p)(z^{-1}-1)+\beta\lambda(z-1))R_{k-1}(u)] \times \exp\{[(\mu-\xi p) \\ &\quad \times (z^{-1}-1)+\beta\lambda(z-1)](t-u)\}\mathrm{d}u. \end{aligned} \tag{3.6}$$

If $\alpha = 2\sqrt{\beta\lambda(\mu-\xi p)}$ and $\gamma = \sqrt{\frac{\beta\lambda}{\mu-\xi p}}$, then using the modified Bessel function of first kind $I_n(.)$ and the Bessel function properties, we get

$$\exp\left\{\left(\beta\lambda z + \frac{\mu-\xi p}{z}\right)t\right\} = \sum_{n=-\infty}^{\infty} (\gamma z)^n I_n(\alpha t). \tag{3.7}$$

Using (3.7) in (3.6), we get

$$\begin{aligned} P(z,t) &= \exp\{[-(\beta\lambda+\mu-\xi p)]t\}\sum_{n=-\infty}^{\infty}(\gamma z)^n I_n(\alpha t) \\ &\quad + \lambda\int_0^t P_{k-1}(u)\exp\{[-(\beta\lambda+\mu-\xi p)](t-u)\} \\ &\quad \times \sum_{n=-\infty}^{\infty}(\gamma z)^n[\gamma^{-1}I_{n-1}(\alpha(t-u)) - I_n(\alpha(t-u))]\mathrm{d}u \\ &\quad + \int_0^t R_{k-1}(u)\exp\{[-(\beta\lambda+\mu-\xi p)](t-u)\} \\ &\quad \times \sum_{n=-\infty}^{\infty}(\gamma z)^n[-\beta\lambda\gamma^{-1}I_{n-1}(\alpha(t-u)) \\ &\quad + (\beta\lambda+\mu-\xi p)I_n(\alpha(t-u)) - (\mu-\xi p)\gamma I_{n+1}(\alpha(t-u))]\mathrm{d}u. \end{aligned} \tag{3.8}$$

Now, comparing the coefficients of z^n on either side of (3.8), we obtain for $n = 1, 2, \ldots$

$$\begin{aligned}P_{n+k-1}(t) &= \exp\{-(\beta\lambda+\mu-\xi p)t\}\gamma^n I_n(\alpha t) + \lambda\int_0^t \exp\{-(\beta\lambda+\mu-\xi p)(t-u)\}\\ &\quad\times\left[I_{n-1}(\alpha(t-u))\gamma^{n-1} - I_n(\alpha(t-u))\gamma^n\right]P_{k-1}(u)\mathrm{d}u\\ &\quad-\int_0^t \exp\{-(\beta\lambda+\mu-\xi p)(t-u)\}R_{k-1}(u)[\lambda I_{n-1}(\alpha(t-u))\gamma^{n-1}\\ &\quad-(\beta\lambda+\mu-\xi p)I_n(\alpha(t-u))\gamma^n + (\mu-\xi p)I_{n+1}(\alpha(t-u)\gamma^{n+1}]\mathrm{d}u.\end{aligned} \tag{3.9}$$

For $n = 0$, we get

$$\begin{aligned}R_{k-1}(t) &= \exp\{-(\beta\lambda+\mu-\xi p)t\}I_0(\alpha t) + \lambda\int_0^t \exp\{-(\beta\lambda+\mu-\xi p)(t-u)\}\\ &\quad\times P_{k-1}(u)\left[I_1(\alpha(t-u))\gamma^{-1} - I_0(\alpha(t-u))\right]\mathrm{d}u\\ &\quad-\int_0^t \exp\{-(\beta\lambda+\mu-\xi p)(t-u)\}R_{k-1}(u)\times[\alpha I_1(\alpha(t-u))\\ &\quad-(\beta\lambda+\mu-\xi p)I_0(\alpha(t-u))]\mathrm{d}u.\end{aligned} \tag{3.10}$$

As $P(z,t)$ does not contain terms with negative powers of z the right hand side of (3.9) with n replaced by $-n$ must be zero. Thus, we obtain

$$\begin{aligned}&\int_0^t \exp\{-(\beta\lambda+\mu-\xi p)(t-u)\}R_{k-1}(u)\,[\beta\lambda I_{n+1}(\alpha(t-u))\gamma^{n-1} - (\beta\lambda+\mu-\xi p)I_n(\alpha(t-u))\gamma^n\\ &\qquad\qquad + (\mu-\xi p)I_{n-1}(\alpha(t-u))\gamma^{n+1}]\mathrm{d}u\\ &= \exp\{-(\beta\lambda+\mu-\xi p)t\}I_n(\alpha t)\gamma^n + \lambda\int_0^t \exp\{-(\beta\lambda+\mu-\xi p)(t-u)\}P_{k-1}(u)\\ &\quad\times[I_{n+1}(\alpha(t-u))\gamma^{n-1} - I_n(\alpha(t-u))\gamma^n]\mathrm{d}u.\end{aligned} \tag{3.11}$$

The usage of (3.11) in (3.9) considerably simplifies the working and results in an elegant expression for $P_n(t)$. This yields, for $n = 1, 2, \ldots$

$$P_{n+k-1}(t) = n\gamma^n\int_0^t \exp\{-(\beta\lambda+\mu-\xi p)(t-u)\}\frac{I_n(\alpha(t-u))}{(t-u)}P_{k-1}(u)\mathrm{d}u. \tag{3.12}$$

The remaining probabilities $P_n(t), n = 0, 1, \ldots, k-1$ can be obtained by solving the equations (2.1) and (2.2). In matrix form, the equations (2.1) and (2.2) can be written as:

$$\frac{\mathrm{d}\mathbf{P(t)}}{\mathrm{d}t} = A\mathbf{P(t)} + \mu P_{k-1}(t)\mathbf{e_{k-1}} \tag{3.13}$$

where the matrix $A = (b_{i,j})_{k-1\times k-1}$ is given as:

$$A = \begin{bmatrix} -(\lambda) & \mu & \cdots & 0\\ \lambda & -(\lambda+\mu) & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & -(\lambda+\mu)\end{bmatrix}$$

$\mathbf{P(t)} = (P_0(t) \;\; P_1(t) \;\; \dots \;\; P_{k-2}(t))^T$, $\mathbf{e_{k-1}} = (0 \;\; 0 \;\; \dots \;\; 1)^T$ is column vector of order $k-1$.

Let $\mathbf{P^*(s)} = (P_0^*(s) \;\; P_1^*(s) \;\; \dots \;\; P_{k-2}^*(s))^T$ denotes the Laplace transform of $\mathbf{P(t)}$. Taking the Laplace transform of equation (3.13) and solving for $\mathbf{P^*(s)}$, we get

$$\mathbf{P^*(s)} = (sI - A)^{-1}\{\mu P_{k-1}^*(s)\mathbf{e_{k-1}} + \mathbf{P(0)}\} \tag{3.14}$$

with $\mathbf{P(0)} = (1 \;\; 0 \;\; \dots \;\; 0)^T$. Thus, only $P_{k-1}^*(s)$ remains to be found. We observe that if $\mathbf{e} = (1 \;\; 1 \;\; \dots \;\; 1)^T_{k-1\times 1}$, then

$$\mathbf{e}^T\mathbf{P^*(s)} + P_{k-1}^*(s) = R_{k-1}^*(s). \tag{3.15}$$

Define

$$f(s) = \left[(s+\beta\lambda+(\mu-\xi p)) - \sqrt{(s+\beta\lambda+(\mu-\xi p))^2-\alpha^2}\right].$$

Taking Laplace transform of (3.10) and solving for $q_{k-1}^*(s)$, we obtain

$$sR_{k-1}^*(s) = 1 + P_{k-1}^*(s)\left[\frac{1}{2}\{f(s)\} - \lambda\right]. \tag{3.16}$$

Using equation (3.16) in (3.15) and simplifying, we get

$$P_{k-1}^* = \frac{1 - s\mathbf{e^T}(sI-A)^{-1}\mathbf{P(0)}}{\{(s+\beta\lambda) - \frac{1}{2}[f(s)] + \mu s\mathbf{e^T}(sI-A)^{-1}\mathbf{e_{k-1}}\}}. \tag{3.17}$$

In equations (3.14) and (3.17), $(sI-A)^{-1}$ has to be found. Let us assume that

$$(sI-A)^{-1} = (b_{ij}^*(s))_{k-1\times k-1}.$$

We note that $(sI-A)^{-1}$ is almost lower triangular. Following [16], we obtain, for $i = 0, 1, \dots, k-2$

$$b_{ij}^*(s) = \begin{cases} \frac{1}{\mu}\frac{u_{k,j+1}(s)u_{i,0}(s) - u_{i,j+1}(s)u_{k,0}(s)}{u_{k,0}(s)}, & j = 0, 1, \dots, k-3 \\ \frac{u_{i,0}(s)}{u_{k,0}(s)}, & j = k-2 \end{cases}. \tag{3.18}$$

where $u_{i,j}(s)$ are recursively given as

$$\begin{aligned} &u_{i,i}(s) = 1, && i = 0, 1, \dots, k-2 \\ &u_{i+1,i}(s) = \tfrac{s+\lambda+\mu}{\mu}, && i = 0, 1, \dots, k-3 \\ &u_{i+1,i-j}(s) = \tfrac{(s+\lambda+\mu)u_{i,i-j} - \lambda u_{i-1,i-j}}{\mu}, && j \le i, i = 1, 2, 3, \dots, k-3 \end{aligned}$$

$$u_{k-1,j}(s) = \begin{cases} [s+\lambda+\mu]u_{k-1,j} - \lambda u_{k-2,j}, & j = 0, 1, \dots, k-3 \\ s+\lambda+\mu, & j = k-2 \end{cases} \tag{3.19}$$

and
$u_{i,j}(s) = 0$, for other i and j. Using these in equation (3.17), we get

$$P_{k-1}^*(s) = \frac{\{1 - s\sum_{i=0}^{k-2} b_{i,0}^*(s)\}}{\{(s+\beta\lambda) - \frac{1}{2}[f(s)] + \mu s\sum_{i=0}^{k-2} b_{i,k-2}^*(s)\}} \tag{3.20}$$

and for $i = 0, 1, \dots, k-2$ from equation (3.14), we get

$$P_i^*(s) = b_{i,0}^*(s) + \mu b_{i,k-2}^*(s)P_{k-1}^*(s). \tag{3.21}$$

We observe that $b^*_{i,j}(s)$ are all rational algebraic functions in s. The cofactor of the (i, j)th element of $(sI - A)$ is a polynomial of degree $k-2-\mid i-j\mid$. Since the characteristic roots of A are all distinct, the inverse transform $b_{i,j}(t)$ of $b^*_{i,j}(s)$ can be obtained by partial fraction decomposition. Let $s_i, i = 0, 1, \dots, k-2$, be the characteristic roots of the matrix A. Then after partial fraction decomposition and simplification, $P^*_{k-1}(s)$ equals to

$$\frac{M^*(s)}{\frac{1}{2}[f(s)]\left[1-\frac{2(\mu-\xi p)(1-\frac{\mu}{\mu-\xi p}N^*(s))}{(s+\beta\lambda+\mu-\xi p)-\sqrt{(s+\beta\lambda+\mu-\xi p)^2-\alpha^2}}\right]} \tag{3.22}$$

where

$$M^*(s) = \sum_{m=0}^{k-2}\frac{M_m}{s-s_m} \tag{3.23}$$

$$N^*(s) = \sum_{m=0}^{k-2}\frac{N_m}{s-s_m} \tag{3.24}$$

with constants M_m and N_m given by

$$M_m = \lim_{s\to s_m}(s-s_m)\left[1-\sum_{l=0}^{k-2}sb^*_{l,0}(s)\right] \tag{3.25}$$

$$N_m = \lim_{s\to s_m}(s-s_m)\left[\sum_{l=0}^{k-2}sb^*_{l,k-2}(s)\right]. \tag{3.26}$$

Hence, (3.22) simplifies into

$$P^*_{k-1}(s) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m}{\Psi}\left(\frac{2\Psi}{\alpha}\right)^{n+1}(n+1)\binom{n}{m}\left(\frac{\mu}{\Psi}\right)^m\left[M^*(s)(N^*(s))^m \frac{[(s+\beta\lambda+\mu-\xi p)-\sqrt{(s+\beta\lambda+\mu-\xi p)^2-\alpha^2}]^{n+1}}{(n+1)\alpha^{n+1}}\right] \tag{3.27}$$

where $\Psi = (\mu - \xi p)$.

Taking Laplace inverse of (3.27), we obtain

$$P_{k-1}(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m}{\Psi}\left(\frac{2\Psi}{\alpha}\right)^{n+1}\left(\frac{\mu}{\Psi}\right)^m(n+1)\binom{n}{m}\left[\int_0^t M(t-u)\int_0^u N^{C(m)}(u-v)\exp\{-(\beta\lambda+\mu-\xi p)v\}\frac{I_{n+1}(\alpha v)}{v}\mathrm{d}u\mathrm{d}v\right]. \tag{3.28}$$

where $N^{C(m)}(t)$ is $m-fold$ convolution of $N(t)$ with itself with $N^{C(0)} = \delta(t)$. Now, the Laplace inverse of equation (3.21) yields,

$$P_i(t) = b_{i,0}(t) + \mu\int_0^t b_{i,k-2}(u)P_{k-1}(t-u)\mathrm{d}u, i = 0, 1, \dots, k-2 \tag{3.29}$$

where $b_{i,j}(t)$ denotes the Laplace inverse of $b^*_{i,j}(s)$ and $b^*_{i,j}(s)$ are the elements of matrix $(sI - A)^{-1}$, $P_{k-1}(u)$ is given by (3.28). Therefore all the state probabilities are obtained explicitly in (3.12), (3.28) and (3.29).

□

3.1. Mean and variance

This section deals with the derivation of mean and variance of the system.

Mean (Expected System Size), $M(t)$

The mean number of customers in the system at time t is given by

$$\begin{aligned} M(t) = E(X(t)) &= m(t) + r(t) \\ &= \sum_{n=1}^{k-1} nP_n(t) + \sum_{n=k}^{\infty} nP_n(t) \end{aligned} \tag{3.30}$$

and

$$M'(t) = \sum_{n=1}^{k-1} nP'_n(t) + \sum_{n=k}^{\infty} nP'_n(t). \tag{3.31}$$

From equations (2.2)–(2.4) after considerable mathematical simplification, the above equation will lead to the following differential equation

$$\begin{aligned} M'(t) = {} & -\mu M(t) + \lambda \sum_{n=0}^{k-1} P_n(t) + \beta\lambda \sum_{n=k}^{\infty} P_n(t) + \lambda k P_{k-1}(t) + \mu \sum_{n=1}^{\infty} nP_{n+1}(t) \\ & + \xi p \left[\sum_{n=k+1}^{\infty} n(n-k)(P_{n+1}(t) - P_n(t)) + \sum_{n=k}^{\infty} nP_{n+1}(t) \right] \end{aligned} \tag{3.32}$$

$$\begin{aligned} M(t) = {} & \lambda \sum_{n=0}^{k-1} \int_0^t P_n(u) \exp(\mu(t-u)) \mathrm{d}u + \beta\lambda \sum_{n=k}^{\infty} \int_0^t P_n(u) \exp(\mu(t-u)) \mathrm{d}u \\ & + \lambda k \int_0^t P_{k-1}(u) \exp(\mu(t-u)) \mathrm{d}u + \mu \sum_{n=1}^{\infty} \int_0^t nP_{n+1}(u) \exp(\mu(t-u)) \mathrm{d}u \\ & + \xi p \sum_{n=k+1}^{\infty} n(n-k) \int_0^t (P_{n+1}(u) - P_n(u)) \exp(\mu(t-u)) \mathrm{d}u \\ & + \xi p \sum_{n=k}^{\infty} n \int_0^t P_{n+1}(u) \exp(\mu(t-u)) \mathrm{d}u \end{aligned} \tag{3.33}$$

where $P_n(t)$ for n=0,1,...,k−2;$P_{k-1}(t)$ and $P_{n+k-1}(t)$ for n=1,2,... are given in equations (3.12), (3.28) and (3.29), respectively.

Variance, V($X(t)$)

The variance of number of customers in the system at time t is given by:

$$\begin{aligned} V(X(t)) &= E(X^2(t)) - [E(X(t))]^2 \\ &= K(t) - [M(t)]^2 \\ &= \sum_{n=1}^{\infty} n^2 P_n(t) - \left[\sum_{n=1}^{\infty} nP_n(t) \right]^2. \end{aligned} \tag{3.34}$$

From equations (2.2)–(2.4) after considerable mathematical simplification the above equation will lead to the following differential equation

$$
\begin{aligned}
K'(t) = {} & -\mu K(t) + \lambda\left[\sum_{n=0}^{k-1} P_n(t) + 2m(t) + k^2 P_{k-1}(t)\right] \\
& + \beta\lambda\left[\sum_{n=k}^{\infty} P_n(t) + 2r(t)\right] + \mu\sum_{n=1}^{\infty} n^2 P_{n+1}(t) \\
& + \xi p\left[\sum_{n=k+1}^{\infty} n^2(n-k)(P_{n+1}(t) - P_n(t)) + \sum_{n=k}^{\infty} n^2 P_{n+1}(t)\right]. \qquad (3.35)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
K(t) = {} & \lambda \int_0^t \left[\sum_{n=0}^{k-1} P_n(u) + 2m(u) + k^2 P_{k-1}(u)\right] \\
& \times \exp(\mu(t-u))\mathrm{d}u + \beta\lambda \int_0^t \left[\sum_{n=k}^{\infty} P_n(u) + 2r(u)\right] \\
& \times \exp(\mu(t-u))\mathrm{d}u + \mu \sum_{n=1}^{\infty} n^2 \int_0^t P_{n+1}(u)\exp(\mu(t-u))\mathrm{d}u \\
& + \xi p \sum_{n=k+1}^{\infty} n^2(n-k) \int_0^t (P_{n+1}(u) - P_n(u))\exp(\mu(t-u))\mathrm{d}u \\
& + \xi p \sum_{n=k}^{\infty} n^2 \int_0^t P_{n+1}(u)\exp(\mu(t-u))\mathrm{d}u. \qquad (3.36)
\end{aligned}
$$

Substituting the above equation in (3.34), we get

$$
\begin{aligned}
V(X(t)) = {} & \lambda \int_0^t \left[\sum_{n=0}^{k-1} P_n(u) + 2m(u) + k^2 P_{k-1}(u)\right] \\
& \times \exp(\mu(t-u))\mathrm{d}u + \beta\lambda \int_0^t \left[\sum_{n=k}^{\infty} P_n(u) + 2r(u)\right] \\
& \times \exp(\mu(t-u))\mathrm{d}u + \mu \sum_{n=1}^{\infty} n^2 \int_0^t P_{n+1}(u)\exp(\mu(t-u))\mathrm{d}u \\
& + \xi p \sum_{n=k+1}^{\infty} n^2(n-k) \int_0^t (P_{n+1}(u) - P_n(u))\exp(\mu(t-u))\mathrm{d}u \\
& + \xi p \sum_{n=k}^{\infty} n^2 \int_0^t P_{n+1}(u)\exp(\mu(t-u))\mathrm{d}u - [M(t)]^2 \qquad (3.37)
\end{aligned}
$$

where $P_n(t)$ for $n = 0, 1, \ldots, k-2$; $P_{k-1}(t)$, $P_{n+k-1}(t)$ for $n = 1, 2, \ldots$ and $M(t)$ are given in equations (3.12), (3.28), (3.29) and (3.33) respectively.

3.2. Numerical Illustration

In this sub-section, we perform the transient numerical analysis of the system. We study the following measures of performances in transient state.

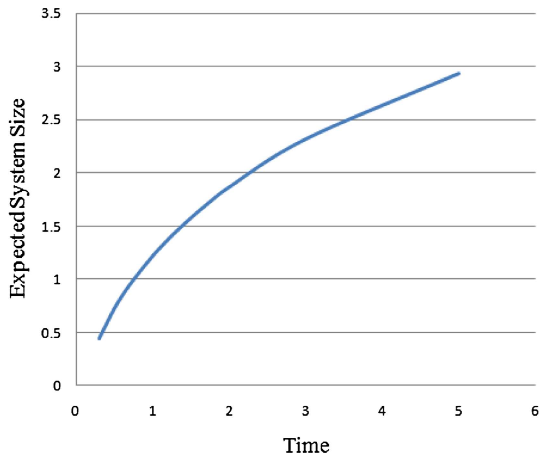

FIGURE 1. Variation of expected system size with respect to time t ($\lambda = 3, \mu = 4, p = 0.4, \xi = 0.1, \beta = 0.4, k = 2$).

(1) **Average Reneging Rate ($R_r(t)$)**

$$R_r = \sum_{n=k}^{N} (n-k)\xi p P_n(t).$$

(2) **Average Retention Rate ($R_R(t)$)**

$$R_R = \sum_{n=k}^{N} (n-k)\xi q P_n(t).$$

The time-dependent variations in various performance measures with respect to the initial system parameters are studied as shown in Figures 1–9. Figure 1 shows that as time varies the average number of customers in the system increases. But after sometime it seems to be constant. In Figures 2–4, the variations in performance measures with respect to the threshold value (k) are presented. As k increases the expected system size increases. But the average reneging rate and the average retention rate shows a decreasing trend with the increase in k. In Figures 5–7, the variations in different measures of performance with the variation in balking probability ($1-\beta$) are presented. We can see that as $1-\beta$ increases all the given performance measures (expected system size, average reneging rate and the average retention rate) decreases. Variations in performance measures with respect to the reneging rate (ξ) is presented in Figures 8 and 9. As ξ increases the average reneging rate as well as the average retention rate increases.

4. Steady-state solution

In steady-state $\lim_{t\to\infty} P_n(t) = P_n$, and $\lim_{t\to\infty} \frac{\mathrm{d}P_n(t)}{\mathrm{d}t} = 0$. Therefore, the equations (2.1)–(2.4) in steady-state becomes:

$$0 = -\lambda P_0 + \mu P_1 \tag{4.1}$$

$$0 = -(\lambda+\mu) P_n + \lambda P_{n-1} + \mu P_{n+1}, \quad 1 \le n \le k-1 \tag{4.2}$$

$$0 = -(\beta\lambda+\mu) P_k + \lambda P_{k-1} + (\mu+\xi p) P_{k+1}, \quad n = k \tag{4.3}$$

$$0 = -(\beta\lambda+\mu+(n-k)\xi p) P_n + \beta\lambda P_{n-1} + (\mu+(n-k+1)\xi p) P_{n+1}, \quad n \ge k+1. \tag{4.4}$$

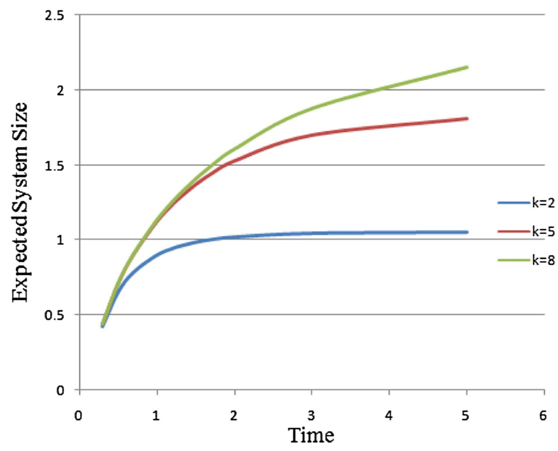

FIGURE 2. Effect of threshold value on expected system size with respect to time for the case $\lambda = 3, \mu = 4, p = 0.4, \xi = 0.1, \beta = 0.4$.

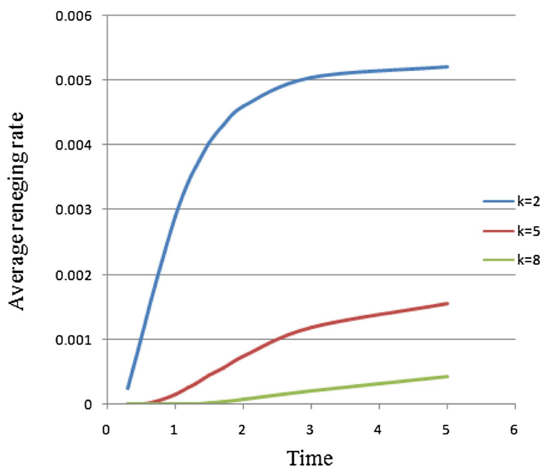

FIGURE 3. Effect of threshold value on average reneging rate with respect to time for the case $\lambda = 3, \mu = 4, p = 0.4, \xi = 0.1, \beta = 0.4$.

On solving equations (4.1)–(4.4) iteratively, we get

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0, \quad 1 \leq n \leq k$$

$$P_n = \frac{(\beta\lambda)^{n-k}}{\prod\limits_{m=k+1}^{n} (\mu + (m-k)\xi p)} \left(\frac{\lambda}{\mu}\right)^k P_0, \quad n \geq k+1.$$

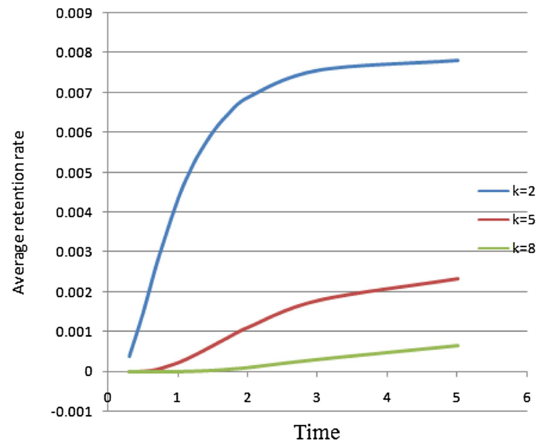

FIGURE 4. Effect of threshold value on average retention rate with respect to time for the case $\lambda = 3, \mu = 4, p = 0.4, \xi = 0.1, \beta = 0.4$.

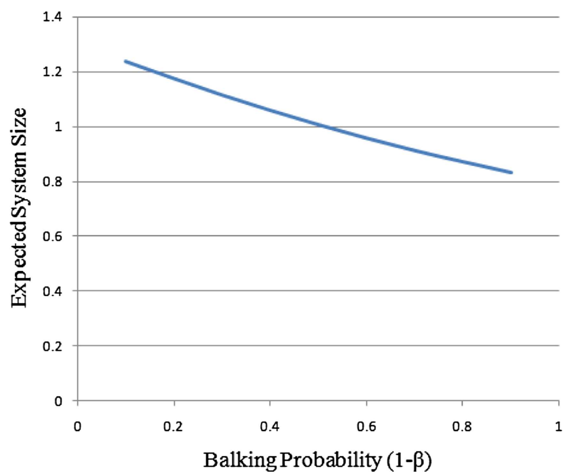

FIGURE 5. Variation of expected system size with the variation in balking probability for the case $\lambda = 3, \mu = 4, p = 0.4, \xi = 0.1, k = 2$.

Using normalization condition, we get

$$P_0 = \left[1 + \sum_{n=1}^{k} \left(\frac{\lambda}{\mu}\right)^n + \sum_{n=k+1}^{\infty} \frac{(\beta\lambda)^{n-k}}{\prod\limits_{m=k+1}^{n} (\mu + (m-k)\xi p)} \left(\frac{\lambda}{\mu}\right)^k \right]^{-1}.$$

5. SPECIAL CASES

Case 5.1. When $\beta = 0$, *i.e.* when there is no balking. The model reduces to a Markovian single server queuing model with reneging and retention.

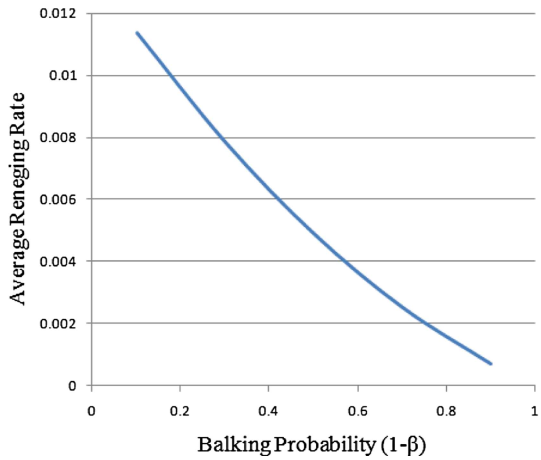

FIGURE 6. Variation of average reneging rate with the variation in balking probability for the case $\lambda = 3, \mu = 4, p = 0.4, \xi = 0.1, k = 2$.

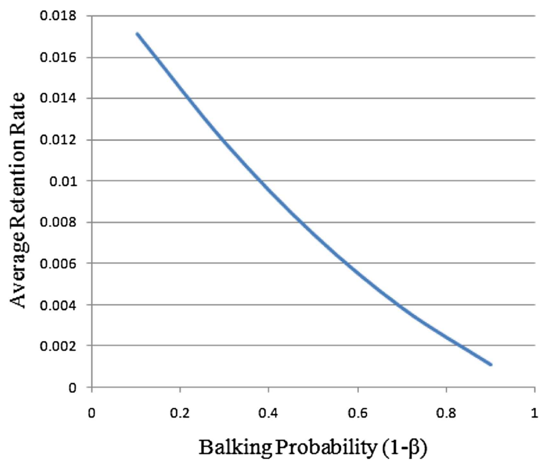

FIGURE 7. Variation of average retention rate with the variation in balking probability for the case $\lambda = 3, \mu = 4, p = 0.4, \xi = 0.1, k = 2$.

For $n = 1, 2, \dots$

$$P_{n+k-1}(t) = n\gamma_1^n \int_0^t \exp\{-(\lambda+\mu-\xi p)(t-u)\} \frac{I_n(\alpha_1(t-u))}{(t-u)} P_{k-1}(u)\mathrm{d}u$$

$$P_{k-1}(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(-1)^m}{\Psi} \left(\frac{2\Psi}{\alpha_1}\right)^{n+1} \left(\frac{\mu}{\Psi}\right)^m (n+1)\binom{n}{m} \left[\int_0^t M(t-u) \int_0^u N^{C(m)}(u-v) \exp\{-(\lambda+\mu-\xi p)v\} \frac{I_{n+1}(\alpha_1 v)}{v} \mathrm{d}u\mathrm{d}v\right]$$

$$P_i(t) = b_{i,0}(t) + \mu \int_0^t b_{i,k-2}(u) P_{k-1}(t-u)\mathrm{d}u, i = 0, 1, \dots, k-2$$

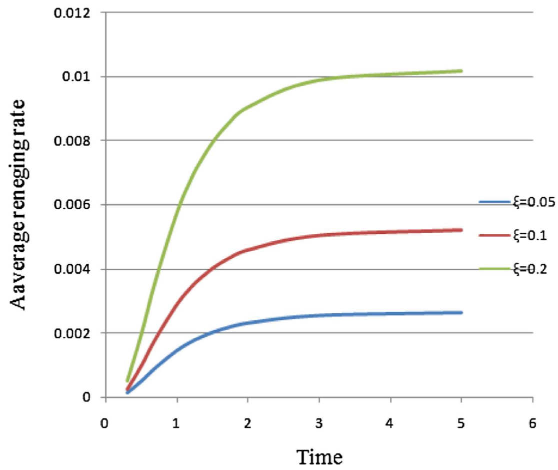

FIGURE 8. Effect of reneging rate on average reneging rate with respect to time for the case $\lambda = 3, \mu = 4, p = 0.4, \xi = 0.1, \beta = 0.4, k = 2$.

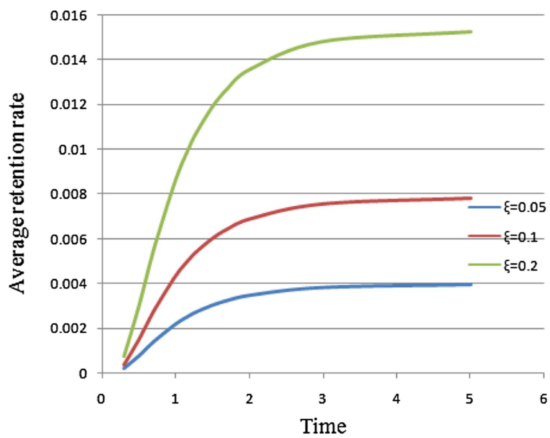

FIGURE 9. Effect of reneging rate on average retention rate with respect to time for the case $\lambda = 3, \mu = 4, p = 0.4, \xi = 0.1, \beta = 0.4, k = 2$.

where $\alpha_1 = 2\sqrt{\lambda(\mu - \xi p)}$,$\gamma_1 = \sqrt{\frac{\lambda}{\mu - \xi p}}$, and $\Psi = (\mu - \xi p)$.

Case 5.2. When $q = 0$, *i.e.* when the retention mechanism is absent. The model reduces to a Markovian single server queuing model with reneging and balking.

For $n = 1, 2, \dots$

$$P_{n+k-1}(t) = n\gamma_1^n \int_0^t \exp\{-((\beta\lambda + \mu - \xi))(t-u)\} \frac{I_n(\alpha_1(t-u))}{(t-u)} P_{k-1}(u) \mathrm{d}u$$

$$P_{k-1}(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(-1)^m}{\Psi} \left(\frac{2\Psi}{\alpha_1}\right)^{n+1} \left(\frac{\mu}{\Psi}\right)^m (n+1) \binom{n}{m} \left[\int_0^t M(t-u) \right.$$
$$\left. \int_0^u N^{C(m)}(u-v) \exp\{-(\beta\lambda + \mu - \xi)v\} \frac{I_{n+1}(\alpha_1 v)}{v} \mathrm{d}u\mathrm{d}v \right]$$

$$P_i(t) = b_{i,0}(t) + \mu \int_0^t b_{i,k-2}(u) P_{k-1}(t-u)\mathrm{d}u, i = 0, 1, \ldots, k-2$$

where $\alpha_1 = 2\sqrt{\beta\lambda(\mu-\xi)}$, $\gamma_1 = \sqrt{\frac{\beta\lambda}{\mu-\xi}}$, and $\Psi = (\mu - \xi)$.

Case 5.3. When there is no reneging, the queuing system reduces to a single server queuing model with balking with

$$P_{n+k-1}(t) = n\gamma_2^n \int_0^t \exp\{-(\beta\lambda+\mu)(t-u)\} \frac{I_n(\alpha_2(t-u))}{(t-u)} P_{k-1}(u)\mathrm{d}u$$

$$P_{k-1}(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(-1)^m}{\mu} \left(\frac{2\mu}{\alpha_2}\right)^{n+1} (n+1) \binom{n}{m} \left[\int_0^t M(t-u) \int_0^u N^{C(m)}(u-v) \exp\{-(\beta\lambda+\mu)v\} \frac{I_{n+1}(\alpha_2 v)}{v} \mathrm{d}u\mathrm{d}v\right]$$

$$P_i(t) = b_{i,0}(t) + \mu \int_0^t b_{i,k-2}(u) P_{k-1}(t-u)\mathrm{d}u, i = 0, 1, \ldots, k-2; n = 1, 2 \ldots.$$

where $\alpha_2 = 2\sqrt{\beta\lambda\mu}$ and $\gamma_2 = \sqrt{\frac{\beta\lambda}{\mu}}$.

6. Conclusions

In this paper the transient as well as steady-state behavior of a single server queuing model with balking, reneging and retention of reneging customers is studied. The mean and variance of the queuing model are obtained. The time-dependent variations in various performance measures with respect to the system parameters are studied. Some important special cases of the queuing model are also derived.

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