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ON AN OPTIMAL REPLENISHMENT POLICY FOR INVENTORY MODELS FOR NON-INSTANTANEOUS DETERIORATING ITEMS WITH STOCK DEPENDENT DEMAND AND PARTIAL BACKLOGGING

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Abstract. This paper proposes a procedure for determining the optimal replenishment policy for the simple inventory model with stock-dependent demand items, non-instantaneous deteriorating items and partial backlogging. The optimal policy is shown to be of a threshold form. That is, (i) if the time of the onset of deterioration is greater than or equal to the time at which partial-backlogging begins in the basic model (with no deterioration), then the optimal policy is determined by the parameters of the basic model, else (ii) The optimal policy corresponds to the unique critical point of the objective function for the model with non-instantaneous deterioration. Moreover, a simple test for deciding in favor of the former model is given. The procedure obtained is simpler and easier to implement than those existing in the literature. Numerical examples are presented for illustration.

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1. Introduction

This paper is concerned with determining the optimal economic order quantity (EOQ) for the basic inventory model with non-instantaneous deteriorating items and stock dependent demand: see [6]. A simple approach for determining the optimal EOQ, along with the various mathematical supporting arguments, is proposed. The model considered in [6] has the particularity that items in stock experience some type of a delay deterioration effect. Additionally, demand for the items is influenced by the amount held in stock. A phenomena well documented in the marketing literature: see for example [4].

In this paper, only the optimal inventory policy for the model considered in [6] will be examined with a view of extending it to models with permissible delay in payment as treated in [5], and to pricing as in [2, 3, 7].

Existing methodology for determining the optimal inventory policies for inventory models with non-instantaneous deteriorating items are based on solving some optimization problems on separate branches of the decision variable. The optimal policy is then picked from the best policy among all branches. Although, this process supplies the optimal policy it is far from being always simple. In this paper we shall present a procedure which gives the optimal branch with only the knowledge of the optimal policy for the inventory model with no deterioration. This is done by comparing the time of the onset of the deterioration with that of the onset

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of backlogging for the model with no deterioration. If the former is greater than the latter then the optimal policy corresponds to the optimal (EOQ) found from the inventory model with no deterioration. Otherwise, the optimal policy corresponds to the optimal (EOQ) for the inventory model with non-instantaneous deteriorating items. Additionally, a procedure for determining this policy is given. This procedure is simple and easy to apply and involves very little computations. Additionally, it is hoped that it could be extended to the models cited above and others.

Key in the analysis of the present work, in contrast to existing work in the literature, is the observation that the objective function of the problem is not only continuous but differentiable on its domain of definition. This as we shall see makes the search for the optimal inventory policy relatively easy.

Before we close this section, we should note that the essence of the procedure proposed in this paper was successfully applied to the basic inventory with non-instantaneous deteriorating items and backlogging in [1]. However, it turns out that the introduction of partial backlogging, in particular, and stock-dependent demand makes the extension of past results not straightforward. The mathematical approach of this paper proposes an alternative treatment to circumvent the problems involved in the extension.

The next section contains the assumptions and notation used throughout the paper. The mathematical model is presented in Section 3. Section 4 contains a compact formulation of the problem of determining the optimal replenishment policy along with the solution of the problem. Section 5 contains illustrative examples of the applicability of the proposed approach. General remarks and possible extensions are found in the conclusion section.

2. Assumptions and notation

The mathematical model of this paper is developed under the same assumptions found in [6]. These will be repeated here for completeness:

(1) A single product is held in stock over an infinite planning horizon.
(2) The planning horizon is made-up of identical cycle of length T, where $T > 0$.
(3) The inventory level is at its maximum level at the beginning of the cycle.
(4) During a cycle the inventory level is affected by demand and possibly deterioration.
(5) Items experience deterioration once they spent at least some known time $\gamma > 0$ in stock.
(6) Deteriorated items are not repaired while in stock.
(7) Shortages are partially backlogged.
(8) The replenishment rate is infinite.
(9) The demand rate, $D(t)$, at some time $t \geq 0$, is given by

$$D(t) = \begin{cases} \alpha + \beta I(t), & I(t) > 0 \\ \alpha, & I(t) \leq 0 \end{cases}$$

where, the β refers to the stock dependent parameter ($\beta \geq 0$), and α is a parameter such that $\alpha > 0$.

(10) The deterioration rate is denoted by $\theta \geq 0$;
the length of time of nonnegative inventory is denoted by $\tau \geq 0$ ($\tau \leq T$);
the backlogging parameter $\delta > 0$;
the holding cost $c_1 > 0$;
the backlogging cost $c_3 > 0$;
the opportunity cost $c_4 > 0$;
the unit cost $c_2 > 0$;
the set-up cost $A > 0$.

3. The inventory model

The inventory model considered in this paper is made up of identical cycles where a typical cycle starts with the maximum inventory level. The inventory level depletes either due to the effect of stock dependent-demand

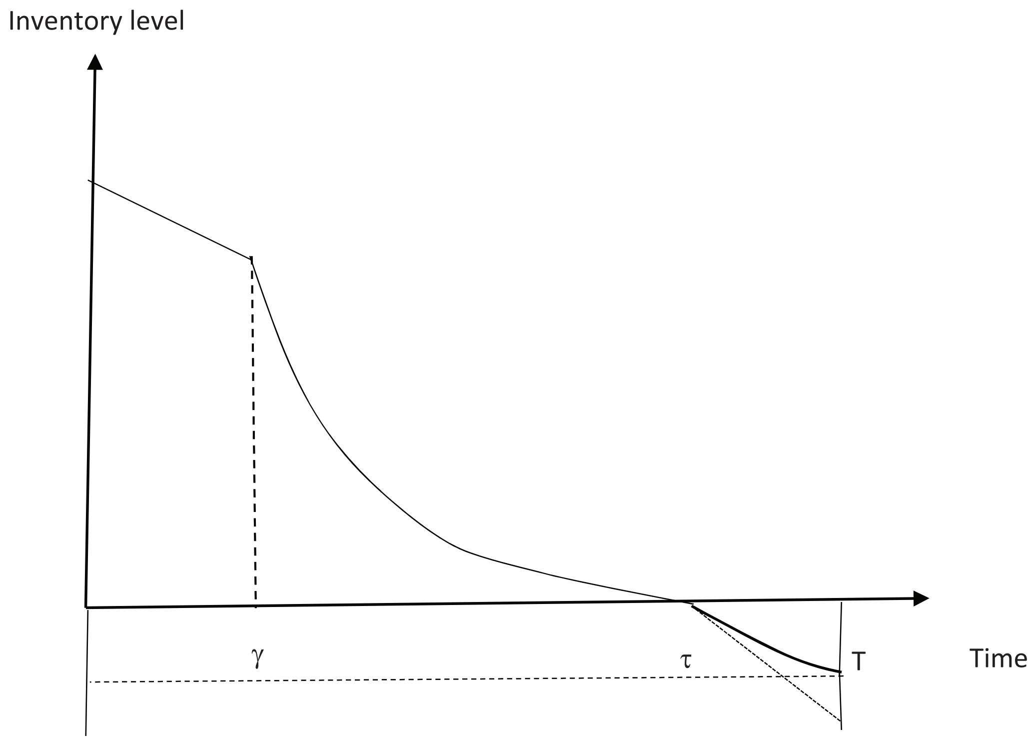

FIGURE 1. The inventory level for during the cycle.

when $\tau \leq \gamma$, or due to the effect of some combination of stock-dependent demand and deterioration when $\tau > \gamma$. Figure 1 depicts a typical changes of the inventory level when $\tau > \gamma$. Shortage occurs on the interval $[\tau, T]$ but only partial backlogged policy is adopted.

Let $I(t)$ represents the inventory level at time t. The change in the inventory level can be described by the following differential equations:

$$I'(t) = -\alpha - \beta I(t), \qquad \text{for} \qquad 0 \leq t \leq \min\{\gamma, \tau\} \tag{3.1}$$

$$I'(t) = -\alpha - (\beta + \theta) I(t), \qquad \text{for} \qquad \min\{\gamma, \tau\} \leq t \leq \tau \tag{3.2}$$

$$I'(t) = -\frac{\alpha}{1 + \delta(T - t)}, \qquad \text{for} \qquad \tau \leq t \leq T \tag{3.3}$$

with $I(\tau) = 0$.

We shall present various pertinent derivations that will be needed later on in this paper for the case $\tau > \gamma$. The case, $\tau \leq \gamma$ can be derived as a special case of the former case.

The solution of (3.1) is:

$$I(t) = \alpha e^{-\beta t} \int_t^\gamma e^{\beta u} \mathrm{d}u + I(\gamma) e^{-\beta(t-\gamma)}, \tag{3.4}$$

where by (3.7) below

$$I(\gamma) = \alpha \int_\gamma^\tau e^{(\beta+\theta)(t-\gamma)} \mathrm{d}t. \tag{3.5}$$

It is easy to see that the amount of inventory on $[0,\gamma]$, $\mathcal{H}(0,\gamma)$, is

$$\begin{aligned}\mathcal{H}(0,\gamma) &:= \frac{\alpha}{\beta}\int_0^\gamma (e^{\beta t}-1)\mathrm{d}t + I(\gamma)\int_0^\gamma e^{-\beta(t-\gamma)}\mathrm{d}t \\ &= \frac{\alpha}{\beta}\int_0^\gamma (e^{\beta t}-1)\mathrm{d}t \\ &\quad + \alpha\left\{\int_0^\gamma e^{-\beta(t-\gamma)}\mathrm{d}t\right\}\left\{\int_\gamma^\tau e^{(\beta+\theta)(t-\gamma)}\mathrm{d}t\right\}, \end{aligned} \tag{3.6}$$

and the solution of (3.2) is:

$$I(t) = \alpha e^{-(\beta+\theta)t}\int_t^\tau e^{(\beta+\theta)u}\mathrm{d}u, \tag{3.7}$$

and the amount of inventory on $[\gamma,\tau]$, $\mathcal{H}(\gamma,\tau)$, is

$$\mathcal{H}(\gamma,\tau) := \frac{\alpha}{\beta+\theta}\int_\gamma^\tau \{e^{(\beta+\theta)(t-\gamma)} - 1\}\mathrm{d}t. \tag{3.8}$$

The solution of (3.3) is:

$$I(t) = \frac{\alpha}{\delta}\log\{1+\delta(T-t)\} + C,$$

where C is a constant such that $I(\tau)=0$. Thus,

$$C = -\frac{\alpha}{\delta}\log\{1+\delta(T-\tau)\}.$$

This yields

$$I(t) = \frac{\alpha}{\delta}\log\left\{\frac{1+\delta(T-t)}{1+\delta(T-\tau)}\right\}. \tag{3.9}$$

Direct computations show that the amount of shortage on $[\tau,T]$, $\mathcal{S}$, is equal to:

$$\mathcal{S} := \frac{\alpha}{\delta}\left[(T-\tau) - \frac{1}{\delta}\log\{1+\delta(T-\tau)\}\right], \tag{3.10}$$

and the amount of lost sales, $\mathcal{L}$, is:

$$\mathcal{L} := \alpha\left[(T-\tau) - \frac{1}{\delta}\log\{1+\delta(T-\tau)\}\right]. \tag{3.11}$$

The amount of deteriorated items is equal to

$$\mathcal{D} := I(\gamma) - \text{amount consumed on } [\gamma,\tau].$$

The amount consumed on this interval is equal to

$$\int_\gamma^\tau \{\alpha + \beta I(t)\}\mathrm{d}t = \alpha(\tau-\gamma) + \frac{\beta\alpha}{\beta+\theta}\int_\gamma^\tau \{e^{(\beta+\theta)(t-\gamma)} - 1\}\mathrm{d}t.$$

Using, (3.5), it follows that

$$\mathcal{D} = \frac{\theta\alpha}{\beta+\theta}\int_\gamma^\tau \{e^{(\beta+\theta)(t-\gamma)} - 1\}\mathrm{d}t. \tag{3.12}$$

4. The optimal inventory policy

Let $R(\tau,T)$ be the total cost per period. This cost is equal to:

$$\begin{aligned}R(\tau,T) &:= \text{set-up cost} + \text{holding cost} + \text{deteriorating cost}\\ &\quad + \text{shortage cost} + \text{opportunity cost}\\ &= A + c_1\{\mathcal{H}(0,\gamma)+\mathcal{H}(\gamma,\tau)\} + c_2\mathcal{D} + c_3\mathcal{S} + c_4\mathcal{L},\end{aligned} \tag{4.1}$$

where $\mathcal{H}$, $\mathcal{D}$, $\mathcal{S}$, and $\mathcal{L}$ are given by ((3.6)–(3.8)), (3.12), (3.10), and (3.11) in that order for the case $\gamma < \tau$. If $\gamma \geq \tau$, then the total cost per period is equal to that in (4.1) with $\theta = 0$.

It follows, by setting $s := \dfrac{c_3}{\delta} + c_4$, that the total cost per period is given by:

$$\begin{aligned}R_0(\tau,T) &:= A + \frac{c_1\alpha}{\beta}\int_0^\tau (e^{\beta t}-1)\mathrm{d}t\\ &\quad + s\alpha\left[(T-\tau) - \frac{1}{\delta}\log\{1+\delta(T-\tau)\}\right],\end{aligned} \tag{4.2}$$

and $\tau \geq \gamma$, the total cost is given by:

$$\begin{aligned}R_\theta(\tau,T) &:= A + \frac{c_1\alpha}{\beta}\int_0^\gamma (e^{\beta t}-1)\mathrm{d}t\\ &\quad + c_1\alpha e^{-\theta\gamma}\left\{\int_0^\gamma e^{-\beta t}\mathrm{d}t\right\}\left\{\int_\gamma^\tau e^{(\beta+\theta)t}\mathrm{d}t\right\}\\ &\quad + \frac{c_2\theta\alpha}{\beta+\theta}\left[\int_\gamma^\tau \left\{e^{(\beta+\theta)(t-\gamma)}-1\right\}\mathrm{d}t\right]\\ &\quad + s\alpha\left[(T-\tau) - \frac{1}{\delta}\ln\{1+\delta(T-\tau)\}\right].\end{aligned} \tag{4.3}$$

It is easy to see that (4.2) and (4.3) coincides when $\theta = 0$. The total cost per unit time is defined as:

$$F(\tau,T) = \frac{R(\tau,T)}{T}, \tag{4.4}$$

where

$$R(\tau,T) = \begin{cases} R_0(\tau,T), & \text{if } \tau < \gamma\\ R_\theta(\tau,T), & \text{if } \tau \geq \gamma\end{cases}.$$

4.1. The case $\theta = 0$

The objective is to determine the minimum of the function

$$F_0(\tau,T) = \frac{R_0(\tau,T)}{T}$$

subject to $T \geq \tau$, and $\tau \geq 0$. Call this problem $\boldsymbol{P}_0$.

For a bivariate function f, let $(\partial_x f)(.,.)$ and $\partial_y f(.,.)$ refer to the partial derivative of f with respect to the first and the second argument respectively. Also, $\partial_x\partial_y f(.,.)$ refers to the cross partial derivatives of f, and $\partial_y^2 f(.,.)$ refers to the second partial derivative with respect to the second argument of f.

Theorem 4.1. *The function of $F_0(\tau,T)$ is differentiable on the set $\Omega = \{(x,y)\in R^2,\ y \geq x > 0\}$. Moreover, $\nabla F_0(\tau,T) = 0$ has a unique solution. This solution minimizes the function $F_0(\tau,T)$ subject to $T \geq \tau \geq 0$.*

Proof. The differentiability of the function F_0 is inferred from that of R_0. Also, $\nabla F_0(\tau, T) = 0$, implies that $(\partial_x F_0)(\tau, T) = 0$, $(\partial_y F_0)(\tau, T) = 0$. It can be shown that $(\partial_x F_0)(\tau, T) = 0$, leads to:

$$\frac{c_1}{\beta s}(e^{\beta\tau} - 1) - \frac{\delta(T-\tau)}{1+\delta(T-\tau)} = 0. \tag{4.5}$$

Write

$$H(\tau) := \frac{c_1}{\beta s}(e^{\beta\tau} - 1), \tag{4.6}$$

to get

$$\frac{\delta(T-\tau)}{1+\delta(T-\tau)} = H(\tau). \tag{4.7}$$

Direct algebra, using (4.6) leads to:

$$T = \tau + \frac{H(\tau)}{\delta\left\{1 - H(\tau)\right\}}. \tag{4.8}$$

Also, note that the left hand-side of (4.7) implies that $0 \leq H(\tau) < 1$. The function H is strictly increasing in τ, with $H(0) = 0$. Therefore there exists a unique root to the equation $H(\tau) = 1$. Denote this root by $\hat{\tau}$. It can be shown that

$$\hat{\tau} = \frac{1}{\beta}\log\left(1 + \frac{\beta s}{c_1}\right). \tag{4.9}$$

The expression (4.8) defines T uniquely as a function of τ. For $\tau \in [0, \hat{\tau})$, write

$$T(\tau) = \tau + \frac{1}{\delta}\frac{H(\tau)}{1 - H(\tau)}. \tag{4.10}$$

Computations show that

$$T'(\tau) = 1 + \frac{1}{\delta}\frac{H'(\tau)}{\left\{1 - H(\tau)\right\}^2},$$

Thus, $T'(\tau) > 1$, since $H'(\tau) = (c_1/s)e^{\beta\tau} > 0$. It follows that T is increasing in τ.

Next, consider the expression $(\partial_y F_0)(\tau, T)$. This is given by:

$$(\partial_y F_0)(\tau, T) = \frac{T(\partial_y R_0)(\tau, T) - R_0(\tau, T)}{T^2}. \tag{4.11}$$

Write $G(\tau)$ as the numerator of the right-hand side of (4.11) and use the fact that T is a function of τ. That is,

$$G(\tau) := T(\tau)(\partial_y R_0)(\tau, T(\tau)) - R_0(\tau, T(\tau)),$$

and

$$G'(\tau) = T(\tau)\left[(\partial_x\partial_y R_0)(\tau, T(\tau)) + (\partial_y^2 R_0)(\tau, T(\tau))T'(\tau)\right], \tag{4.12}$$

since $(\partial_x F_0)(\tau, T(\tau)) = 0$ results in $(\partial_x R_0)(\tau, T(\tau)) = 0$.

The definition of R_0 in (4.2) gives

$$(\partial_y R_0)(\tau, T(\tau)) = \frac{s\alpha\delta(T-\tau)}{1+\delta(T-\tau)}$$

$$(\partial_x\partial_y R_0)(\tau, T(\tau)) = -\frac{s\alpha\delta}{\left\{1+\delta(T-\tau)\right\}^2}$$

$$(\partial_y^2 R_0)(\tau, T(\tau)) = \frac{s\alpha\delta}{\left\{1+\delta(T-\tau)\right\}^2}.$$

Thus,

$$G'(\tau) = \frac{s\alpha\delta}{\{1+\delta(T-\tau)\}^2} T'(\tau)\{T'(\tau)-1\}.$$

Hence, $G'(\tau) > 0$, since $T'(\tau) > 1$.

Note that, $T(0) = 0$, $G(T(0) = -R_0(T(0), 0) = -A < 0$. Moreover, it can be seen from (4.7) and (4.8) that as $\tau \longrightarrow \hat{\tau}^-$, $T(\tau) \longrightarrow \infty$. It can also be shown that $G(T(\tau))$ is asymptotically equivalent to $\frac{s\alpha}{\delta}\log\{1+\delta(T-\tau)\}$ which goes to ∞ as $T(\tau) \longrightarrow \infty$ $(\tau \longrightarrow \hat{\tau}^-)$. It follows by the intermediate value theorem that there exist a unique $\tau^* \in (0, \hat{\tau})$, such that $G(\tau^*) = 0$. Therefore, $(\tau^*, T(\tau^*)) \in \Omega$, and is the unique solution of $\nabla F_0(\tau, T) = 0$. This solution is a minimizer of F_0 since F_0 is unbounded above. This completes the proof. □

Remark 4.2. The optimal value $\tau^*$is such that $\tau^* < \hat{\tau}$, where $\hat{\tau}$ is given by (4.9).

4.1.1. The case $\theta \neq 0$

The objective is to determine the minimum of the function

$$F_\theta(\tau, T) = \frac{R_\theta(\tau, T)}{T} \tag{4.13}$$

subject to $T \geq \tau$, and $\tau \geq \gamma$. Call this problem $\boldsymbol{P}_\theta$.

Theorem 4.3. *The function of $F_\theta(\tau, T)$ is differentiable on the set $\Gamma = \{(x, y) \in R^2,\ y \geq x \geq \gamma\}$. Moreover, $\nabla F_\theta(\tau, T) = 0$ has either (i) a unique solution which corresponds to the minimizer of the function $F_\theta(\tau, T)$ on Γ, else (ii) no solution, and the minimum of $F_\theta(\tau, T)$ on Γ occurs at γ.*

Proof. The differentiability of the function F_θ is inferred from that of R_θ. Also, $\nabla F_\theta(\tau, T) = 0$, implies that $(\partial_x F_\theta)(\tau, T) = 0$, $(\partial_y F_\theta)(\tau, T) = 0$. It can be shown that $(\partial_x F_\theta)(\tau, T) = 0$, leads to:

$$\begin{aligned} c_1\alpha e^{-\theta\gamma}\left\{\int_0^\gamma e^{-\beta t}\mathrm{d}t\right\} e^{(\beta+\theta)\tau} + \frac{(c_1+\theta c_2)\alpha}{\beta+\theta}\left\{e^{(\beta+\theta)(\tau-\gamma)} - 1\right\} \\ + s\alpha\left\{-1 + \frac{1}{1+\delta(T-\tau)}\right\} = 0. \end{aligned} \tag{4.14}$$

Rearranging all terms in (4.14) and after some simplification we get:

$$\begin{aligned} \frac{\delta(T-\tau)}{1+\delta(T-\tau)} = \frac{c_1}{s} e^{-\theta\gamma}\left\{\int_0^\gamma e^{-\beta t}\mathrm{d}t\right\} e^{(\beta+\theta)\tau} \\ + \frac{c_1+\theta c_2}{s(\beta+\theta)}\left\{e^{(\beta+\theta)(\tau-\gamma)} - 1\right\}. \end{aligned} \tag{4.15}$$

Set

$$Q(\tau) := \frac{c_1}{s} e^{-\theta\gamma}\left\{\int_0^\gamma e^{-\beta t}\mathrm{d}t\right\} e^{(\beta+\theta)\tau} + \frac{c_1+\theta c_2}{s(\beta+\theta)}\left\{e^{(\beta+\theta)(\tau-\gamma)} - 1\right\}. \tag{4.16}$$

Note that the left-hand side of (4.15) implies that (a) $0 \leq Q(\tau) < 1$ is required for (4.15) to make sense, (b) $Q(\gamma) = H(\gamma)$ (H is defined in (4.6)), (c) the function Q is strictly increasing in τ on the set Γ, (d) $Q(\tau) \to \infty$, as $\tau \to \infty$.

(i) Assume that $Q(\gamma) < 1$, it follows that there exist a unique $\tilde{\tau} > \gamma$ such that $Q(\tilde{\tau}) = 1$, and $\tilde{\tau} < \hat{\tau}$ (the root of $H(\tau) = 1$: see (4.9)), and $\gamma < \hat{\tau}$.

Simple algebra using (4.15) and (4.16) leads to:

$$T = \tau + \frac{Q(\tau)}{\delta\{1 - Q(\tau)\}} \tag{4.17}$$

For $\tau \in [\gamma, \tilde{\tau})$, the value of T in (4.17) is uniquely defined as a function of τ. Write $T := T(\tau)$ and differentiate to get

$$T'(\tau) = 1 + \frac{1}{\delta} \frac{Q'(\tau)}{\{1 - Q(\tau)\}^2},$$

Thus, T is increasing in τ since $Q'(\tau) > 0$. Also, $T'(\tau) > 1$. Also,

$$(\partial_y F_\theta)(\tau, T) = \frac{T(\partial_y R_\theta)(\tau, T) - R_\theta(\tau, T)}{T}. \tag{4.18}$$

Let $P(\tau)$ be the numerator of the right-hand side of (4.18)

$$P(\tau) := T(\partial_y R_\theta)(\tau, T) - R_\theta(\tau, T)$$

Next, use the fact that T is a function of τ ($T = T(\tau)$) and differentiate to get

$$P'(\tau) = T'(\tau) \left[(\partial_x \partial_y R_\theta)(\tau, T(\tau)) + (\partial_y^2 R_\theta)(\tau, T(\tau)) T'(\tau) \right]. \tag{4.19}$$

Now, use the fact

$$\begin{aligned} (\partial_y R_\theta)(\tau, T(\tau)) &= s\alpha \left[1 - \frac{1}{1 + \delta \{T(\tau) - \tau\}} \right], \\ (\partial_x \partial_y R_\theta)(\tau, T(\tau)) &= \frac{-s\alpha}{[1 + \delta \{T(\tau) - \tau\}]^2}, \\ (\partial_y^2 R_\theta)(\tau, T(\tau)) &= \frac{s\alpha}{[1 + \delta \{T(\tau) - \tau\}]^2}, \end{aligned}$$

to infer that

$$P'(\tau) = \frac{s\alpha}{[1 + \delta \{T(\tau) - \tau\}]^2} T'(\tau) \left[\{-1 + T'(\tau)\} \right].$$

Now, recall that

$$P(\tau) = \frac{s\alpha T(\tau)}{[1 + \delta \{T(\tau) - \tau\}]} - R_\theta(\tau, T(\tau)). \tag{4.20}$$

Also, note that the values of T at γ in (4.8) and (4.17) coincide and that $R_\theta(\gamma, T(\gamma)) = R_0(\gamma, T(\gamma))$. The last equality follows since R is continuous on the line segment $\tau = \gamma$. Therefore, $P(\gamma) = G(\gamma) < 0$. It can also be shown that as $\tau \to \tilde{\tau}^-$, $T \to \infty$, and that asymptotically

$$P(\tau) \approx \frac{s\alpha}{\delta} \log\left[1 + \{T(\tau) - \tau\}\right] \to \infty.$$

Hence, there exist a unique solution $\tau^* \in [0, \tilde{\tau})$ such that $P(\tau^*) = 0$ with $(\tau^*, T(\tau^*)) \in \Gamma$ and solves $\nabla F_0(\tau, T) = 0$. This solution is a minimizer of F_θ since F_θ is unbounded above. This finishes part (i) of the lemma.

(ii) If $H(\gamma) \geq 1$, then the minimizer of $F_\theta(\tau, T)$ on Γ cannot be a critical point. Then, it must be the boundary point with $\tau = \gamma$, and the optimal T is found from minimizing $F_0(\gamma, T)$. This completes the proof of the theorem.

□

4.2. The optimal policy

We shall next consider the problem of minimizing F given by (4.4) subject to $T \geq \tau \geq 0$. Call this problem $\boldsymbol{P}$. The next lemma is related to the differentiability of the function F.

Lemma 4.4. *The function F defined in (4.4) is differentiable on the set $\Omega = \{(x, y), y \geq x \geq 0, y \neq 0\}$.*

Proof. The differentiability of the function F on Ω except on the line segment $x = \gamma$ is clear. Direct computations show that

$$\begin{aligned}(\partial_x R_0)(\gamma^-, T) &= \frac{c_1\alpha}{\beta}(e^{\beta\tau} - 1) - \frac{s\alpha T(\tau)}{1 + \delta\{T(\tau) - \tau\}} \\ &= (\partial_x R_\theta)(\gamma^+, T),\end{aligned}$$

and

$$\begin{aligned}(\partial_y R_0)(\gamma^-, T) &= \frac{s\alpha T(\tau)}{1 + \delta\{T(\tau) - \tau\}} \\ &= (\partial_y R_\theta)(\gamma^+, T),\end{aligned}$$

which leads to the required result. □

The next theorem provides a simple way of identifying the optimal solution of F on Ω.

Theorem 4.5. *The optimal inventory policy occurs at the critical point of the function F. If $(\tau^*, T(\tau^*))$ is the optimal solution of $\boldsymbol{P}_0$ and $\tau^* \leq \gamma$, then $(\tau^*, T(\tau^*))$ is the unique solution for Problem $\boldsymbol{P}$. Else, the minimum of Problem $\boldsymbol{P}$ occurs at the critical point of P_θ.*

Proof. The result is immediate from Lemma 4.4; Theorems 4.1, 4.3, and by noting that for $T \geq \tau \geq \gamma$, $F_0(\tau, T) \leq F_\theta(\tau, T)$ with equality only when $\tau = \gamma$. □

The results of the previous analysis suggest the following simple procedure for identifying the optimal inventory policy.

For a given γ, find $(\tau^*, T(\tau^*))$ (the optimal solution of $\boldsymbol{P}_0$). (a) The optimal policy is given by $(\tau^*, T(\tau^*))$ if $\tau^* \leq \gamma$, or (c) The optimal policy (τ, T) is the unique root of the nonlinear equations $\nabla F_\theta(\tau, T) = 0$.

Remark 4.6.

(1) Finally, note that the proofs of Theorems 4.1 and 4.3 contain a univariate procedure for determining the solution of $\nabla F_\theta(\tau, T) = 0$, for $\theta = 0$, and $\theta \neq 0$ (when it exists). Furthermore, if $H(\tau) \geq 1$, then $\tau^* \leq \gamma$.

(2) Note that the paper [6] contains a test, based on the sign of $P(\gamma)$ in (4.20), for the existence of the critical point for problem P_θ. The test appears to be correct but the proof of the validity of the test is based on the assumption that the optimal $\tau \leq \gamma$. An assumption that cannot be made a priori. Moreover, Example 5.2 in [6] (this will be repeated below) suggest a procedure put forward cannot always lead to the optimal inventory policy.

(3) Note that paper [6] considered only the problem of minimizing F_θ, defined in (4.13), as treated in Theorem 2. They considered the sign of the expression

$$\begin{aligned}\Delta := c_1\frac{\delta\gamma - 1}{\delta}(e^{\beta\gamma} - 1) - \frac{s}{\delta}\ln\left\{1 - \frac{c_1}{\beta s}(e^{\beta\gamma} - 1)\right\} \\ - \frac{A}{\alpha} - \frac{c_1}{\beta^2}(e^{\beta\gamma} - \beta\gamma - 1).\end{aligned}$$

This expression can be shown to be equivalent to considering the sign of $P(\gamma)$ defined in (4.20). Theorem 2 in [6] states that if $\Delta \leq 0$, then the minimizer of F_θ in Γ is a critical point. Else if $\Delta > 0$, the optimal solution occurs at γ. It is not difficult to construct examples where Δ is not even defined. This occurs when $1 - \frac{c_1}{\beta s}(e^{\beta\gamma} - 1) \leq 0$. See Example 5.3 below.

5. Numerical examples

This section contains three examples with a view to show the applicability of the procedure suggested in this paper. The first two examples are found in [6].

Example 5.1. Let $A = 250$, $c_1 = 0.5$, $c_2 = 1.5$, $c_3 = 2.5$, $c_4 = 2$, $\delta = 2$, $\alpha = 2$, $\beta = 0.1$, $\theta = 0.08$, and $\gamma = 1/12 = 0.0833$. The optimal inventory policy of model P_0 has $\tau^* = 3.30773 > \gamma$. Therefore, the optimal inventory policy is the minimizer of P_θ. The solution is $(1.03338, 1.16866)$ which agrees with that given in [6].

Example 5.2. Let $A = 50$, $c_1 = 0.5$, $c_2 = 1.5$, $c_3 = 2.5$, $c_4 = 2$, $\delta = 2$, $\alpha = 1000$, $\beta = 0.1$, $\theta = 0.08$, and $\gamma = 6/12 = 0.5$. The optimal inventory policy of model P_0 has $\tau^* = 0.423954 < \gamma$. Therefore, the optimal inventory policy is the minimizer of P_0. The solution is $(0.423954, 0.459645)$ with objective function 216.535. Moreover, the optimal solution for P_θ is $(0.5, 0.53619)$ with objective value 219.356, which does not agree with that given in [6]. It is also clear from this example that as long as $\gamma > 0.423954$, the optimal inventory policy remains unchanged.

Example 5.3. Let $A = 250$, $c_1 = 0.5$, $c_2 = 1.5$, $c_3 = 2.5$, $c_4 = 2$, $\delta = 2$, $\alpha = 2$, $\beta = 0.1$, $\theta = 0.08$, and $\gamma = 1$. Here, the test in [6] is inconclusive as Δ is not defined. However, the procedure suggested in this paper is able to find the optimal inventory policy. The optimal solution of P_0 has $\tau^* = 0.21879 < \gamma$. Therefore, the optimal inventory policy is the minimizer of P_0. The solution is $(0.21879, 2688.59)$ with objective function 0.00244882.

6. Conclusion

The present paper revisited the classical (EOQ) inventory model for noninstantaneous deteriorating items proposed in [6]. A simple procedure for determining the optimal inventory policy which minimizes the total cost per unit time is suggested. This procedure exploited the fact that the objective function of the problem is differentiable on its domain of definition. It was shown that deterioration affects the optimal policy only if the time of the onset of deterioration, γ, is less than the optimal time of the start of partial backlogging when deterioration is absent. Otherwise, the optimal policy corresponds to the (EOQ) of the inventory model with no deterioration. In other words, for relatively large γ, the deterioration cost has no effect on the optimal policy. Moreover, it was observed that the analysis presented in [6] appears to assume that $\gamma < \tau$. An assumption that cannot be made a priori as it is easy to construct examples where the assumption is violated. Example 5.3 shows this possibility.

Finally, it is conjectured that the basic idea of exploiting the fact that the objective function of the present problem is differentiable to determine the optimal inventory can be extended to inventory models with permissible delay in payment, pricing and possibly more.

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