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OPTIMIZING VENDOR-MANAGED INVENTORY SYSTEMS WITH LIMITED STORAGE CAPACITY AND PARTIAL BACKORDERING UNDER STOCHASTIC DEMAND

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Abstract. A supply chain member's coordination is a challenging issue and a key factor of success in business markets. In vendor-managed inventory systems, the vendor makes replenishment decisions at the site of buyer by which the supply chain can be coordinated more efficiently. Two integrated vendor managed inventory systems under continuous review and periodic review replenishment policies are developed considering partial backordering and limited storage capacity at the buyer's side. Furthermore, traditional retailer managed inventory systems under the same settings are developed to compare against the integrated systems. Efficient algorithms are presented to derive the optimal values of decision variables. Finally, numerical experiments and comprehensive sensitivity analysis are used to show the applicability and efficiency of the proposed VMI systems.

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1. Introduction

Proper managing of inventory systems can result in reducing total cost of supply chains. Several partners such as retailers and vendors are involved in supply chains, in which they need to be coordinated with each other to gain more profit. Thus, coordination is a challenging issue in supply chain especially in inventory management systems under which enhancing cost-efficiency of chains can be obtained [48]. Coordinating supply chains may be achieved by using information technology (IT) and electronic data interchange (EDI), because vendors can receive point-of-sale data [39]. Therefore, it is vital to manage supply chains properly to enhance competitive advantages of commercial supply chains via integrating vendors and retailers. As a result of integration of these parts of supply chains, vendor is able to determine the most appropriate order quantity, which should be produced, and deliver them to retailers [38].

Two common concepts exist in literature to manage inventory systems: traditional retailed inventory (RMI) systems and vendor-managed inventory (VMI). In RMI systems, retailers make decision about their own replenishments variables. These decisions not only increase the total cost of a chain, but also reduce chain performances. For overcoming these difficulties, integrating supply chains has been a topical issue among researchers and practitioners. VMI is widely known as coordinating program among suppliers and buyers (*i.e.* retailers). In VMIs

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systems the vendor makes centralized replenishment decisions, such as deciding on timing and quantities [17] for buyers in an integrated viewpoint by which a supply chain can be coordinated [16, 46]. In VMI systems, the vendor is responsible to place orders for buyers and to ship items [47]. In this light, enhancing service levels, ordering cost reduction, brand improvements, more exact sales predicting, and precise production planning are some well-known advantages of VMI systems. Companies such as Wal-Mart, Intel, and Shell Chemical have employed VMI systems because of its benefits [8].

In VMI, the vendor is likely to ship more quantities to buyers as a result of employing centralized decisions. In a same way, storage capacity at the site of buyers is not unlimited. Thus, the vendor is restricted from shipping large quantities and he should keep inventory levels below a pre-agreed storage capacity [17, 40]. Glock [26] stated that it is essential to develop integrated inventory models and VMI systems which are applicable in real-life problems. A significant point for developing inventory systems is considering partial backordering (PBO) situation. Under PBO, an inventory system may encounter with backorders and lost-sales simultaneously, which is more applicable in reality than restricting a system with one of the backorders or lost-sales cases. To respond to such practices, it is vital to develop VMI systems with considering of limited storage capacity (LSC) at the site of buyer while PBO is permitted.

Benefits of employing VMI systems in comparison with traditional RMI systems are discussed above. There are many successful paradigms resulting from the adoption of the VMI strategies as mentioned before. Contributing in VMI systems with developing more analytical models, which are closed to real world can be significant point for commercial supply chains [13, 26, 51].

Two main well-known realistic assumptions for inventory systems are LSC and PBO. LSC is a valid assumption, as well as PBO, which have not been involved in many previous studies in which VMI systems are developed. In this paper, a single-vendor single-buyer problem is studied to integrate the production-inventory systems under VMI policy. The significant contribution of this paper is presenting two novel VMI systems with both continuous replenishment review system (r, Q) and periodic replenishment review system (R, T) considering LSC and PBO simultaneously. Also, two RMI systems in the same settings are presented in order to make comparison between effects of implementing the proposed VMI systems and traditional RMI systems. Theorems are established to solve the problems and necessary propositions are defined and proved by which the conditions for convexities are achieved. The closed-form optimal solutions of decision variables are obtained in both (r, Q) and (R, T) review systems. Furthermore, efficient iterative algorithms are presented to determine optimal values of decision variables with low computation complexities. The proposed algorithms can be implemented with commercial software such as MATLAB. In brief, main contributions of this paper can be summarized as developing VMI systems under PBO and LSC; Formulating the problem under (r, Q) and (R, T) replenishment review systems; Obtaining closed-form of optimal decision variables with proved convexities; Proposing iterative algorithms for solving the proposed VMI systems under PBO and LSC; Developing counterpart RMI systems to make comparison between RMIs and VMIs.

Remainder of this paper is structured as follows. Literature review is provided in Section 2. Developed models are presented in Section 3. The proposed VMI problems under (r, Q) and (R, T) as two replenishment review systems are presented in Sections 3.1 and 3.2, respectively. Their RMI counterparts are presented in Sections 3.3 and 3.4. Comprehensive numerical experiments and sensitivity analysis are provided in Section 4. Managerial implications are provided in Section 5 and conclusion and future works are performed in Section 6.

2. Literature review

As mentioned before, VMI is able to enhance the performance of a chain. For instance, Dong and Xu [18] discussed applicability and profitability of VMI systems in long and short-term periods. Mishra and Raghunathan [43] represented that VMI systems are capable of increasing competition among companies and manufacturers. Claassen *et al.* [15] studied impact of VMI systems on supply chain performances. They stated improving customer's satisfaction level, controlling a whole supply chain, and cost reduction are advantages of VMI systems. Yao *et al.* [54] examined VMI systems and expressed that replenishment quantities would be typically reduced in a chain after implementing VMI.

Several related works exist in the literature aiming to integrate the whole chain. Cetinkaya and Lee [12] studied on VMI systems to coordinate transportation scheduling and inventory management decisions and [3] proposed an optimization solution procedure for solving the model presented by Cetinkaya and Lee [12]. Fry *et al.* [22] presented a single-vendor single-buyer problem in which vendor should be penalized if buyer's inventory level exceeds from a pre-agreed range. Ouyang *et al.* [45] developed the previous work when shortage is permitted. They presented VMI systems for stochastic and deterministic lead-times. Choi *et al.* [14] developed an inventory model under VMI in order to determine the vendor service levels. Jaruphongsa *et al.* [35] presented a lot-sizing inventory problem while considering time windows to meet demands. Jaruphongsa *et al.* [36] extended their previous work in which warehouses encounter limited storage capacities. Ben-Daya and Hariga [7] studied a single-vendor single-buyer problem where demand through the lead-time is stochastic. Also, they considered that lead-time might have been varied linearly with lot-size. Obviously, assuming demands as stochastic variables are more realistic than considering them as deterministic parameters. Hsiao [32] extended the previous work with allowing two reorder points and service levels. Ben-Daya and Hariga [7] and Hsiao [32] developed their mathematical models under continuous review system (r, Q). Afterwards, Yedes *et al.* [55] reformulated the previous works for a lot-for-lot problem and made a comparison among the achieved results and previous researchers' results. Zhou and Wang [58] proposed an integrated production-inventory system with single-vendor single-buyer for deteriorating goods with shortages. Cardenas-Barron *et al.* [11] presented an algorithm to solve a multi-constraint and multi-product VMI system under economic order quantity. It is noteworthy to mention that in vendor-managed consignment inventory, the buyer does not charge the vendor or pay for delivered goods until they are sold or used [20, 29]. Bazan *et al.* [6] developed a consignment stock under VMI when vendor might produce defective items while these items may need to be scrapped or reworked. Omar and Supadi [44] developed a single-vendor single-buyer problem when the demand rate would be decreased linearly during finite time horizon. Also, Glock [26] reviewed works published in joint economic lot sizing (JELS) concepts and stated that inventory models need to be developed practically to be consistent with real world applications. For instance, entering partial back-ordering and limited storage capacities to the problems are more realistic than assuming infinite volume of warehouses. Thus, to be more consistent with real world applications, Mandal and Khan [42] developed a joint economic lot-sizing model under parameter uncertainties because parameters might be imprecise in real world. Afterwards, Sadjadi *et al.* [49] proposed other joint economic lot-sizing model while addressing budget limitations and stated that budget limitation should be considered as a real life problem. Masud *et al.* [1] proposed an efficient solution algorithm to determine optimal decisions for an integrated reliable production-inventory model. Braglia *et al.* [4] developed a VMI with considering consignment agreement and introducing physical space occupation at the buyer as a capacity limitation. Bylka and Górny [9] presented a CS for integrating a two-level supply chain when lot sizes can be equal or non-equal. Jaggi *et al.* [34] developed a CS for integrating single-vendor single-retailer problem in which backorders are allowed and payment delays are permitted for buyer. Khan *et al.* [39] proposed a CS agreement for single-vendor and single-buyer problem where defective items might be produced. Afterwards, Giri *et al.* [24] extended Khan *et al.* [39]'s work for unequal-sized deliveries. Another significant study is presented by Cai *et al.* [10] for multi-vendor and single-buyer where substitutable brands are investigated under VMI using game theory. Notably, limited storage capacity for buyer is not considered in their problem. Zahran and Jaber [57] proposed a CS contract for multi-vendor and multi-buyer when upstream can suggest using delay-in-payments to downstream supply chain partners. They proposed four coordination scenarios with combination of consignment and traditional inventory policies. Also, backorders and limited storage capacity are not considered in their study. Kaasgari *et al.* [38] presented a VMI with considering discount and perishable products for the problem of single-vendor and multi-retailers. They devised two meta-heuristic algorithms to solve their proposed model. Verma and Chatterjee [52] developed a problem of single-vendor and multi-retailers when the replenishment quantity for retailers is limited by an upper bound. They presented a deterministic model and proposed a heuristic algorithm to solve the model.

Jauhari *et al.* [37] investigated on a single-vendor single-buyer problem with considering freight rate discount under stochastic demand but buyer's space limitation is not considered in their study. Hemmati *et al.* [30] proposed a CS with considering price- and stock- dependent demand with a deteriorating item for the problem of single-vendor single-buyer. Braglia *et al.* [5] proposed an inventory replenishment policy for an integrated supply chain including single buyer and single supplier when backorders and lost sales are permitted under stochastic demand. Islam and Hoque [33] developed a single buyer single vendor in an integrated supply chain under consignment policy and considered numerous factors which are applicable in real world such as capacity constraint, work-in-process inventory, shipping time, and etc. Also, interested readers may refer to Sarker [50] in which consignment policy works are reviewed and a comparative perspective is made.

Considering LSC at the buyers' sides is more realistic than considering existing unlimited capacity in different types of inventory models. For instance, Hoque and Goyal [31] presented an integrated model for single-vendor single-buyer problem in which buyer encounters capacity constraint of the transport equipment used to transfer batches from the vendor to the buyer. Dye *et al.* [19] developed a deterministic inventory model with capacity constraint. Also, Feng *et al.* [21] developed an economic lot-sizing problem with stochastic demand and space limitation for buyer. Liu [41], and Giri and Moon [25] proposed economic lot-sizing problem with deterministic demand and capacity constraints for the buyer. It should be noted that Liu [41] deals with time-varying inventory capacity, while Giri and Moon [25] with static. Almehdawe and Mantin [2] exploited Stackelberg game to coordinate single-vendor and multi-buyer when vendor encounters capacity constraint. Hariga [27] proposed a stochastic inventory model in which buyer has limited space to store items. In another similar work, this model is extended when vendor is encountered capacity constraint [28]. Giri *et al.* [23] presented a JELS model with single-vendor single-buyer in which storage capacity at the vendor's side and deterministic demand are considered. These mentioned works are not under VMI and CS policies, but there are more VMI problems in which space limitation is considered. For instance, Jaruphongsa *et al.* [36] has presented a two-echelon lot-sizing model with considering limited storages for warehouses. Darwish and Odah [17] considered an upper bound for buyer's inventory level for their VMI problem. Hariga *et al.* [29] presented a VMI based on economic order quantity (EOQ) model when shipment frequencies can be unequal. Yi and Sarker [56] proposed a CS-VMI system for coordinating single-vendor and single-buyer with considering space limitation at buyer's side and permitting backorders. They assumed demand follows a normal probability distribution, but they could not obtain closed-form of decision variables and tried to solve their model with meta-heuristic algorithms. In addition, their work lacks of proving convexity conditions of objective function. Also, Lee *et al.* [40] presented VMI inventory systems using an EOQ where LSC at the buyer's side limits the vendor to ship large quantities. Also, they assumed shortages are allowed and the vendor would be penalized when the buyer encountered stockouts.

3. Models development

VMI systems would be applied to inventory management systems to coordinate between suppliers and buyers (*i.e.* retailers). For this purpose, centralized decisions will be made by vendor assuming the vendor have access to the necessary information of buyer such as replenishment decision parameters. The problem is to formulate a single-vendor and single-buyer VMI system under continuous review system (r, Q) and periodic review (R, T), which are well-known as two popular replenishment policies, with permitting PBO and considering LSC at the buyer's side. These two main assumptions are not considered in most of the developed inventory systems (*i.e.* RMI and VMI) which are reviewed in Section 2. In the point of applicability, considering PBO and LSC are more practical. As mentioned before, PBO means lost-sales and backorders are permitted in an inventory management system and a certain ratio of orders encounters backorders and others encounter lost-sales. Also, it is vital to consider the storage capacity volume at the buyer's side that needs to hold goods, because an obtained solution might be infeasible for the buyer due to replenishment exceeding storage capacity.

Following assumptions are used to develop VMI and RMI systems under (r, Q) and (R, T) review systems. It is noteworthy to mention that the RMIs are developed to make comparison between the applicability of VMI and RMI systems when PBO is allowed and storage capacity is limited.

Assumptions

- Single-vendor, single-buyer and single-item situation is considered.
- Vendor is aware of buyer's replenishment decision parameters (*i.e.* inventory level, demand, etc.).
- Production is permitted at vendor's side with the finite rate of P which is greater than buyer demand ($P > D$).
- One machine is available to vendor for producing the orders' items.
- Buyer's procurement lead-time varies linearly with lot sizes and shipping time.
- (r, Q) and (R, T) replenishment policies can be used to review the inventory levels.
- Demand follows from normal distribution.
- Storage capacity for buyer is limited.
- Backorders and lost-sales are permitted for the buyer (PBO is permitted).

Parameters

D	Demand rate of buyer (units/year)
P	Production rate of vendor where $P > D$ (units/year)
A_s	Fixed ordering cost of vendor ($)
A_B	Fixed ordering cost of buyer ($)
h_s	Unit inventory holding cost of vendor ($/units)
h_B	Unit inventory holding cost of buyer ($/units)
α	Fraction of the demand which will be backordered ($0 \leq \alpha \leq 1$)
π_1	Unit backordering cost of buyer ($/units)
π_2	Unit lost-sale cost of buyer ($/units)
b	Shipping time from vendor to buyer
$\bar{b}$	Average backlogged quantity of buyer (units)
SS	Safety stock of buyer (units)
μ	Demand during lead-time for buyer (units)
σ	Standard deviation of demand through lead-time for buyer (units)
σ_L	Standard deviation of demand through lead-time (units)
k_p	Inverse of service level (p) in normal distribution
$\text{Gu}_(k_p)$	Right hand unit of normal linear-loss integral
$l(Q)$	Buyer's procurement lead-time under $(r, Q)(l(Q) = Q/P + b)$
$l(\text{DT}) + T$	Buyer's procurement lead-time under $(R, T)(l(\text{DT}) + T = \text{DT}/P + T + b)$
A	Percentages of orders encountered with backorders
M	Storage capacity of buyer (Units)
$\overline{F}(x)$	complementary cumulative distribution of x
KB	Total cost of buyer in traditional inventory system ($)
KS	Total cost of vendor in traditional inventory system ($)
TC	Total cost of chain ($)

Decision variables

R	Maximum level of inventory under (R, T) policy (units)
r	Reorder point in (r, Q) policy (units)
Q_1	Quantity of items shipped from vendor to buyer under VMI and (r, Q) when LSC is not activated (units)
$\overline{Q}_1$	Quantity of items shipped from vendor to buyer under VMI and (r, Q) when LSC is activated (units)
Q_2	Quantity of items shipped from vendor to buyer under RMI and (r, Q) when LSC is not activated (units)
$\overline{Q}_2$	Quantity of items shipped from vendor to buyer under RMI and (r, Q) when LSC is activated (units)
T_1	cycle time under VMI and (R, T) when LSC is not activated
$\overline{T}_1$	cycle time under VMI and (R, T) when LSC is activated
T_2	cycle time under RMI and (R, T) when LSC is not activated
$\overline{T}_2$	cycle time under RMI and (R, T) when LSC is activated
T	Time-period or cycle time

3.1. VMI under (r, Q) with PBO and LSC

In VMI systems, the vendor is responsible to make replenishment decisions of buyer and all of the costs (such as ordering cost, holding costs, backordering costs, and lost-sale costs) will be carried to the vendor. Therefore, we have $\mathrm{KB}_{\mathrm{VMI}}^{r,Q} = 0$ meaning that total cost of the buyer is negligible in VMI systems [46, 54]. Additionally, total cost of chain is $\mathrm{TC}_{\mathrm{VMI}}^{r,Q} = \mathrm{KB}_{\mathrm{VMI}}^{r,Q} + \mathrm{KS}_{\mathrm{VMI}}^{r,Q} = \mathrm{KS}_{\mathrm{VMI}}^{r,Q}$, in which $\mathrm{KS}_{\mathrm{VMI}}^{r,Q}$ is the total cost of the vendor. Let denote the cost of chain under backorders and lost-sales with $\mathrm{TC}_{\mathrm{VMIBO}}^{r,Q}$ and $\mathrm{TC}_{\mathrm{VMILS}}^{r,Q}$, respectively.

$$\mathrm{TC}_{\mathrm{VMIBO}}^{r,Q} = \overbrace{\frac{D(A_B + A_S)}{Q}}^{\text{Fixed ordering cost}} + \overbrace{h_B\left(\frac{Q}{2} + \mathrm{SS}\right)}^{\text{Buyer's holding cost in BO}} + \overbrace{h_S\frac{Q}{2}\left(1 - \frac{D}{P}\right)}^{\text{Vendor's holding cost}} + \overbrace{\pi_1\frac{D}{Q}\bar{b}(r, l(Q))}^{\text{Backordering cost}} \tag{3.1}$$

$$\mathrm{TC}_{\mathrm{VMILS}}^{r,Q} = \overbrace{\frac{D(A_B + A_S)}{Q}}^{\text{Fixed ordering cost}} + \overbrace{h_B\left(\frac{Q}{2} + \mathrm{SS} + \bar{b}(r, l(Q))\right)}^{\text{Buyer's holding cost in lost-sales}} + \overbrace{h_S\frac{Q}{2}\left(1 - \frac{D}{P}\right)}^{\text{Vendor's holding cost}} + \overbrace{\pi_2\frac{D}{Q}\bar{b}(r, l(Q))}^{\text{lost-sale cost}}. \tag{3.2}$$

The first terms of equations (3.1) and (3.2) represent the fixed ordering costs associating to buyer and vendor. The second terms denote holding costs of buyer containing costs relating to the average quantity of orders and safety stocks. It is well-known that the amount of lost-sale will be involved when calculating the buyer's holding cost (Eq. (3.2)). The third terms are vendors holding costs and the fourth terms are backordered costs and lost-sale cost in both equations (3.1) and (3.2), respectively.

As $1 - \alpha$ percentage of orders encounter lost sales, total cost of chain under PBO can be calculated as $\mathrm{TC}_{\mathrm{VMI}}^{r,Q} = \alpha\mathrm{TC}_{\mathrm{VMIBO}}^{r,Q} + (1-\alpha)\mathrm{TC}_{\mathrm{VMILS}}^{r,Q}$ under (r, Q) policy. Thus, we obtain total cost of chain under VMI and PBO using equation (3.3) as below.

$$\begin{aligned}\mathrm{TC}_{\mathrm{VMI}}^{r,Q} = {} & \frac{D(A_B + A_S)}{Q} + h_B\left(\frac{Q}{2} + \mathrm{SS}\right) + (1-\alpha)h_B\bar{b}(r, l(Q)) + h_S\frac{Q}{2}\left(1 - \frac{D}{P}\right) + \alpha\pi_1\frac{D}{Q}\bar{b}(r, l(Q)) \\ & + (1-\alpha)\pi_2\frac{D}{Q}\bar{b}(r, l(Q)).\end{aligned} \tag{3.3}$$

Demand through the lead-time is distributed normally $x \sim N\left(Dl\left(Q\right), \sigma^2 l\left(Q\right)\right)$, where $l\left(Q\right)$ is the buyer's procurement lead-time and is equal to $\frac{Q}{P} + b$ which varies linearly with lot-sizes plus shipping time. The parameter b would be inspired by delay times of shipping goods from vendor to buyer because of moving, inspections, and etc.

It is known that $\bar{b}\left(r\right) = \sigma\sqrt{Q/P + b}\,\mathrm{Gu}(k_p)$ (see Appendix A for proof). Thus, total cost of chain can be reformulated as presented in equation (3.4):

$$\begin{aligned}\mathrm{TC}_{\mathrm{VMI}}^{r,Q} = {} & \frac{D(A_B + A_S)}{Q} + h_B\frac{Q}{2} + h_B\mathrm{SS} + (1-\alpha)h_B\sigma\sqrt{Q/P + b}\,\mathrm{Gu}(k_p) + h_S\frac{Q}{2}\left(1 - \frac{D}{P}\right) \\ & + \left(\alpha\pi_1 + (1-\alpha)\pi_2\right)\frac{D}{Q}\sigma\sqrt{Q/P + b}\,\mathrm{Gu}(k_p).\end{aligned} \tag{3.4}$$

Equation (3.4) shows the total cost of chain in which α precent of shortages would be back-ordered and $1 - \alpha$ precent would be lost-sale. Hence, we formulate the PBO for the considered problem, but LSC is not involved in the formulations yet. As mentioned before, the buyer has a LSC, which the vendor should consider this point when places orders. Therefore, storage capacity of buyer must be less than its capacity (M). The main problem is as follows:

$$\begin{aligned} & \min\ \mathrm{TC}_{\mathrm{VMI}}^{r,Q} \\ & \mathrm{st}: Q + \mathrm{SS} \le M \text{ or } Q + r - \mu_L \le M. \end{aligned} \tag{3.5}$$

For solving the above mathematical model, first we calculate the value of optimal decision variables without considering the constraint. If the solution satisfies the constraint $(Q + r - \mu_L \le M)$, the optimal values for decision variables are obtained. Otherwise, the solution procedure will continue considering the constraint.

Proposition 3.1. $\mathrm{TC}_{\mathrm{VMI}}^{r,Q}$ *is continues, strictly increasing, and concave in* r *and* Q *when* $\beta+\eta>\gamma$ *where* $\beta=8DPA_B\left(bP+Q\right)\sqrt{b+\frac{Q}{P}}+8DPA_S\left(bP+Q\right)\sqrt{b+\frac{Q}{P}},\gamma=Q^3h_B\sigma\left(\left(1-\alpha\right)\mathrm{Gu}(k_p)+k_p\right)$*, and* $\eta=DGu(k_p)\sigma\left(\alpha\pi_1+\pi_2\left(1-\alpha\right)\right)\left(8b^2P^2+12bPQ+3Q^2\right)$.

Proof of Proposition 3.1. See Appendix B. □

It should be noted that if a problem does not satisfy conditions of Proposition 3.1, decision makers need to utilize other optimization techniques such as using optimization software and solving the problem with different solvers.

For solving the model as a non-constrained model we need the partial derivative with respect to r and Q as shown in equations (3.6) and (3.8). Note that $\frac{\partial\bar{b}(r,l(Q))}{\partial r}=-\overline{F}(r)$ ([46], Appendix B)

$$\frac{\partial\mathrm{TC}_{\mathrm{VMI}}^{r,Q}}{\partial r}=h_B+(\alpha\pi_1+(1-\alpha)\pi_2)\,\frac{D}{Q}\left[-\overline{F}(r)\right]+(1-\alpha)h_B\left[-\overline{F}(r)\right]=0. \tag{3.6}$$

Thus the reorder point (r) from its complementary cumulative distribution for the (r,Q) policy can be calculated using equation (3.7):

$$\overline{F}(r)=\frac{h_BQ}{\left(\alpha\pi_1+(1-\alpha)\pi_2\right)D+(1-\alpha)h_BQ}. \tag{3.7}$$

Totally, to calculate the optimal value of decision variable r partial derivative of cost function with respect to r is used and it is equalled to zero to optimize the function. Therefore, we obtained equation (3.6). Also, according to Appendix B of [46] we know $\frac{\partial\bar{b}(r,l(Q))}{\partial r}=-\overline{F}(r)$. After some transformations, we obtained equation (3.7) as the complementary cumulative distribution. Thus, using $\overline{F}(r)$ we can obtain r as explained in algorithms. The safety stock can be calculated with $\mathrm{SS}=k_p\sigma_L$, where k_p is inverse of service level and σ_L is standard deviation of demand through lead-time which can be calculated using $\sigma_L=\sigma\sqrt{Q/P+b}$. Therefore, $\mathrm{SS}=k_p\sigma\sqrt{Q/P+b}$ is replaced in total cost wherever is needed. Consequently, for deriving Q we have:

$$\begin{aligned}\frac{\partial\mathrm{TC}_{\mathrm{VMI}}^{r,Q}}{\partial Q}=&-\frac{D(A_B+A_S)}{Q^2}+\frac{h_B}{2}+\frac{h_Bk_p\sigma}{2P\sqrt{Q/P+b}}+\frac{h_S}{2}\left(1-\frac{D}{P}\right)+\frac{(1-\alpha)h_B\sigma\mathrm{Gu}(k_p)}{2P\sqrt{Q/P+b}}\\&+(\alpha\pi_1+(1-\alpha)\pi_2)\left[\frac{D\sigma\mathrm{Gu}(k_p)}{2PQ\sqrt{Q/P+b}}-\frac{D}{Q^2}\times\sigma\sqrt{Q/P+b}\mathrm{Gu}(k_p)\right]=0.\end{aligned} \tag{3.8}$$

Thus, the order quantity (Q) can be calculated using equation (3.9):

$$Q_1=\sqrt{\frac{2D(A_B+A_S)+2D\left(\alpha\pi_1+(1-\alpha)\pi_2\right)\sigma\sqrt{Q/P+b}\mathrm{Gu}(k_p)}{h_B+\frac{h_Bk_p\sigma}{P\sqrt{Q/P+b}}+h_S(1-\frac{D}{P})+\frac{(1-\alpha)h_B\sigma\mathrm{Gu}(k_p)}{P\sqrt{Q/P+b}}+(\alpha\pi_1+(1-\alpha)\pi_2)\frac{D\sigma\mathrm{Gu}(k_p)}{PQ\sqrt{Q/P+b}}}}. \tag{3.9}$$

Since Q is a function of an initial Q, an efficient algorithm should be developed to determine the optimal value of Q. Initial value of order quantity (Q_0) can be obtained from equation (3.10) with assuming that backorders and lost-sales are not permitted (PBO is not allowed) in cost functions ($\pi_1=\pi_2=0$ and $\bar{b}(r,l(Q))=0$).

$$Q_0=\sqrt{\frac{D(A_B+A_S)}{\frac{h_B}{2}+\frac{h_S}{2}\left(1-\frac{D}{P}\right)}}. \tag{3.10}$$

Theorem 3.2. (a) *If* $Q_1+r-\mu_L\le M$*, then* r *and* Q_1 *are optimal for* VMI *system. Hence* $Q^*_{\mathrm{VMI}}=Q_1$ *and* $r^*_{\mathrm{VMI}}=r$.

(b) *If* $Q_1+r-\mu_L>M$ *then* Q_1 *will be assumed as initial order quantity and will be updated by equation (3.9) and denoted by* $\overline{Q}_1$*. Therefore,* $Q^*_{\mathrm{VMI}}=\overline{Q}_1$ *and* $r^*_{\mathrm{VMI}}=M+\mu_L-Q^*_{\mathrm{VMI}}$.

Proof of Theorem 3.2.

(a) It has proven with the unconstrained problem before.
(b) If $Q_1 + r - \mu_L > M$ then we should have $Q_1 + r - \mu_L = M$ which results in $r = M + \mu_L - Q_1$.

Therefore, Q_1 will be considered as another initial order quantity Q_0 and after reformulating the reorder point, k_p and $\mathrm{Gu}(k_p)$ needs to be updated. Where $k_p = \left[r - D\left(\frac{Q_0}{P} + b\right)\right]/\sigma\sqrt{\frac{Q_0}{P} + b}$ (see Appendix A for proof. Just simply insert $\left(\frac{Q_0}{P} + b\right)$ instead of $l(Q)$ in Appendix A). Afterwards, equation (3.9) can be utilized to recalculate the order quantity when LSC is activated.

As mentioned before Q_1 is a function of an initial value of Q in equation (3.9). For this purpose, equation (3.10) may be utilized to determine the initial value of Q. Thus, we need to develop an efficient algorithm in order to compute their optimal values. Algorithm 1 is presented not only for calculating the decision variables, but also for showing the optimization procedure for the developed VMI systems with PBO and LSC. ☐

Algorithm 1:

$Q_0 \leftarrow \sqrt{\frac{D(A_B+A_S)}{\frac{h_B}{2}+\frac{h_S}{2}(1-\frac{D}{P})}}$, compute $\bar{F}$ and find r, compute k_p and $\mathrm{Gu}(k_p)$
  Calculate Q_1
While $\mid Q_1 - Q_0 \mid = 0$ **do**
    $Q_0 \leftarrow Q_1$
    Compute $\bar{F}$ and find r
    $k_p \leftarrow \frac{r-D\left(\frac{Q_0}{P}+b\right)}{\sigma\sqrt{\frac{Q_0}{P}+b}}$
    $\mathrm{Gu}(k_p) \leftarrow \phi(k_p) - k\overline{\phi}(k_p)$
    Calculate Q_1
End while
If $Q_1 + r - \mu_L \leq M$
    $Q^*_{\mathrm{VMI}} \leftarrow Q_1$
    $r^*_{\mathrm{VMI}} \leftarrow r$
    Calculate total cost of chain
Else
    $Q_0 \leftarrow Q_1$
    $r = M + \mu_L - Q_1$
    $k_p \leftarrow \frac{r-D\left(\frac{Q_0}{P}+b\right)}{\sigma\sqrt{\frac{Q_0}{P}+b}}$
    $\mathrm{Gu}(k_p) \leftarrow \phi(k_p) - k_p\overline{\phi}(k_p)$
    Recalculate Q_1 and denote it with $\overline{Q}_1$
    $Q^*_{\mathrm{VMI}} \leftarrow \overline{Q}_1$
    $r^*_{\mathrm{VMI}} \leftarrow M + \mu_L - Q^*_{\mathrm{VMI}}$
    Calculate total cost of chain
End if

3.2. VMI under (R, T) with PBO and LSC

We denote the total cost of chain under (R, T) policy with $\mathrm{TC}^{R,T}_{\mathrm{VMI}}$. As mentioned before, vendor accounts for replenishing inventory at buyer's side and incurs all of the involved costs where cost of the buyer is negligible ($\mathrm{KB}^{R,T}_{\mathrm{VMI}} = 0$) and $\mathrm{TC}^{R,T}_{\mathrm{VMI}} = \mathrm{KB}^{R,T}_{\mathrm{VMI}} + \mathrm{KS}^{R,T}_{\mathrm{VMI}} = \mathrm{KS}^{R,T}_{\mathrm{VMI}}$ [46, 54]. As explained before, total cost of chain under

PBO is a convex combination of costs associating with backordered ($\mathrm{TC}^{R,T}_{\mathrm{VMIBO}}$) and lost-sales ($\mathrm{TC}^{R,T}_{\mathrm{VMILS}}$). Therefore, we have:

$$\mathrm{TC}^{R,T}_{\mathrm{VMIBO}} = \overbrace{\frac{(A_B + A_S)}{T}}^{\text{Fixed ordering cost}} + \overbrace{h_B\left(\frac{\mathrm{DT}}{2} + \mathrm{SS}\right)}^{\text{Buyer's holding cost}} + \overbrace{h_S\frac{\mathrm{DT}}{2}\left(1 - \frac{D}{P}\right)}^{\text{Vendor's holding cost}} + \overbrace{\pi_1\frac{\bar{b}(r, l(\mathrm{DT}) + T)}{T}}^{\text{Backording cost}} \tag{3.11}$$

$$\mathrm{TC}^{R,T}_{\mathrm{VMILS}} = \overbrace{\frac{(A_B + A_S)}{T}}^{\text{Fixed ordering cost}} + \overbrace{h_B\left(\frac{\mathrm{DT}}{2} + \mathrm{SS} + \bar{b}(r, l(\mathrm{DT}) + T)\right)}^{\text{Buyer's holding cost}} + \overbrace{h_S\frac{\mathrm{DT}}{2}\left(1 - \frac{D}{P}\right)}^{\text{Vendor's holding cost}} + \overbrace{\pi_2\frac{\bar{b}(r, l(\mathrm{DT}) + T)}{T}}^{\text{Lost-sale cost}}. \tag{3.12}$$

The first terms of equations (3.11) and (3.12) show the fixed ordering costs which should be paid in whole cycles. The second terms, are buyer's holding costs including costs which should be paid for holding average order quantities, safety stocks, and average amount of backlogged in lost-sale. The third terms are vendor's holding cost and the last terms are cost of backorders and lost-sales.

To obtain the total cost of chain under PBO, equations (3.11) and (3.12) should be combined using the formulation of $\mathrm{TC}^{R,T}_{\mathrm{VMI}} = \alpha\mathrm{TC}^{R,T}_{\mathrm{VMIBO}} + (1-\alpha)\mathrm{TC}^{R,T}_{\mathrm{VMILS}}$. Thus, total cost of chain under PBO is obtained as presented in equation (3.13):

$$\begin{aligned}\mathrm{TC}^{R,T}_{\mathrm{VMI}} = \frac{(A_B + A_S)}{T} &+ h_B\frac{\mathrm{DT}}{2} + h_B\mathrm{SS} + (1-\alpha)h_B\bar{b}(r, l(\mathrm{DT}) + T) + h_S\frac{\mathrm{DT}}{2}\left(1 - \frac{D}{P}\right) \\ &+ \alpha\pi_1\frac{\bar{b}(r, l(\mathrm{DT}) + T)}{T} + (1-\alpha)\pi_2\frac{\bar{b}(r, l(\mathrm{DT}) + T)}{T}.\end{aligned} \tag{3.13}$$

Demand through lead-time follows normal probability distribution $x \sim N\left(D\left(l\left(\mathrm{DT}\right) + T\right), \sigma^2\left(l\left(\mathrm{DT}\right) + T\right)\right)$. Also, lead-time in (R, T) replenishment review system varies linearly with cycle time and can be calculated with $l\left(\mathrm{DT}\right) + T + b = \frac{\mathrm{DT}}{P} + T + b$. On the other hand we know $\bar{b}\left(R, l\left(\mathrm{DT}\right) + T\right) = \sigma\sqrt{(\mathrm{DT}/P) + T + b}\mathrm{Gu}(k_p)$ (Proof is similar to Appendix A). Also, we know $\mathrm{SS} = k_p\sigma_{L+T} = k_p\sigma\sqrt{(\mathrm{DT}/P) + T + b}$ under (R, T) policy. Hence, the total cost of chain under (R, T) is reformulated as equation (3.14).

$$\begin{aligned}\mathrm{TC}^{R,T}_{\mathrm{VMI}} = \frac{(A_B + A_S)}{T} &+ h_B\frac{\mathrm{DT}}{2} + h_Bk_p\sigma\sqrt{(\mathrm{DT}/P) + T + b} + h_S\frac{\mathrm{DT}}{2}\left(1 - \frac{D}{P}\right) \\ &+ (\alpha\pi_1 + (1-\alpha)\pi_2)\frac{\sigma\sqrt{(\mathrm{DT}/P) + T + b}\mathrm{Gu}(k_p)}{T} + (1-\alpha)h_B\sigma\sqrt{(\mathrm{DT}/P) + T + b}\mathrm{Gu}(k_p).\end{aligned} \tag{3.14}$$

After adding the LSC constraint to problem, the main problem with considering LSC is as follows:

$$\begin{aligned}&\min\ \mathrm{TC}^{R,T}_{\mathrm{VMI}} \\ &\mathrm{st}:\ \mathrm{DT} + \mathrm{SS} \leq M or \mathrm{DT} + R - \mu_{L+T} \leq M.\end{aligned} \tag{3.15}$$

To solve the above model, first the unconstrained model should be solved. If the obtained optimal values of decision variables satisfy the constraint, the calculated value for decision variables are optimal for the constrained problem as well. Therefore, first we try to derive the optimal value of decision variables (R, T) without considering the constraint. For this purpose the derivation technique would be utilized. Therefore, the maximum level of inventory in (R, T) policy can be derived using equation (3.16). It is noteworthy to mention that $\frac{\partial\bar{b}(r, l(\mathrm{DT}) + T)}{\partial r} = -\overline{F}(R)$ (proof is similar to Appendix B of [46]).

$$\frac{\partial\mathrm{TC}^{R,T}_{\mathrm{VMI}}}{\partial R} = h_B - (\alpha\pi_1 + (1-\alpha)\pi_2)\frac{\overline{F}(R)}{T} - (1-\alpha)h_B\overline{F}(R) = 0. \tag{3.16}$$

Thus, the complementary cumulative distribution of maximum level of inventory can be calculated using equation (3.17)

$$\overline{F}(R) = \frac{h_B T}{\alpha\pi_1 + (1-\alpha)\pi_2 + (1-\alpha)h_B T}. \tag{3.17}$$

Safety stock can be computed using equation $\mathrm{SS} = k_p \sigma_{L+T}$ in which σ_{L+T} is the standard deviation during lead-time in (R,T) policy. As demand during the lead-time follows $x \sim N\left(D\left(l\left(\mathrm{DT}\right)+T\right), \sigma^2\left(l\left(\mathrm{DT}\right)+T\right)\right)$, we have $\sigma_{L+T} = \sigma\sqrt{l(\mathrm{DT})+T+b} = \sigma\sqrt{\mathrm{DT}/P+T+b}$.

Afterwards, the total cost of chain is derived according to T to find the optimal value of cycle time:

$$\begin{aligned}\frac{\partial \mathrm{TC}_{\mathrm{VMI}}^{R,T}}{\partial T} = &-\frac{(A_B+A_S)}{T^2} + \frac{h_B D}{2} + \frac{k_p h_B \sigma}{2\sqrt{(\mathrm{DT}/P)+T+b}} + \frac{h_S D}{2}\left(1-\frac{D}{P}\right) \\ &+ (1-\alpha)h_B\sigma\frac{\mathrm{Gu}(k_p)}{2\sqrt{(\mathrm{DT}/P)+T+b}} \\ &+ (\alpha\pi_1+(1-\alpha)\pi_2)\left[\frac{\sigma\mathrm{Gu}(k_p)}{2T\sqrt{(\mathrm{DT}/P)+T+b}} - \frac{\sigma\sqrt{(\mathrm{DT}/P)+T+b}\,\mathrm{Gu}(k_p)}{T^2}\right].\end{aligned} \tag{3.18}$$

Therefore, the cycle time (T) can be computed using equation (3.19) as below:

$$T_1 = \sqrt{\frac{2(A_B+A_S) + 2\left(\alpha\pi_1+(1-\alpha)\pi_2\right)\sigma\sqrt{(\mathrm{DT}/P)+T+b}\,\mathrm{Gu}(k_p)}{h_B D + h_S D(1-\frac{D}{P}) + \frac{k_p h_B \sigma}{\sqrt{(\mathrm{DT}/P)+T+b}} + \frac{(1-\alpha)h_B\sigma\mathrm{Gu}(k_p)}{\sqrt{(\mathrm{DT}/P)+T+b}} + \frac{(\alpha\pi_1+(1-\alpha)\pi_2)\sigma\mathrm{Gu}(k_p)}{T\sqrt{(\mathrm{DT}/P)+T+b}}}}. \tag{3.19}$$

As the optimal cycle time is a function of an initial T, an efficient algorithm needs to be developed to calculate the optimal value of T. Initial value of cycle time (T_0) can be computed by equation (3.2) which is achieved without considering PBO and LSC ($\pi_1 = \pi_2 = 0$ and $\overline{b}(r, l(\mathrm{DT})+T) = 0$) in the developed VMI system:

$$T_0 = \sqrt{\frac{2(A_B+A_S)}{h_B + h_S(1-D/P)}}. \tag{3.20}$$

Proposition 3.3. $\mathrm{TC}_{\mathrm{VMI}}^{R,T}$ *is continues, strictly increasing, and concave in* R *and* T *when* $\omega+\psi > \nu+\kappa$ *where* $\omega = 8\left(A_B+A_s\right)\left(b+\mathrm{DT}/P+T\right)^{3/2}$, $\psi = 8\mathrm{Gu}(k_p)\sigma\left(\alpha\pi_1+\pi_2\left(1-\alpha\right)\right)\left(b+\mathrm{DT}/P+T\right)^2$, $\nu = 4\mathrm{Gu}(k_p)\sigma T\left(\alpha\pi_1-\pi_2\left(\alpha-1\right)\right)\left(b+\mathrm{DT}/P+T\right)$, *and* $\kappa = \left(\left(1-\alpha\right)\mathrm{Gu}(k_p)\sigma+\sigma k_p\right)T^3 h_B - \mathrm{Gu}(k_p)\sigma T^2\left(\alpha\pi_1+\pi_2\left(1-\alpha\right)\right)$.

Proof of Proposition 3.3. See the Appendix B. □

Theorem 3.4. (a) *If* $\mathrm{DT}_1 + R - \mu_{L+T} \le M$ *then* $T_{RT}^* = T_1$ *and* $R_{\mathrm{VMI}}^* = R$.

(b) *If* $\mathrm{DT}_1 + R - \mu_{L+T} > M$ *then* T_1 *will be considered as initial cycle time. Afterwards, optimum cycle time can be recalculated using equation (3.19) and will be denoted by* $\overline{T}_1$. *Hence,* $T_{\mathrm{VMI}}^* = \overline{T}_1$ *and* $R_{\mathrm{VMI}}^* = M + \mu_{L+T} - \mathrm{DT}_{\mathrm{VMI}}^*$.

Proof of Theorem 3.4. (a) It has proven with the unconstrained problem before.

(b) If $\mathrm{DT}_1 + R - \mu_{L+T} > M$ then we have $\mathrm{DT}_1 + R - \mu_{L+T} = M$, and accordingly $R = M + \mu_{L+T} - \mathrm{DT}_1$.

Then T_1 would play the role of another initial cycle time T_0 and we know that with changing the R, inverse of service level $\left(k_p = \left(R - D\left(\frac{\mathrm{DT}_0}{P} + T_0 + b\right)\right)/\sigma\sqrt{\left(\frac{\mathrm{DT}_0}{P} + T_0 + b\right)}\right)$ will be altered and $\mathrm{Gu}(k_p)$ needs to be updated. Afterwards, optimal cycle time should be recalculated using equation (3.19).

Optimization procedure is presented through an efficient algorithm as follows in order to solve the proposed VMI systems under (R, T) policy with considering PBO and LSC.

Algorithm 2:

$T_0 \leftarrow \sqrt{\frac{2(A_B+A_S)}{h_B+h_S(1-\frac{D}{P})}}$, Compute $\bar{F}$ and find R, compute k_p and $\mathrm{Gu}(k_p)$
Calculate T_1
While $\mid T_1 - T_0 \mid = \mathbf{0}$ **do**
  Compute $\bar{F}$ and find R
  $k_p \leftarrow (r - D\,(\mathrm{DT}_0/P + T_0 + b)) \,/ \left(\sigma\sqrt{\mathrm{DT}_0/P + T_0 + b}\right)$
  $\mathrm{Gu}(k_p) \leftarrow \phi(k_p) - k\overline{\phi}(k_p)$
  $T_0 \leftarrow T_1$
  Calculate T_1
End while
If $\mathrm{DT}_1 + R - \mu_{L+T} \leq M$
  $T^*_{\mathrm{VMI}} \leftarrow T_1$
  $R^*_{\mathrm{VMI}} \leftarrow R$
  Calculate total cost of chain
Else
  $T_0 \leftarrow T_1$
  $R \leftarrow M + \mu_{L+T} - \mathrm{DT}_1$
  $k_p \leftarrow (r - D\,(\mathrm{DT}_0/P + T_0 + b)) \,/ \sigma\sqrt{\mathrm{DT}_0/P + T_0 + b}$
  $\mathrm{Gu}(k_p) \leftarrow \phi(k_p) - k\overline{\phi}(k_p)$
  Recalculate T_1 and denote it with $\overline{T}_1$
  $T^*_{\mathrm{VMI}} \leftarrow \overline{T}_1$
  $R^*_{\mathrm{VMI}} \leftarrow M + \mu_{L+T} - \mathrm{DT}^*_{\mathrm{VMI}}$
  Calculate total cost of chain
End if

A comprehensive flowchart for solving the proposed VMI systems is provided in Figure 1. □

3.3. Traditional RMI under (r, Q) with PBO and LSC

The buyer is responsible to make orders in traditional RMI system under (r, Q) replenishment review policy. It means that the buyer determines his own optimal reorder point (r) and order quantity (Q) which are not necessary the optimal value for the vendor. Let us denote the total cost of buyer in RMI under (r, Q) with $\mathrm{KB}^{r,Q}_{\mathrm{RMI}}$ and cost of buyer with considering backordering and lost-sales with $\mathrm{KB}^{r,Q}_{\mathrm{RMIBO}}$ and $\mathrm{KB}^{r,Q}_{\mathrm{RMILS}}$, respectively. According to the definitions, we have:

$$\mathrm{KB}^{r,Q}_{\mathrm{RMIBO}} = \overbrace{\frac{DA_B}{Q}}^{\text{Fixed ordering cost}} + \overbrace{h_B\left(\frac{Q}{2} + \mathrm{SS}\right)}^{\text{Buyer's holding cost}} + \overbrace{\pi_1 \frac{D}{Q}\bar{b}(r, l(Q))}^{\text{Backordering cost}} \tag{3.21}$$

$$\mathrm{KB}^{r,Q}_{\mathrm{RMILS}} = \overbrace{\frac{DA_B}{Q}}^{\text{Fixed ordering cost}} + \overbrace{h_B\left(\frac{Q}{2} + \mathrm{SS} + \bar{b}(r, l(Q))\right)}^{\text{Buyer's holding cost}} + \overbrace{\pi_2 \frac{D}{Q}\bar{b}(r, l(Q))}^{\text{Lost-sale cost}}. \tag{3.22}$$

The first terms of equations (3.21) and (3.22) denote the fixed ordering cost that the buyer has to pay in cycles. Their second terms, represent buyer's holding cost associated with holding average quantity orders, average safety stocks, and average backlogged in lost-sale. Finally, the third terms calculate backordering and

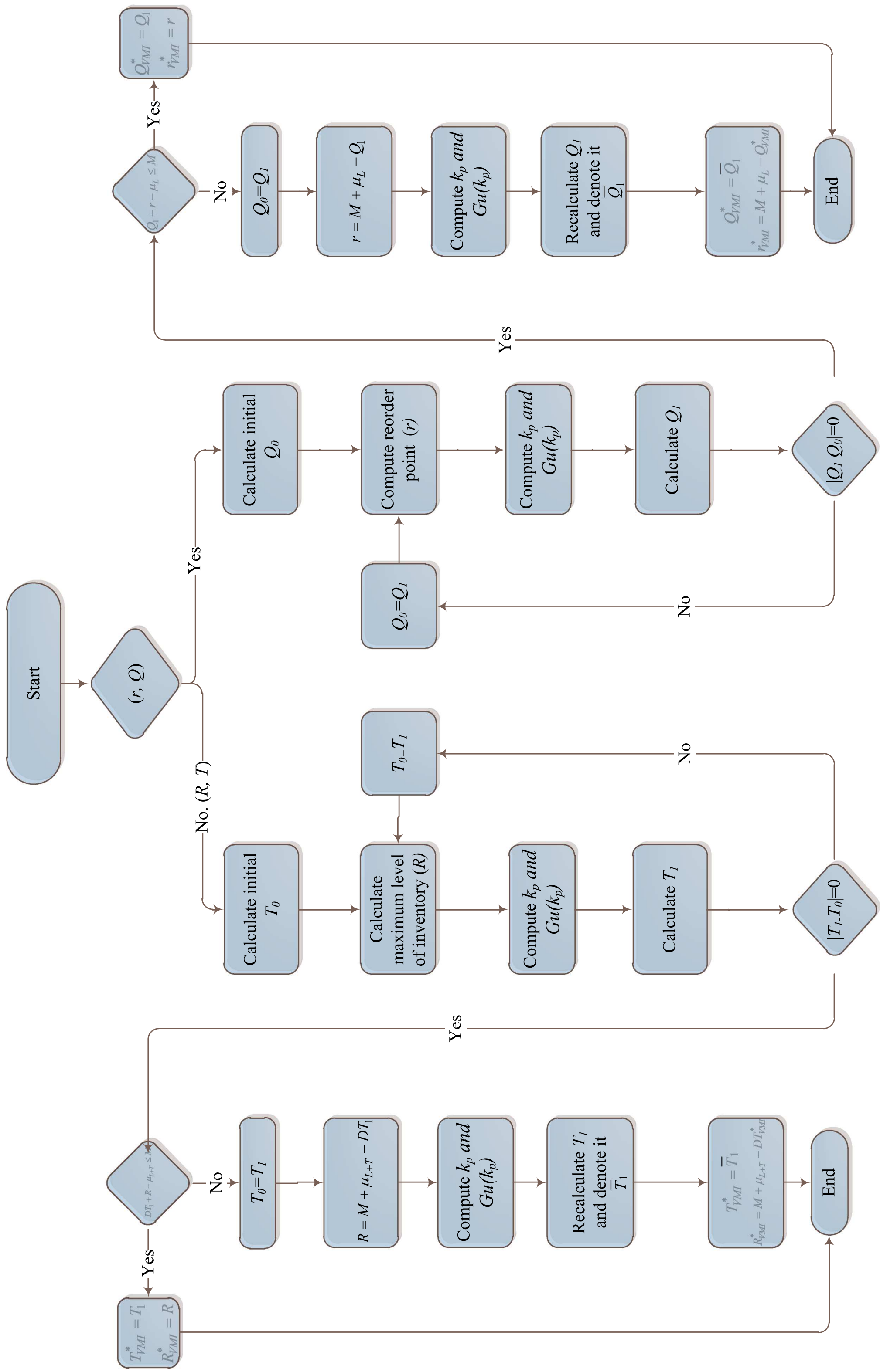

FIGURE 1. Flowchart for solving developed VMI under (r, Q) and (R, T) with LSC and PBO.

lost-sale costs. Thus, for calculating the total cost of retailer with partial backordering ($\mathrm{KB}^{r,Q}_{\mathrm{RMI}} = \alpha \mathrm{KB}^{r,Q}_{\mathrm{RMIBO}} + (1-\alpha)\mathrm{KB}^{r,Q}_{\mathrm{RMILS}}$) we have:

$$\mathrm{KB}^{r,Q}_{\mathrm{RMI}} = \frac{DA_B}{Q} + h_B\frac{Q}{2} + h_B\mathrm{SS} + (1-\alpha)h_B\bar{b}(r,l(Q)) + \alpha\pi_1\frac{D}{Q}\bar{b}(r,l(Q)) + (1-\alpha)\pi_2\frac{D}{Q}\bar{b}(r,l(Q)). \tag{3.23}$$

Since demand through the lead-time follows $x \sim N\left(Dl\left(Q\right), \sigma^2 l\left(Q\right)\right)$ and $\bar{b}\left(r, l\left(Q\right)\right) = \sigma\sqrt{Q/P+b}\,\mathrm{Gu}(k_p)$, and $\mathrm{SS} = r - \mu_L$ the total cost of retailer can be reformulated as equation (3.24):

$$\begin{aligned}\mathrm{KB}^{r,Q}_{\mathrm{RMI}} = \frac{DA_B}{Q} + h_B\frac{Q}{2} + h_B\mathrm{SS} + (1-\alpha)h_B\sigma\sqrt{Q/P+b}\,\mathrm{Gu}(k_p)\\ + \left(\alpha\pi_1 + (1-\alpha)\pi_2\right)\frac{D}{Q}\sigma\sqrt{Q/P+b}\,\mathrm{Gu}(k_p).\end{aligned} \tag{3.24}$$

The main problem, which considers LSC and PBO simultaneously, is as follows:

$$\begin{aligned}&\min\ \mathrm{KB}^{r,Q}_{\mathrm{RMI}}\\ &\mathrm{st:}\ Q + \mathrm{SS} \le M \text{ or } Q + r - \mu_L \le M.\end{aligned} \tag{3.25}$$

As explained before, for solving the constrained managed inventory systems, the unconstrained problem should be solved first. If the obtained results satisfy the problem (3.25), the final optimal values are obtained as well. Otherwise the solution procedure should be continued considering the constraint.

Proposition 3.5. $\mathrm{KB}^{r,Q}_{\mathrm{RMI}}$ *is continues, strictly increasing, and concave in* r *and* Q *when* $\upsilon + \varsigma > \vartheta$ *where* $\upsilon = 8DPA_B\left(bP+Q\right)\sqrt{b+\frac{Q}{P}}$, $\varsigma = \mathrm{Gu}(k_p)\sigma D\left(\pi_1\alpha + (1-\alpha)\pi_2\right)\left(8b^2P^2 + 12bPQ + 3Q^2\right)$, *and* $\vartheta = (1-\alpha)\,\mathrm{Gu}(k_p)\sigma Q^3 h_B$.

Proof of Proposition 3.5. See Appendix B for proof. □

For obtaining the reorder point (r) the above formulation is derived with respect to r and the complementary cumulative distribution at $\widehat{r}$ is obtained as follows:

$$\overline{F}(\widehat{r}) = \frac{h_BQ}{\left(\alpha\pi_1 + (1-\alpha)\pi_2\right)D + (1-\alpha)h_BQ}. \tag{3.26}$$

As it is known $\mathrm{SS} = k_p\sigma\sqrt{\frac{Q}{P}+b}$, the total cost of retailer is derived with respect to Q to obtain the order quantity as below:

$$\begin{aligned}\frac{\partial \mathrm{KB}^{r,Q}_{\mathrm{RMI}}}{\partial Q} = -\frac{DA_B}{Q^2} + \frac{h_B}{2} + \frac{h_Bk_p\sigma}{2\sqrt{PQ}} + \frac{(1-\alpha)h_B\sigma\mathrm{Gu}(k_p)}{2P\sqrt{Q/P+b}}\\ + \left(\alpha\pi_1 + (1-\alpha)\pi_2\right)\left[\frac{D\sigma\mathrm{Gu}(k_p)}{2PQ\sqrt{Q/P+b}} - \frac{D}{Q^2}\times\sigma\sqrt{Q/P+b}\,\mathrm{Gu}(k_p)\right].\end{aligned} \tag{3.27}$$

Hence, the optimal order quantity for the unconstrained problem can be computed using equation (3.28):

$$Q_2 = \sqrt{\frac{2DA_B + 2D\sigma\left(\alpha\pi_1 + (1-\alpha)\pi_2\right)\sqrt{Q/P+b}\,\mathrm{Gu}(k_p)}{h_B + \frac{h_Bk_p\sigma}{\sqrt{PQ}} + \frac{(1-\alpha)h_B\sigma\mathrm{Gu}(k_p)}{P\sqrt{Q/P+b}} + \left(\alpha\pi_1 + (1-\alpha)\pi_2\right)\frac{D\sigma\mathrm{Gu}(k_p)}{PQ\sqrt{Q/P+b}}}}. \tag{3.28}$$

The initial value of Q in the above equation can be calculated as follows:

$$Q_0 = \sqrt{\frac{2DA_B}{h_B}}. \tag{3.29}$$

It is noteworthy to mention that the initial value of Q is obtained with solving the RMI problem (Eq. (3.23)) when PBO and LSC is not permitted ($\pi_1 = \pi_2 = 0$ and $\bar{b}(r,l(Q)) = 0$).

Theorem 3.6. (a) *If* $Q_2 + r - \mu_L \leq M$ *then* $Q^*_{\text{RMI}} = Q_2$ *and* $r^*_{\text{RMI}} = r$

(b) *If* $Q_2 + r - \mu_L > M$ *then* Q_2 *plays the role of initial order quantity. Then optimal order quantity will be recalculated using equation (3.28) and will be denoted by* $\overline{Q}_2$. *Therefore,* $Q^*_{\text{RMI}} = \overline{Q}_2$ *and* $r^*_{\text{RMI}} = M + \mu_L - Q^*_{\text{RMI}}$.

Proof of Theorem 3.6.

(a) It has proven with the unconstrained problem before.

(b) If $Q_2 + r - \mu_L > M$ then we should have $Q_2 + r - \mu_L = M$. Hence, we have $r = M + \mu_L - Q_2$. Then Q_2 can be considered as a new initial order quantity Q_0 and one should update the inverse level of service level $\left(k_p = \left(r - D\left(\frac{Q_0}{P} + b\right)\right) / \left(\sigma\sqrt{\frac{Q_0}{P} + b}\right)\right)$, and obtain $\text{Gu}(k_p)$ and recalculate the order quantity using equation (3.28).

Total cost of chian including costs of retailer and vendor is obtained as follows:

$$\text{TC}^{r,Q}_{\text{RMI}} = \frac{DA_B}{Q} + h_B\frac{Q}{2} + h_B\text{SS} + \frac{DA_s}{Q} + h_s\frac{Q}{2}\left(1 - \frac{D}{P}\right) + \alpha\pi_1\frac{D}{Q}\bar{b}(r, l(Q)) + (1-\alpha)\pi_2\frac{D}{Q}\bar{b}(r, l(Q)). \quad (3.30)$$

It is noteworthy to mention that Algorithm 1 should be adopted to determine the optimal decision variables for the developed traditional RMI under (r, Q) policy under LSC and PBO. □

3.4. Traditional RMI under (R, T) with PBO and LSC

Total cost of buyer under (R, T) policy is denoted with $\text{KB}^{R,T}_{\text{RMI}}$. Cost of retailer under fully backordering and fully lost-sales are denoted with $\text{KB}^{R,T}_{\text{RMIBO}}$ and $\text{KB}^{R,T}_{\text{RMILS}}$, respectively. Therefore, costs of retailer under full backordering and full lost-sales can be formulated as follows:

$$\text{KB}^{R,T}_{\text{RMIBO}} = \overbrace{\frac{A_B}{T}}^{\text{Fixed ordering cost}} + \overbrace{h_B\left(\frac{\text{DT}}{2} + \text{SS}\right)}^{\text{Buyer's holding cost}} + \overbrace{\pi_1\frac{\bar{b}(r, l(\text{DT}) + T)}{T}}^{\text{Backordering cost}} \quad (3.31)$$

$$\text{KB}^{R,T}_{\text{RMILS}} = \overbrace{\frac{A_B}{T}}^{\text{Fixed ordering cost}} + \overbrace{h_B\left(\frac{\text{DT}}{2} + \text{SS} + \bar{b}(r, \text{DT} + T)\right)}^{\text{Buyer's holding cost}} + \overbrace{\pi_2\frac{\bar{b}(r, l(\text{DT}) + T)}{T}}^{\text{Lost-sale cost}}. \quad (3.32)$$

The first terms of equations (3.31) and (3.32) calculate fixed ordering costs for buyer. The second terms denote buyer's holding cost including average order quantities and average safety stocks. In addition, average of backlogged should be involved for calculating holding costs in lost-sales. As mentioned in previous sections, the $1 - \alpha$ precent of demands encounter lost-sales. Thus, the total cost of chain can be determined using $\text{KB}^{R,T}_{\text{RMI}} = \alpha\text{KB}^{R,T}_{\text{RMIBO}} + (1-\alpha)\text{KB}^{R,T}_{\text{RMILS}}$.

$$\begin{aligned}\text{KB}^{R,T}_{\text{RMI}} = {} & \frac{A_B}{T} + h_B\frac{\text{DT}}{2} + h_B\text{SS} + (1-\alpha)h_B\bar{b}(r, l(\text{DT}) + T) \\ & + \alpha\pi_1\frac{\bar{b}(r, l(\text{DT}) + T)}{T} + (1-\alpha)\pi_2\frac{\bar{b}(r, l(\text{DT}) + T)}{T}. \end{aligned} \quad (3.33)$$

As demand during lead-time follows normal probability distribution $\left(x \sim N\left(D\left(l\left(\text{DT}\right) + T\right), \sigma^2\left(l\left(\text{DT}\right) + T\right)\right)\right)$, total cost of buyer can be reformulated as equation (3.34). After some algebra we have:

$$\begin{aligned}\text{KB}^{R,T}_{\text{RMI}} = {} & \frac{A_B}{T} + h_B\frac{\text{DT}}{2} + h_B\text{SS} + (1-\alpha)h_B\sigma\sqrt{(\text{DT}/P) + T + b}\text{Gu}(k_p) \\ & + (\alpha\pi_1 + (1-\alpha)\pi_2)\frac{\sigma\sqrt{(\text{DT}/P) + T + b}\text{Gu}(k_p)}{T}, \end{aligned} \quad (3.34)$$

where $\mathrm{SS} = k_p\sigma\sqrt{(\mathrm{DT}/P) + T + b}$. Therefore, the main problem for the developed RMI systems with PBO and LSC is as follows:

$$\begin{aligned} &\min\ \mathrm{KB}_{\mathrm{RMI}}^{R,T} \\ &\mathrm{st}:\ \mathrm{DT} + \mathrm{SS} \leq M \text{ or } \mathrm{DT} + R - \mu_{L+T} \leq M. \end{aligned} \tag{3.35}$$

First, the above problem should be solved without considering the LSC. If the obtained solutions satisfy the LSC, optimal value of decision variables are achieved as well. Otherwise, the solution procedure continues considering LSC as follows.

Proposition 3.7. $\mathrm{KB}_{\mathrm{RMI}}^{R,T}$ *is continues, strictly increasing, and concave in R and T when $\chi + \zeta + \phi > 0$ where* $\chi = 8PA_B\left(bP + T\left(D + P\right)\right)\sqrt{b + \frac{\mathrm{DT}}{P} + T}$, $\phi = T^3 h_B\left(D + P\right)^2 \sigma\left(\left(\alpha - 1\right)\mathrm{Gu}(k_p) - k_p\right)$ $\zeta = \mathrm{Gu}(k_p)\left(\alpha\pi_1 + (1-\alpha)\pi_2\right)\sigma\left(8b^2P^2 + 12bPT\left(D + P\right) + 3T^2\left(D + P\right)^2\right)$.

Proof of Proposition 3.7. See Appendix B for proof. □

For solving the unconstrained problem, first the complementary cumulative distribution at point R can be calculated using equation (3.36) which is similar to the procedure used in Section 4.2:

$$\overline{F}(R) = \frac{h_B T}{\alpha\pi_1 + (1-\alpha)\pi_2 + (1-\alpha)h_B T}. \tag{3.36}$$

Using the first derivative of total cost of buyer with respect to T we have:

$$\begin{aligned} \frac{\partial \mathrm{KB}_{\mathrm{RMI}}^{R,T}}{\partial T} = &-\frac{A_B}{T^2} + \frac{h_B D}{2} + \sigma\left(\frac{D}{P} + 1\right)\frac{k_p h_B}{2\sqrt{(\mathrm{DT}/P) + T + b}} + (1-\alpha)h_B\sigma\left(\frac{D}{P} + 1\right)\frac{\mathrm{Gu}(k_p)}{2\sqrt{(\mathrm{DT}/P) + T + b}} \\ &+ \left(\alpha\pi_1 + (1-\alpha)\pi_2\right)\left[\sigma\left(\frac{D}{P} + 1\right)\frac{\mathrm{Gu}(k_p)}{2T\sqrt{(\mathrm{DT}/P) + T + b}} - \frac{\sigma\sqrt{(\mathrm{DT}/P) + T + b}\,\mathrm{Gu}(k_p)}{T^2}\right]. \end{aligned} \tag{3.37}$$

Therefore, the cycle time for the unconstrained traditional RMI system is:

$$T_2 = \sqrt{\frac{2A_B + 2\left(\alpha\pi_1 + (1-\alpha)\pi_2\right)\sigma\sqrt{(\mathrm{DT}/P) + T + b}\,\mathrm{Gu}(k_p)}{h_B D + \sigma\left(\frac{D}{P} + 1\right)\left[\frac{k_p h_B}{\sqrt{(\mathrm{DT}/P)+T+b}} + (1-\alpha)h_B\frac{\mathrm{Gu}(k_p)}{\sqrt{(\mathrm{DT}/P)+T+b}} + \left(\alpha\pi_1 + (1-\alpha)\pi_2\right)\frac{\mathrm{Gu}(k_p)}{T\sqrt{(\mathrm{DT}/P)+T+b}}\right]}}. \tag{3.38}$$

The initial value for cycle time is determined as follows, which is the optimal value for the unconstrained problem without considering PBO and LSC:

$$T_0 = \sqrt{\frac{2A_B}{h_B}}. \tag{3.39}$$

Theorem 3.8. *(a) If $\mathrm{DT}_2 + R - \mu_{L+T} \leq M$ then $T^*_{RT} = T_2$ and $R^*_{\mathrm{RMI}} = R$*

(b) If $\mathrm{DT}_2 + R - \mu_{L+T} > M$ then T_2 will be considered as initial cycle time. Hence, the optimal cycle time can be recalculated using equation (3.38) and denoted with $\overline{T}_2$. Hence, $T^*_{\mathrm{RMI}} = \overline{T}_2$ and $R^*_{\mathrm{RMI}} = M + \mu_{L+T} - \mathrm{DT}^*_{\mathrm{RMI}}$.

Proof of Theorem 3.8.

(a) It has proven with the unconstrained problem before.
(b) If $\mathrm{DT}_2 + R - \mu_{L+T} > M$ then we should have $\mathrm{DT}_2 + R - \mu_{L+T} = M$, Thus T_2 can be considered as the new initial cycle time T_0 and service level (k_p) and $\mathrm{Gu}(k_p)$ should be updated. Afterwards, we can recalculate the cycle time using equation (3.38) with the new initial cycle time and denote that with $\overline{T}_2$. Finally, the maximum level of i nventory can be calculated using $R = M + \mu_{L+T} - D\overline{T}_2$.

For solving the decision variable, Algorithm 2 can be used for the developed RMI systems as well as the VMI system. Notably, total cost of chain for the developed RMI system under (R,T) policy can be calculated as follows:

$$\mathrm{TC}_{\mathrm{RMI}}^{R,T} = \frac{A_B}{T} + h_B\frac{\mathrm{DT}}{2} + h_B\mathrm{SS} + \frac{A_s}{T} + h_s\frac{\mathrm{DT}}{2}(1-\frac{D}{P}) + (1-\alpha)h_B\bar{b}(r,l(\mathrm{DT})+T) + \alpha\pi_1\frac{\bar{b}(r,l(\mathrm{DT})+T)}{T} + (1-\alpha)\pi_2\frac{\bar{b}(r,l(\mathrm{DT})+T)}{T}\cdot \tag{3.40}$$

□

4. Computational and practical results

Efficient algorithms are developed in previous sections to ensure that the proposed VMI and RMI systems would be solved in reasonable time and low complexity, especially for large-scale inventory problems. Notably, computational complexities and required time are small for the presented algorithms. All the algorithms are implemented and coded in well-known commercial software MATLAB R2010a running on laptop with Core i5 CPU and 4 GB of RAM.

To represent the applicability of the proposed VMI systems accounting for PBO and LSC, and show efficacy of the proposed algorithms, numerical experiments are provided in this section inspired from a real case of Iranian automaker and parts. In Section 4.1 decision variables associated with numerical examples are calculated under different storage capacities and results are discussed. In Section 4.2 sensitivity analysis is implemented on the key parameters to show how decisions would be varied with changing parameters and how one can improve the supply chains performances.

4.1. Numerical experiment

An illustrative example is presented in this section based on a real case inspired from an automaker and parts companies located in Iran. The considered company produces lighting systems and audio systems. Its customer is one of the Iranian automotive manufacturers. PBO and LSC are the main challenges which automakers and parts industries should cope with them. For instance the automotive manufacturer may be provided with another automaker and part company in shortages. Therefore the considered automaker and parts provider encounter PBO. The Iranian automaker supply required parts from single-vendor (*i.e.* part manufacturer). Thus, coordinating between the single-vendor and single-buyer is vitally important and the proposed VMI systems can be helpful. Demand for automakers depends on a variety of socioeconomic criteria, such as income, age, saving, season, preferences, and etc., which might be varied during a time horizon. Thus, the automaker encounters stochastic demands apparently. The objective of implementing VMI in this automaker is reducing inventory and warehousing costs, enhancing supply chain performances by optimizing the chain and keeping tighter control on inventory. Figure 2 illustrates the chain for the considered automaker and parts provider.

Thus, the following parameters are used for numerical experiment: $h_B = \$30$ per unit and per month, $A_B = \$100$ per order, $h_s = \$65$ per unit and per month, $A_s = \$80$ per order, $D = 250\,000$ units per month, $\sigma^2 = 20000$ units per month, $P = 350\,000$ units per month, $\pi_1 = \$180$ per unit backordered and per month, $\pi_2 = \$200$ per unit lost-sales and per month, ratio of backordered demands are equal to $\alpha = 80\%$, and buyer's procurement lead-time is equal to 0.1 month. Each of developed VMI and RMI systems are solved under different storage capacity levels and results are provided in Tables 1 and 2.

From Table 1, the LSC is activated from capacity volume 2000–6000 in the developed VMI and RMI systems under (r,Q). It is concluded from Table 1 that the developed VMI system under (r,Q) is more beneficial than its counterpart RMI system. Improving cost-efficiency after implementing VMI systems are calculated and results are reported. More storage capacities result in higher reorder points and fewer order quantities with fewer total cost of chain.

The proposed VMI and RMI systems under (R,T) system are solved with the provided example and results are shown in Table 2. According to the obtained results, cycle time would be reduced while the LSC is activated

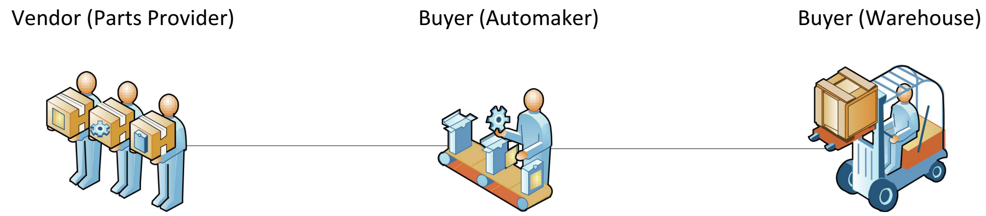

FIGURE 2. Supply chain network for the considered automaker and parts provider.

TABLE 1. Results for developed VMI and RMI under (r, Q) review system.

M	VMI				RMI				Cost
	r^*_{VMI}	Q^*_{VMI}	$\text{TC}^{r,Q}_{\text{VMI}}$ $(\times 10^7)$	LSC	r^*_{RMI}	Q^*_{RMI}	$\text{TC}^{r,Q}_{\text{RMI}}$ $(\times 10^7)$	LSC	reduction
2000	18 884	28 405	1.5964	✓	17 540	33 111	1.8126	✓	11.93%
3000	19 917	28 290	1.3974	✓	18 557	33 050	1.6397	✓	14.78%
4000	20 961	28 136	1.1982	✓	19 578	32 975	1.4667	✓	18.31%
5000	22 023	27 919	0.9988	✓	20 606	32 880	1.2936	✓	22.78%
6000	23 118	27 587	0.7992	✓	21 642	32 754	1.1205	✓	28.67%

because of limited storage of buyer. On the other hand maximum level of inventories increases with increasing the capacity storage, which leads to place more order quantities in less cycle time and fewer safety stocks. Cost reduction column, which is provided in Table 2, demonstrates that cost-efficiency of the proposed VMI is better than its counterpart RMI. Also, convergence of algorithms is tested through several test instances and results are reported in Table 3. Results represents that algorithms are capable of converging to final solutions efficiently.

Applicability and efficacy of the proposed algorithms for solving the developed VMI and RMI under (r, Q) and (R, T) review systems with PBO and LSC are examined in this section via the provided illustrative example. Also, it is represented that how the integrated decisions in VMI systems can enhance the performances of a chain when the problem encounters LSC and PBO.

4.2. Sensitivity analysis

To study how decision variables vary when key parameters change, a comprehensive sensitivity analysis is implemented. Key parameters include buyer holding cost, buyer ordering costs, supplier holding cost, supplier ordering cost, demand, and backordering rates. Parameters are varied with 5% in each iteration and the presented VMI and RMI systems are solved. The obtained optimal value of replenishment variables are presented in Tables 4 and 5. Also, the cost reductions as a result of implementing VMI systems are calculated. Figure 3 is provided to show how order quantities vary for (r, Q) policy upon changing the key parameters. Similar results for the (R, T) review system are shown in Figure 4. It is noteworthy to mention that in all of the numerical experiments in this section, the storage capacity (M) of buyer assumed to be equal to 2000 units by which LSC constraint is activated.

4.2.1. Effects of supplier's and buyer's holding costs

Figure 3(a) and (b) provides the general trend of changing order quantities when buyer and supplier unit holding costs vary. From the mentioned figures, increasing buyer holding cost leads to decrease the order quantities in (r, Q) policy under VMI and RMI. Because the buyer has to pay more total holding costs for inventories and prefer to reduce the ordering costs by increasing reordering points. Also, it is obvious from Figure 3(b) that varying unit holding cost for the supplier has no effect on order quantities in RMI system. In fact, the buyer

TABLE 2. Results for developed VMI and RMI under (R,T) review system.

M	VMI				RMI				Cost
	R^*_{VMI}	T^*_{VMI}	$\text{TC}^{R,T}_{\text{VMI}}\ (\times 10^6)$	LSC	R^*_{RMI}	T^*_{RMI}	$\text{TC}^{R,T}_{\text{RMI}}\ (\times 10^2)$	LSC	reduction
2000	47 462	0.1146	3.8205	✓	45 178	0.1018	4.5060	✓	15.21%
3000	48 391	0.1142	3.4416	✓	46 159	0.1017	4.1058	✓	16.18%
4000	49 298	0.1137	3.0618	✓	47 136	0.1016	3.7055	✓	17.37%
5000	50 171	0.1130	2.6809	✓	48 106	0.1014	3.3051	✓	18.89%
6000	50 986	0.1119	2.2979	✓	49 067	0.1012	2.9045	✓	20.88%

TABLE 3. Convergence test for developed algorithms

Test	VMI		RMI	
problem	# of iterations	Time (s)$\times 10^{-2}$	# of iterations	Time (s)
1	4	1.541	3	1.414
2	4	1.802	4	1.654
3	6	2.163	7	1.958
4	5	2.639	6	2.429
5	8	3.273	7	3.003
6	10	4.091	10	3.755
7	10	4.827	11	4.427
8	11	5.743	11	5.298
9	13	6.951	14	6.380
10	14	8.341	14	7.656

makes decisions in RMI system without considering and involving supplier's conditions and decisions would be made for optimizing the cost of buyer which are not essentially optimal for the supplier.

As a result from Table 4, in both of the VMI and RMI systems, reordering point would be increased with increasing buyer's unit holding cost, because backorders and lost-sales would be prevented as much as possible. As mentioned, supplier's unit holding cost has not any effect on reordering points in RMI systems. In addition, order quantities in VMI system are lower than that of RMI system. In VMI systems the vendor tends to place lower order quantities for buyer with larger reorder point to benefit from implementing VMI system.

Thus, vendor prefer to send lower order quantities to buyer as much as possible with larger reorder points when buyer encounters LSC, which is evident in Table 1 in a constant unit holding cost. Notably, order quantities and reorder points may act vice versa because of LSC. Because ordering more quantities requires less safety stocks, by which the storage capacity does not exceed its limitation.

Figure 4(a) and (b) represents the behaviour of optimal cycle times in (R,T) replenishment review system when the unit holding costs varies. In the proposed VMI system with LSC, increasing unit holding costs results in decreasing cycle times, which means the vendor place orders for the buyer in shorter periods. Therefore, the average of order quantities in (R,T) replenishment review systems would be reduced with decreasing cycle time. Increasing buyer unit holding cost leads to decreasing the maximum level of inventories (R) and reducing cycle times, and consequently reducing average of order quantities. Hence, the chain would not encounter lost-sales and backorders as much as possible. Notably, supplier's unit holding cost changes has not any effect on the decisions in RMI systems because the buyer has made decisions without considering supplier's viewpoint. It is noteworthy to mention that, cycle times in VMI systems are longer than RMI systems. It means vendor is able

Table 4. Changes of decision variables based on changing key parameters under (r, Q).

	Changes	VMI			RMI			Cost
		r^*_{VMI}	Q^*_{VMI}	$\text{TC}^{r,Q}_{\text{VMI}} \times 10^7$	r^*_{RMI}	Q^*_{RMI}	$\text{TC}^{r,Q}_{\text{RMI}} \times 10^7$	reduction
	−10%	18 708	29 021	1.6279	17 197	34 310	1.8655	12.74%
	−5%	18 798	28 707	1.6119	17 375	33 689	1.8383	12.32%
h_B	0%	18 884	28 405	1.5964	17 540	33 111	1.8126	11.93%
	5%	18 967	28 116	1.5815	17 694	32 571	1.7883	11.56%
	10%	19 046	27 837	1.5670	17 839	32 065	1.7652	11.23%
	−10%	18 779	28 773	1.6134	17 540	33 111	1.8126	10.99%
	−5%	18 832	28 587	1.6048	17 540	33 111	1.8126	11.46%
h_s	0%	18 884	28 405	1.5964	17 540	33 111	1.8126	11.93%
	5%	18 935	28 228	1.5882	17 540	33 111	1.8126	12.38%
	10%	18 984	28 055	1.5801	17 540	33 111	1.8126	12.83%
	−10%	18 887	28 105	1.5958	17 543	33 099	1.8120	11.93%
	−5%	18 886	28 300	1.5961	17 541	33 105	1.8123	11.93%
A_B	0%	18 884	28 405	1.5964	17 540	33 111	1.8126	11.93%
	5%	18 883	28 610	1.5967	17 538	33 117	1.8129	11.93%
	−10%	18 886	27 997	1.5959	17 540	33 111	1.8126	11.96%
	−5%	18 885	28 201	1.5962	17 540	33 111	1.8126	11.94%
A_s	0%	18 884	28 405	1.5964	17 540	33 111	1.8126	11.93%
	5%	18 883	28 609	1.5967	17 540	33 111	1.8126	11.91%
	10%	18 882	29 013	1.5969	17 540	33 111	1.8126	11.91%
	−10%	14 999	26 602	1.3575	13 087	31 957	1.5842	14.31%
	−5%	16 913	27 494	1.4746	15 289	32 544	1.6977	13.14%
D	0%	18 884	28 405	1.5964	17 540	33 111	1.8126	11.93%
	5%	20 914	29 343	1.7231	19 835	33 661	1.9289	10.67%
	10%	23 005	30 312	1.8549	22 173	34 194	2.0464	9.36%
	−10%	18 908	28 252	1.5820	17 568	32 991	1.7971	11.97%
	−5%	18 896	28 324	1.5892	17 554	33 061	1.8048	11.95%
α	0%	18 884	28 405	1.5964	17 540	33 111	1.8126	11.93%
	5%	18 872	28 547	1.6036	17 525	33 171	1.8204	11.91%
	10%	18 861	28 688	1.6108	17511	33 221	1.8282	11.89%

to provide buyer with more average order quantities in VMI systems as a result of longer cycle time and lower backlogged and ordering costs in an integrated viewpoint.

4.2.2. Effects of supplier's and buyer's ordering costs

Figure 3(c) and (d) demonstrate how order quantities might be changed when buyer's and supplier's unit ordering costs vary in (r, Q) review system. Increasing unit ordering costs of buyer and supplier results in increasing order quantities for buyer in VMI system, which helps to reduce total ordering cost of chain and backlogged costs. Optimal order quantities may be increased as much as storage capacity of the buyer permits. Therefore, reorder points would be reduced because lower reorder point results in fewer order frequencies. Thus, the vendor tries to place orders for the buyer with more quantities in fewer order frequencies. Similar to the analysis for the holding costs, optimal quantity orders in VMI system is lower than that of RMI system under (r, Q) policy because vendor makes integrated decisions which helps to coordinate chain. On other hand the unit ordering cost of buyer affect the optimal decision variables in RMI under (r, Q) policy which encounters buyer larger quantities.

Figure 4(c) and (d) are used to show the results of (R, T) replenishment review system when buyer's and supplier's unit ordering costs change. Increasing ordering costs of buyer and supplier leads to increase the cycle

TABLE 5. Changes of decision variables based on changing key parameters under (R, T).

	Changes	VMI			RMI			Cost
		R^*_{VMI}	T^*_{VMI}	$\text{TC}^{R,T}_{\text{VMI}} \times 10^6$	R^*_{RMI}	T^*_{RMI}	$\text{TC}^{R,T}_{\text{RMI}} \times 10^6$	reduction
h_B	−10%	47 997	0.1176	3.8736	45 940	0.1061	4.6054	15.89%
	−5%	47 723	0.1161	3.8465	45 544	0.1038	4.5544	15.54%
	0%	47 462	0.1146	3.8205	45 178	0.1018	4.5060	15.21%
	15%	47 212	0.1132	3.7955	44 838	0.0999	4.4599	14.90%
	20%	46 973	0.1119	3.7716	44 521	0.0981	4.4158	14.59%
h_s	−10%	47 780	0.1164	3.8347	45 178	0.1018	4.5060	14.90%
	−5%	47 619	0.1155	3.8275	45 178	0.1018	4.5060	15.06%
	0%	47 462	0.1146	3.8205	45 178	0.1018	4.5060	15.21%
	15%	47 310	0.1137	3.8137	45 178	0.1018	4.5060	15.36%
	20%	47 161	0.1129	3.8070	45 178	0.1018	4.5060	15.51%
A_B	−10%	47 455	0.1141	3.8193	45 171	0.1014	4.5048	15.22%
	−5%	47 458	0.1143	3.8199	45 175	0.1016	4.5054	15.22%
	0%	47 462	0.1146	3.8205	45 178	0.1018	4.5060	15.21%
	5%	47 466	0.1149	3.8211	45 182	0.1020	4.5066	15.21%
	10%	47 470	0.1152	3.8217	45 186	0.1022	4.5072	15.21%
A_s	−10%	47 456	0.1125	3.8196	45 178	0.1018	4.5060	15.23%
	−5%	47 459	0.1134	3.8200	45 178	0.1018	4.5060	15.22%
	0%	47 462	0.1146	3.8205	45 178	0.1018	4.5060	15.21%
	15%	47 465	0.1156	3.8210	45 178	0.1018	4.5060	15.20%
	20%	47 468	0.1167	3.8215	45 178	0.1018	4.5060	15.19%
D	−10%	40 908	0.1134	3.4261	39 967	0.1069	4.0305	15.00%
	−5%	44 109	0.1139	3.6227	42 555	0.1043	4.2678	15.12%
	0%	47 462	0.1146	3.8205	45 178	0.1018	4.5060	15.21%
	5%	50 981	0.1155	4.0199	47 836	0.0995	4.7451	15.28%
	10%	54 681	0.1165	4.2210	50 526	0.0973	4.9851	15.33%
α	−10%	47 391	0.1142	5.1873	45 115	0.1014	6.2943	17.59%
	−5%	47 427	0.1144	4.5047	45 147	0.1016	5.4010	16.60%
	0%	47 462	0.1146	3.8205	45 178	0.1018	4.5060	15.21%
	5%	47 498	0.1148	3.1347	45 210	0.102	3.6093	13.15%
	10%	47 533	0.115	2.4474	45 242	0.1022	2.7110	9.72%

time in VMI systems. Therefore, vendor is able to provide buyer with longer periods and consequently with larger average of order quantities in order to reduce total ordering costs of the chain. Also, for controlling the LSC of buyer the vendor has to increase the maximum level of inventory and cycle times. Hence the buyer is able to stock inventories in accordance to the space limitations. Besides, similar effect can be obtained in RMI systems under (R, T) with increasing unit ordering cost for the buyer. On the other hand changing the supplier's unit ordering costs does not vary the optimal value of cycle times and maximum level of inventories.

4.2.3. Effects of buyer's demand

Figures 3(e) and 4(e) are drawn to show the changes of decision variable based on changes in buyer's demand in (r, Q) and (R, T) replenishment review systems, respectively. From Figure 3(e), increasing buyer's demand leads to increase the order quantities for both of the VMI and RMI systems under (r, Q) policy. It is obvious that for meeting more demands the vendor has to place larger order quantities for buyer in VMI systems. Also, in RMI system the buyer has to place more order quantities which is similar to analysis of VMI system, but his decision is not essentially optimal for the chain and increases total cost of chain. Another significant point is that, increasing demands increases demands during lead-times (μ_L) in (r, Q) review systems. Therefore, vendor

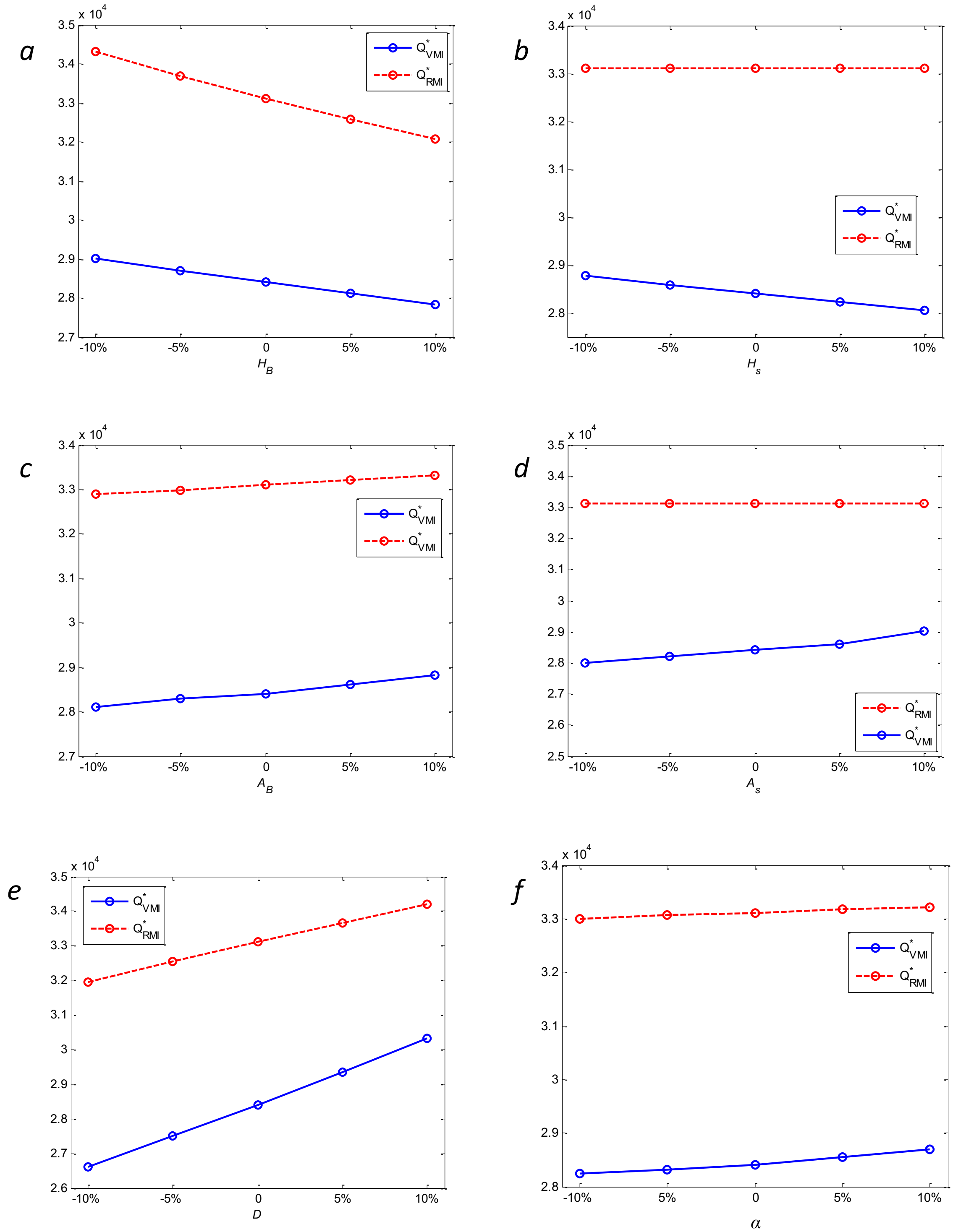

FIGURE 3. Changes in order quantities based on changing key parameters in (r, Q) system $(M = 2000)$.

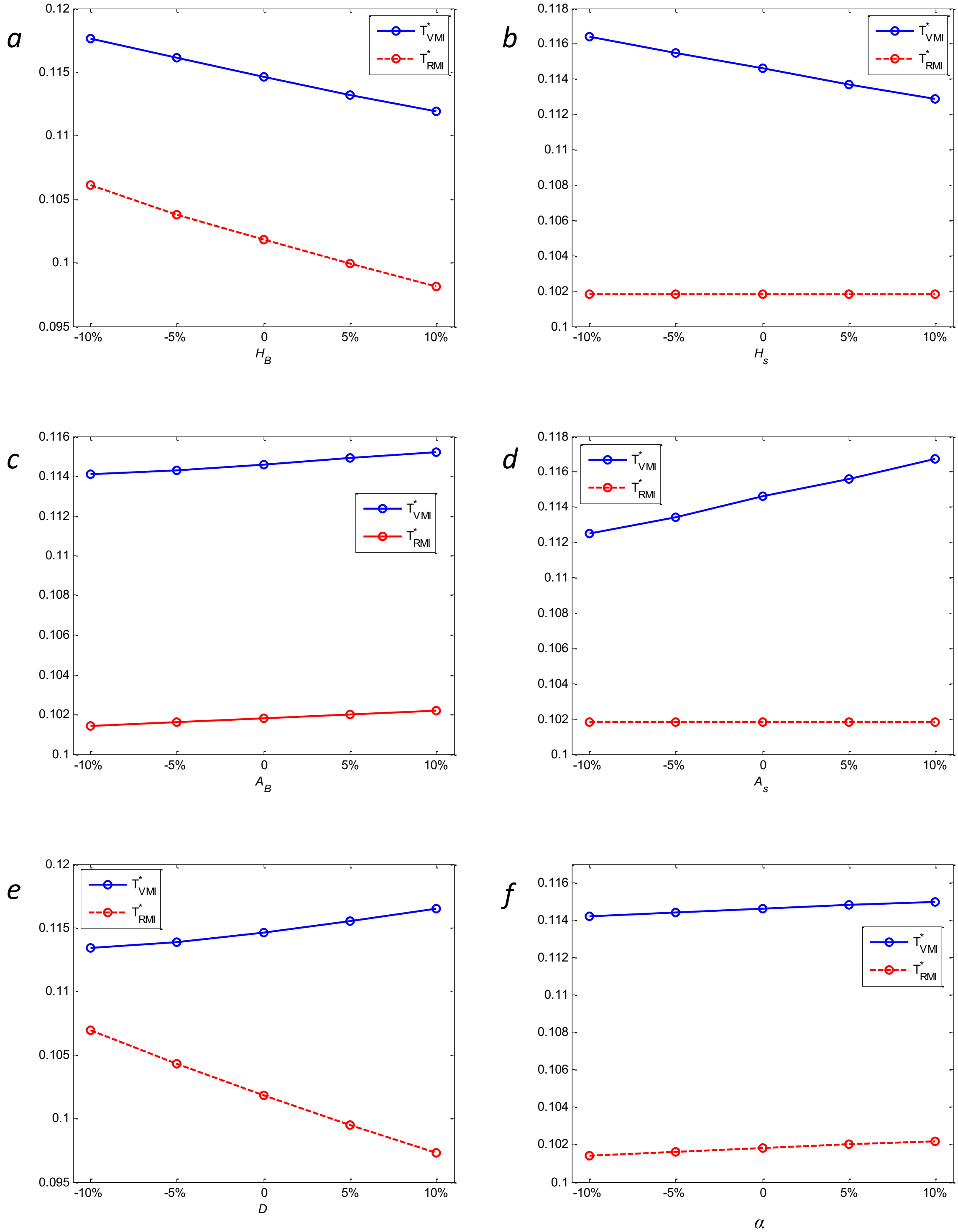

FIGURE 4. Changes in cycle times based on changing key parameters in (R, T) systems ($M = 2000$).

in VMI system and buyer in RMI systems decide to increase their reorder points by which they are able to place orders sooner with lower safety stocks. This circumstance enables them to order more quantities in VMI and RMI systems. Note that VMI systems are more sensitive than RMI systems to increasing demands and cost reduction is remarkable after implementing VMI systems.

As can be concluded from Figure 4(e) increasing in buyer's demand rate leads to increase the optimal cycle time in VMI under (R, T) policy and increase maximum level of inventory. It means, the average order quantities would be increased with increasing demand. Hence, the vendor is capable of managing placing larger order quantities. On the other hand, increase in demand leads to reduce optimal cycle time in RMI systems under (R, T) replenishment review system. Thus, buyer makes orders in fewer order frequencies with determining more maximum level of inventories. Shorter cycle times results in smaller averages of order quantities. On the other hand, increasing in demands causes to increase demands during lead-times (μ_{L+T}) in (R, T) review systems. Afterwards, buyer is able to stock more inventories at its own site leading to increase maximum level of inventories. From Figure 4(e) cycle times would be reduced during increasing in demand under RMI, because the buyer prefer to place larger order quantities in shorter cycle time to meet demand. But it is not optimal for the chain and imposes more costs to the chain in RMI systems. In addition, optimal cycle times in VMI systems under (R, T) is longer than that of RMI systems where the vendor acts in integrated view and makes optimal decisions with considering the whole chain.

4.2.4. Effects of backordering rate

As mentioned before, α precent of demands will be backordered and $1 - \alpha$ precent will be lost-sale. It is beneficial to help decision makers to show how then can improve performance of chain with altering backordering rate. As can be observed from Table 4, with reducing the backordering rate, total cost of chain under (r, Q) would be reduced and the chain tends to increase lost-sale to be more profitable. It is because of that the buyer has LSC and it has to encounter LSC constraint for meeting backorders but in lost-sale, the chain has not to satisfy the LSC constraint as much as possible. Also, Table 5 shows that with increasing the backordering rate under (R, T) policy, total cost of chain will be reduced significantly. Therefore, letting demands be backordered is more beneficial than letting them be lost-sale when (R, T) replenishment review system is applied, because decision maker should order for items more conservatism and has to increase the maximum level of inventory by which the backlogged level decreased. Consequently, cost of backlogs will be reduced in (R, T). Figure 3(f) shows the behaviour of order quantities of VMI and RMI with changing backordering rate. With increasing backordering rate, order quantities would be increased under (r, Q) in VMI and RMI policies, since the chain has to meet more unmet demands in future. This behaviour leads to increase the average on hand inventory and the total cost of chain. In addition, Figure 4(f) represents the effect of varying backordering rate on cycle time under (R, T) in VMI and RMI policies. With increasing backordering rate, cycle times would be increased and the chain has to order more quantities.

4.2.5. Cost reductions

Cost reductions after implementing VMI systems are calculated when key parameters are changed. Tables 4 and 5 reveal cost reductions in the presented VMI systems under (r, Q) and (R, T) replenishment review systems. As can be concluded from the both figures, cost reductions are positive in both settings meaning implementing VMI systems are beneficial when the buyer encounters LSC and partial backordering by which performance of commercial supply chains can be improved. When buyer's demand is high, cost savings are more significant after implementing VMI systems under (R, T) than that of (r, Q) policies. Similar results can be obtained about holding cost of buyer for VMI under (r, Q) review system. In this way, implementing VMI systems under (r, Q) and (R, T) policies, when ordering cost for buyer and holding cost for supplier increase, results in positive cost reductions but fewer than that of previous parameters. Thus, positive effect of employing VMI systems with LSC and PBO is proved through the sensitivity analysis and numerical experiments (see Tabs. 4 and 5). Totally, it is concluded that total cost of chain with VMI and (R, T) polices is lower than that of (r, Q) with considering LSC and PBO. It may imply that because of the LSC, the buyer should make decisions himself. In this paper it is shown that, sharing information of buyer with vendor not only about replenishment parameters, but also about storage capacities can result in more profitable decision for the chain.

5. Managerial implications

According to the comprehensive sensitivity analysis conducted and behaviour of inventory in proposed VMI and RMI systems, the main managerial implications are extracted and listed below:

- All the firms with single-vendor single-buyer, which encounters with LSC and PBO can get benefit and enhance the performance of chain by implementing the proposed VMI systems.
- If firms has the ability to select between implementing (r, Q) and (R, T) policies with same settings studied in this paper, it is better to implement (R, T), because total cost of chain under (R, T) would be less than that of (r, Q).
- Cost reductions after implementing proposed VMI systems in (R, T) policy are more than (r, Q) policy.
- Letting unmet demand be lost-sale in (r, Q) replenishment system is more beneficial than letting them be backordered for chains where buyer has LSC and planned backorders may be applied. Also, letting unmet demand be back-ordered is more beneficial than letting them be lost-sale in (R, T) replenishment system. Overall, if the chain uses (R, T) it is profitable to increase the backordering rate and if it utilizes (r, Q), it is desirable to decrease the backordering rate.

6. Conclusion

In VMI systems the vendor is responsible to make replenishment decisions by which help the chain to be more coordinated and beneficial. For this purpose, buyer provides vendor with online inventory information such as inventory level, demands, costs, and etc. Advantages of implementing VMI systems in superior to RMI systems are well-discussed in the literature. PBO and LSC are two main assumptions in many real-world cases which should be addressed while developing VMI systems. In VMI systems, vendor is more intended to place large quantity of orders for buyer as a result of making centralized decisions. On the other hand, buyer might encounter with limited storage capacities. Also, considering partial backordering where a certain ratio of demand encounter with backorder and others encounter with lost-sale is more realistic.

In this paper, two VMI systems for single-vendor and single-buyer with considering PBO and LSC are developed which account for stochastic demands. VMI systems are developed based on continuous review system (r, Q) and periodic review systems (R, T). Convexity conditions and required solution procedures are provided in order to solve the proposed VMI systems. Also, efficient algorithms are presented which are easy to implement for large scale inventory problems. In addition, counterpart RMI systems under (r, Q) and (R, T) replenishment review systems with considering PBO and LSC are developed to compare VMI systems with RMI systems.

A comprehensive numerical example inspired from a real case of Iranian automaker and parts providers is provided to represent the applicability and efficacy of the proposed VMI systems and presented algorithms. Also, sensitivity analyses are provided on key parameters and significant differences between VMI and RMI systems are discussed in (r, Q) and (R, T) review systems. According to the obtained results of numerical example and sensitivity analysis, superiority of the proposed VMI systems to their counterpart RMI systems can be concluded. It is demonstrated that the cost-efficiency of the chain can be enhanced after implementing the proposed VMI systems. Future work may consider backordering rate as a decision variables or a decreasing function of waiting time. Besides, some essential parameters such as demand and shipping times might be encountered uncertainties. Some coordination mechanism can be applied in the model in order to make centralized decisions which are profitable for buyer and vendor.

Appendix A.

For calculating backorders $\bar{b}(r)$ in (r, Q) inventory review systems, suppose that demand in lead time is distributed normally and follows $x \sim N\left(Dl\left(Q\right), \sigma^2 l\left(Q\right)\right)$. Thus, probability function of its normal distribution is as follows:

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}\sqrt{l(Q)}}\mathrm{e}^{-1/2\left(\frac{x-Dl(Q)}{\sigma\sqrt{l(Q)}}\right)^2}. \tag{A.1}$$

As mentioned before $l(Q)=Q/P+b$. Therefore, buyer's backorders during the lead time can be calculated as below:

$$\bar{b}\left(r,l\left(Q\right)\right)=\int_r^\infty (x-r)f(x)\mathrm{d}x. \tag{A.2}$$

Thus, we have:

$$\bar{b}\left(r,l\left(Q\right)\right)=\int_r^\infty \frac{(x-Dl(Q))}{\sqrt{2\pi}\sigma\sqrt{l\left(Q\right)}}\mathrm{e}^{-\frac{1}{2}\left(\frac{x-Dl(Q)}{\sigma\sqrt{l(Q)}}\right)^2}\mathrm{d}x \tag{A.3}$$

To calculate the equation we have:

$$\frac{x-Dl(Q)}{\sigma\sqrt{l\left(Q\right)}}=u\rightarrow \mathrm{d}x=\sigma\sqrt{l\left(Q\right)}\mathrm{d}u \tag{A.4}$$

$$\frac{r-D\left(l\left(Q\right)\right)}{\sigma\sqrt{l(Q)}}=k_p\rightarrow r=k_p\sigma\sqrt{l(Q)}+Dl\left(Q\right). \tag{A.5}$$

Finally we can reformulate the backorders during lead time as follows:

$$\bar{b}\left(r,l\left(Q\right)\right)=\sigma\sqrt{l\left(Q\right)}\int_k^\infty (u-k)\,\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}(u)^2}\mathrm{d}u=\sigma\sqrt{l\left(Q\right)}\mathrm{Gu}(k_p)=\sigma\sqrt{Q/P+b}\mathrm{Gu}(k_p). \tag{A.6}$$

Where right hand unit of normal linear-loss integral can be determined by $\mathrm{Gu}(k)=\varphi(k)-k\overline{\varphi}(k)$. Similar procedures can be performed to calculate the backorders during lead-time in (R,T) policy. Note that in (R,T) review systems we have $l(\mathrm{DT})+T=\frac{\mathrm{DT}}{P}+T+b$.

Appendix B.

Proof of proposition 3.1. To proof the Proposition 3.1, we should prove how the hessian matrix of $\mathrm{TC}_{\mathrm{VMI}}^{r,Q}$ is positive. For this purpose, we have:

$$H^{(1)}=\begin{bmatrix}\frac{\partial^2\mathrm{TC}_{\mathrm{VMI}}^{r,Q}}{\partial^2 Q} & \frac{\partial^2\mathrm{TC}_{\mathrm{VMI}}^{r,Q}}{\partial Q\partial r}\\ \frac{\partial^2\mathrm{TC}_{\mathrm{VMI}}^{r,Q}}{\partial^2 r} & \frac{\partial^2\mathrm{TC}_{\mathrm{VMI}}^{r,Q}}{\partial r\partial Q}\end{bmatrix} \tag{B.1}$$

where

$$\frac{\partial^2\mathrm{TC}(r,Q)}{\partial^2 Q}=\frac{\begin{pmatrix}8DPA_B\left(bP+Q\right)\sqrt{b+\frac{Q}{P}}+8DPA_S\left(bP+Q\right)\sqrt{b+\frac{Q}{P}}-Q^3h_B\sigma\left(\left(1-\alpha\right)\mathrm{Gu}(k_p)+k_p\right)\\ +D\mathrm{Gu}(k_p)\sigma\left(\alpha\pi_1+\pi_2\left(1-\alpha\right)\right)\left(8b^2P^2+12bPQ+3Q^2\right)\end{pmatrix}}{4PQ^3\left(bP+Q\right)\sqrt{b+\frac{Q}{P}}} \tag{B.2}$$

$$\frac{\partial^2\mathrm{TC}_{\mathrm{VMI}}^{r,Q}}{\partial Q\partial r}=0 \tag{B.3}$$

$$\frac{\partial^2\mathrm{TC}_{\mathrm{VMI}}^{r,Q}}{\partial^2 r}=\frac{-\left(\left(\alpha\pi_1+(1-\alpha)\pi_2\right)D+(1-\alpha)h_B\right)}{Q}f(r) \tag{B.4}$$

$$\frac{\partial^2\mathrm{TC}_{\mathrm{VMI}}^{r,Q}}{\partial r\partial Q}=\frac{\left(\alpha\pi_1+(1-\alpha)\pi_2\right)D}{Q^2}\overline{F}(r). \tag{B.5}$$

Let us define:

$$\beta = 8DPA_B\left(bP+Q\right)\sqrt{b+\frac{Q}{P}} + 8DPA_S\left(bP+Q\right)\sqrt{b+\frac{Q}{P}} \tag{B.6}$$
$$\gamma = Q^3 h_B \sigma\left(\left(1-\alpha\right)\mathrm{Gu}(k_p)+k_p\right) \tag{B.7}$$
$$\eta = DGu(k_p)\sigma\left(\alpha\pi_1+\pi_2\left(1-\alpha\right)\right)\left(8b^2P^2+12bPQ+3Q^2\right). \tag{B.8}$$

Therefore, the first and second principal minors of $H^{(1)}$ is determined by equations (B.9) and (B.10), respectively:

$$\left|H_{11}^{(1)}\right| = \frac{\partial^2 \mathrm{TC}_{\mathrm{VMI}}^{r,Q}}{\partial^2 Q} > 0 \to \beta+\eta>\gamma \tag{B.9}$$

$$\left|H_{22}^{(2)}\right| = \frac{\left(\alpha\pi_1+(1-\alpha)\pi_2\right)D\overline{F}(r)}{Q^2} \times \frac{\begin{pmatrix} 8DPA_B\left(bP+Q\right)\sqrt{b+\frac{Q}{P}}+8DPA_S\left(bP+Q\right)\sqrt{b+\frac{Q}{P}}-Q^3h_B\sigma\left(\left(1-\alpha\right)\mathrm{Gu}(k_p)+k_p\right) \\ +DGu(k_p)\sigma\left(\alpha\pi_1+\pi_2\left(1-\alpha\right)\right)\left(8b^2P^2+12bPQ+3Q^2\right)\end{pmatrix}}{4PQ^3\left(bP+Q\right)\sqrt{b+\frac{Q}{P}}}. \tag{B.10}$$

As the second principal minor of Hshould be positive, we have:

$$\left|H_{22}^{(2)}\right| > 0 \to \beta+\eta>\gamma. \tag{B.11}$$

□

Proof of Proposition 3.3. To proof the Proposition 3.3, there is a need to find conditions which the hessian matrix of $\mathrm{TC}_{\mathrm{VMI}}^{R,T}$ is positive:

$$H^{(2)} = \begin{bmatrix} \frac{\partial^2 \mathrm{TC}_{\mathrm{VMI}}^{R,T}}{\partial^2 T} & \frac{\partial^2 \mathrm{TC}_{\mathrm{VMI}}^{R,T}}{\partial T\partial R} \\ \frac{\partial^2 \mathrm{TC}_{\mathrm{VMI}}^{R,T}}{\partial^2 R} & \frac{\partial^2 \mathrm{TC}_{\mathrm{VMI}}^{R,T}}{\partial R\partial T}\end{bmatrix} \tag{B.12}$$

where

$$\frac{\partial^2 \mathrm{TC}_{\mathrm{VMI}}^{R,T}}{\partial^2 T} = \frac{\begin{pmatrix} 8\left(A_B+A_s\right)\left(b+\frac{\mathrm{DT}}{P}+T\right)^{3/2}+8\mathrm{Gu}(k_p)\sigma\left(\alpha\pi_1+\pi_2\left(1-\alpha\right)\right)\left(b+\frac{\mathrm{DT}}{P}+T\right)^2 \\ -4L\sigma T\left(\alpha\pi_1-\pi_2\left(\alpha-1\right)\right)\left(b+\frac{\mathrm{DT}}{P}+T\right)-\left(\left(1-\alpha\right)\mathrm{Gu}(k_p)\sigma+\sigma k_p\right)T^3h_B \\ -\mathrm{Gu}(k_p)\sigma T^2\left(\alpha\pi_1+\pi_2\left(1-\alpha\right)\right)\end{pmatrix}}{4T^3\left(b+\frac{\mathrm{DT}}{P}+T\right)^{3/2}} \tag{B.13}$$

$$\frac{\partial^2 \mathrm{TC}_{\mathrm{VMI}}^{R,T}}{\partial T\partial R} = 0 \tag{B.14}$$

$$\frac{\partial^2 \mathrm{TC}_{\mathrm{VMI}}^{R,T}}{\partial^2 R} = -\frac{\left(\alpha\pi_1+(1-\alpha)\pi_2+(1-\alpha)h_B\right)f(R)}{T} \tag{B.15}$$

$$\frac{\partial^2 \mathrm{TC}_{\mathrm{VMI}}^{R,T}}{\partial R\partial T} = \frac{\left(\alpha\pi_1+(1-\alpha)\pi_2\right)\overline{F}(R)}{T^2}. \tag{B.16}$$

Following notations are defined to continue the formulations:

$$\omega = 8\left(A_B + A_s\right)\left(b + \frac{\mathrm{DT}}{P} + T\right)^{3/2} \tag{B.17}$$

$$\psi = 8\mathrm{Gu}(k_p)\sigma\left(\alpha\pi_1 + \pi_2\left(1-\alpha\right)\right)\left(b + \frac{\mathrm{DT}}{P} + T\right)^{2} \tag{B.18}$$

$$\nu = 4\mathrm{Gu}(k_p)\sigma T\left(\alpha\pi_1 - \pi_2\left(\alpha-1\right)\right)\left(b + \frac{\mathrm{DT}}{P} + T\right) \tag{B.19}$$

$$\kappa = \left(\left(1-\alpha\right)\mathrm{Gu}(k_p)\sigma + \sigma k_p\right)T^3 h_B - \mathrm{Gu}(k_p)\sigma T^2\left(\alpha\pi_1 + \pi_2\left(1-\alpha\right)\right). \tag{B.20}$$

As the first principal minor of $H^{(2)}$should be positive, we have:

$$21\left|H_{11}^{(2)}\right| = \frac{\partial^2 \mathrm{TC}(R,T)}{\partial^2 T} > 0 \rightarrow \omega + \psi > \nu + \kappa. \tag{B.21}$$

And the second principal minor of $H^{(2)}$ is as follows:

$$\left|H_{22}^{(2)}\right| = \frac{\left(\alpha\pi_1 + (1-\alpha)\pi_2\right)\overline{F}(R)}{4T^5\left(b + \frac{\mathrm{DT}}{P} + T\right)^{3/2}}\begin{pmatrix} 8\left(A_B + A_s\right)\left(b + \frac{\mathrm{DT}}{P} + T\right)^{3/2} + 8\mathrm{Gu}(k_p)\sigma\left(\alpha\pi_1 + \pi_2\left(1-\alpha\right)\right)\left(b + \frac{\mathrm{DT}}{P} + T\right)^{2} \\ -4L\sigma T\left(\alpha\pi_1 - \pi_2\left(\alpha-1\right)\right)\left(b + \frac{\mathrm{DT}}{P} + T\right) - \left(\left(1-\alpha\right)\mathrm{Gu}(k_p)\sigma + \sigma k_p\right)T^3 h_B \\ -\mathrm{Gu}(k_p)\sigma T^2\left(\alpha\pi_1 + \pi_2\left(1-\alpha\right)\right) \end{pmatrix} \tag{B.22}$$

As the second principal minor $H_{22}^{(2)}$should be positive, we have:

$$\left|H_{22}^{(2)}\right| > 0 \rightarrow \omega + \psi > \nu + \kappa. \tag{B.23}$$

□

Proof of Proposition 3.5. $H^{(3)}$ denotes the hessian matrix of $\mathrm{KB}_{\mathrm{RMI}}^{r,Q}$ which its first and second minors should be positive:

$$H^{(3)} = \begin{bmatrix} \frac{\partial^2 \mathrm{KB}_{\mathrm{RMI}}^{r,Q}}{\partial^2 Q} & \frac{\partial^2 \mathrm{KB}_{\mathrm{RMI}}^{r,Q}}{\partial Q \partial r} \\ \frac{\partial^2 \mathrm{KB}_{\mathrm{RMI}}^{r,Q}}{\partial^2 r} & \frac{\partial^2 \mathrm{KB}_{\mathrm{RMI}}^{r,Q}}{\partial r \partial Q} \end{bmatrix} \tag{B.24}$$

where we have:

$$\frac{\partial^2 \mathrm{KB}_{\mathrm{RMI}}^{r,Q}}{\partial^2 Q} = \frac{\left(8DPA_B\left(bP + Q\right)\sqrt{b + \frac{Q}{P}} + \mathrm{Gu}(k_p)\sigma\left(\left(\pi_1\alpha + (1-\alpha)\pi_2\right)D\left(8b^2P^2 + 12bPQ + 3Q^2\right) - \left(1-\alpha\right)Q^3 h_B\right)\right)}{\left(4PQ^3\left(bP + Q\right)\sqrt{b + \frac{Q}{P}}\right)} \tag{B.25}$$

$$\frac{\partial^2 \mathrm{KB}_{\mathrm{RMI}}^{r,Q}}{\partial Q \partial r} = 0 \tag{B.26}$$

$$\frac{\partial^2 \mathrm{KB}_{\mathrm{RMI}}^{r,Q}}{\partial^2 r} = \frac{-\left(\left(\alpha\pi_1 + (1-\alpha)\pi_2\right)D + (1-\alpha)h_B\right)}{Q}f(r) \tag{B.27}$$

$$\frac{\partial^2 \mathrm{KB}_{\mathrm{RMI}}^{r,Q}}{\partial r \partial Q} = \frac{\left(\alpha\pi_1 + (1-\alpha)\pi_2\right)D}{Q^2}\overline{F}(r). \tag{B.28}$$

Let us define:

$$\upsilon = 8DPA_B\,(bP+Q)\sqrt{b+\frac{Q}{P}} \tag{B.29}$$

$$\varsigma = \mathrm{Gu}(k_p)\sigma D\,(\pi_1\alpha+(1-\alpha)\pi_2)\left(8b^2P^2+12bPQ+3Q^2\right) \tag{B.30}$$

$$\vartheta = (1-\alpha)\,\mathrm{Gu}(k_p)\sigma Q^3 h_B \tag{B.31}$$

As the first principal minor should be positive, we have:

$$\left|H_{11}^{(3)}\right| > 0 \rightarrow \upsilon+\varsigma > \vartheta. \tag{B.32}$$

In addition, the second principal can be calculated as follows:

$$\left|H_{22}^{(3)}\right| = \frac{(\alpha\pi_1+(1-\alpha)\pi_2)\,D\overline{F}(r)}{Q^2} \times \frac{\left(8DPA_B\,(bP+Q)\sqrt{b+\frac{Q}{P}}+\mathrm{Gu}(k_p)\sigma\begin{pmatrix}(\pi_1\alpha+(1-\alpha)\pi_2)\,D\left(8b^2P^2+12bPQ+3Q^2\right)\\ -(1-\alpha)\,Q^3h_B\end{pmatrix}\right)}{\left(4PQ^3\,(bP+Q)\sqrt{b+\frac{Q}{P}}\right)}. \tag{B.33}$$

As the second principal should be positive, we have:

$$\left|H_{22}^{(3)}\right| > 0 \rightarrow \upsilon+\varsigma > \vartheta. \tag{B.34}$$

□

Proof of Preposition 3.7. $H^{(4)}$ as the hessian matrix of $\mathrm{KB}_{\mathrm{RMI}}^{R,T}$needs to be positive:

$$H^{(4)} = \begin{bmatrix}\frac{\partial^2\mathrm{KB}_{\mathrm{RMI}}^{R,T}}{\partial^2T} & \frac{\partial^2\mathrm{KB}_{\mathrm{RMI}}^{R,T}}{\partial T\partial R}\\ \frac{\partial^2\mathrm{KB}_{\mathrm{RMI}}^{R,T}}{\partial^2R} & \frac{\partial^2\mathrm{KB}_{\mathrm{RMI}}^{R,T}}{\partial R\partial T}\end{bmatrix} \tag{B.35}$$

where we have:

$$\frac{\partial^2\mathrm{KB}_{\mathrm{RMI}}^{R,T}}{\partial^2T} = \frac{1}{4P^2T^3\left(b+\frac{\mathrm{DT}}{P}+T\right)^{3/2}}\left(\begin{array}{l}8PA_B\,(bP+T\,(D+P))\sqrt{b+\frac{\mathrm{DT}}{P}+T}\\ +\left(\begin{array}{l}\mathrm{Gu}(k_p)\,(\alpha\pi_1+(1-\alpha)\pi_2)\,\sigma\begin{pmatrix}8b^2P^2+12bPT\,(D+P)\\ +3T^2\,(D+P)^2\end{pmatrix}\\ +T^3h_B\,(D+P)^2\,\sigma\,((\alpha-1)\,\mathrm{Gu}(k_p)-k_p)\end{array}\right)\end{array}\right) \tag{B.36}$$

$$\frac{\partial^2\mathrm{KB}_{\mathrm{RMI}}^{R,T}}{\partial T\partial R} = 0 \tag{B.37}$$

$$\frac{\partial^2\mathrm{KB}_{\mathrm{RMI}}^{R,T}}{\partial^2R} = -\frac{(\alpha\pi_1+(1-\alpha)\pi_2+(1-\alpha)h_B)\,f(R)}{T} \tag{B.38}$$

$$\frac{\partial^2\mathrm{KB}_{\mathrm{RMI}}^{R,T}}{\partial R\partial T} = \frac{(\alpha\pi_1+(1-\alpha)\pi_2)\,\overline{F}(R)}{T^2}. \tag{B.39}$$

Let assume:

$$\chi = 8PA_B\,(bP+T\,(D+P))\sqrt{b+\frac{\mathrm{DT}}{P}+T} \tag{B.40}$$

$$\zeta = \mathrm{Gu}(k_p)\,(\alpha\pi_1+(1-\alpha)\pi_2)\,\sigma\left(8b^2P^2+12bPT\,(D+P)+3T^2\,(D+P)^2\right) \tag{B.41}$$

$$\phi = T^3h_B\,(D+P)^2\,\sigma\,((\alpha-1)\,\mathrm{Gu}(k_p)-k_p)\,. \tag{B.42}$$

As the first principal minor of $H^{(4)}$ should be positive, we have:

$$\left|H_{11}^{(4)}\right| > 0 \rightarrow \chi + \zeta + \phi > 0 \tag{B.43}$$

Also, the second principal minor of $H^{(4)}$ is obtained as follows:

$$\left|H_{22}^{(4)}\right| = \frac{(\alpha\pi_1 + (1-\alpha)\pi_2)\,\overline{F}(R)}{T^2} \times \frac{\left(\begin{array}{l} 8PA_B\,(bP + T\,(D+P))\sqrt{b + \frac{\mathrm{DT}}{P} + T} \\ + \left(\begin{array}{l} \mathrm{Gu}(k_p)\,(\alpha\pi_1 + (1-\alpha)\pi_2)\,\sigma \left(\begin{array}{l} 8b^2P^2 + 12bPT\,(D+P) \\ +3T^2\,(D+P)^2 \end{array}\right) \\ +T^3 h_B\,(D+P)^2\,\sigma\,((\alpha-1)\,\mathrm{Gu}(k_p) - k_p) \end{array}\right) \end{array}\right)}{4P^2T^3\left(b + \frac{\mathrm{DT}}{P} + T\right)^{3/2}}. \tag{B.44}$$

For positivity of the second principal minor, we have:

$$\left|H_{22}^{(4)}\right| > 0 \rightarrow \chi + \zeta + \phi > 0. \tag{B.45}$$

□

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