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INVESTMENT FOR PROCESS QUALITY IMPROVEMENT AND SETUP COST REDUCTION IN AN IMPERFECT PRODUCTION PROCESS WITH WARRANTY POLICY AND SHORTAGES

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Abstract. Cost reduction for setup and improvement of processes quality are the main target of this research along with free minimal repair warranty for an imperfect production System. This paper deals with the effect of setup cost reduction and process quality improvement on the optimal production cycle time for an imperfect production process with free product minimal repair warranty. Here the production system is subject to a random breakdown from an *controlled* system to an *out-of-control* state. Shortages are fully backlogged. The main target to minimize the total cost by simultaneously optimizing the production run time, setup cost, and process quality. A solution algorithm with some numerical experiments are provided such as the proposed model can illustrate briefly. Sensitivity analysis section is decorated for the optimal solution of the model with respect to major cost parameters of the system are carried out, and the implications of the analysis are discussed.

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1. Introduction

Most of the existing production models considered perfect machine throughout the production process. But if the machine operates for a long-time, it may transfer from an *in-control* state to the *out-of-control* state, where the manufacturing system produces defective/imperfect items. These items are usually reworked at a fixed cost to restore to the original quality. Production models with unreliable machines have been considered by some researchers. Porteus [1] discussed an imperfect production process with a significant relationship between quality

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and lot size. Groenevelt *et al.* [2] addressed an inventory model which study the effects of failure of machine, *i.e.*, machine breakdowns and corrective maintenance for the economic lot sizing decisions.

In another study, Groenevelt *et al.* [3] obtained an optimal production lot size and safety-stock level, when the repair time follows a general probability distribution and the time to failure is exponentially distributed. Hariga and Ben-Daya [4] extended Groenevelt *et al.*'s [2] model by assuming the random deterioration process to follow a general probability distribution. Kim and Hong [5] calculate the production run length, where process produced defective items. Lee *et al.* [6] developed an inventory model in order to improve the quality of products and to reduce the proportion of defective items. Hou [7] developed an economic production quantity (EPQ) model with an imperfect production process in which setup cost and process quality were functions of capital expenditure. Sarkar and Saren [8] developed a product inspection policy for an imperfect production system with inspection errors and warranty cost.

Goyal and Cárdenas-Barrón [9] developed an production-inventory model to determine both the optimal lot size and the cost of manufacturing process in an imperfect production system. Sana *et al.* [10] extended the production model for an imperfect production system where defective items are sold at a reduced-price. Cárdenas-Barrón [11] proposed a multi-stage inventory model on optimal batch sizing with a rework process. Sana [12] developed a production-inventory model. In this model he determine the optimal product reliability along with production rate, that maximized the total profit for any imperfect production system. An integrated production-inventory model was developed by Sana [13], where three-layer supply chain is associated with perfect and imperfect products.

Sarkar [14] developed an production model where demand depended on price and advertisement for an imperfect production process under the effect of inflation. Sarkar and Sarkar [15] explained a controlled economic manufacturing quantity model with probabilistic deterioration. Euler-Lagrange method was implied to solve the model. method. Sarkar *et al.* [16] developed an inventory model with selling-price and time-dependent demand for an imperfect production process. An integrated-inventory model was establish by Ouyang *et al.* [17] developed with a two-part trade-credit in an imperfect production process. Most recently, this research work is extended by Sarkar *et al.* [18] by considering random defective rate, rework process for clarify the imperfect items, and variable backorders.

Deb and Chaudhuri [19] were among the first to incorporate shortage into inventory model. Without using the calculations for optimality from differential calculus, Cárdenas-Barrón [20] explained an EPQ model with shortages by using basic algebraic procedure. Sana *et al.* [21] investigated a production model with deteriorating items and trended demand with shortages.

Later, Cárdenas-Barrón [22] presented an economic production quantity (EPQ) model with rework process for imperfect items in a single-stage manufacturing system where planned backorders takes place. An imperfect product production system was developed by Sana [23], in which he considered shortages due to regular preventive maintenance and sold the product with a free minimal repair warranty. This model was extended by Das Roy and Sana [24] by considering price sensitive stochastic demand. In the same direction a two-echelon supply chain model was developed by Modak *et al.* [25], where demand was depended on price, quality and warranty. Cárdenas-Barrón [26] derived inventory models with two backorder costs using analytical geometry and algebra. A stock-dependent inventory model was developed by Sarkar and Sarkar [27] where quality of the product is improved under the consideration of partial backlogging and deterioration. Recently, Sarkar *et al.* [28] described a two-echelon supply chain model for deteriorating items. Rather then, two continues investment are used to improved production quality and to reduced setup cost. In this direction Dey *et al.* [29] developed a production model where demand is price dependent and lead time demand follows a poisson distributed demand. An inventory model with price discount offer was developed by Sarkar *et al.* [30].

An imperfect production model was developed by Sarkar *et al.* [31], in which they considered that demand is time dependent. In the same direction Sarkar *et al.* [34] developed a stock dependent inventory model, where they considered that production process become imperfect. Reliability of any production process is one of the most important factor discussed by Sarkar *et al.* [32]. A time varying deterioration inventory model was introduced by Sana [33], in this model he also considered partial backorder.

TABLE 1. Author(s) contributions table.

Author(s)	Quality improvement	Warranty policy	Imperfect production	Shortages	Setup cost reduction
Ouyang *et al.* [17]	√	NA	NA	NA	NA
Cárdenas-Barrón [20]	NA	NA	NA	√	NA
Sana [23]	NA	√	√	NA	NA
Sarkar *et al.* [28]	√	NA	NA	NA	√
Dey *et al.* [29]	√	NA	√	NA	√
Porteus [35]	√	NA	NA	NA	√
Sarkar and Moon [38]	NA	NA	√	NA	√
This model	√	√	√	√	√

Notes. √ indicates the keyword contains in the research and NA indicates that the research idea does not exists.

All of the above mentioned models treated setup cost as a fixed constant. In most of the practical situations, the cost for setup can be reduced through different types of policy such as training of worker, change of procedural, and specialized equipment acquisition. In this direction, Porteus [35] introduced the concept of reducing the setup cost on the classical economic order quantity (EOQ) model. Ouyang *et al.* [36] extended a continuous review inventory model with lead time and ordering cost reduction. Pan and Lo [37] discussed a continuous review inventory model to show the curve-effect on setup cost reduction. Sarkar and Majumder [39] investigated an integrated vendor-buyer supply chain model where setup cost for vendor is reduced. Recently, Sarkar and Moon [38] developed an improved inventory model with setup cost reduction as a decision variable. In that study, they compared their model with two existing models to compare the effects of variable setup cost for vendor.

Major contributions of various author(s) are given in Table 1.

In this proposed model, an effort has been made to investigate the effect of setup cost reduction and process quality improvement for an imperfect production system with allowable shortages; products are sold with some free minimal repair warranty cost. The intention is to minimize the total relevant cost of the system. A simple and efficient algorithm is developed to find the optimality of the solutions. Numerical example is elaborated to illustrate the proposed model. The paper is designed as: in the next section, problem definition, basic assumptions and notation are provided. In Section 3, the proposed model is formulated. Numerical example is presented in Section 4. Some managerial insights are discussed in Section 6. Finally, concluding remarks are provided.

2. PROBLEM DEFINITION, ASSUMPTIONS AND NOTATION

This section contain problem definition, assumptions followed by notation are used such as it is much easier to understand the model.

2.1. Problem definition

In a long run production process due to machinery problem or labour problem or some other problem system moves from controlled to uncontrolled state. In this uncontrolled state process can produced defective items. An effort has been made to improve the process quality by a continuous investment and reduced the setup cost which is directly effect to reduced the total system cost with allowable shortages. As the system may produced imperfect products in *out-of-control* state, thus maintain the brand image of the company and for customer satisfaction products are sold with some free minimal repair warranty cost. The intention is to minimize the total relevant cost of the system. A simple and efficient algorithm is developed to find the optimality of the solutions. Numerical example is elaborated to illustrate the proposed model.

2.2. Notation

Decision variables	
t	production run time (time unit)
K	reduced setup cost per setup (\$/setup)
ξ	reduced percentage of defective items produced in an *out-of-control* state
Parameters	
D	annual demand rate (units/year)
P	annual production rate (units/year)
K_0	initial setup cost per setup (\$/setup)
ξ_0	original percentage of defective items produced in an out-of-control state
H_c	holding cost (\$/unit/year)
η_c	rework cost (\$/unit)
w_c	minimum repair cost for warranty (\$/unit)
B_c	backorder cost (\$/unit)
w	warranty period (time unit)
X	an elapsed time until production process shifts (year)
$f(x)$	probability density function of X
δ	annual fractional cost of capital investment (/\$/day)
$g_1(\gamma)$	failure rate function of perfect items
$g_2(\gamma)$	failure rate function of imperfect items, where $g_2(\gamma) > g_1(\gamma)$
T_1	production time when backorder is replenished (year)
T_2	production time when inventory builds up (year)
T_3	time period in which no production occurs and inventory depletes (year)
T_4	time period in which no production occurs and shortage occurs (year)

2.3. Assumptions

To developed this inventory model, one can assumed as follows,

(1) Production will start in an *in-control* state, and produced perfect items.
(2) When a production goes through a long run production process, then production process may transfer from an *in-control* state to an *out-of-control* state. If the production process transferred to an *out-of-control* state, it will remain same there till the end of a production run, and production will continue at a fixed proportion of the produced items is defective.
(3) All defective items which are produced, detected after the end of the production cycle, and a rework cost is incorporated for defective items.
(4) The production process is brought back to again in the *in-control* state with each setup.
(5) A logarithmic investment function is used to reduce the setup cost and to improve the process quality.
(6) The free minimal repair warranty (FRW) policy is considered to formulate the model.
(7) Shortages are allowed and are fully backlogged.
(8) The demand rate is constant and deterministic.

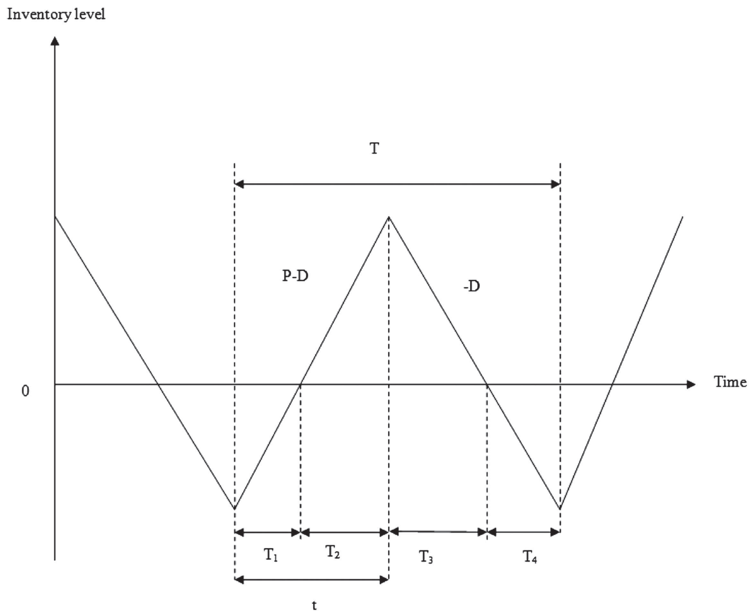

FIGURE 1. Logistic diagram of the proposed model.

3. Formulation of the model

A production-inventory model is considered with backorder in an imperfect production process. The model is divided into four major phases. Phase 1 is the backorder replenishment period, phase 2 is the inventory building period, the inventory depletion period is phase 3, and phase 4 represents the shortage period. The time duration of the ith phase is indicated by T_i.

Before formulating the cost functions, one can first clarify the inter relationship between T_1, T_2, T_3, T_4 and t. From Figure 1, the following results are obtained

$$T = T_1 + T_2 + T_3 + T_4$$
$$t = T_1 + T_2 = \frac{D}{P}T$$
$$T_2 = t - T_1$$
$$T_3 = \frac{(P-D)(t-T_1)}{D}$$
$$T_4 = \frac{P-D}{D}T_1.$$

Now, the total cost per unit time consists of the following values.

3.1. Setup cost

To start a production process first need to setup the process, to set the whole production process some cost is needed which is known as setup cost. The setup cost per unit time in this model calculate as follows:

$$\text{SC} = \frac{K}{T} = \frac{KD}{Pt}.$$

3.2. Holding cost

The total inventory of the model per cycle is given by the upper triangular area above the time axis in Figure 1. It is note that the duration of time with the positive level of inventory is

$$T_2 + T_3 = t - T_1 + \frac{(P-D)(t-T_1)}{D} = \frac{P}{D}(t-T_1)\cdot$$

Hence, the average inventory level is

$$\begin{aligned}
&= \frac{[(P-D)t - (P-D)T_1]\,[T - (T_1+T_4)]}{2T} \\
&= \frac{[(P-D)(t-T_1)]\,[T_2+T_3]}{2T} \\
&= \frac{D[(P-D)\frac{P}{D}(t-T_1)^2]}{2Pt} \\
&= \frac{(P-D)(t-T_1)^2}{2t}\cdot
\end{aligned}$$

After produced the products or to produced products some space is needed. To keep or holds the products some cost is needed, which are known as holding cost. Thus, the cost for holds the goods per unit time is given by

$$\mathrm{HC} = \frac{H_c(P-D)(t-T_1)^2}{2t}\cdot$$

3.3. Backording cost

Due to huge demand of a particular product, some time producer faces shortages conditions. That is demand of the product is high but manufacturer has not enough product. In this situation customers have two option, they wait for that product or they can take it from elsewhere. Thus manufacturer faces loss in both cases, they loss the brand image of company as well as direct profit.

Thus, the total shortage in the system per cycle is given by the lower triangular in Figure 1 below the time axis. Therefore, the average shortage level is

$$\begin{aligned}
&= \frac{T_1(P-D)T_1}{2T} + \frac{T_4 D T_4}{2T} \\
&= \frac{(P-D)T_1^2}{2t}\cdot
\end{aligned}$$

The backordering cost per unit time is given by

$$\mathrm{SC} = \frac{B_c(P-D)T_1^2}{2t}\cdot$$

3.4. Reworking cost

As in the long process production process moves to *out-of-control* state and can produced defective products. To kept the brand image of the company they spend some money to make those imperfect items, perfect.

The number of defective items N in a production cycle is given by

$$N = \begin{cases} 0 & \text{if } X \geq t, \\ \xi P(t-X) & \text{if } X < t. \end{cases}$$

Therefore, the expected defective items in each cycle of production is

$$E(N) = \int_0^t \xi P(t-x)f(x)\mathrm{d}x = \frac{\xi P}{\lambda}(\mathrm{e}^{-\lambda t} + \lambda t - 1).$$

and the rework cost per unit time is

$$= \frac{\eta_c E(N)}{T} = \frac{\eta_c D}{Pt}E(N) = \frac{D\xi\eta_c}{\lambda t}(\mathrm{e}^{-\lambda t} + \lambda t - 1).$$

The fraction of reworked items of the total produced is

$$R(t) = \frac{E(N)}{Pt} = \frac{\xi}{\lambda t}(\mathrm{e}^{-\lambda t} + \lambda t - 1).$$

3.5. Free minimal warranty cost

To kept the brand image of company and satisfy their customers companies gave some time period in which if there is any problem in the perticular product, they rework that free of cost, which time period is called warranty period. Thus for this warranty, companies spend some cost.

Suppose that $g_1(\gamma)$ and $g_2(\gamma)$ are the hazard rate for a conforming and nonconforming item, respectively. Further, $g_2(\gamma) > g_1(\gamma)$, which implies that a nonconforming item is more positively to fail than a conforming item. The expected number of minimal repairs for a conforming and nonconforming item within the warranty period w is $\int_0^w g_1(\gamma)\mathrm{d}\tau$ and $\int_0^w g_2(\gamma)\mathrm{d}\tau$, respectively. Under the free-repair warranty policy, where all failures occurring during warranty period w are rectified through minimal repair. The hazard rate for that item which is repaired remains constant as that just before failure.

The probability of a product failing within the warranty period $[0, w]$ is

$$W = [1 - R(t)]\int_0^w g_1(\gamma)\mathrm{d}\tau + R(t)\int_0^w g_2(\gamma)\mathrm{d}\tau = [1 - R(t)]A_1 + R(t)A_2,$$

where $A_1 = \int_0^w g_1(\gamma)d\tau$ and $A_2 = \int_0^w g_2(\gamma)\mathrm{d}\tau$.

Therefore, the expected minimal free repair cost per unit time for warranty is

$$\mathrm{WC} = \frac{w_c PtW}{T} = \frac{w_c DPt\,\{[1 - R(t)]A_1 + R(t)A_2\}}{Pt} = w_c D\,\{A_1 + R(t)[A_2 - A_1]\} = w_c D\left[A_1 + \frac{\xi}{\lambda t}(\mathrm{e}^{-\lambda t} + \lambda t - 1)[A_2 - A_1]\right].$$

The expected total cost $\mathrm{ETC}(t, T_1)$ is the sum of setup cost, holding cost, backordering cost, warranty cost, and rework cost, *i.e.*,

$$\mathrm{ETC}(t, T_1) = \frac{KD}{Pt} + \frac{1}{2t}\left[(H_c + B_c)(P - D)T_1^2\right] + H_c(P - D)\left(\frac{t}{2} - T_1\right) + w_c D\left[A_1 + \frac{\xi}{\lambda t}(\mathrm{e}^{-\lambda t} + \lambda t - 1)[A_2 - A_1]\right] + \frac{D\xi\eta_c}{\lambda t}(\mathrm{e}^{-\lambda t} + \lambda t - 1). \quad (3.1)$$

Taking first- and second-order derivatives of equations (3.1), one can obtain

$$\frac{\partial \mathrm{ETC}(t, T_1)}{\partial T_1} = \frac{1}{t}(H_c + B_c)(P - D)T_1 - H_c(P - D) \tag{3.2}$$

$$\frac{\partial^2 \mathrm{ETC}(t, T_1)}{\partial T_1^2} = \frac{1}{t}(H_c + B_c)(P - D) > 0. \tag{3.3}$$

From equation (3.3), it is noticed that $\mathrm{ETC}(t, T_1)$ is convex with respect to T_1. Thus, equating equation (3.2) to zero, one can find the optimal production time, when backorder is replenished as

$$T_1 = \left(\frac{H_c}{H_c + B_c}\right) t. \tag{3.4}$$

Using equations (3.4) and (3.1) becomes

$$\begin{aligned}\mathrm{ETC}(t) = {} & \frac{KD}{Pt} + \frac{H_c(P - D)t}{2}\left(\frac{B_c}{H_c + B_c}\right) + \frac{D\xi\eta_c}{\lambda t}(\mathrm{e}^{-\lambda t} + \lambda t - 1) \\ & + w_c D\left[A_1 + \frac{\xi}{\lambda t}(\mathrm{e}^{-\lambda t} + \lambda t - 1)[A_2 - A_1]\right].\end{aligned} \tag{3.5}$$

3.6. Investment in setup cost reduction and process quality improvement

The setup cost and quality are always not fixed, some time considered this as a decision variables then the control of quality level and setup cost is consummated by varying of capital investment allocated for improvement of the quality level and reduction of the setup cost. That is, by introducing capital investment, it is possible to reduce the setup cost and to increase the quality level. In the existing literature, several investment functions have been used to investigate the effects of setup cost reduction and process quality improvement in which the logarithmic investment function is widely used since it is unchanged with the Japanese experience as introduced in Hall [43]. The relation between setup cost and the investment function, which is considered by Porteus [35] is

$$\xi_K(K) = a \ln\left(\frac{K_0}{K}\right) \qquad 0 < K \le K_0,$$

where $a = \frac{1}{\Delta}$, and $\Delta =$ percentage decrease in K per dollar increase in $\xi_K(K)$.
Similarly, the relationship between process quality ξ and capital investment in process quality improvement $\xi_\xi(\xi)$ is considered by

$$\xi_\xi(\xi) = b \ln\left(\frac{\xi_0}{\xi}\right) \qquad 0 < \xi \le \xi_0,$$

where $b = \frac{1}{\Delta}$ and $\Delta =$ percentage decrease in ξ per dollar increase in $\xi_\xi(\xi)$.
Therefore, including these two investment cost functions, the expected total cost function in equation (3.5) becomes

$$\begin{aligned}\mathrm{ETC}(t, K, \xi) &= \mathrm{ETC}(t) + \delta\left[a \ln\left(\frac{K_0}{K}\right) + b \ln\left(\frac{\xi_0}{\xi}\right)\right] \\ &= \frac{KD}{Pt} + \frac{H_c(P - D)t}{2}\left(\frac{B_c}{H_c + B_c}\right) + \frac{D\xi\eta_c}{\lambda t}\left(\mathrm{e}^{-\lambda t} + \lambda t - 1\right) \\ &\quad + w_c D\left[A_1 + \frac{\xi}{\lambda t}\left(\mathrm{e}^{-\lambda t} + \lambda t - 1\right)[A_2 - A_1]\right] \\ &\quad + \delta\left[a \ln\left(\frac{K_0}{K}\right) + b \ln\left(\frac{\xi_0}{\xi}\right)\right].\end{aligned} \tag{3.6}$$

If $w = 0$ and $B_c \to \infty$, *i.e.*, the products are sold without warranty and shortages are not allowed. In this case, (3.6) will reduce to Hou [7], when an elapsed time until the production system enters into an *out-of-control* state is exponentially distributed.

If $w = 0$, $K^* = K_0$, and $\xi^* = \xi_0$, then (3.6) can be reduce to (3.1) Chung and Hou [40], when the elapsed time until the production system enters into an *out-of-control* state is exponentially distributed. Further, if $B_c \to \infty$, then (3.6) will reduce to (3.2) in Kim and Hong [5].

Therefore, this model is a generalized model of Kim and Hong [5], Chung and Hou [40], and Hou [7].

In order to optimise the cost minimization problem in equation (3.6), the first and second order derivatives of $\mathrm{ETC}(t, K, \xi)$ with respect to K, ξ, *i.e.*,

$$\frac{\partial \mathrm{ETC}(t,K,\xi)}{\partial K} = \frac{D}{Pt} - \frac{\delta a}{K} \tag{3.7}$$

$$\frac{\partial^2 \mathrm{ETC}(t,K,\xi)}{\partial K^2} = \frac{\delta a}{K^2} > 0 \tag{3.8}$$

$$\frac{\partial \mathrm{ETC}(t,K,\xi)}{\partial \xi} = \frac{D}{\lambda t}\left(\mathrm{e}^{-\lambda t} + \lambda t - 1\right)\left[w_c(A_2 - A_1) + \eta_c\right] - \frac{\delta b}{\xi} \tag{3.9}$$

$$\frac{\partial^2 \mathrm{ETC}(t,K,\xi)}{\partial \xi^2} = \frac{\delta b}{\xi^2} > 0. \tag{3.10}$$

From equations (3.8) and (3.10), it is easy to see that $\mathrm{ETC}(t, K, \xi)$ is convex in K and ξ. Therefore, the optimal K and ξ can be obtained by equating (3.7) and (3.9) to zero, *i.e.*,

$$K = \frac{a\delta Pt}{D} \tag{3.11}$$

$$\xi = \frac{\delta b}{D\left[w_c(A_2 - A_1) + \eta_c\right]\left\{1 + \frac{1}{\lambda t}(\mathrm{e}^{-\lambda t} - 1)\right\}}. \tag{3.12}$$

Using (3.11) and (3.12), equation (3.6) becomes

$$\mathrm{ETC}(t) = \delta\left[a \ln\left(\frac{K_0 D}{a\delta Pt}\right) + b\ln\left(\frac{\xi_0 D\left[w_c(A_2 - A_1) + \eta_c\right]\left\{1 + \frac{1}{\lambda t}(\mathrm{e}^{-\lambda t} - 1)\right\}}{\delta b}\right)\right] + \delta a + \frac{H_c(P-D)t}{2}\left(\frac{B_c}{H_c + B_c}\right) + w_c D A_1 + \delta b. \tag{3.13}$$

Differentiating equation (3.13), with respect to t results in

$$\frac{\mathrm{dETC}(t)}{\mathrm{d}t} = \frac{H_c(P-D)}{2}\left(\frac{B_c}{H_c+B_c}\right) - \frac{\delta a}{t} + \frac{\delta b}{t(\mathrm{e}^{-\lambda t} + \lambda t - 1)}\left(1 - \mathrm{e}^{-\lambda t} - \lambda t\mathrm{e}^{-\lambda t}\right). \tag{3.14}$$

Let $f(t) = t\frac{\mathrm{dETC}(t)}{\mathrm{d}t}$.

Then, t is the optimal solution of $\mathrm{ETC}(t)$ if and only if $f(t^*) = 0$.

Equation (3.15) yields

$$f(t) = \frac{H_c(P-D)}{2}\left(\frac{B_c}{H_c+B_c}\right)t - \delta a + \frac{\delta b}{(\mathrm{e}^{-\lambda t} + \lambda t - 1)}\left(1 - \mathrm{e}^{-\lambda t} - \lambda t\mathrm{e}^{-\lambda t}\right). \tag{3.15}$$

According to Porteus [35], the cost in practice incurred by setups is higher than that the cost incurred by producing a reworked item when the optimal capital investments are made. Therefore, the assumption $\delta a > \delta b$ is practical. It is difficult to find the closed form solution for t^*. In Theorem 3.1, the lower and upper bounds of t^* are developed, by which the bounds of t^* can be acquired. These limitations can facilitate the finding process of t^*. The following result states that the bisection method depend on the intermediate value theorem is used to find the value of t^*.

Theorem 3.1. $t_1 < t^* < t_2$ *where*

$$t_1 = \frac{2\delta(a-b)}{H_c(P-D)}\left(\frac{H_c+B_c}{B_c}\right) \text{ and } t_2 = \frac{2\delta a}{H_c(P-D)}\left(\frac{H_c+B_c}{B_c}\right).$$

Proof. See Appendix A.

The following lemmas are needed to prove Theorem 3.1. □

Lemma 3.2. (a) $1 - \mathrm{e}^{-\lambda t} - \lambda t e^{-\lambda t} > 0$, for all $t > 0$.
(b) $\lambda t e^{-\lambda t} + 2\mathrm{e}^{-\lambda t} + \lambda t - 2 > 0$, for all $t > 0$.
(c) $\mathrm{e}^{-\lambda t} + \lambda t - 1 > 0$, for all $t > 0$.

Proof. See Appendix A.

Therefore, after obtaining t^*, the setup cost reduction by optimal capital investments and process quality improvement can be obtained by the following equations

$$\xi^*_K(K) = a \ln\left(\frac{DK_0}{\delta a P t^*}\right) \tag{3.16}$$

and

$$\xi^*_\xi(\xi) = b \ln\left(\frac{D\xi_0\left[w_c(A_2-A_1)+\eta_c\right]\left\{1+\frac{1}{\lambda t^*}(\mathrm{e}^{-\lambda t^*}-1)\right\}}{\delta b}\right). \tag{3.17}$$

Using Theorem 3.1, the following algorithm can be developed to obtain the optimal production run time, setup cost, and process quality. □

Algorithm 3.3.

Step 1. Let $\epsilon > 0$, and set $t_{\text{lower}} = t_1$ and $t_{\text{upper}} = t_2$.
Step 2. Set $t_{\text{opt}} = \frac{t_{\text{lower}}+t_{\text{upper}}}{2}$.
Step 3. If $|f(t_{\text{opt}})| < \epsilon$, follow Step 5. Otherwise, follow Step 4.
Step 4. If $f(t_{\text{opt}}) > 0$, set $t_{\text{upper}} = t_{\text{opt}}$. If $f(t_{\text{opt}}) < 0$, set $t_{\text{lower}} = t_{\text{opt}}$. Then follow Step 2.
Step 5. Using the value of t_{opt} in equations (3.11) and (3.12), solve $K^*(t_{\text{opt}})$ and $\xi^*(t_{\text{opt}})$, respectively.
Step 6.1. If $K^*(t_{\text{opt}}) \leq K_0$, then $K_{\text{opt}} = K^*(t_{\text{opt}})$, the setup cost reduction investment is obtained using equation (3.16). Otherwise, set $K_{\text{opt}} = K_0$, and no capital investment is required to reduce the setup cost.
Step 6.2. If $\xi^*(t_{\text{opt}}) \leq \xi_0$, then $\xi_{\text{opt}} = \xi^*(t_{\text{opt}})$, the process quality improvement investment cost is obtained using equation (3.17). Otherwise, set $\xi_{\text{opt}} = \xi_0$, and no capital investment is required to improve the process quality.
Step 7. $K^* = K_{\text{opt}}$, $\xi^* = \xi_{\text{opt}}$, and $t^* = t_{\text{opt}}$, are the optimal setup cost, and process quality, production run length respectively.

4. Numerical example

Example 4.1. The following parametric values are considered in appropriate units: $K = \$100$/setup, $P = 15\,000$ units/year, $D = 5000$ units/year, $H_c = \$1.5$/unit/year, $B_c = \$3$/unit, $w_c = \$20$/unit, $\eta_c = \$30$/unit, $\delta = 0.12$ per dollar per day, $\xi = 0.2$, $a = 1450$, $b = 30$, $\lambda = 0.2$, $\beta_1 = \beta_2 = 2, \xi_1 = 1/36, \xi_2 = 1/12, h_1(t) = \xi_1^{\beta_1}\beta_1 t^{\beta_1 - 1}$, and $h_2(t) = \xi_2^{\beta_2}\beta_2 t^{\beta_2 - 1}$. The optimal solution is $t^* = 0.034$ year, $\xi^* = 0.0069$, $K^* = \$17.79$, and the corresponding expected total cost is $\mathrm{ETC}^* = \$969.159$/year (see Fig. 2).

Further, the cases are solved when $\lambda = 0.03, 0.05, 0.1, 0.2, 0.3$, and 0.4. The optimal solutions are summarized in Table 2.

The optimal production run length and setup cost under various warranty periods and the corresponding expected total cost per unit time are summarized in Table 2.

5. Sensitivity analysis

This section describe the effects of changes in major cost parameters such as K, H_c, w_c, B_c, η_c, and δ on optimal total cost. By changing the values of parameters by −50%, −25%, +25%, and +50% one can find sensitivity

Table 2. Optimal solutions for Example 4.1.

λ	Investment model						Without investment model		Saving (%)
	t^* (year)	ξ^*	K^* (\$/setup)	$\xi^*(\xi)$	$\xi^*(K)$	$\mathrm{ETC}(\cdot)$(\$)	t^* (year)	$\mathrm{ETC}_1(\cdot)$(\$)	
0.03	0.0340802	0.0462	17.780	43.96	2503.48	962.34	0.0781	1161.64	17.16
0.05	0.0340804	0.0277	17.790	59.27	2503.48	964.17	0.0761	1185.10	18.64
0.10	0.0340808	0.0139	17.790	80.05	2503.46	966.67	0.0715	1241.10	22.11
0.20	0.0340816	0.0069	17.791	100.82	2503.42	969.16	0.0644	1343.78	27.88
0.30	0.0340824	0.0046	17.791	112.94	2503.39	970.61	0.0592	1436.89	32.45
0.40	0.0340833	0.0035	17.792	121.54	2503.35	971.65	0.0550	1522.69	36.19

Notes. Savings $= \{[\mathrm{ETC}_1(\cdot) - \mathrm{ETC}(\cdot)]/\mathrm{ETC}_1(\cdot)\} \times 100\%$.

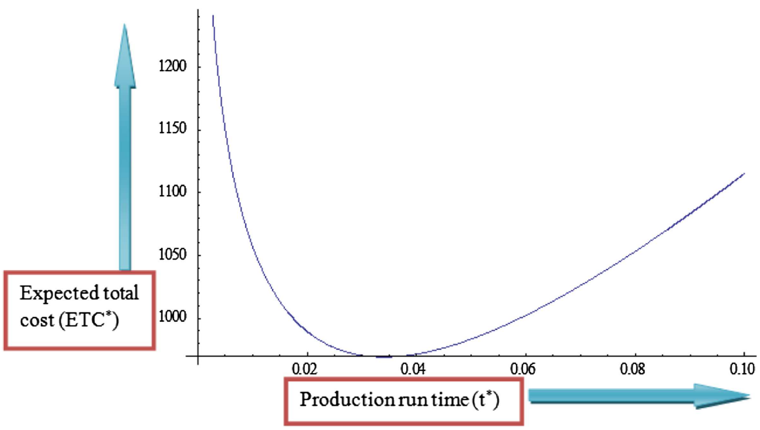

Figure 2. Graphical representation of expected cost per unit time (ETC^*) *versus* production run time (t^*).

TABLE 3. Effect of changes in the parameters on expected total cost.

Parameters	Changes (in %)	PCI in $\mathrm{ETC}_2^*(\cdot)$	Parameters	Changes (in %)	PCI in $\mathrm{ETC}_2^*(\cdot)$
	-50%	-12.44%		-50%	-05.06%
	-25%	-05.16%		-25%	-01.85%
K_0	$+25\%$	$+4.01\%$	B_c	$+25\%$	$+01.21\%$
	$+50\%$	$+7.28\%$		$+50\%$	$+02.07\%$
	-50%	-08.98%		-50%	-00.25%
	-25%	-03.53%		-25%	-00.10%
H_c	$+25\%$	$+02.52\%$	η_c	$+25\%$	$+00.08\%$
	$+50\%$	$+04.41\%$		$+50\%$	$+00.15\%$
	-50%	-15.93%		-50%	-21.63%
	-25%	-07.96%		-25%	-09.29%
w_c	$+25\%$	$+07.96\%$	δ	$+25\%$	$+07.02\%$
	$+50\%$	$+15.93\%$		$+50\%$	$+12.24\%$

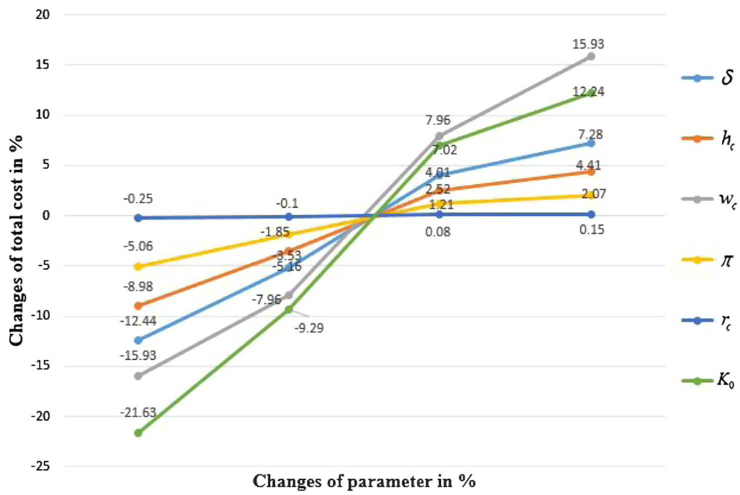

FIGURE 3. Graphical representation of effect of parametric value in total cost.

analysis, change the value of one parameter at a time while other parameters are unchanged. Depending upon the sensitivity analysis of key parameters, the results are presented in Table 3.

The percentage cost increase (PCI) is defined as follows:

$$\mathrm{PCI} = \frac{\mathrm{ETC}^* - \mathrm{ETC}_2^*}{\mathrm{ETC}_2^*} \times 100\%.$$

From Table 3, the discussion of sensitivity analysis of the key parameters is as follows

- If the setup cost increases, then the expected total cost also increases. From Table 3, it is observed that negative change in setup cost is more sensitive than it's positive change.
- Increasing the value of the holding cost increases the expected total cost. From Table 3, it is noticed that a negative change in holding cost results a greater reduction in expected total cost than does a positive change. Thus, it can be concluded that the holding cost is more sensitive to negative change than positive change with regard to expected total cost.

- If the warranty cost increases while all parameters unalter, the expected total cost tends to increase. The negative and positive changes in warranty cost gives approximately the same amount of change in expected total cost function.
- Increasing value of backorder cost and rework cost increases the expected total cost. These two cost parameters are less sensitive than the other parameters.
- If the annual fractional cost of capital investment is increased, then the expected total cost is also increased. Negative change in this parameter has greater effect on expected total cost than does positive change in this parameter. Therefore, capital investment is the most sensitive parameter among all.

6. Managerial insights

- The option of investment to reduce setup cost and increase process quality, as specified in equations (3.5) and (3.6) are considered in this model. The optimal investment strategy can be explicitly discussed. That is, the optimal process quality and setup cost are determined as shown in equations (3.11) and (3.12), respectively. Therefore, the benefits of setup cost reduction and improvement of process quality improvement which is discussed by Porteus [35] and Hong *et al.* [42] can be achieved using the optimal lot sizing policies.
- Equations (3.16) and (3.17) might lead to negative values of $\xi_K^*(K)$ and $\xi_K^*(\xi)$. This result can be presentd as an indicator that a reduction of the existing technologies ($K > K_0$ and $\xi > \xi_0$) is necessary. In practical situations, such a reduction might not be possible. Therefore, it is necessary to introduce two critical parameters $\tilde{a}$ and $\tilde{b}$, such that, for $a > \tilde{a}$ and $b > \tilde{b}$, no investment should occur. By following equations (3.16) and (3.17), it can be easily stated that, for a positive capital investment, $\frac{DK_0}{\delta a P t^*} > 1$, and $\frac{D\xi_0[w_c(A_2-A_1)+\eta_c]\left\{1+\frac{1}{\lambda t^*}(\mathrm{e}^{-\lambda t^*}-1)\right\}}{\delta b} > 1$; otherwise, the length of production run would be t_0 (the optimal length of production run in the classical EMQ model). Thus, $\tilde{a}$ and $\tilde{b}$ are derived as:

$$\tilde{a} = \frac{DK_0}{\delta P t_0} \tag{6.1}$$

and

$$\tilde{b} = \frac{D\xi_0\left[w_c(A_2-A_1)+\eta_c\right]\left\{1+\frac{1}{\lambda t_0}(\mathrm{e}^{-\lambda t_0}-1)\right\}}{\delta}. \tag{6.2}$$

- From equations (6.1) and (6.2), $\tilde{a}$ and $\tilde{b}$ are proportional to the initial setup cost K_0 and the initial process quality ξ_0, respectively. This implies that the larger its initial value, the greater are $\tilde{a}$ and $\tilde{b}$, and the more profitable is an investment to reduce the setup cost and to improve the process quality. The denominator of equation (6.1) decreases strictly monotonous in δ and Pt_0. Therefore, investment in setup cost reduction should be made when there are low interest rate δ, small initial lot size, and higher demand rate D. Similarly, the denominator of equation (6.2) decreases strictly monotonous in δ. Hence, the smaller the interest rate, the greater $\tilde{b}$ becomes. In addition, when there are high rework η_c and warranty w_c cost, high demand rate D, and high initial production run length t_0, an investment in the process quality improvement should made.

7. Conclusions

This model investigated the effects of setup cost reduction and process quality improvement on optimal production run length. The main theme of this study was to minimize the expected total cost based on the production run length, setup cost, and process quality as decision variables. An effective algorithm is developed to determine the optimal decision variables and the corresponding expected total cost. From equation (3.15), one can find that the value of the production run length depends on a and b. *i.e.*, the optimal production run length was dependent on setup cost reduction and process quality improvement. The effect of capital investment was supported by the numerical examples. From the view of real life situation, this strategy is valid and useful to the business's competitive advantages. The numerical results proved that the savings of expected total annual cost are realized, when the reduction of setup cost and process for improving the quality could be achieved through extra investments.

In future, the model can be extended to consider preventive maintenance (see Sana [23]). Instated of single item, consideration of multi product or assembled product another interesting research direction. Quality of any product obviously effect in any business industry, thus upgradation of quality is another effective research (see Sarkar *et al.* [41]). In current situation effect of carbon emission during transportation is take a vital place. Thus, in this environment concuss situation reduction of carbon emission cost is a very realistic research.

Appendix A. Proof of Lemma 3.2 and Theorem 3.1

Proof of Lemma 3.2

(a) Let, $g(y) = 1 - \mathrm{e}^{-\lambda y} - \lambda y e^{-\lambda y} > 0$. Then $g'(y) = \lambda^2 y e^{-\lambda y} > 0$, for all $y > 0$. Therefore, $g(y)$ is an increasing function of x, for all $y > 0$. Consequently, $g(y) > g(0) = 0$ for all $y > 0$. One can have $1 - \mathrm{e}^{-\lambda y} - \lambda y e^{-\lambda y} > 0$ for all $y > 0$.

(b) Let, $m(y) = \lambda y e^{-\lambda y} + 2\mathrm{e}^{-\lambda y} + \lambda y - 2 > 0$. Then, $m'(y) = \lambda(1 - \mathrm{e}^{-\lambda y} - \lambda y e^{-\lambda y}) > 0$ (using (a)). Hence $m(y)$ is an increasing function of y for all $y > 0$. Therefore, $m(y) > m(0) = 0$ for all $y > 0$. Consequently, $\lambda y e^{-\lambda y} + 2\mathrm{e}^{-\lambda y} + \lambda y - 2 > 0$, for all $y > 0$.

(c) Let, $h(y) = \mathrm{e}^{-\lambda y} + \lambda y - 1 > 0$. Then, $h'(y) = \lambda(1 - \mathrm{e}^{-\lambda y}) > 0$, hence $h(y)$ is an increasing function of y for all $y > 0$. Therefore, $h(y) > h(0) = 0$ for all $y > 0$. We have $\mathrm{e}^{-\lambda y} + \lambda y - 1 > 0$ for all $y > 0$.

Proof of Theorem 3.1

One can see that $f(t)$ in (3.16) is a continuous function with $\lim_{t\to 0^+} f(t) = -\delta(a-b) < 0$ and $\lim_{t\to\infty} f(t) = \infty > 0$. Further

$$
\begin{aligned}
& f(t_2) = \delta b(1 - \mathrm{e}^{-\lambda t_2} - \lambda t_2 \mathrm{e}^{-\lambda t_2}) \\
& \text{As, } 1 - \mathrm{e}^{-\lambda t_2} - \lambda t_2 \mathrm{e}^{-\lambda t_2} > 0,\ f(t_2) > 0 = f(t^*) \\
\text{and } & f(t_1) = -\delta b + \frac{\delta b}{(\mathrm{e}^{-\lambda t_1} + \lambda t_1 - 1)} \left(1 - \mathrm{e}^{-\lambda t_1} - \lambda t_1 \mathrm{e}^{-\lambda t_1}\right) \\
& = -\frac{\delta b(\lambda t_1 \mathrm{e}^{-\lambda t_1} + 2\mathrm{e}^{-\lambda t_1} + \lambda t_1 - 2)}{\mathrm{e}^{-\lambda t_1} + \lambda t_1 - 1} \\
& \text{Since } (\lambda t_1 \mathrm{e}^{-\lambda t_1} + 2\mathrm{e}^{-\lambda t_1} + \lambda t_1 - 2) > 0, \mathrm{e}^{-\lambda t_1} + \lambda t_1 - 1 > 0 \\
& f(t_1) < 0 = f(t^*).
\end{aligned}
$$

Hence, the result $t_1 < t^* < t_2$ follows.

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