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REPLENISHMENT OF IMPERFECT ITEMS IN AN EOQ INVENTORY MODEL WITH PARTIAL BACKORDERING

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Abstract. This paper deals with an inventory model in which a percent of the items in the lot is imperfect. The supplier is far from the buyer. After the reception of the order, immediately the products are inspected and imperfect items are identified. Due to the fact that supplier is located at long distance and the demand is needed to cover, the imperfect items are replenished by perfect ones from a local supplier at higher cost. In addition, the imperfect items are withdrawn and sold at a salvaged price as second-degree items. The shortage is allowed and partially backordered. The following three cases are considered: Case I. The reordered items are received when inventory level is zero; Case II. The reordered items are received when the backordered quantity is equal to the imperfect items quantity; and Case III. The reordered items are received when shortage is still remained. These cases are studied and analyzed in detail. In each case, the aim is to obtain the optimal value of the length period and the percent of period duration in which the inventory level is positive. A numerical example is presented to show the applicability of proposed inventory model. The results show that Case I has the lowest holding and shortage cost, so the total benefit is higher than the other two cases.

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1. Introduction

One century ago, the economic order quantity (EOQ) inventory model was proposed by Harris [5]. It is well known that the EOQ inventory model answers to one of the most important and relevant question in the inventory system, which is: how many products must be ordered? In the inventory literature, there exist several variants of the EOQ inventory model. It is worth to mentioning that one basic assumption in EOQ inventory model is that purchased products have perfect quality and satisfy the buyer's standards. However, in real world situations this assumption is typically violated.

The presence of imperfect items has received a lot of attention by many researchers and academicians. For example, Porteus [18] studies the effect of imperfect items when production line is out of control. Similarly,

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Rosenblatt and Lee [22] investigate the economic production cycles with imperfect items considering that the percentage of defectives is dependent of lot size. Their main conclusion is that producing/ordering in smaller lots reduces the percentage of defectives per lot. From a different perspective, Salameh and Jaber [24] develop an interesting extension of the EOQ inventory model that considers imperfect items. Salameh and Jaber [24]'s paper has been significantly noted in literature and continues as a center of interest for researchers involved in inventory and logistics fields (see *e.g.* Khan *et al.* [8]). Basically, Salameh and Jaber [24] develop a new variant of EOQ inventory model where a shipment contains a random percentage of defective items. Their inventory model assumes that upon receiving a lot, it is completely screened at an inspection rate, which is greater than the demand rate to ensure no shortages. By the end of the screening period, the items are classified as perfect and imperfect quality items. Salameh and Jaber [24] mention that an imperfect item is not necessarily defective, but it has a quality that is suitable for sale in a secondary market with a lower price. Salameh and Jaber [24]'s work suggests ordering larger lots contrary to what the traditional EOQ inventory model recommends. Their work has been (1) corrected by Cárdenas-Barrón [1] and Maddah and Jaber [11], (2) simplified by Goyal and Cardenas-Barron [4] and (3) critiqued by Papachristos and Konstantaras [15].

A comprehensive review of variants of the Salameh and Jaber [24]'s inventory model is given in Khan *et al.* [8]. They classify extensions of the Salameh and Jaber [24]'s inventory model in six sections: (1) EOQ/EPQ, (2) shortage, (3) fuzzy, (4) supply chain, (5) quality, and (6) others. Other researchers such as Sarkar [25], Ouyang and Chang [14] extend Salameh and Jaber [24]'s inventory model to consider payments. Recently, Paul *et al.* [16] present a joint replenishment problem to determine the ordering policy for multiple items having a certain percentage of defective products with price discount.

Rezaei [20] extends Salameh and Jaber [24]'s inventory model with shortages which are fully backordered. Conversely, Yu *et al.* [33] develop an inventory model with partial backordering and lost sales. Wee *et al.* [32] present an EPQ inventory model which on-going deterioration causes partial backordering. Afterwards, Wee *et al.* [31] propose an EPQ inventory model with fully backordered shortage, which poor-quality items occur during production. In the same year, Eroglu and Ozdemir [3] give an extension of Salameh and Jaber [24]'s inventory model by permitting the shortages to be fully backordered. Later, Chang and Ho [2] revisit the study of Wee *et al.* [31] and apply the well-known renewal reward theorem to derive a new expected net profit by exact close-form solution. Roy *et al.* [23] discuss different causes of shortages in the EOQ inventory model with defective items. Wahab *et al.* [30] determine the economic order quantity for a coordinated two-level international supply considering shortage and environmental effects. After, Konstantaras *et al.* [9] mention that the inspection of products and the positive relation between seller and purchaser create knowledge and experience that decrease shortage and proportion of imperfect items. Using the fuzzy theory, Liu and Zheng [10] extend the EOQ inventory model with imperfect items and shortages when inspection errors occur. Hsu and Hsu [6] derive an EOQ inventory model with imperfect quality items, inspection errors, shortage backordering, and sales returns. Rad *et al.* [19] propose an EOQ inventory model that combines operations and pricing decisions in a two-stage supply chain when the demand rate is dependent on the price, and shortage is permitted. Moussawi-Haidar *et al.* [12] investigate the effect of deterioration on the instantaneous replenishment. Skouri *et al.* [26] propose an inventory system with allowed shortages.

A common assumption in the papers before mentioned is that at the end of screening period or the end of cycle, imperfect items are withdrawn from the inventory system and the imperfect items are sold in single batch at discount rate. In this direction, Jaber *et al.* [7] propose a new approach to Salameh and Jaber [24]'s inventory model in which shortages are not allowed. They assume that there exist a distant supplier that provides the goods. The lot at the reception are inspected and imperfect items are withdrawn. The imperfect items can be repaired by local repair store or replaced by perfect ones from local supplier. Both of two cases impose extra charges to inventory system (repair cost or purchasing cost of an emergency order). Moroever, the imperfect items are sold on a second-degree market at lower price. Taleizadeh *et al.* [28] extend an EOQ inventory model in which imperfect items are replenished through a local supplier when shortage is permitted. In a subsequent paper, Taleizadeh *et al.* [29] derive and EPQ inventory model with assumption that the rework is outsourced considering shortage. Table 1 shows a brief description of research works related to imperfect quality items.

Table 1. Description of research works related to imperfect quality or defective items.

Author(s)	EOQ/EPQ	Imperfect/ Defective/ Deteriorating item	Replenishment of imperfect/ Defective items	Backordering	Brief description
Salameh and Jaber [24]	EOQ	Imperfect	No	No	Present an EOQ inventory model with imperfect quality items; where the imperfect quality items are suitable for sale in a secondary market at a lower price
Cárdenas-Barrón [1]	EOQ	Imperfect	No	No	Provides a correction to the Salameh and Jaber [24]'s inventory model
Maddah and Jaber [11]	EOQ	Imperfect	No	No	Correct a weakness in the Salameh and Jaber [24]'s inventory model with unreliable supply, characterized by a random fraction of imperfect quality items and a screening process
Goyal and Cardenas-Barron [4]	EOQ	Imperfect	No	No	Propose a simple method to determine the lot size for the Salameh and Jaber [24]'s inventory model
Papachristos and Konstantaras [15]	EOQ	Imperfect	No	No	Conditions in Salameh and Jaber [24] are not enough to avoid shortages. Then several conditions are added to guarantee that the shortage does not occur
Sarkar [25]	EOQ	Imperfect	No	No	Derives an inventory model with delay in payment with stock dependent demand when shortages are not permitted in an imperfect production system
Dey and Giri [13]	EOQ/EPQ	Defective	No	Fully	develops a single-vendor single-buyer integrated production-inventory model with stochastic demand and imperfect production process
Ouyang and Chang [14]	EPQ	Imperfect	No	Fully	Introduce an EPQ inventory model with delay in payment when shortages are fully backordered in an imperfect production system
Moussawi-Haidar *et al.* [12]	EOQ	Imperfect	No	Fully	Formulate a modified EOQ inventory model for an imperfect quality items with unreliable supply
Paul *et al.* [16]	EOQ	Imperfect	No	No	Two scenarios with and without price discount are presented to analyze the effect of the percentage of imperfect items on the ordering policy
Rezaei [20]	EOQ	Imperfect	No	Fully	Develops an EOQ inventory model for imperfect items considering backorders
Yu *et al.* [33]	EOQ	Imperfect	No	Partial	Determine the ordering policy for a deteriorating item with imperfect quality including partial backordering
Wee *et al.* [32]	EOQ	Imperfect	No	Fully	Build an integrated production-inventory model for deteriorating items with imperfect quality allowing shortages
Wee *et al.* [31]	EOQ	Imperfect	No	Fully	Propose an inventory model for items taking into account imperfect quality and shortage backordering
Eroglu and Ozdemir [3]	EOQ	Imperfect	No	Fully	Derive an EOQ inventory model taking into consideration defective items and shortages
Chang and Ho [2]	EOQ	Imperfect	No	Fully	Revision of Wee *et al.* [31]'s inventory model by well-known renewal-reward theorem to obtain exact closed-form solutions

TABLE 1. continued.

Author(s)	EOQ/EPQ	Imperfect/ Defective/ Deteriorating item	Replenishment of imperfect/ Defective items	Backordering	Brief description
Roy *et al.* [23]	EOQ	Imperfect	No	Partial	Introduce an EOQ inventory model with imperfect quality items and partial backlogging
Wahab *et al.* [30]	EPQ	Imperfect	No	No	Present some EOQ inventory models for a coordinated two-level international supply chain
Konstantaras *et al.* [9]	EOQ	Imperfect	No	Fully	Develop some inventory models for imperfect quality items including shortages and considering learning in inspection
Liu and Zeng [10]	EOQ	Imperfect	No	Fully	Formulate a fuzzy EOQ inventory model with imperfect items, shortages and inspection errors
Hsu and Hsu [6]	EOQ	Imperfect	No	Fully	Derive an EOQ inventory model with imperfect quality items, inspection errors, shortage backordering, and sales returns
Taleizadeh *et al.* [27]	EOQ	Deteriorating	No	Fully	Revisit the fuzzy rough EOQ inventory model for deteriorating items taking into account quantity discount and prepayment
Rad *et al.* [19]	EOQ/EPQ	Imperfect	No	Fully	Optimize the inventory and sales decisions in a two-stage supply chain with imperfect production and backorders
Moussawi-Haidar *et al.* [12]	EOQ	Imperfect	No	Fully	Study the impact of deterioration on EOQ inventory model with imperfect quality items considering without and with backordering
Skouri *et al.* [26]	EOQ	Defective	No	Fully	Derive an EOQ inventory model with backordering when there is a rejection of defective supply batches
Jaber *et al.* [7]	EOQ	Imperfect	Yes	No	Build an EOQ inventory model for imperfect items with buy and repair options
Taleizadeh *et al.* [28]	EOQ	Imperfect	Yes	Partial	Introduce an EOQ inventory model with reparation of imperfect products and partial backordering
Taleizadeh *et al.* [29]	EPQ	Imperfect	Yes	Fully	Outsourcing rework of imperfect items with backordering
This paper	EOQ	Imperfect	Yes	Partial	Proposes an EOQ inventory model for the replenishment of imperfect items with partial backordering

This paper allows shortage which is partially backordered whereas Jaber *et al.* [7]'s inventory model does not consider this feature. In addition, imperfect items are replenished by perfect items, which are purchased from local supplier. The replenishment of perfect items that substitute the imperfect items can occur according to three cases which are defined in Section 2.

The rest of this paper is organized as follows. Section 2 presents the problem definition. Section 3 derives mathematical models, proves the concavity of profit functions and determines the optimal value for decision variables. Section 4 solves a numerical example. Section 5 provides a sensitivity analysis. Section 6 gives some managerial insights. Section 7 presents a conclusion and future research directions.

2. Problem definition

Nowadays, supply chains are affected on global trade. In modern supply chains, the buyers to prepare raw material for products, firstly investigate in global bourses and negotiate with the best supplier independently where it is located. Therefore, the global scale businesses make the supply chain so complex.

When the products are purchased from a vendor, which is located far away from the buyer, then risk of having imperfect items in the lot is higher due to several reasons such as transportation problems. Therefore, lots can contain a percentage of products that does not meet the quality standards. So, in order to care brand reputation and sell products with acceptable quality, it is necessary to conduct an inspection process with the intention of ensuring the quality of products. Thus, a percentage of the product are identified as imperfect products. The demand rate is constant and shortage is allowed. After the lot is received, firstly, the backordered demand is covered and then all items are screened carefully in order to identify the imperfect items. Since products are valuable, imperfect items must be substituted by perfect items. Due to the fact that vendor is far away from buyer therefore the rejection of imperfect items to vendor is not allowed. Consequently, at the end of screen period, imperfect items are withdrawn as a single lot, which is sold in a secondary market at a lower salvaged price (c_s). Besides, perfect items are purchased from local supplier at a higher cost (c_E) with the aim to replace the imperfect items and satisfy the demand. Purchasing cost (c_E) from local supplier is higher than initial purchasing cost (c_u). Furthermore, initial purchasing cost (c_u) is higher than salvaged price (c_s).

When the new purchased items that substitute the imperfect items are added to inventory system, the inventory level is either zero or negative. The following three cases can occur: (I) the reordered items are received when the inventory level is zero, (II) the reordered items are received when the backordered quantity is equal to the imperfect items quantity and (III) the reordered items are received when shortage is remained. The main aim of the inventory model is to determine the optimal value for the duration period (T), percentage of the duration period (F) for which the inventory level is positive and the economic order quantity (Q) in order to maximize the total profit.

3. Mathematical modeling

In view of that after receipt reordered items, the inventory level would not be positive, the following three cases are considered: (I) the reordered items are received when the inventory level is zero, (II) the reordered items are received when the backordered quantity is equal to the imperfect items quantity, and (III) the reordered items are received when shortage is still remained. Both demand and inspection rate are constant and known. The inspection rate (x) is greater than demand rate (D); *i.e.* $x > D$. At the beginning of period, the inventory level is $I_{\max}$ (see Fig. 1). In order to identify the imperfect items, all products are inspected carefully according to the inspection rate x and its duration t_i is given by $t_i = I_{\max}/x$. The proportion of items that is imperfect (ρ) and its probability density function are determined at the end of screen period. The imperfect items identified are withdrawn and then they are sold at salvaged price (c_s). On the other hand, perfect items are purchased from local supplier at a higher cost. The salvaged price (c_s) is less than initial purchase cost, and purchase cost (c_u) is less than emergency cost (c_E), mathematically speaking: $c_s < c_u < c_E$. The buyer is interested in maximizing the total profit and also in determining the best choice between three cases. Table 2 presents the parameters and variables used in the development of this paper.

3.1. Case I. The reordered items are received when inventory level is zero

At the beginning of cycle, the inventory level is maximum ($I_{\max}$) and it is equal to FTD. Immediately, all products are inspected at an inspection rate x with duration t_i. Thus, $t_i = FTD/x$. The symbol ρ represents the percent of items that are imperfect. Therefore, the imperfect items quantity is given by ρFTD. These imperfect items are withdrawn and sold at salvaged price c_s. In order to replace the imperfect items, perfect items are purchased from a local supplier at purchasing cost of c_E. In this case, the reordered items arrived to store when the inventory level is zero. So, after adding the products to inventory system, the inventory level is ρFTD. At

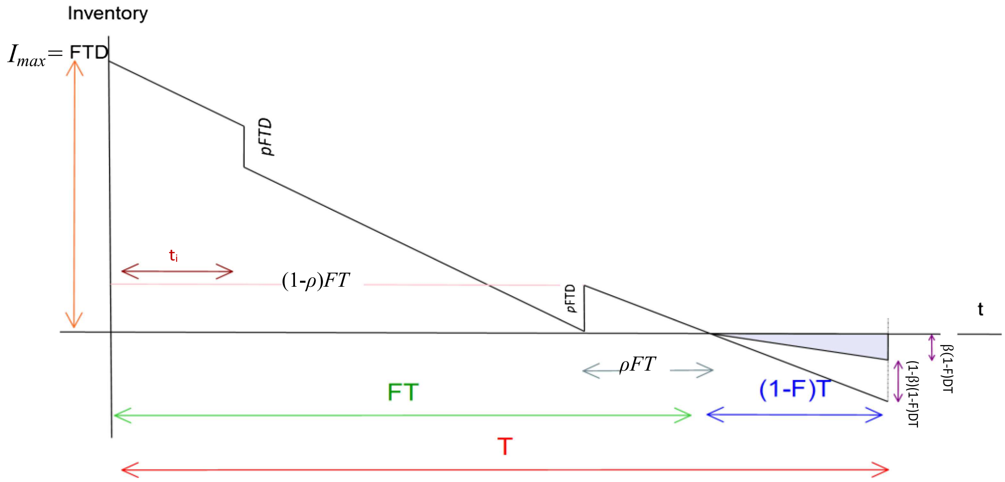

FIGURE 1. Inventory level in Case I.

TABLE 2. Description of parameters and variables.

	Parameters
t_i	Inspection time of products (time unit)
D	Demand rate (units/ time unit)
x	Inspection rate (units/ time unit)
ρ	Fraction of imperfect items
$f(\rho)$	Probability density function of imperfect items; ρ
g	Cost of lost sales (\$/unit)
β	Fraction of backordered lost sales
k	Buyer's ordering cost (\$/order)
h_E	Holding cost of emergency purchased unit (\$/unit/ time unit)
h	Holding cost (\$/unit/ time unit)
c_E	Unit purchasing cost of an emergency order (\$/unit)
c_u	Unit cost (\$/unit)
c_s	Salvaged price (\$/unit)
π	Backordered cost (\$/unit)
P	Unit price (\$/unit)
$E[.]$	Expected value of a random variable
	Decision variables
T	Cycle time (time unit)
F	Percentage of duration in which inventory level are positive
	Dependent variable
Q	Order quantity (units)

the end of cycle, the shortage is $(1\text{-}F)TD$. The symbol β represents the percent of shortage that is backordered and the rest is lost sales. The backordered demand has backordered charges at π and lost sales charges per lost sale unit is g. Figure 1 illustrates the behavior of inventory for the Case I.

The ordered quantity per cycle is given by $Q = FTD + \beta(1-F)TD$. The total holding cost (HC) is calculated as:

$$\text{HC} = h\left(\frac{(1-\rho)^2F^2TD}{2} + \frac{\rho T(FD)^2}{x}\right) + h_E\frac{(\rho F)^2DT}{2}. \tag{3.1}$$

Where h is the holding cost of products and h_E is the holding cost of emergency ordered items. The shortage cost (SC) is determined as:

$$\mathrm{SC} = \pi\frac{\beta(1-F)^2TD}{2} + g(1-\beta)(1-F)D. \tag{3.2}$$

Where π is the backordered cost and g is the lost sales cost and β is the percentage of backordered quantity. Thus, the total profit (TP) is equal to the total revenue less the total cost. The $\mathrm{TP}(T,F)$ is expressed as:

$$\begin{aligned}\mathrm{TP}(T,F) = P(FD+\beta(1-F)D) + c_s\rho FD - \Bigg[\frac{k}{T} + c_u\left(FD+\beta(1-F)D\right) + c_iFD + c_E\rho FD\\ + h\left(\frac{(1-\rho)^2F^2TD}{2} + \frac{\rho T(FD)^2}{x}\right) + h_E\frac{(\rho F)^2DT}{2} + \pi\frac{\beta(1-F)^2TD}{2} + g(1-\beta)(1-F)D\Bigg]. \end{aligned}\tag{3.3}$$

Where P is unit price of sales, c_s is salvage price, k is fixed ordered cost, c_u is initial purchased cost per unit, c_i is inspection cost per unit, and c_E is the emergency purchased cost. The main objective is to find the optimal value for cycle period (T) and percentage of duration of the cycle period in which the inventory level is positive. Also, the optimal ordered quantity is determined with the optimal cycle period. In order to optimize TP (T,F), firstly the concavity of profit function must be proved. To simplify the equation, the digits +1 and −1 are added to the ordered quantity and the $\mathrm{TP}(T,F)$ is changed to:

$$\begin{aligned}\mathrm{TP}(T,F) = P(+1-1+FD+\beta(1-F)D) + c_s\rho FD - \Bigg[\frac{k}{T} + c_u\left(+1-1+FD+\beta(1-F)D\right)\\ + c_iFD + c_E\rho FD + h\left(\frac{(1-\rho)^2F^2TD}{2} + \frac{\rho T(FD)^2}{x}\right) + h_E\frac{(\rho F)^2DT}{2}\\ + \pi\frac{\beta(1-F)^2TD}{2} + g(1-\beta)(1-F)D\Bigg].\end{aligned}\tag{3.4}$$

Then the total profit function is simplified as:

$$\begin{aligned}\mathrm{TP}(T,F) = PD(1-(1-F)(1-\beta)) + c_s\rho FD - \Bigg[\frac{k}{T} + c_uD\left(1-(1-F)(1-\beta)\right)\\ + c_iFD + c_E\rho FD + h\left(\frac{(1-\rho)^2F^2TD}{2} + \frac{\rho T(FD)^2}{x}\right) + h_E\frac{(\rho F)^2DT}{2}\\ + \pi\frac{\beta(1-F)^2TD}{2} + g(1-\beta)(1-F)D\Bigg].\end{aligned}\tag{3.5}$$

By defining $c_d = (P+g-c_u)$ and $c_k = c_E - c_s$ then $\mathrm{TP}(T,F)$ is rewritten as:

$$\begin{aligned}\mathrm{TP}(T,F) = D(P-c_u) - \Bigg\{\frac{k}{T} + c_iFD + h\left(\frac{(1-\rho)^2F^2TD}{2} + \frac{\rho T(FD)^2}{x}\right) + h_E\frac{(\rho F)^2DT}{2}\\ + \pi\frac{\beta(1-F)^2TD}{2} + c_d(1-\beta)(1-F)D + c_k\rho FD\Bigg\}\end{aligned}\tag{3.6}$$

$$\begin{aligned}\mathrm{TP}(T,F) = D(P-c_u) - \Bigg\{c_dD(1-\beta) + \frac{1}{T}(k) + F\left(c_iD + c_k\rho D - c_dD(1-\beta)\right)\\ + T\left(\frac{\pi\beta D}{2}\right) - FT(\pi\beta D) + F^2T\left(\frac{hD(1-\rho)^2}{2} + \frac{\rho hD^2}{x} + \frac{h_E\rho^2D}{2} + \frac{\pi\beta D}{2}\right)\Bigg\}.\end{aligned}\tag{3.7}$$

Since $D(P-c_u)$ is a constant term, the TP (T,F) is maximized by the pair (T,F), if and only if the cost $\{N(T,F)\}$ is minimized. Then the $N(T,F)$ is expressed as:

$$N(T,F) = c_d D(1-\beta) + \frac{1}{T}(k) + F\left(c_i D + c_k \rho D - c_d D(1-\beta)\right) + T(\frac{\pi\beta D}{2}) - FT(\pi\beta D) + F^2 T\left(\frac{hD(1-\rho)^2}{2} + \frac{\rho h D^2}{x} + \frac{h_E \rho^2 D}{2} + \frac{\pi\beta D}{2}\right). \tag{3.8}$$

Now, the $N(T,F)$ is rewritten as follows:

$$N(T,F) = G_0 + \frac{1}{T}(G_1) + T(G_2 - G_4 F + G_5 F^2) + G_3 F \tag{3.9}$$

where:

$$\begin{aligned}
G_0 &= c_d D(1-\beta)\\
G_1 &= k\\
G_2 &= \tfrac{\pi\beta D}{2}\\
G_3 &= c_i D + c_k \rho D - c_d D(1-\beta)\\
G_4 &= 2G_2 = \pi\beta D\\
G_5 &= \tfrac{hD(1-\rho)^2}{2} + \tfrac{\rho h D^2}{x} + \tfrac{h_E \rho^2 D}{2} + \tfrac{\pi\beta D}{2}.
\end{aligned}$$

It is clear that G_1, G_2, G_4 and G_5 are positive. The term $c_d D(1-\beta)$ is a constant therefore it can be eliminated from $N(T,F)$. Hence $N(T,F)$ is expressed in a compact form as:

$$\mathrm{N}(T,F) = \frac{1}{T}G_1 + T\lambda(F) + \gamma(F) \tag{3.10}$$

where $\lambda(F) = G_2 - G_4 F + G_5 F^2 = G_2 - 2G_2 F + G_5 F^2$ and $\gamma(F) = G_3 F$.

$N(T,F)$ is optimized according to Pentico and Drake [17]'s optimization process. The first derivative of $N(T,F)$ with respect to T is as follows

$$\frac{\partial N(T,F)}{\partial T} = 0, \rightarrow \frac{-1}{T^2}G_1 + \lambda(F) = 0$$
$$T(F) = \sqrt{\frac{G_1}{\lambda(F)}}. \tag{3.11}$$

Remember that G_1 is positive. Therefore, it is sufficient to prove that $\lambda(F)$ for all values of F is positive:

$$\lambda(F) = G_2 - G_4 F + G_5 F^2 \rightarrow \Delta = 4(-G_2)^2 - 4G_2 G_5 = 4G_2(G_2 - G_5)$$
$$\Rightarrow \frac{\pi\beta D}{2} - \left(\frac{hD(1-\rho)^2}{2} + \frac{\rho h D^2}{x} + \frac{h_E \rho^2 D}{2} + \frac{\pi\beta D}{2}\right) < 0. \tag{3.12}$$

Since the equation (3.12) is negative, the discriminant of $\lambda(F)$ is negative consequently $\lambda(F)$ does not have root. Also, $\lambda(0) = G_2 > 0$ and $\lambda(F)$ is a continuous function, therefore $\lambda(F)$ for all values of F is always positive. By substitution of T into equation (3.9), the $N(F)$ is written as:

$$N(F) = 2\sqrt{G_1 \lambda(F)} + \gamma(F).$$

Note that $N(F)$ is continuous when F [0, 1]. Notice that there exist one or more local minimums, where the smallest of them is the global minimum of the total cost. The first and second derivatives of $N(F)$ with respect to F are given below

$$\frac{\partial N\left(F\right)}{\partial F} = \sqrt{G_1}\frac{\lambda'(F)}{\sqrt{\lambda(F)}} + \gamma'(F) \text{ and } \lambda(F) > 0 \tag{3.13}$$

$$\frac{\partial^2 N\left(F\right)}{\partial F^2} = \sqrt{G_1}\frac{\lambda''(F)\sqrt{\lambda(F)} - \lambda'(F)\frac{1}{2\sqrt{\lambda(F)}}}{\lambda(F)}$$

$$\frac{\partial^2 N\left(F\right)}{\partial F^2} = \sqrt{G_1}\frac{2\lambda''(F)\lambda(F) - (\lambda'(F))^2}{2\sqrt{(\lambda(F))^3}}\cdot \tag{3.14}$$

For proving that N is convex, it was shown that equation (3.14) is positive, $\lambda(F)$ and G_1 are positive, thus the numerator is always positive:

$$\begin{aligned}
&2\lambda''(F)\lambda(F) - (\lambda'(F))^2 \geq 0\\
&2(2G_5)(G_2 - 2G_2F + G_5F^2) - (-2G_2 + 2G_5F)^2 \geq 0\\
&4G_5G_2 - 8G_5G_2F + 4G_5^2F^2 - G_4^2 - 4G_5^2F^2 + 8G_2G_5F \geq 0\\
&4G_5G_2 - 4G_2^2 \geq 0\\
&4G_2(G_5 - G_2) \geq 0\\
&\tfrac{hD(1-\rho)^2}{2} + \tfrac{\rho hD^2}{x} + \tfrac{h_E\rho^2 D}{2} + \tfrac{\pi\beta D}{2} - \tfrac{D\pi\beta}{2} \geq 0\\
&\text{and}\\
&\tfrac{hD(1-\rho)^2}{2} + \tfrac{\rho hD^2}{x} + \tfrac{h_E\rho^2 D}{2} > 0.
\end{aligned}$$

Consequently, $N(T,F)$ is a convex function. If first derivatives of $N(F,T)$ respect T and F are setting to zero then the optimal solutions are obtained easily. As per equation (3.13):

$$\frac{\partial N(T,F)}{\partial F} = 0$$

$$G_3 - 2G_2T + 2G_5TF = 0$$

$$F(T) = \frac{2G_2T - G_3}{2G_5T} = \frac{G_2}{G_5} - \frac{G_3}{2G_5T}$$

$$F(T) = \frac{\pi\beta - c_i - c_k\rho + c_s(1-\beta)}{\left(h(1-\rho)^2 + \frac{2\rho hD}{x} + h_E\rho^2 + \pi\beta\right)T}\cdot \tag{3.15}$$

By substitution of *F(T)* (Eq. (3.15)) into equation (3.11) one obtains

$$T = \sqrt{\frac{G_1}{G_2(1 - 2(\frac{2G_2T-G_3}{2G_5T}) + G_5\left(\frac{2G_2T-G_3}{2G_5T}\right)^2}}$$

$$T = \sqrt{\frac{G_1G_5 - \frac{1}{4}G_3^2}{G_2G_5 - G_2^2}} \text{ and } G_1G_5 - \frac{1}{4}G_3^2 > 0$$

$$T = \sqrt{\frac{k\left(\frac{h(1-\rho)^2}{2} + \frac{\rho hD}{x} + \frac{h_E\rho^2}{2} + \frac{\pi\beta}{2}\right) - \frac{D}{4}\left(c_i + c_k\rho - c_d(1-\beta)\right)^2}{\frac{\pi\beta}{2}\left(\frac{hD(1-\rho)^2}{2} + \frac{\rho hD^2}{x} + \frac{h_E\rho^2 D}{2} + \frac{\pi\beta D}{2}\right)}}\cdot \tag{3.16}$$

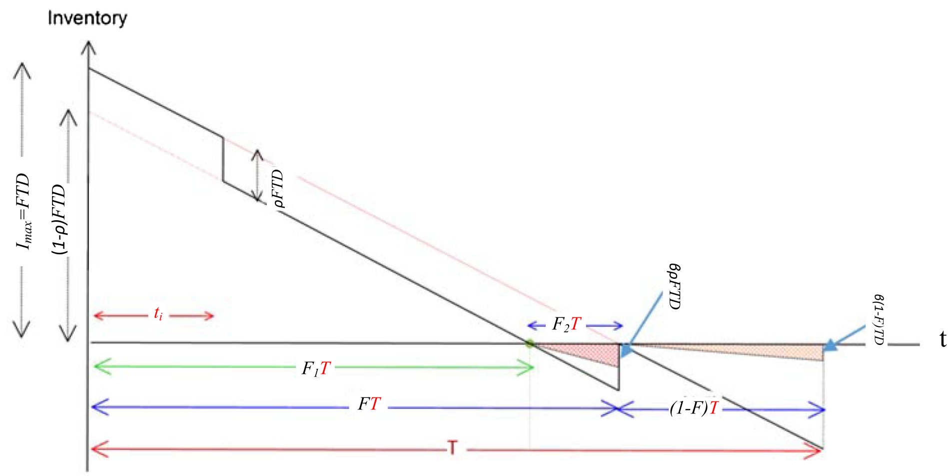

FIGURE 2. Inventory level in Case II.

The expected value of random variable (ρ) is substituted into equations (3.15) and (3.16)

$$T = \sqrt{\frac{k\left(\frac{h(1-E(\rho))^2}{2} + \frac{E(\rho)hD}{x} + \frac{h_E E(\rho^2)}{2} + \frac{\pi\beta}{2}\right) - \frac{D}{4}\left(c_i + c_k E(\rho) - c_d(1-\beta)\right)^2}{\frac{\pi\beta}{2}\left(\frac{hD(1-E(\rho))^2}{2} + \frac{E(\rho)hD^2}{x} + \frac{h_E E(\rho^2)D}{2} + \frac{\pi\beta D}{2}\right)}} \tag{3.17}$$

$$F(T) = \frac{\pi\beta - c_i - c_k E(\rho) + c_d(1-\beta)}{\left(h(1-E(\rho))^2 + \frac{2E(\rho)hD}{x} + h_E E(\rho^2) + \pi\beta\right)T}. \tag{3.18}$$

Let $w_1 = k\left(\frac{h(1-E(\rho))^2}{2} + \frac{E(\rho)hD}{x} + \frac{h_E E(\rho^2)}{2} + \frac{\pi\beta}{2}\right) - \frac{D}{4}\left(c_i + c_k E(\rho) - c_d(1-\beta)\right)^2$. If $w_1 > 0$, T^* is equal T in equation (3.17). On the other hand, if $w_1 \leq 0$, T^* is equal to zero, meaning that the optimal value of shortage is infinite, this means that no inventory level exists.

3.2. Case II. The reordered items are received when the backordered quantity is equal to the imperfect items quantity

Remember that F is the percentage of length duration in which the inventory level is positive. Therefore, the inventory level at the beginning of cycle is FTD and these products are inspected at an inspection rate x during t_i ($t_i = FTD/x$). There exists ρ percent of items, which is imperfect (ρFTD). At the end of cycle, the imperfect items are withdrawn and sold at salvaged price of c_s. These imperfect items are substituted from a local supplier at an emergency cost of c_E. The reordered items arrive to store when the shortage is equal to ρFTD. After adding these products to inventory system then inventory level reaches zero. The shortage is $(1-F)$ TD. A β percent of shortage is backordered and the rest is considered as lost sale. The backordered demand has a cost of π and the cost per lost sales unit is g. Figure 2 shows the behavior of inventory for the Case II.

The $F_i (i = 1, 2)$is in the interval [0,1] and F is determined as $F_1 + F_2$, where $F_1 = (1-\rho)F$ and $F_2 = \rho F$. The ordered quantity per cycle is $F_1TD + \beta F_2TD + \beta(1-F)TD = F_1TD + \beta(1-F_1)TD$. The holding cost is equal to:

$$\mathrm{HC} = h\left(\frac{(1-\rho)^2F^2TD}{2} + \frac{\rho F^2TD^2}{x}\right). \tag{3.19}$$

The backordered cost (SC) is:

$$\mathrm{SC} = \pi\frac{\beta\rho^2F^2TD}{2} + \pi\frac{\beta(1-F)^2TD}{2} + g(1-\beta)F_2D + g(1-\beta)(1-F)D$$
$$\mathrm{SC} = \pi\frac{\beta\rho^2F^2TD}{2} + \pi\frac{\beta(1-F)^2TD}{2} + g(1-\beta)(1-F_1)D. \tag{3.20}$$

Therefore, the total profit is defined:

$$\mathrm{TP}(T,F) = P(F_1D + \beta(1-F_1)D) + c_s\rho FD - \Bigg\{\frac{k}{T} + c_u(FD + \beta(1-F)D) + c_iFD + c_E\rho FD + h\left(\frac{(1-\rho)^2F^2TD}{2} + \frac{\rho F^2TD^2}{x}\right) + \pi\frac{\beta\rho^2F^2TD}{2} + \pi\frac{\beta(1-F)^2TD}{2} + g(1-\beta)(1-F_1)D\Bigg\}. \tag{3.21}$$

Considering $F_1 + \beta(1-F_1) = 1 - 1 + F_1 + \beta(1-F_1) = 1-(1-F_1)(1-\beta)$, $c_d = P + g - c_u$ and $c_k = c_E - c_s$. Remember that c_E is greater than c_s. Thus, the profit function is given by:

$$\mathrm{TP}(T,F) = (P-c_u)D - \Bigg\{c_dD(1-\beta) + \frac{k}{T} + c_iFD + h\left(\frac{(1-\rho)^2F^2TD}{2} + \frac{\rho F^2TD^2}{x}\right) + \pi\frac{\rho^2F^2TD}{2} + \pi\frac{\beta(1-F)^2TD}{2} - c_dD(1-\beta)F_1 + c_k\rho FD\Bigg\}. \tag{3.22}$$

Obviously, the profit function is maximized when $N(T,F)$ is minimized. Where $N(T,F)$ is given by:

$$N(T,F) = c_dD(1-\beta) + \frac{k}{T} + c_iFD + h\left(\frac{(1-\rho)^2F^2TD}{2} + \frac{\rho F^2TD^2}{x}\right) + \pi\frac{\beta\rho^2F^2TD}{2} + \pi\frac{\beta(1-F)^2TD}{2} - c_dD(1-\beta)(1-\rho)F + c_k\rho FD$$

Now, $N(T,F)$ is rewritten as:

$$N(T,F) = c_dD(1-\beta) + \frac{1}{T}(k) + F(c_iD + c_k\rho D - c_dD(1-\beta)(1-\rho)) - FT(\pi\beta D) + T(\frac{\pi\beta D}{2}) + F^2T\left(\frac{h(1-\rho)^2D}{2} + \frac{\rho hD^2}{x} + \frac{\pi\beta\rho^2D}{2} + \frac{\pi\beta D}{2}\right)$$
$$N(T,F) = G_0 + \frac{1}{T}(G_1) + T(G_2 - G_4F + G_5F^2) + G_3F \tag{3.23}$$

where

$$G_0 = c_dD(1-\beta)$$
$$G_1 = k$$
$$G_2 = \frac{\pi\beta D}{2}$$
$$G_3 = D(c_i + c_k\rho - c_d(1-\beta)(1-\rho)$$
$$G_4 = 2G_2 = \pi\beta D$$
$$G_5 = \frac{h(1-\rho)^2D}{2} + \frac{\rho hD^2}{x} + \frac{\pi\beta\rho^2D}{2} + \frac{\pi\beta D}{2}\cdot$$

It is easy to see that $(P-c_u)D$ is a constant term and it can be eliminated from the total profit $\mathrm{TP}(T,F)$. Notice that $G_{1,}G_{2,}G_4$ and G_5 are positive. Therefore, the function $N(T,F)$ is a convex function. By taking the first derivative of $N(T,F)$ with respect T and F. Then, making them to equal zero the optimal values for decision variables are found.

The $N(T,F)$ is rewritten as:

$$N(T,F)=\frac{1}{T}G_1+T\lambda(F)+\gamma(F) \tag{3.24}$$

where

$$\lambda(F)=G_2-G_4F+G_5F^2=G_2-2G_2F+G_5F^2$$
$$\gamma(F)=G_3F.$$

Taking the first derivative of $N(T,F)$ (Eq. (3.24)) with respect to F,

$$\frac{\partial N(T,F)}{\partial F}=0\rightarrow G_3-2G_2T+2G_5TF=0$$
$$F(T)=\frac{2G_2T-G_3}{2G_5T}=\frac{G_2}{G_5}-\frac{G_3}{2G_5T}. \tag{3.25}$$

By substituting of G_2, G_3, G_5 into equation (3.25) then the optimal value for F is obtained:

$$F(T)=\frac{\pi\beta T-(c_i+c_k\rho-c_dD(1-\beta))}{\left(h(1-\rho)^2+\frac{2\rho D}{x}+\pi\beta\rho^2+\pi\beta\right)T}. \tag{3.26}$$

Also, taking the first derivative of $N(T,F)$ (Eq. (3.24)) with respect to T:

$$\frac{\partial N(T,F)}{\partial T}=0\rightarrow\frac{-1}{T^2}G_1+\lambda(F)=0$$
$$T(F)=\sqrt{\frac{G_1}{\lambda(F)}}$$
$$T(F)=\sqrt{\frac{G_1}{G_2-2G_2F+G_5F^2}}. \tag{3.27}$$

Substituting F (Eq. (3.26)) into equation (3.27) yields to:

$$T=\sqrt{\frac{G_1}{G_2(1-2(\frac{2G_2T-G_3}{2G_5T})+G_5\left(\frac{2G_2T-G_3}{2G_5T}\right)^2}}$$
$$T=\sqrt{\frac{G_1G_5-\frac{1}{4}G_3^2}{G_2G_5-G_2^2}}$$
$$T=\sqrt{\frac{k\left(\frac{h(1-\rho)^2}{2}+\frac{\rho hD}{x}+\frac{\pi\beta\rho^2}{2}+\frac{\pi\beta}{2}\right)-\frac{D}{4}\left(c_i+c_k\rho-c_d(1-\beta)\right)^2}{\frac{\pi\beta}{2}\left(\frac{h(1-\rho)^2D}{2}+\frac{\rho hD^2}{x}+\frac{\pi\beta\rho^2D}{2}\right)}}. \tag{3.28}$$

The expected value of random variable (ρ) is substituted into equations (3.28) and (3.26),

$$T=\sqrt{\frac{k\left(\frac{h(1-E(\rho))^2}{2}+\frac{E(\rho)hD}{x}+\frac{\pi\beta E(\rho^2)}{2}+\frac{\pi\beta}{2}\right)-\frac{D}{4}\left(c_i+c_kE(\rho)-c_d(1-\beta)\right)^2}{\frac{\pi\beta}{2}\left(\frac{h(1-E(\rho))^2D}{2}+\frac{E(\rho)hD^2}{x}+\frac{\pi\beta E(\rho^2)D}{2}\right)}}. \tag{3.29}$$

and

$$F(T)=\frac{\pi\beta T-(c_i+c_kE(\rho)-c_dD(1-\beta))}{\left(h(1-E(\rho))^2+\frac{2E(\rho)D}{x}+\pi\beta E(\rho^2)+\pi\beta\right)T}. \tag{3.30}$$

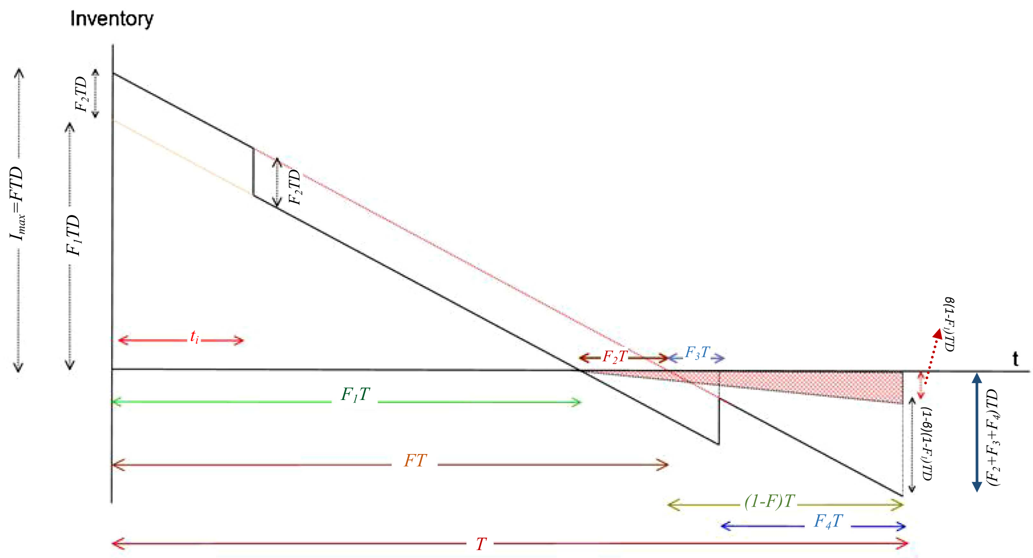

FIGURE 3. Inventory level in Case III.

Considering $w_2 = k\left(\frac{h(1-E(\rho))^2}{2} + \frac{E(\rho)hD}{x} + \frac{\pi\beta E(\rho^2)}{2} + \frac{\pi\beta}{2}\right) - \frac{D}{4}\left(c_i + c_k E(\rho) - c_d(1-\beta)\right)^2$.

If $w_2 > 0$ then T^* is equal T in equation (3.29). If $w_2 \leq 0$ then T^* is equal zero, it means that the optimal value of shortage is infinite, and no inventory cycle exists.

3.3. Case III. The reordered items are received when shortage is still remained

The inventory lever for Case III is depicted in Figure 3. Here, the duration length is divided into 4 parts. The parts are represented by F_i where $i = 1, 2, 3, 4$. Also, F is equal $F_1 + F_2$. The decision variable F represents the percentage of length duration in which the inventory level is positive. Here, 1-$F = F_3 + F_4$. The inventory level at the beginning of cycle is FTD, the goods are inspected at an inspection rate x with a duration of t_i where $t_i = FTD/x$. The ρ represents the percentage of items that are imperfect ($\rho FTD = F_2 TD$). The imperfect items are withdrawn at the end of cycle and they are sold at salvaged price of c_s. The imperfect items are substituted by perfect ones that are bought in a local supplier at a cost of c_E. The reordered items arrive to store when the shortage level is $(F_2 + F_3)\ TD$.

After adding goods to inventory system, the inventory level is still negative, it means that the shortage is still remained. Therefore, at the end of cycle, the shortage is $(F_2 + F_3 + F_4)\ TD$. The β percent of shortage is backordered and the rest is considered as lost sale. The backordered demand has a backordered cost of π and the lost sale cost per unit is g.

The holding cost is calculated as

$$\mathrm{HC} = h\left(\frac{(1-\rho)^2 F^2 TD}{2} + \frac{\rho(F)^2 TD^2}{x}\right). \tag{3.31}$$

The shortage cost is given as

$$\mathrm{SC} = \pi\frac{(F_2 + F_3 + F_4)(F_3 + F_4)\beta TD}{2} + g(1-\beta)(F_3 + F_4)D. \tag{3.32}$$

Total quantity of goods that is needed per cycle is $(F_1+F_2)TD+\beta(F_3+F_4)TD = FTD+\beta(1-F)TD$. The total profit is the total revenue less the total cost. Thus,

$$\begin{aligned}\mathrm{TP}(T,F) &= PD((F_1+F_2)+\beta(F_3+F_4)) + c_s\rho FD - \left\{\frac{k}{T} + c_uD\left((F_1+F_2)D+\beta(F_3+F_4)D\right) + c_i(F_1+F_2)D\right.\\ &\quad + c_E\rho FD + h\left(\frac{(1-\rho)^2F^2TD}{2}+\frac{\rho(F)^2TD^2}{x}\right) + \pi\frac{(F_2+F_3+F_4)(F_3+F_4)\beta TD}{2}\\ &\quad \left. + g(1-\beta)(F_3+F_4)D\right\} \end{aligned} \tag{3.33}$$

$$\begin{aligned}\mathrm{TP}(T,F) &= PD(F+\beta(1-F)) + c_s\rho FD - \left\{\frac{k}{T} + c_uD\left(F+\beta(1-F)\right) + c_iFD + c_E\rho FD\right.\\ &\quad \left. + h\left(\frac{(1-\rho)^2F^2TD}{2}+\frac{\rho(F)^2TD^2}{x}\right) + \pi\frac{(1-F_1)(1-F)\beta TD}{2} + g(1-\beta)(1-F)D\right\}. \end{aligned} \tag{3.34}$$

Firstly, the concavity of the total profit function is proved. Note that:

$$\begin{aligned} F+\beta(F_3+F_4) &= 1-1+F+\beta(1-F) = 1-(1-\beta)(1-F)\\ F_2+F_3+F_4 &= 1-F_1 = 1-(1-\rho)F\\ c_d &= P+g-c_u\\ c_k &= c_E-c_s, c_E > c_s. \end{aligned}$$

The total profit function is simplified as below:

$$\begin{aligned}\mathrm{TP}(T,F) &= (P-c_u)D - \left\{\frac{k}{T} + c_iFD + c_k\rho FD + h\left(\frac{(1-\rho)^2F^2TD}{2}+\frac{\rho(F)^2TD^2}{x}\right)\right.\\ &\quad \left. + \left(1-(1-\rho)F\right)(1-F)\frac{\pi\beta TD}{2} + c_dD(1-\beta)(1-F)\right\}. \end{aligned} \tag{3.35}$$

It is easy to see that the total profit function is maximized when $N(T,F)$ is minimized where $N(T,F)$ is given below:

$$\begin{aligned} N(T,F) &= \frac{k}{T} + c_iFD + c_k\rho FD + h\left(\frac{(1-\rho)^2F^2TD}{2}+\frac{\rho(F)^2TD^2}{x}\right) + \left(1-(1-\rho)F\right)(1-F)\frac{\pi\beta TD}{2}\\ &\quad + c_dD(1-\beta)(1-F)\\ N(T,F) &= c_dD(1-\beta) + \frac{1}{T}(k) + T(\frac{\pi\beta D}{2}) + F(c_iD+c_k\rho D - c_dD(1-\beta)) - FT\left(\frac{\pi\beta D(2-\rho)}{2}\right)\\ &\quad + FT^2\left(\frac{(1-\rho)^2hD}{2}+\frac{\rho hD^2}{x}+\frac{\pi\beta D(1-\rho)}{2}\right). \end{aligned}$$

$N(T,F)$ is expressed as

$$N(T,F) = G_0 + \frac{1}{T}(G_1) + T(G_2 - G_4F + G_5F^2) + G_3F \tag{3.36}$$

where

$$
\begin{aligned}
G_0 &= c_d D(1-\beta) \\
G_1 &= k \\
G_2 &= \frac{\pi\beta D}{2} \\
G_3 &= D(c_i + c_k\rho - c_d(1-\beta)) \\
G_4 &= \frac{\pi\beta D(2-\rho)}{2} \\
G_5 &= \frac{(1-\rho)^2 hD}{2} + \frac{\rho h D^2}{x} + \frac{\pi\beta D(1-\rho)}{2}\cdot
\end{aligned}
$$

Notice that G_1, G_2 ,G_4 and G_5 are positive. Furthermore, $c_d D(1-\beta)$ is a constant term and can be eliminated from $N(T,F)$. Therefore, equation (3.36) is rewritten as

$$\mathrm{N}(T,F) = \frac{1}{T}G_1 + T\lambda(F) + \gamma(F). \tag{3.37}$$

Where $\lambda(F) = G_2 - G_4 F + G_5 F^2$ and $\gamma(F) = G_3 F$. Firstly, $N(T,F)$ is derived with respect to T and then equally to zero the first derivative $\frac{\partial N(T,F)}{\partial T} = 0 \to \frac{-1}{T^2}G_1 + \lambda(F) = 0$

$$T(F) = \sqrt{\frac{G_1}{\lambda(F)}}\cdot \tag{3.38}$$

The numerator of T in equation (3.38) is always positive so the denominator must be positive too. When $\lambda(0) = 0T$ is not defined. Also, $T(F)$ has no solution when the discriminant of $\lambda(F)$ is negative.

$$
\begin{aligned}
\lambda(F) &= G_2 - G_4 F + G_5 F^2 \\
\Delta &= (-G_4)^2 - 4G_2 G_5
\end{aligned}
$$

By substituting of G_4, G_2 and G_5 one obtains

$$
\begin{aligned}
&\Rightarrow \left(-\tfrac{\pi\beta D(2-\rho)}{2}\right)^2 - 4\tfrac{\pi\beta D}{2}\left(\tfrac{(1-\rho)^2 hD}{2} + \tfrac{\rho h D^2}{x} + \tfrac{\pi\beta D(1-\rho)}{2}\right) < 0 \\
&\Rightarrow -\left(\tfrac{\pi\beta D}{2}\right)\left[\tfrac{(1-\rho)^2 hD}{2} + \tfrac{\rho h D^2}{x} + \tfrac{\pi\beta D}{2}(-4 + 4\rho - \rho^2 + 4 - 4\rho)\right] < 0 \\
&\Rightarrow \tfrac{\pi\beta D^2}{2}\left[\tfrac{(1-\rho)^2 h}{2} + \tfrac{\rho h D}{x} - \tfrac{\pi\beta\rho^2}{2}\right] > 0.
\end{aligned}
$$

Considering that the following constrain is satisfied

$$\frac{(1-\rho)^2 h}{2} + \frac{\rho h D}{x} > \frac{\pi\beta\rho^2}{2}\cdot \tag{3.39}$$

Thus, it follows that the discriminant of $\lambda(F)$ is always positive, therefore T defined in equation (3.38) is meaningful. By substituting T into $N(F,T)$, then $N(F)$ is written as

$$\mathrm{N}(F) = 2\sqrt{G_1\lambda(F)} + \gamma(F). \tag{3.40}$$

Taking first and second derivatives of $N(F)$ with respect F one obtains

$$\begin{aligned}\frac{\partial N\left(F\right)}{\partial F} &= \sqrt{G_1}\frac{\lambda'(F)}{\sqrt{\lambda(F)}}+\gamma'(F) \\ \frac{\partial^2 N\left(F\right)}{\partial F^2} &= \sqrt{G_1}\frac{\lambda''(F)\sqrt{\lambda(F)}-(\lambda'(F))^2\frac{1}{2\sqrt{\lambda(F)}}}{\lambda(F)} \\ \frac{\partial^2 N\left(F\right)}{\partial F^2} &= \sqrt{G_1}\frac{2\lambda''(F)\lambda(F)-(\lambda'(F))^2}{2\sqrt{(\lambda(F))^3}}.\end{aligned} \tag{3.41}$$

G_1 is always positive and according to assumption (3.39) then λ(F) is always greater than zero. In order to prove $N(F)$ is convex it is enough to show that the second derivative of $N(F)$ is positive. Therefore, it is sufficient to proof that $2\lambda''(F)\lambda(F)-\lambda'^2(F)\geq 0$. Substituting the corresponding expression of G_is

$$\begin{aligned}2(2G_5)(G_2-G_4F+G_5F^2)-(-G_4+2G_5F)^2 &\geq 0 \\ 4G_2G_5-4G_4G_5F+4G_5^2F^2-G_4^2-4G_5^2F^2+4G_4G_5F &\geq 0 \\ 4G_2G_5-G_4^2 &\geq 0 \\ 4\frac{\pi\beta D}{2}\left(\frac{(1-\rho)^2hD}{2}+\frac{\rho hD^2}{x}+\frac{\pi\beta D(1-\rho)}{2}\right)-\left(\frac{\pi\beta D(2-\rho)}{2}\right)^2 &\geq 0 \\ \frac{\pi\beta D}{2}\left(\frac{(1-\rho)^2hD}{2}+\frac{\rho hD^2}{x}-\frac{\pi\beta D\rho^2}{2}(-4+4\rho+4-4\rho+\rho^2)\right) &\geq 0.\end{aligned}$$

Therefore, $N(F)$ is convex. By taking partial derivative of the $N(T,F)$ (Eq. (3.37)) with respect to T and F and setting them equal to zero; thus the optimal solutions are obtained as bellow

$$\begin{aligned}\frac{\partial N\left(F\right)}{\partial F} &= \sqrt{G_1}\frac{\lambda'(F)}{\sqrt{\lambda(F)}}+\gamma'(F) \\ \frac{\partial N\left(F\right)}{\partial F} &= -G_4T+2G_5FT+G_3=0 \\ F(T) &= \frac{G_4T-G_3}{2G_5T}\end{aligned} \tag{3.42}$$

and

$$F(T)=\frac{\pi\beta(2-\rho)T+2c_d(1-\beta)-2c_i-2c_k\rho}{4\left(\frac{(1-\rho)^2h}{2}+\frac{\rho hD}{x}+\frac{\pi\beta(1-\rho)}{2}\right)T} \tag{3.43}$$

$$T(F)=\sqrt{\frac{G_1}{G_2-G_4F+G_5F^2}}. \tag{3.44}$$

By substituting F (Eq. (3.42)) into equation (3.44):

$$T = \sqrt{\frac{G_1}{G_2 - G_4(\frac{G_4T-G_3}{2G_5T}) + G_5\left(\frac{G_4T-G_3}{2G_5T}\right)^2}}$$

$$T = \sqrt{\frac{4G_1G_5 - G_3^2}{4G_2G_5 - G_4^2}} \tag{3.45}$$

$$T = \sqrt{\frac{4k\left(\frac{(1-\rho)^2hD}{2} + \frac{\rho hD^2}{x} + \frac{\pi\beta D(1-\rho)}{2}\right) - (D(c_i + c_k\rho - c_d(1-\beta)))^2}{4\frac{\pi\beta D}{2}\left(\frac{(1-\rho)^2hD}{2} + \frac{\rho hD^2}{x} + \frac{\pi\beta D(1-\rho)}{2}\right) - \left(\frac{\pi\beta D(2-\rho)}{2}\right)^2}}$$

$$T = \sqrt{\frac{4k\left(\frac{(1-\rho)^2h}{2} + \frac{\rho hD}{x} + \frac{\pi\beta(1-\rho)}{2}\right) - D\left(c_i + c_k\rho - c_d(1-\beta)\right)^2}{4\frac{\pi\beta}{2}\left(\frac{(1-\rho)^2hD}{2} + \frac{\rho hD^2}{x} - \frac{\pi\beta D\rho^2}{2}\right)}}. \tag{3.46}$$

Expected value of random variable (ρ) is substituted into equations (3.43) and (3.46), hence:

$$T = \sqrt{\frac{4k\left(\frac{(1-E(\rho))^2h}{2} + \frac{E(\rho)hD}{x} + \frac{\pi\beta(1-E(\rho))}{2}\right) - D\left(c_i + c_kE(\rho) - c_d(1-\beta)\right)^2}{4\frac{\pi\beta}{2}\left(\frac{(1-E(\rho))^2hD}{2} + \frac{E(\rho)hD^2}{x} - \frac{\pi\beta D(E(\rho))^2}{2}\right)}} \tag{3.47}$$

$$F(T) = \frac{\pi\beta(2-E(\rho))T + 2c_d(1-\beta) - 2c_i - 2c_kE(\rho)}{4\left(\frac{(1-E(\rho))^2h}{2} + \frac{E(\rho)hD}{x} + \frac{\pi\beta(1-E(\rho))}{2}\right)T}. \tag{3.48}$$

Considering $w_3 = k\left(\frac{(1-E(\rho))^2h}{2} + \frac{E(\rho)hD}{x} + \frac{\pi\beta(1-E(\rho))}{2}\right) - \frac{D}{4}\left(c_i + E(\rho)c_k - c_d(1-\beta)\right)^2$.

If $w_3 > 0$ then T^* is given by equation (3.47). If $w_3 \le 0$ then T^* is equal zero, this means that the optimal value of shortage is infinite, and no inventory cycle exists. Also, the condition of Case III is given by $\frac{(1-\rho)^2h}{2} + \frac{\rho hD}{x} > \frac{\pi\beta\rho^2}{2}$, so if this condition is violated, means backordered cost is very huge, in this situation shortage is meaningless and therefore no shortages are permitted. In this circumstance, the economic ordered quantity is determined with Jaber *et al.* [7]'s inventory model which is as follows

$$Q^* = \sqrt{\frac{2kD}{h(1 - 2E(\rho) + E(\rho^2) + 2E(\rho)\frac{D}{x}) + h_EE(\rho^2)}}. \tag{3.49}$$

4. Numerical example

This section illustrates the use of the three cases with a numerical example. The optimal values of the decision variables and the lot size are calculated. Table 3 provides the data taken from the Salameh and Jaber [24]. Table 4 presents data taken from Jaber *et al.* [7]. Table 5 shows additional data for the numerical example.

According to Table 3 the distribution of percentage of imperfect items is assumed that follows a uniform distribution $[U \sim (0, 0.04)]$. Thus, the expected value of the defective items (ρ) is

$$E(\rho) = \int_a^b \rho f(\rho)\mathrm{d}\rho = \int_a^b \rho\frac{1}{b-a}\mathrm{d}\rho = \frac{b+a}{2} = \frac{0.04+0}{2} = 0.02.$$

TABLE 3. Data taken from Salameh and Jaber [24].

Symbol	Value	Units
D	50 000	units/year
x	175 200	units/year
P	50	\$/unit
k	100	\$/order
c_u	25	\$/unit
c_s	20	\$/unit
h	5	\$/unit/year
c_i	0.5	\$/unit
ρ	$U \sim (0, 0.04)$	
$f(\rho)$	$1/(0.04 - 0)$	

TABLE 4. Date extracted from Jaber *et al.* [7].

Symbol	Value	Units
c_E	40	\$/unit
h_E	8	\$/unit/ year

TABLE 5. Additional data.

Description	Symbol	Value	Units
Backordered cost per unit	π	20	\$/unit
Lost sales cost per unit	g	0.5	\$/unit
Percentage of backordered demand	β	97%	

Moreover, the $E(\rho^2)$ is

$$E(\rho^2) = \text{var}(\rho) + E^2(\rho) = \frac{(b-a)^2}{12} + E^2(\rho) = \frac{(0.04)^2}{12} + (0.02)^2 = 0.0005333333.$$
$$(1 - E(\rho))^2 = 1 - 2E(\rho) + E(\rho^2) = 1 - 2(0.02) + 0.000533 = 0.9605333355.$$

Firstly, the significance of each profit functions is verified:

$w_1 = 985.3880 > 0$

$w_2 = 931.1284 > 0$

$w_3 = 965.7747 > 0$.

For Case III the following constraint $\frac{(1-E(\rho))^2 h}{2} + \frac{E(\rho) h D}{x} - \frac{\pi \beta E(\rho^2)}{2} = 2.4247 > 0$ must be satisfied. Therefore, all of conditions are satisfied. Hence, the optimal value for the decision variables and the economic order quantity for each case are given in Table 6.

From Table 6 it can be observed that the best optimal policy is Case I which has the highest profit per year rather than Case II and Case III. Note that Case II is second option.

TABLE 6. Optimal value for the decision variables and the economic order quantity.

	TP	T	F	Q
Case I	1200732.887	0.0289	60.70 %	1428.138
Case II	1200277.629	0.0281	57.88 %	1385.718
Case III	1200667.453	0.0286	60.70 %	1414.757

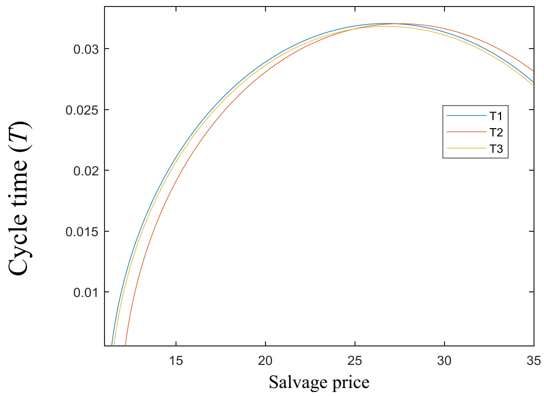

FIGURE 4. Cycle time against salvage price (c_s) for the three cases.

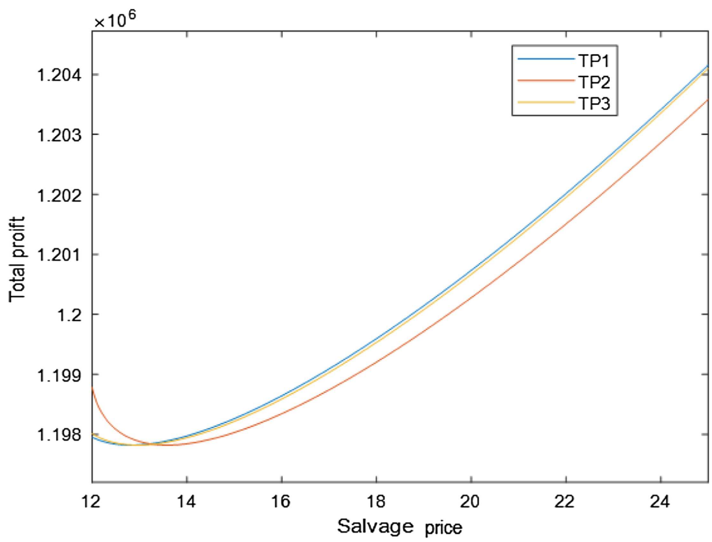

FIGURE 5. Behavior of total profit for the three cases against salvage price (c_s).

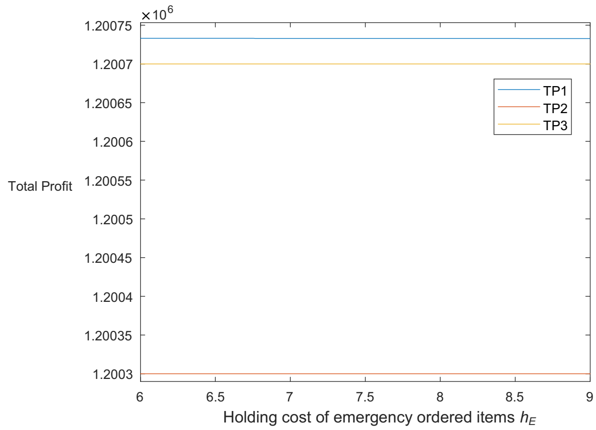

FIGURE 6. Behavior of total profit for the three cases against holding cost of emergency ordered items.

5. Sensitivity analysis

The behavior of cycle time against salvage price for the three cases are shown on Figure 4. It is easy to see that when salvage price less than about 12 the cycle time is zero therefore the strategy replenishment is not considered; in other words no quantity of goods is ordered.

The behavior of total profit for the three cases against salvage price is shown on Figure 5. Notice that the total profit of Case I for various value of salvage price mostly is higher than other two cases.

Figure 6 shows the behavior of total profit against to holding cost of emergency ordered items. It can be observed that the total profit is constant in the three cases.

6. Managerial insights

This paper is useful for managers of companies that purchase raw material for an assembly line from a distant supplier. For example the companies that provide final products such as automobile and household appliances manufacturers. Therefore, due to large distance that exists between supplier and buyer, if a proportion of purchased products is detected as imperfect, then the replenishment of them is not possible. Thus, instead of reordering perfect products, the imperfect products can be replaced by perfect ones by purchasing them at local supplier at a higher cost. In addition, the imperfect items are sold on salvage price as second-degree items. Furthermore, in many situations, these companies face shortages. In this direction, this paper develops a new variant of the inventory model of Salameh and Jaber [24] that considers shortages and these are partially backordered. In this respect, three cases have been studied. The model in Case I has the lowest holding and shortage cost rather than Case II and Case II, as a result the total benefit in Case I is higher than the other two cases.

7. Conclusion and further research directions

This paper develops an inventory model in which a percent of the items in the lot is imperfect and these imperfect items are substituted by perfect items which are bought in a local supplier at higher cost. The imperfect items are withdrawn and sold at a salvaged price. Three cases are considered and the optimal solution for each case is derived. In order to show the application of the proposed inventory model a numerical example is solved. Future research directions that researchers and academicians can explore are as follows: (1) consider stochastic demand, (2) multiproduct case, (3) trade credit, (4) repairing the defective items in local repair store.

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