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TWO PHASES INVENTORY MODEL WITH VARIABLE CYCLE LENGTH UNDER DISCOUNT POLICY

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Abstract. This study deals with single stage inventory model where two phases are involved in an inventory cycle. In the first phase of the cycle, demand depends on both of inventory level and selling price while in the second, the demand depends on price only. Discount policy in selling price is offered in the second phase and inventory level at the end of the cycle is taken to be zero. Two models have been constructed on infinite time horizon. In the first model the demand rate is taken as the sum of two linear functions of inventory level and selling price and, in the second model, it is taken as a product of two power functions of inventory level and selling price. Our objective is to maximize average profit by considering ordering lot size and selling price as decision variables. Numerical examples of each model have been provided. The optimality criteria for the solutions are also checked by both graphically and numerically. Sensitivity analysis for different parameters in both models has been discussed in details to check the feasibility of the models.

Mathematics Subject Classification. 90B05.

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1. Introduction

Many inventory models have been constructed by taking demand constant over the inventory cycle. But in real life situation it is not always possible. The natural trend in the market is that people is attracted by a large display of commodity. Due to this reason number of shopping malls is increasing day by day. Presently the market situation is highly competitive due to different media. Customers generally collect the feedback of the product of the same category of different brands and then they take the decision. The choice of people is very sensitive on price and stock. So it is a huge problem in this competitive market for the retailer who sells the product of a particular brand. For example, who sells jeans of a particular brand faces this type of problem like Killer, Pepe, Denim, etc. The problem is that if the stock of the product decreases, then generally customers are switching over to the other brand. Then, the buyer has to think about the situation to clear his remaining stock. In that situation, he may offer discount policy on the products to attract the customers.

Keywords. Two component of demand, discount policy, price, variable cycle length.

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Two components of demand in two phases in an inventory cycle may be observed by a retailer who sells a particular brand of a product like shoes, jeans, suiting and shirting, electronic gadgets, etc. Here initially demand is sensitive on both stock and price. But as cycle time progresses, stock of the products decrease and customers' choice is going to be limited. After a certain level of stock of products, inventory level may not grow interest to enhance the demand. For that situation, the retailer has to find new strategy to clear the rest of stock. As a result, the retailer can offer discount policy to empty the stock. So in the second phase, the demand of the products depends generally price only.

Another interesting example of the above situation is the industries who are manufacturing shoes of a particular brand like Bata, Adidas, Reebok, Puma, etc. When the inventory cycle starts, there is the largest display of shoes. As time progresses in the cycle, inventory level decreases due to demand. After a certain period of time, the manufacturer has to consider two situations. The first one, he will go for further replenishment of stock. The second one, he will give some discount to reduce the stock. There are many constraints for the first situation. Limitation of capital, holding cost, new products are available, huge competition among several brands, and uncertain nature of people's demand may be those constraints. If stock of shoes remains for a considerable amount of time, the glossiness of shoes reduces which may lead to a constraint for the people's choice. It should be beneficial for the manufacturer if he goes for the second situation. The customer may show eagerness to buy shoe of reputed brand name at a discounted price. Our models are also useful for those retailers who sell dress materials of particular brands like Raymond, Grasim, Vimal, etc.

2. Literature survey

It is a common belief that large piles of the displayed goods always attract people to buy. So the demand rate of a product may be depended on its stock level. This idea was first taken into consideration in an inventory model by Gupta and Vrat [10]. In that model consumption rate was a function of initial stock level. Later, Baker and Urban [6] developed an inventory model where the demand rate was a function of instantaneous inventory level. After that, Datta and Pal [7] presented an inventory model in which demand rate was depended on the instantaneous inventory level until a given time when the inventory level down to a certain level. They also assumed that the demand rate was constant for the remaining period. Urban [33] extended this model by relaxing terminal condition of zero inventory at the end of the cycle. Paul *et al.* extended Datta and Pal's model [7] by inclusion of shortage. Padmanabhan and Vrat [20] constructed an inventory model for perishable items, where the demand rate was depending linearly on the instantaneous inventory level. They also studied the model for three situations: without backlogging, complete backlogging and partial backlogging. A comprehensive review of literature of two types of inventory models was presented by Urban [34] where the demand rate was depended on the initial inventory level and instantaneous inventory level. He also showed the difference and similarities between two concepts. An EPQ model with stock and price dependent demand is considered by Teng and Chang [30]. They imposed an upper bound on display stock. Hou and Lin [13] formulated an inventory model over a finite horizon taking stock dependent demand. In that model shortage was partially backlogged at a variable rate. An inventory model with stock dependent demand and variable holding cost was developed by Alfares [4]. An inventory model with shortage where demand is dependent on stock and price was considered by Jain *et al.* [14]. Zhou *et al.* [42] presented an inventory model where supply chain coordination was considered and demand was a power function of the inventory level. In that model, inflation and time value of money was considered. Wu [36] developed an optimal replenishment policy for non- instantaneous deteriorating items with inflation for deteriorating items with stock dependent demand and shortages partially backlogged was developed by Yang *et al.* [38]. An inventory model for perishable items with stock dependent demand was developed by Heish and Dye [11]. In that model limitation of display space was also considered. An extension of the model of You and Heish [40] incorporating deterioration was done by Tripathy *et al.*

[32]. Duan *et al.* [8] presented an inventory model with stock dependent demand for perishable items. In the backlogging model, the partial backlogging rate was depended on both waiting time and the amount of product already backlogged. Sarkar [26] considered an inventory model for deteriorating products with stock dependent demand. In that model time varying backlogging and deteriorating rate also had been taken in consideration. Krommyda *et al.* [15] constructed an inventory model with stock dependent demand. In that model demand was satisfied by two mutually substitutable products. An inventory model for deteriorating items, where demand is a linear function of stock along with trade credit policy was developed by Pervin *et al.* [25]. In that model holding cost was a monotonic increasing function of time and price discount policy adopted on backordered items.

Abad [1, 2] considered the pricing and lot-sizing problem. An inventory model where the demand rate was a linear function of selling price along with quantity discount and partial backordering was considered by Wee [35]. Papachristos and Skouri [23] analyzed an inventory model where the demand rate was a convex decreasing function of the selling price together with the variable backlogging rat. There are many works with price dependent demand. Works of Yang [39], Mukhopadhyay *et al.* [19], Teng *et al.* [31], Abad [3] are noteworthy. An integrated inventory model with price-sensitive demand was developed by Lin and Ho [16]. In that model quantity discount policy was also applied. An inventory model for perishable items, where the demand rate was a quadratic decreasing function of selling price over a finite time horizon was developed by Sana [27]. Shi *et al.* [28] considered an inventory model where demand was taken to be both linearly price sensitive and additive stochastic. Pal *et al.* [22] developed an imperfect production, inventory model with two cycles where demand decreases quadratic ally with respect to the selling price. Alfares *et al.* [5] developed an inventory model with price dependent demand. In that model holding cost was taken to be a linear increasing function of storage time and the purchased cost was a decreasing step function of order size.

You and Heish [40] presented an inventory model for a seasonal item over a finite planning horizon. Demand rate was taken to be dependent on price and stock. Dye and Heish [9] considered an inventory model for deteriorating items, where demand was dependent on both price and stock. In that model finite planning horizon and limitation on display items also considered. Soni [29] formulated an inventory model with stock and price dependent demand for non-instantaneous deteriorating items. In that model inventory deteriorates at a constant rate after a fixed time period. Pal *et al.* [22] developed an inventory model with stock and price dependent demand under inflation along with a permissible delay in payments. Some authors Zhang *et al.* [41], Lu *et al.* [17] use Pontryagin's maximum principle to solve the problem of maximization of profit in an inventory model with stock and price dependent demand. An inventory model for a seasonal perishable product is developed by Mishra *et al.* [18]. In that model deterioration rate is controlled by some investment in preservation technology.

In the following, Table 1 provides a comparison of this work with past researchers' works.

2.1. Problem definition

Here two single stage inventory models are developed with two components of demand in two phases in a single inventory cycle (see Fig 1). Demand rate is depended on both stock and selling price for the first phase of the inventory cycle. When the stock level reaches from initial inventory level Q to inventory level μQ (where μ is a positive proper fraction) the second phase starts. In the second phase demand rate depends on selling price only and the supplier offer a discount in selling price to exhaust the remaining stock. In the first model demand rate is the sum of two linear functions in selling price and stock for the first cycle. While in the second model, demand rate is the product of two power function in selling price and stock for the first cycle. Both models have been framed by taking time horizon infinite. Our objective is to maximize average profit function when both order quantity and selling price are decision variables.

TABLE 1. A comparison of the present work with the related previous works.

Literature	Nature of inventory EOQ/EPQ	Two component of demand when there is no shortage/inventory level is non-negative	Demand rate	Discount in selling price
Datta and Pal [7]	EOQ	Yes	Power function of stock level	No
Padmanabhan and Vrat [20]	EOQ	No	Linear function of stock level	No
Papachristos and Skouri [23]	EOQ	No	Convex decreasing function of selling price	No
Teng and Chang [30]	EPQ	No	Sum of two function: non-linear function of price and linear in stock	No
Wu *et al.* [36]	OQ	No	Linear function of stock level	No
Hou and Lin [13]	EOQ	No	Sum of two functions: Linear function of price and stock; Power function of price and linear function of stock	No
Jain *et al.* [14]	EPQ	No	Sum of price and stock	No
You and Heish [40]	EOQ	No	Sum of two function: linear function in selling price and stock level	No
Sana [27]	EOQ	No	Quadratic function of selling price	No
Duan *et al.* [8]	EOQ	No	Linear function of stock level	No
Tripathy *et al.* [32]	EOQ	No	Sum of two linear function in stock and price	No
Sarkar [26]	EOQ	No	Linear function of stock level	No
Pal *et al.* [21]	PQ	No	Quadratic function of price	Yes
Yang [37]	EOQ	No	Power function of stock level	No
Krommyda *et al.* [15]	EOQ	No	Linear function of stock level	No
Alfares and Ghaithan [5]	EOQ	No	Linear function of selling price	No
Lu *et al.* [17]	EOQ	No	Product of two linear function of stock level and selling price	No
Mishra *et al.* [18]	EOQ	No	Model 1: Sum of two function: linear function of selling price & stock level Model 2: Sum of two function: power function of selling price & linear function of stock level	No
Hemmati *et al.* [12]	EPQ	No	Sum of two linear function of stock level and selling price	No
Pervin *et al.* [25]	EOQ	No	Linear function of stock	Yes
This Paper	EOQ	Yes	Model 1: Sum of two function: linear function of selling price & stock level Model 2: Product of two function: Power function of selling price and stock level	Yes

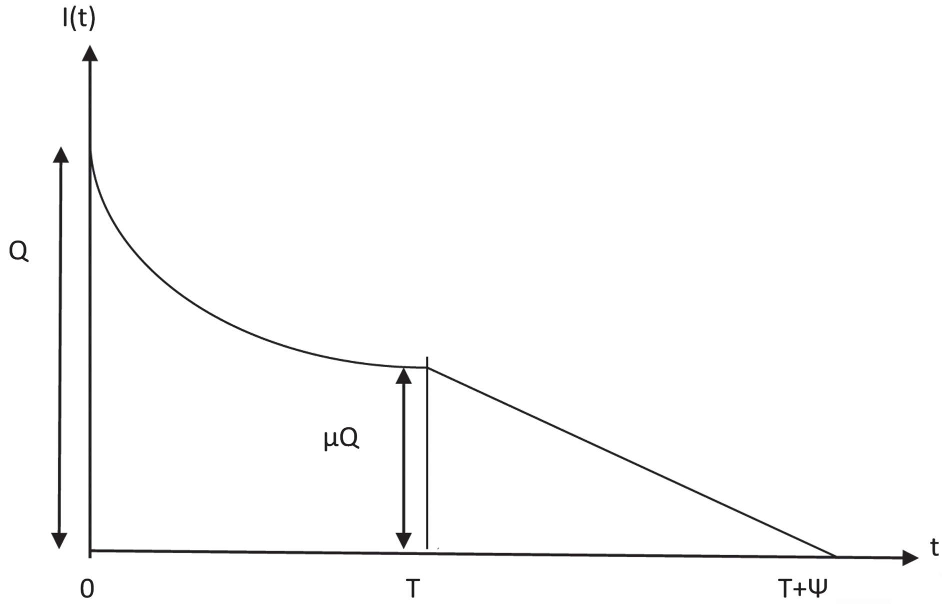

FIGURE 1. Inventory level *vs.* time.

3. Notations

The following notations are used to develop the model

Q	Ordering lot size
s	Selling price per unit item
k	Set up cost per order
h	Holding cost per unit per unit time
c	Purchase cost per unit
$I(t)$	Inventory level at time t
T	Time schedule for the first cycle
Ψ	Time schedule for the second cycle
α	Scale parameter of the demand
β	Shape parameter of the demand
P	Average Profit per unit item.

4. Assumptions

The following assumptions are made to develop the model

(i) The inventory system involves only one item and shortage is not permitted.
(ii) Time horizon is infinite.
(iii) Models are developed over two phases within a cycle of period $[0, T+\Psi]$. The first phase is $[0, T]$ and the second phase is $[T, T+\Psi]$,bothof the models have two phases in the same inventory cycle.
(iv) Discount in selling price is offered during the second phase of the inventory cycle. Selling price of each quantity for the first phase is s and that is for the second phase is θs where $(1-\theta)\,100$ is the discounting percentage on the selling price.

(v) Demand rate (r) is dependent on the stock and the selling price during the period $[0, T]$. For the first model, the demand rate $r = a - bs + \gamma I(t)$ [40] is considered where a, b, γ all are positive such that r always remains nonnegative. In the second model, demand rate is considered as $r = \alpha \{I(t)\}^{\beta} s^{-\gamma}$ with $\alpha > 0, 0 < \beta < 1, \gamma > 0$.

(vi) In the second phase of the inventory cycle for both the models, the demand rate is considered as only sensible on selling price during the period $[T, T + \Psi]$. For the first model, the demand rate is assumed as $r = a_1 - b_1\theta s$ where $a_1 > 0, b_1 > 0, 0 < \theta < 1$ and $s < \frac{a_1}{b_1\theta}$ and for the second model, the demand rate is considered as $r = \alpha_1 (\theta s)^{-\gamma_1}$ with $\alpha_1 > 0, \gamma_1 > 0$. Here, $1 - \theta$ is considered as the discount rate on the selling price. As the cycle progresses, the inventory level decreases. Initially, when cycle of a new lunched product is started, the demand rate generally depends on displayed stock and selling price. After a certain period of time, information about the products as well as feedback about the products is well known to the customers. So, generally, demand rate is not too much depending on stock after a certain time. Then the demand rate may depend on selling price only. During this time, discount in selling price is offered to the customers to increase the demand rate.

(vii) The inventory level $I(t)$ is taken to be a continuous function at time $t = T$, the point where demand changes from one type to another type.

5. Model description

In this model, an inventory model is developed where two types of demand patters are considered. Initially, the model starts with an ordering lot Q and after scheduling time $T + \Psi$, the inventory level goes to zero. After the first phase of time length T, when the inventory level reaches at μQ, where $0 < \mu < 1$, the second phase will start.

In the first model, the demand rate is considered as $r = a - bs + \gamma I(t)$ in the initial phase and in the second phase, the demand rate is assumed as $r = a_1 - b_1\theta s$. For the second model, demand rate is $r = \alpha \{I(t)\}^{\beta} s^{-\gamma}$ in the first phase and in the second phase $[T, T + \Psi]$, the demand rate is $r = \alpha_1 (\theta s)^{-\gamma_1}$. Discount in selling price is offered only in the second cycle.

5.1. Model I

The differential equation governing inventory level for the time period $\lceil 0, T \rceil$ is given by

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -\{a - bs + \gamma I(t)\} \text{ Subject to } I(0) = Q, I(T) = \mu Q \text{ where } 0 < \mu < 1.$$

Solution of the differential equation using initial condition $I(0) = Q$ is given by

$$I(t) = Qe^{-\gamma t} - \frac{(a - bs)}{\gamma}\left\{1 - \mathrm{e}^{-\gamma t}\right\}. \tag{5.1}$$

Using the condition $I(T) = \mu Q$, the equation (5.1) reduces to

$$\frac{a - bs}{\gamma Q} = \frac{\mathrm{e}^{-\gamma T} - \mu}{1 - \mathrm{e}^{-\gamma T}}. \tag{5.2}$$

The simplification form of equation (5.2) is

$$1 - \mathrm{e}^{-\gamma T} = \frac{(1 - \mu)\gamma Q}{a - bs + \gamma Q}. \tag{5.3}$$

Simplifying the above relation (5.3), we get time schedule for the first phase

$$T = \frac{1}{\gamma}\mathrm{Log}\left(\frac{a - bs + \gamma Q}{a - bs + \mu\gamma Q}\right). \tag{5.4}$$

Now holding cost for the period $[0,T]$ is given by

$$h\int_0^T I(t)\,\mathrm{d}t = h\left[\frac{(a-bs+\gamma Q)\left(1-\mathrm{e}^{-\gamma T}\right)}{\gamma^2}-\frac{(a-bs)\,T}{\gamma}\right] = h\left[\frac{(1-\mu)\,Q}{\gamma}-\frac{(a-bs)\,T}{\gamma}\right]\text{[Using (5.3)]}\cdot \tag{5.5}$$

The differential equation governing inventory level for the *period* $[T, T+\Psi]$ is given by

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -\left(a_1 - b_1\theta s\right)\ \text{Subject to } I(T)=\mu Q, I(T+\Psi)=0$$

Now, the solution of the differential equation is

$$I(t) = \left(a_1 - b_1\theta s\right)(T+\Psi-t)\,. \tag{5.6}$$

Using the condition $I(T)=\mu Q$ and equation (5.6), we get time schedule for the second phase

$$\Psi = \frac{\mu Q}{a_1 - b_1\theta S}\cdot \tag{5.7}$$

Now holding cost for the period $[T, T+\Psi]$ is given by

$$h\int_T^{T+\Psi} I(t)\,\mathrm{d}t = h\left[\frac{\mu^2 Q^2}{2\left(a_1-b_1\theta s\right)}\right]\cdot \tag{5.8}$$

The total cost of the model is the addition of the purchase cost, holding cost, and set up cost, *i.e.*,

$$\text{total cost} = cQ + h\left[\frac{(1-\mu)\,Q}{\gamma}-\frac{(a-bs)\,T}{\gamma}\right]+h\frac{\mu^2Q^2}{2\left(a_1-b_1\theta S\right)}+k\ \text{[using (5.5) and (5.8)]}. \tag{5.9}$$

The gained revenue of the model during period $[0, T+\Psi]$ is

$$(Q-\mu Q)\,s + \mu Q\theta s \quad \text{where } \ 0<\theta<1. \tag{5.10}$$

Therefore, total profit of the model during period $[0, T+\Psi]$ for this case is Revenue $-$ Total cost.
The average profit per unit item during period $[0, T+\Psi]$ for this case is $P=$ Total Profit$/Q$.
Using equation (5.9) and (5.10), the average profit can be expressed as

$$P = P(Q,s) = (1-\mu)\,s + \mu\theta s + \frac{hT}{\gamma Q}(a-bs) - \frac{h}{\gamma}(1-\mu) - \frac{h\mu^2 Q}{2\left(a_1-b_1\theta s\right)} - \frac{k}{Q} - c. \tag{5.11}$$

Now, our objective is to find the optimal decisions on ordering lot size and selling price such that the average profit will be maximized.
For optimality test of the average profit function, necessary conditions are $\frac{\partial P}{\partial Q}=0$, $\frac{\partial P}{\partial s}=0$.
Differentiating the average profit function with respect to order quantity, we get

$$\frac{\partial P}{\partial Q} = \frac{h}{\gamma Q}(a-bs)\frac{\partial T}{\partial Q} - \frac{hT}{\gamma Q^2}(a-bs) - \frac{h\mu^2}{2\left(a_1-b_1\theta s\right)} + \frac{k}{Q^2}$$

$$\Rightarrow \frac{\partial P}{\partial Q} = \frac{h}{\gamma Q}(a-bs)\left[\frac{1}{a-bs+\gamma Q} - \frac{\mu}{a-bs+\mu\gamma Q}\right] - \frac{hT}{\gamma Q^2}(a-bs) - \frac{h\mu^2}{2\left(a_1-b_1\theta s\right)} + \frac{k}{Q^2}\cdot \tag{5.12}$$

$$\text{Now } \frac{\partial P}{\partial Q} = 0 \text{ gives } \frac{(a-bs)(1-\mu)\,Q}{(a-bs+\gamma Q)(a-bs+\mu\gamma Q)} - \frac{\mu^2Q^2\gamma}{2\left(a_1-b_1\theta s\right)(a-bs)} + \frac{k\gamma}{h(a-bs)} = T. \tag{5.13}$$

Differentiating the average profit function with respect to selling price, we get

$$\frac{\partial P}{\partial s}=1-\mu+\mu\theta-\frac{hTb}{\gamma Q}+\frac{h}{\gamma Q}(a-bs)\frac{\partial T}{\partial s}-\frac{h\mu^2 Qb_1\theta}{2(a_1-b_1\theta s)^2}$$

$$\Rightarrow\frac{\partial P}{\partial s}=1-\mu+\mu\theta-\frac{hTb}{\gamma Q}+\frac{h}{\gamma Q}(a-bs).\frac{b}{\gamma}\left[\frac{1}{a-bs+\mu\gamma Q}-\frac{1}{a-bs+\gamma Q}\right]-\frac{h\mu^2 Qb_1\theta}{2(a_1-b_1\theta s)^2}. \quad (5.14)$$

$$\text{Now, }\frac{\partial P}{\partial s}=0\text{ gives }(1-\mu+\mu\theta)\frac{\gamma Q}{hb}+\frac{(a-bs)(1-\mu)Q}{(a-bs+\mu\gamma Q)(a-bs+\gamma Q)}-\frac{\mu^2Q^2\gamma b_1\theta}{2b(a_1-b_1\theta s)^2}=T. \quad (5.15)$$

Let $G=(1-\mu+\mu\theta)\frac{\gamma Q}{hb}+\frac{(a-bs)(1-\mu)Q}{(a-bs+\mu\gamma Q)(a-bs+\gamma Q)}-\frac{\mu^2Q^2\gamma b_1\theta}{2b(a_1-b_1\theta s)^2}-T$.

Eliminating T from (5.13) and (5.15), the expression can be written as

$$Q^2A_2+QB_2(s)-C_2(s)=0 \quad (5.16)$$

where

$$A_2=h\mu^2\gamma(a_1b-ab_1\theta),\ B_2(s)=2(a_1-b_1\theta s)^2(a-bs)(1-\mu+\mu\theta)\gamma\text{ and }C_2(s)=2kb\gamma(a_1-b_1\theta s)^2.$$

Solution of equation (5.16) is

$$Q=\frac{-B_2(s)\pm\Delta(s)}{2A_2(s)},\text{where }\Delta(s)=\sqrt{B_2^2(s)+4A_2C_2(s)}. \quad (5.17)$$

Now it is clear that $C_2(s)$ is a non-negative quantity. If $s<\frac{a}{b}$ then $B_2(s)$ is also non-negative as μ is a positive proper fraction. Now $\Delta_2(s)$ is a non-negative quantity for real values of Q. If $B_2(s)<\Delta(s)$, A_2 must be positive *as* $\frac{\Delta^2(s)-B_2^2(s)}{4C_2(s)}=A_2$.

Hence, the positive value of Q is $\frac{-B_2(s)+\Delta(s)}{2A_2}$.

Substituting $Q=\frac{-B_2(s)+\Delta(s)}{2A_2}$ in equation (5.15), the equation can be written as

$$-\frac{1}{8}\frac{b_1\gamma(B_2-\Delta)^2\theta\mu^2}{A_2^2b(a_1-b_1s\theta)^2}+2\frac{A_2(a-bs)(B_2-\Delta)(-1+\mu)}{(2aA_2-2A_2bs+\gamma(-B_2+\Delta))(2aA_2-2A_2bs+\gamma(-B_2+\Delta)\mu)}+\frac{1}{2}\frac{\gamma(-B_2+\Delta)(1+(-1+\theta)\mu)}{A_2bh}-\frac{1}{\gamma}\text{Log}\left[\frac{2aA_2-2A_2bs+\gamma(-B_2+\Delta)}{2aA_2-2A_2bs+\gamma(-B_2+\Delta)\mu}\right]=0. \quad (5.18)$$

Now the equation (5.18) is a nonlinear equation in one variable s. The solution of the equation (5.18) gives the value of s. Substituting value of s in (5.17), the value of Q is calculated. Now, we have to check the second order condition of the optimality test of the average profit function.

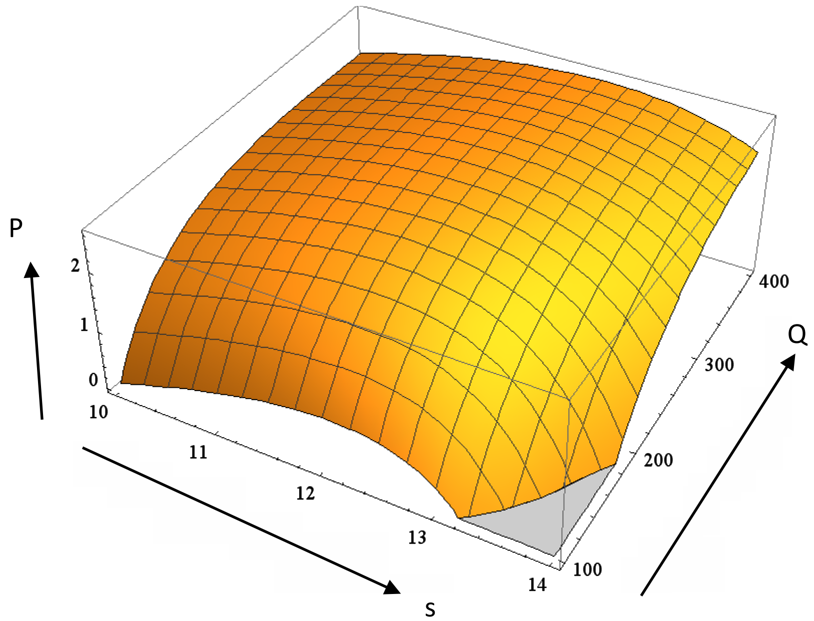

FIGURE 2. Average profit per unit item *vs.* ordering lot-size and selling price for Model-I.

Let us find all types of second order derivatives of average profit function with respect to the ordering lot size and the selling price

$$\frac{\partial^2 P}{\partial Q^2} = -\frac{2k}{Q^3} - \frac{h\mu(a-bs)^2(1-\mu)}{Q(a-bs+Q\gamma)(a-bs+Q\mu\gamma)^2} - \frac{h(a-bs)^2(1-\mu)}{Q(a-bs+Q\gamma)^2(a-bs+Q\gamma\mu)} - \frac{2h(a-bs)^2(1-\mu)}{\gamma Q^2(a-bs+Q\gamma)(a-bs+Q\mu\gamma)} + \frac{2h(a-bs)T}{\gamma Q^3}$$

$$\frac{\partial^2 P}{\partial s^2} = h\left[-\frac{b_1^2 Q\theta^2\mu^2}{(a_1-b_1 s\theta)^3}\right] + h\left[\frac{b^2(-a+bs)(-1+\mu)}{\gamma(a-bs+Q\gamma)(a-bs+Q\gamma\mu)^2}\right] + h\left[\frac{b^2(-a+bs)(-1+\mu)}{\gamma(a-bs+Q\gamma)^2(a-bs+Q\mu\gamma)}\right] + h\left[\frac{2b^2(-1+\mu)}{\gamma(a-bs+Q\gamma)(a-bs+Q\mu\gamma)}\right]$$

$$\frac{\partial^2 P}{\partial Q\partial s} = -\frac{hb_1\theta\mu^2}{2(a_1-b_1 s\theta)^2} + \frac{hb(-1+\mu)(a-bs)^2}{Q\gamma(a-bs+Q\gamma)(a-bs+Q\mu\gamma)^2} - \frac{hb(a-bs)^2(-1+\mu)}{Q\gamma(a-bs+Q\gamma)^2(a-bs+Q\gamma\mu)} + \frac{2hb(a-bs)(-1+\mu)}{Q\gamma(a-bs+Q\gamma)(a-bs+Q\mu\gamma)} + \frac{hb(a-bs)(-1+\mu)(-a+bs+Q\gamma\mu)}{Q\gamma(a-bs+Q\gamma)(a-bs+Q\gamma\mu)^2} + \frac{hbT}{Q^2\gamma}.$$

The solution will be optimal if the corresponding Hessian matrix of the profit function is negative definite, *i.e.*, if all the eigenvalues of the Hessian matrix of the profit function $\begin{pmatrix} \frac{\partial^2 P}{\partial Q^2} & \frac{\partial^2 P}{\partial Q\partial s} \\ \frac{\partial^2 P}{\partial Q\partial s} & \frac{\partial^2 P}{\partial s^2} \end{pmatrix}$ are negative.

5.1.1. *Algorithm for the solution procedure*

An algorithm for the numerical solution of the Model I is given below

Step 1: Assign values to all parameters like $a, b, \gamma, \mu, h, k, c, a_1, b_1, \theta$.
Step 2: Find A_2, $B_2(s)$, $C_2(s)$.
Step 3: Find $\Delta(s)$. Then find Q as a function of s.

Step 4: Putting Q, calculate G.
Step 5: Solve $G = G(s) = 0$ and find the solution $s = s^*$.
Step 6: Putting $s = s^*$ in Q, obtain the value $Q = Q^*$.
Step 7: Find the eigenvalues of the Hessian matrix of the profit function at (Q^*, s^*).
Step 8: If all the eigenvalues of the Hessian matrix are negative, then the solution of the model is the optimal solution. Else go to step 1 and assign new values to parameters repeat consecutive steps.

5.1.2. Numerical example

Here one example is considered with the values of parameters in appropriate units

$$a = 200, b = 15, \gamma = 0.2, \mu = 0.6, h = \$2 \text{ per unit per unit time}, k = \$500, c = \$2 \text{ per unit}, a_1 = 400, b_1 = 30, \theta = 0.7.$$

Then the optimal results for the model are: average profit per unit item $P^* = \$2.65$ per unit item, selling price $s^* = \$12.33$, ordering lot size $Q^* = 326.51$ units, initial demand rate for the first phase of the inventory cycle = 80.35 units per unit time, demand rate for the second phase of the inventory cycle = 141.07 units per unit time, time schedule for the first phase of the inventory cycle = $T^* = 1.96$ time unit, time schedule for the second phase $\Psi^* = 1.39$ time unit. These results are optimal since eigenvalues of the Hessian matrix $\begin{pmatrix} \frac{\partial^2 P}{\partial Q^2} & \frac{\partial^2 P}{\partial Q \partial s} \\ \frac{\partial^2 P}{\partial Q \partial s} & \frac{\partial^2 P}{\partial s^2} \end{pmatrix}$ for $Q = Q^*, s = s^*$ are $-0.35, -0.000017$.

So the objective function is concave and unimodal function. the optimality of the model is also checked by graphically (see Fig. 2).

5.2. Sensitivity analysis

It is the process through which we can test the robustness of the results of our model. If there is any error in the model this process helps to detect it. We can understand relationships between input variables (here parameters) and output variables (here decision variables) through it. It also frames, bridge from modelers to decision makers. So it is mandatory for a mathematical model.

5.2.1. Sensitivity analysis for Model-I

From the above table, we observe the following characteristic of the parameters:

When holding cost (h) increases, both decision variables selling price (s) and ordering lot size (Q) decreases. The time period for the first phase (T) decreases as the lot size decreasing rate is more than the changes of the initial demand rate of the first phase. The time period for the second phase (Ψ) also decreases since the lot size decreases along with the demand rate for that phase increases. Average profit (P) decreases due to the joint effect of decrease of decision variables.

If setup cost (k) increases, lot size increases much more than selling price here. The time period for the first phase (T) increases due to major increase in lot size with respect to initial demand rate of that phase. Average profit (P) decreases due to a major increase in the time period for the second phase (Ψ) .

With the increasing values of the coefficient parameter of the inventory level γ, both decision variable increases. Change in lot size is much more than the selling price. The time period for the second phase (Ψ) increases since the demand rate for the second period decreases. Average profit (P) increases for joint increase of both decision variables.

When μ is increasing, the selling price decreases and the ordering lot size first increases then decreases. This is quite reasonable since μQ must be a finite quantity which is to be sold in the second phase of the inventory cycle. The time period for the first phase decreases as the selling quantity $(1 - \mu) Q$ decreases. Average profit increases due to joint effect of s and Q.

Ordering lot size decreases much more than the selling price when θ increases. The time period for the first phase (T) decreases as the lot size decreasing rate is more than the changes of the initial demand rate of the first phase. The time period for the second phase (Ψ) increases as the demand rate for that period decreases. Average profit (P) increases as discount decreases while θ increases.

Both the decision variables increase when the parameter a increases. Also in this case, the time schedule of the second phase (Ψ) increases much more with respect to the time schedule of the first phase (T). Average profit (P) increases for joint increase of both decision variables.

Increasing value of b causes major decreasing in selling price. Then the time period of second phase (Ψ) decreases much more with respect to the time period for the first phase (T). This is due to huge increasing of the demand rate for the second phase. Average profit decreases for the joint effect of both decision variables.

Parameter	Value of the parameter	% change in Selling Price (s)	% change in Ordering lot size (Q)	% change of Initial rate for demand the first phase	% change of demand rate for the second phase	% change in T	% change in Ψ	% lchange in profit (P)
h	1.8	2.92	14.20	4.82	−5.36	11.73	20.86	15.84
	1.9	1.39	6.44	2.06	−2.53	5.61	9.35	7.54
	2.1	−1.22	−5.40	−1.58	2.23	−4.59	−7.91	−7.17
	2.2	−2.27	−9.97	−2.87	4.17	−8.67	−13.66	−13.96
k	450	−.49	−8.63	−5.90	.89	−3.06	−9.35	6.03
	475	−.24	−4.26	−2.90	.45	−1.53	−4.32	3.01
	525	.24	4.20	2.86	−.45	2.04	9.35	−2.64
	550	.49	8.33	5.65	−.89	3.51	14.39	−5.66
γ	.18	−1.21	−9.35	−8.42	5.21	−3.06	−13.66	−8.67
	.19	−.57	−4.90	−4.50	2.67	−1.53	−7.19	−4.52
	.21	1.54	5.29	5.04	−2.82	1.53	7.91	4.52
	.22	3.24	10.79	10.30	−5.95	3.06	17.98	9.05
μ	.56	−1.21	−.77	2.17	2.23	8.67	−9.35	−1.13
	.58	−.56	−.28	1.08	1.04	4.59	−5.04	−.38
	.62	.56	.06	−1.26	−1.26	−4.08	4.31	.75
	.64	1.21	−.10	−2.87	−2.23	−8.67	8.63	1.13
θ	.66	.49	5.17	3.09	9.65	2.55	−4.31	−7.92
	.68	.24	2.59	1.56	4.81	1.53	−2.87	−3.77
	.72	−.24	−2.60	−1.56	−4.78	−1.02	2.16	3.77
	.74	−.57	−5.20	−2.92	−9.38	−2.55	4.31	7.54
a	190	−5.76	−2.34	−1.09	10.57	−1.02	−11.51	−21.89
	195	−2.92	−1.18	−.46	5.36	−.51	−6.47	−10.94
	205	2.92	1.20	.48	−5.36	1.02	6.47	10.94
	210	5.75	2.43	1.17	−10.57	2.04	14.39	21.89
b	13	18.89	6.41	−1.79	−34.68	10.71	62.59	59.62
	14	8.67	2.61	−1.16	−15.93	5.10	21.58	27.54
	16	−7.46	−1.84	1.48	13.69	−3.57	−13.67	−23.77
	17	−14.03	−3.14	3.36	25.75	−7.14	−23.02	−44.15
a_1	360	−2.75	−14.32	−5.29	−23.29	−11.22	11.51	−10.57
	380	−1.30	−7.50	−3.11	−11.80	−5.10	5.04	−4.90
	420	1.30	8.55	3.97	11.80	5.61	−2.88	4.15
	440	2.68	18.94	9.23	23.44	11.22	−3.60	7.92
b_1	26	2.84	17.90	8.01	19.95	11.73	−2.16	7.17
	28	1.30	7.90	3.43	10.02	5.61	−2.16	3.77
	32	−1.30	−6.65	−2.41	−9.70	−5.10	3.60	−4.15
	34	−2.76	−12.39	−3.75	−18.74	−10.71	7.91	−5.68

Increasing rate of ordering lot size is more than selling price when the value of the parameter a_1 increases. The time period of the first phase (T) increases as the lot size increasing rate is much more than that of the initial demand rate for the first phase. The time period for the second phase (Ψ) decreases while average profit increases.

As b_1 increases, ordering lot size decreases much more than the selling price. The time period for the second phase (Ψ) increases while the time period of the first phase (T) decreases. Average profit (P) decreases due to joint decrease of selling price and ordering lot size.

5.3. Managerial insight

On the basis of sensitivity analysis manager/decision maker of a company get some necessary information about his future steps to make his organization profitable. He also gets an idea on which topic he has to be alert.

5.3.1. Managerial Insight for the Model-I

In this model, holding cost and set up cost/order cost should be decreased to get more profit per unit quantity. The quantity which is going to be selling in the second phase of the inventory cycle is more than the first phase, but the ordering lot size should be small to get more profit. The discount should be also less to obtain a profitable model. To get much profit, the coefficient parameter associated with inventory level must be increased.

5.4. Model-II

The differential equation governing inventory level for the period $[0, T]$ is given by

$\frac{\mathrm{d}I}{\mathrm{d}t} = -\alpha \{I(t)\}^{\beta} s^{-\gamma}$ where $\alpha > 0, 0 < \beta < 1, \gamma > 0$ subject to $I(0) = Q, I(T) = \mu Q$ where $0 < \mu < 1$.
Now the solution of the differential equation with the initial condition $I(0) = Q$ is

$$I(t) = \left[Q^{1-\beta} - \alpha(1-\beta) s^{-\gamma} t\right]^{\frac{1}{1-\beta}}. \tag{5.19}$$

Using the condition $I(T) = \mu Q$ and equation (5.17), the time schedule for the first cycle is

$$T = \frac{\left[Q^{1-\beta} - (\mu Q)^{1-\beta}\right] s^{\gamma}}{\alpha(1-\beta)}. \tag{5.20}$$

Now holding cost for the period $[0, T]$ is given by $h \int_0^T I(t)\,\mathrm{d}t$.
Using equations (5.19) and (5.20), the holding cost for the period $[0, T]$ is

$$h \int_0^T I(t)\,\mathrm{d}t = \frac{hs^{\gamma}}{\alpha(2-\beta)} \left[Q^{2-\beta} - (\mu Q)^{2-\beta}\right]. \tag{5.21}$$

The differential equation governing inventory level for the period $[T, T+\Psi]$ is $\frac{\mathrm{d}I}{\mathrm{d}t} = -\alpha_1 (\theta s)^{-\gamma_1}$ where $\alpha_1 > 0, \gamma_1 > 0$, Subject to $I(T) = \mu Q, I(T+\Psi) = 0$ with $0 < \theta < 1$.
Solving the differential equation with the help of terminal condition $I(T+\Psi) = 0$, the inventory level is

$$I(t) = \alpha_1 (\theta s)^{-\gamma_1} (T + \Psi - t). \tag{5.22}$$

Using the condition $I(T) = \mu Q$, the time schedule for the second cycle is

$$\Psi = \frac{\mu Q (\theta s)^{\gamma_1}}{\alpha_1}. \tag{5.23}$$

Holding cost for the period $[T, T+\Psi]$ is given by $h \int_T^{T+\Psi} I(t)\,\mathrm{d}t$.
The equations (5.20) and (5.21) give

$$h \int_T^{T+\Psi} I(t)\,\mathrm{d}t = \frac{h\mu^2 Q^2 (\theta s)^{\gamma_1}}{2\alpha_1}. \tag{5.24}$$

The total cost of the model is addition of the purchase cost, holding cost, and set up cost, *i.e.*,

$$\text{total cost} = k + cQ + \frac{hs^{\gamma}}{\alpha(2-\beta)} \left[Q^{2-\beta} - (\mu Q)^{2-\beta}\right] + \frac{h\mu^2 Q^2 (\theta s)^{\gamma_1}}{2\alpha_1} \text{ [using equations (5.21) and (5.24)]}.$$

The revenue gain of the model during period $[0, T+\Psi]$ is $= (Q - \mu Q)s + \mu Q \theta s$.
Therefore, total profit of the model during period $[0, T+\Psi]$ for this case is Revenue – Total cost.
The average profit per unit item during period $[0, T+\Psi]$ for this case is $P =$ Total Profit$/Q$.

$$\Rightarrow P = (1-\mu)s + \mu\theta s - \frac{k}{Q} - c - \frac{hs^{\gamma}}{\alpha(2-\beta)} \left[Q^{1-\beta} - \mu^{2-\beta} Q^{1-\beta}\right] - \frac{h\mu^2 Q (\theta s)^{\gamma_1}}{2\alpha_1}. \tag{5.25}$$

Now, our objective is to find the optimal decisions on ordering lot size and selling price such that the average profit will be maximized.

For maximization test of the profit function, necessary conditions are given by $\frac{\partial P}{\partial Q} = 0$, $\frac{\partial P}{\partial s} = 0$. Differentiating the average profit function with respect to order quantity, we get

$$\frac{\partial P}{\partial Q} = \frac{k}{Q^2} - \frac{h\mu^2 (\theta s)^{\gamma_1}}{2\alpha_1} - \frac{hs^{\gamma}\left(1 - \mu^{2-\beta}\right)(1-\beta) Q^{-\beta}}{\alpha(2-\beta)}.$$

Solution of $\frac{\partial P}{\partial Q} = 0$ gives

$$-As^{\gamma} Q^{-\beta} - Bs^{\gamma_1} + \frac{k}{Q^2} = 0 \tag{5.26}$$

where $A = \frac{h(1-\beta)\left(1-\mu^{2-\beta}\right)}{\alpha(2-\beta)}$ and $B = \frac{h\mu^2 \theta^{\gamma_1}}{2\alpha_1}$, $G = -As^{\gamma} Q^{-\beta} - Bs^{\gamma_1} + \frac{k}{Q^2}$.

Again differentiating average profit function with respect to selling price

$$\frac{\partial P}{\partial s} = 1 - \mu + \mu\theta - \frac{h\gamma s^{\gamma-1} Q^{1-\beta}\left(1-\mu^{2-\beta}\right)}{\alpha(2-\beta)} - \frac{h\mu^2\gamma_1\theta^{\gamma_1} Q s^{\gamma_1-1}}{2\alpha_1}.$$

Now the solution of $\frac{\partial P}{\partial s} = 0$ gives

$$\frac{Xs}{Q} - Ys^{\gamma}Q^{-\beta} - Zs^{\gamma_1} = 0 \tag{5.27}$$

where $X = 1-\mu+\mu\theta$, $Y = \frac{h\gamma\left(1-\mu^{2-\beta}\right)}{\alpha(2-\beta)}$, $Z = \frac{h\mu^2\gamma_1\theta^{\gamma_1}}{2\alpha_1}$.

Eliminating $s^{\gamma}Q^{-\beta}$ from (5.26) and (5.27), we have

$$LQ^2 s^{\gamma_1} + MsQ - N = 0 \tag{5.28}$$

where $L = BY - AZ, M = AX, N = kY.$

Substituting values of A, B, X, Y, Z in the above expressions, we get

$$L = \frac{h^2\mu^2\theta^{\gamma_1}\left(1-\mu^{2-\beta}\right)}{2\alpha\alpha_1(2-\beta)}\left\{\gamma - \gamma_1(1-\beta)\right\}, M = \frac{h(1-\beta)\left(1-\mu^{2-\beta}\right)}{\alpha(2-\beta)}(1-\mu+\mu\theta), N = \frac{kh\gamma\left(1-\mu^{2-\beta}\right)}{\alpha(2-\beta)}.$$

As $0 < \mu, \theta, \beta < 1, \alpha > 0$, M and N are positive quantity. If $\frac{\gamma}{\gamma_1} > 1-\beta$ then L becomes positive.

Now the solution for positive value of Q from the equation (5.28) is

$$Q = \frac{-Ms + \sqrt{(Ms)^2 + 4LNs^{\gamma_1}}}{2Ls^{\gamma_1}}. \tag{5.29}$$

Now substituting Q in (5.26), the equation reduces to

$$\frac{4L^2ks^{2\gamma_1}}{\left(-Ms + \sqrt{(Ms)^2 + 4LNs^{\gamma_1}}\right)^2} - \frac{hs^{\gamma_1}\theta^{\gamma_1}\mu^2}{2\alpha_1} - \frac{2^{\beta}hs^{\gamma}\left(\frac{-Ms+\sqrt{(Ms)^2+4LNs^{\gamma_1}}}{Ls^{\gamma_1}}\right)^{-\beta}(-1+\beta)\mu^{-\beta}\left(-\mu^2+\mu^{\beta}\right)}{\alpha(2-\beta)} = 0. \tag{5.30}$$

Now the equation (5.30) is a nonlinear equation in one variable s. If we solve it, we will get the value of s. Substituting value of s in (5.29) we will get the value of Q. Now, we have to check the second order condition of the optimality test of the average profit function.

Now considering all second order derivatives of average profit function with respect to order quantity and selling price

$$\frac{\partial^2 P}{\partial Q^2} = -\frac{2k}{Q^3} + \frac{hQ^{-1-\beta}s^{\gamma}(1-\beta)\beta\left(1-\mu^{2-\beta}\right)}{\alpha(2-\beta)}$$

$$\frac{\partial^2 P}{\partial s^2} = \frac{h\gamma(1-\gamma)s^{\gamma-2}Q^{1-\beta}\left(1-\mu^{2-\beta}\right)}{\alpha(2-\beta)} + \frac{h\mu^2Q\theta^{\gamma_1}\gamma_1(1-\gamma_1)s^{\gamma_1-2}}{2\alpha_1}$$

$$\frac{\partial^2 P}{\partial Q\partial s} = -\frac{h\gamma_1\theta^{\gamma_1}\mu^2 s^{\gamma_1-1}}{2\alpha_1} - \frac{hQ^{-\beta}s^{\gamma-1}(1-\beta)\gamma\left(1-\mu^{2-\beta}\right)}{\alpha(2-\beta)}.$$

The solution will be optimal if the corresponding Hessian matrix of the profit function is negative definite, *i.e.*, if all the eigenvalues of the Hessian matrix $\begin{pmatrix} \frac{\partial^2 P}{\partial Q^2} & \frac{\partial^2 P}{\partial Q\partial s} \\ \frac{\partial^2 P}{\partial Q\partial s} & \frac{\partial^2 P}{\partial s^2} \end{pmatrix}$ are negative.

5.4.1. Algorithm for the solution procedure

Algorithm for the numerical solution of the Model II is given as follows

Step 1: Assign values to the parameters: $\alpha, \alpha_1, \beta, \gamma, \gamma_1, k, \mu, h, \theta$
Step 2: Calculate L, M, N
Step 3: Calculate Q as a function of s
Step 4: Find A, B

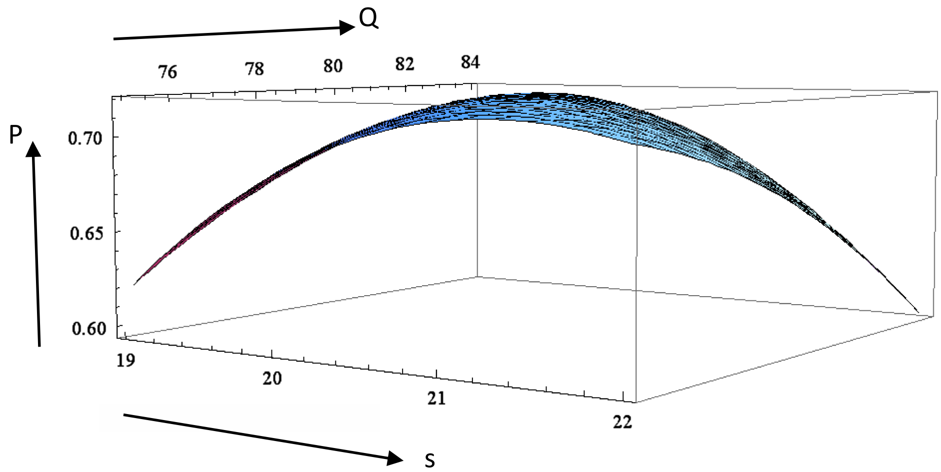

FIGURE 3. A Average profit per unit item *vs.* ordering lot-size and selling.

Step 5: Calculate G
Step 6: Solve $G = G(s) = 0$ and find the solution $s = s^*$
Step 7: Putting $s = s^*$ in Q, obtain the value $Q = Q^*$.
Step 8: Find the eigenvalues of the Hessian matrix of the profit function at (Q^*, s^*).
Step 9: If all the eigenvalues of the Hessian matrix are negative then the solution of the model is the optimal solution. Else go to Step 1 and assign new values to parameters repeat consecutive steps.

5.4.2. Numerical example

We consider the values of the parameters in appropriate units as follows:

$c = \$1$ per unit, $\gamma = 2, \beta = 0.4, \gamma_1 = 1.75, \alpha = 150, \alpha_1 = 250, k = \$500, \mu = 0.6, h = \$0.45$ per unit per unit time, and $\theta = 0.7$.

Then the optimal results for the model are: average profit per unit item $= P^* = \$0.72$ per unit item, selling price $= s^* = \$20.32$, ordering lot size $= Q^* = 79.66$ units, initial demand rate for the first cycle $= 2.09$ units per unit time, demand rate for the second cycle $= 2.40$ units per unit time, time schedule for the first cycle $= T^* = 16.75$ time units, time schedule for the second cycle $= \Psi^* = 19.91$ time units.

These results are optimal as the eigenvalues of the Hessian matrix $\begin{pmatrix} \frac{\partial^2 P}{\partial Q^2} & \frac{\partial^2 P}{\partial Q \partial s} \\ \frac{\partial^2 P}{\partial Q \partial s} & \frac{\partial^2 P}{\partial s^2} \end{pmatrix}$.

For $Q = Q^*, s = s^*$ are $-0.039, -0.000304$. So the objective function is concave and unimodal function. The graphical representation of the profit function with respect to lot size and the selling price is given in Figure 3. Here, Figure 3 shows clearly that the objective function is a concave function.

5.4.3. Sensitivity analysis for Model-II

From the above table, we observe the following characteristic of the parameters:

When holding cost (h) increases, selling price decreases while ordering lot size increases. Initial demand rate for the first phase increases due to the increasing lot size and decreasing selling price. Since the initial demand rate for the first phase increases more than ordering lot size, time schedule (T) for the first phase decreases. The demand rate for the second phase also increases as selling price decreases. As a consequence of this, time schedule (Ψ) for the second phase decreases and per unit average profit decreases.

Ordering lot size increases while selling price decreases with a higher set up cost (k). Consequently, the initial demand rate increases for the first phase of the inventory cycle. The variation in initial demand rate for the first cycle is much more than the variation in ordering lot size. As a result, the time period (T) for the first phase decreases. Due to the increasing demand rate for the second phase, time schedule (Ψ) for the second phase also decreases. Discount period increases with respect to the first phase. This effect along with decrease in selling price causes reduction in average profit.

Parameter	Value of the parameter	% change in Selling Price (s)	% change in Ordering lot size (Q)	% change of Initial demand rate for the first phase	% change of demand rate for the second phase	% change in T	% change in Ψ	% change in Average profit (P)
h	0.448	1.62	−1.43	−3.83	−2.92	2.33	1.41	5.55
	0.449	0.79	−0.72	−1.91	−1.25	1.33	.70	2.78
	0.451	−0.79	0.73	1.91	1.25	−1.34	−0.60	−2.78
	0.452	−1.57	1.46	3.83	2.91	−2.33	−1.30	−5.55
k	485	7.48	−9.03	−16.75	−12.08	9.13	3.26	27.78
	495	2.46	−3.10	−6.22	−4.17	2.98	1.16	8.34
	505	−2.36	3.16	6.22	4.17	−2.92	−1.05	−8.34
	510	−4.67	6.44	12.91	8.75	−5.67	−2.06	−16.67
μ	0.56	−6.05	7.54	16.75	11.67	2.57	−9.99	13.89
	0.58	−3.25	3.89	8.61	5.83	1.13	−5.17	6.94
	0.62	3.83	−4.17	−8.61	−6.67	−0.72	5.83	−6.94
	0.64	8.37	−8.61	−17.70	−13.33	−1.01	12.25	−12.50
θ	0.66	−10.38	5.29	32.06	34.16	−12.54	−14.11	−29.17
	0.68	−5.31	7.27	14.83	15.42	−6.50	−7.28	−15.28
	0.72	5.61	−6.77	−12.92	−13.75	6.92	7.28	15.28
	0.74	11.47	−13.00	−23.92	−25.00	14.20	15.97	31.94
β	0.38	−18.80	20.24	48.80	43.75	−19.76	−16.47	−63.89
	0.39	−9.60	9.30	21.53	19.17	−10.09	−8.34	−33.34
	0.41	9.90	−7.99	−16.75	−15.42	10.50	8.68	36.11
	0.42	20.37	−14.91	−29.66	−27.92	21.43	17.73	73.61
α	140	−18.26	21.31	51.20	42.08	−19.64	−14.72	−51.39
	145	−9.40	9.29	22.49	18.75	−10.15	−7.48	−26.39
	155	9.99	−8.71	−17.70	−15.42	10.80	7.88	29.16
	160	20.57	−16.40	−31.58	−27.92	22.39	15.97	59.72
α_1	240	−2.76	1.97	6.70	0.83	−4.36	1.20	−15.28
	245	−1.33	0.95	3.35	0.42	−2.09	0.65	−8.34
	255	1.28	−0.89	−2.87	−0.42	1.97	−0.60	6.94
	260	2.50	−1.72	−5.26	−0.83	4.00	−1.25	13.88
γ	1.98	25.94	−20.22	−38.28	−33.33	29.79	19.48	58.33
	1.99	11.86	−10.41	−21.05	−17.92	13.49	9.04	26.39
	2.01	−10.04	10.95	24.88	20.41	−11.34	−7.78	−23.61
	2.02	−18.70	22.44	55.02	43.75	−21.07	−14.72	−43.06
γ_1	1.73	5.26	−4.15	−11.00	−3.75	7.94	−0.65	19.44
	1.74	2.71	−2.16	−5.74	−2.08	4.05	−0.15	9.72
	1.76	−2.71	2.33	6.69	2.08	−4.05	0.15	−9.72
	1.77	−5.51	4.88	13.87	5.00	−8.18	0.15	−19.44

With the increasing values of μ, selling price increases and ordering lot size decreases. Consequently, the initial demand rate for the first phase decreases. Therefore, the time schedule (T) for the first phase decreases as the variation of ordering lot is less than the variation of the initial demand rate. The time schedule (Ψ) for the second phase increases significantly. The selling quantity for the first phase is $(1-\mu)\,Q$ and for the second phase is μQ. As μ increases the selling quantity for the first phase is going to be smaller and for the second phase it is going to be larger. The average profit decreases as discount period increases along with selling quantity.

When θ increases, selling price increases while lot size decreases. The time period of the first phase increases as the initial demand decreases much more than lot size. Discount period, *i.e.*, the second phase also increases. The average per unit profit increases as selling price becomes high and discount is low.

As β increases, selling price increases while ordering lot size decreases. The time period of the first phase (T) increases as initial demand decreases much more than lot size. Decrease in demand rate causes longer second phase (Ψ). It should be noted that as β increases the rate of increasing of discount period is less with respect to that of the first period. This effect along with increasing selling price causes major increase in average profit.

Selling price increases, but both ordering lot size and initial demand rate for the first cycle decreases when α increases. The decreasing rate of initial demand for the first period is higher than the decreasing rate of ordering lot size. So the time schedule (T) for the first period increases. Increasing selling price causes decreased at the demand rate for the second cycle. As a consequence, the time period for the second phase (Ψ) increases. The time schedule for the first phase increases faster than the second phase. This together with increasing the selling price causes higher average profit.

As α_1 increases, selling price increases while ordering lot size and initial demand rate for the first cycle decrease. The time schedule (T) for the first phase increases as the initial demand rate decreases more than ordering lot size. A change in demand rate for the second cycle is very small. The average per unit profit increases.

With the increasing values of γ, the selling price decreases and ordering lot size increases. Consequently, the initial demand rate for the first phase increases. The initial demand rate for the first phase increases more than ordering lot size. As a result, the time schedule (T) for the first period decreases. The demand rate for the second phase of the inventory cycle increases as selling price decreases. As a consequence of this, the time schedule (Ψ) for the second phase decreases. It should be noted that the decreasing rate of the first phase is more than the decreasing rate of the second phase. This effect together with the decrease in selling price causes a major reduction in average profit.

As $\gamma_1$increases, selling price decreases while ordering lot size and the initial demand rate for the first phase increase. The increasing rate of the initial demand rate for the first phase of the inventory cycle is much more than the increasing rate of the ordering lot size. As a result, the time schedule (T) decreases. The demand rate for the second phase of the inventory cycle increases since selling price decreases. As γ_1 increases, discount period (Ψ) increases much more with respect to no discount period (T). This incident along with decrease in selling price causes decrease in average profit.

5.4.4. Managerial insights of Model-II

Here, the average profit per quantity increases when the holding cost and set up/order cost decrease. Again, the quantity which is going to be sold in the first phase should be larger to get more profit, *i.e.*, less quantity to be sold in the second phase of the inventory cycle with a small amount of discount. Increasing values of scale and the shape parameter associated with inventory level in demand rate for the first phase of the inventory cycle are responsible for more and more profit. The decreasing value of the parameter associated with selling price in demand rate for the first phase of the inventory cycle is required for more profit. The model is more sensitive with respect to these three parameters.

5.5. Summary of sensitivity analysis of both models together with comparison

From the sensitivity analysis of the Model-I, the following conclusions are given:

(i) The per unit average profit is increasing with decreasing values *of* h, k, b, b_1 and with increasing values of γ,μ,θ,a,a_1.
(ii) When the parameters k, γ,a, a_1 are increasing, the ordering lot size also increases, but the ordering lot size decreases when the parameters h, μ,θ, $b, b_1$are varying positively.
(iii) With the higher values of the parameters k, γ,μ, a, a_1, the selling price increases, but the selling price decreases with the positive changes of the parameters h, θ,b_1.
(iv) Time horizon for both the first and the second phase increases with the increasing value k, γ, a and decreasing value of h, b. The time schedule for the first phase decreases and the same for the second phase increases when the parameters μ, θ, b_1 varying positively.

As we go through the sensitivity analysis of Model-II, we observed that the average profit per unit item increases as holding cost, setup/order cost decreases and value of θ increases. This result is similar to Model-I. The result is different when the value of μ increases. In the Model-II, when scale parameter α and the shape parameter β increases average profit increases largely and when γ increases average profit decreases significantly. The parameters α, β, γ are most sensitive parameters with respect to model-II.

6. Conclusion

It is a well-known fact that large display of stock always motivates customers and demand is generally influenced by selling price. So both the stock level and selling price take a vital role in the determination of people's choice. Due to huge competition in the market the retailer who sells the product of a particular type, may face problem when stock level decreases. After a certain period of time, information about the products/feedback about the products is well known to the customers. So, generally, demand rate is not too much depending on stock after a certain time. In this situation retailer should go for a discount policy to clear his remaining stock. Depending on this idea, we have formulated our inventory model with two phases in an inventory cycle. In the first phase, the demand rate is influenced by both stock level and price. In the second phase demand rate is dependent on price only with discount policy. Here, we have analyzed the two models with respect to the ordering lot size of raw material and the selling price of the retailer such that the average profit of the models would be optimal. We have also discussed the situation when the retailer starts the second phase with discount policy.

Two models have been developed for two types of demand pattern. In the first model demand rate for the first cycle is taken as a sum of two linear functions of selling price and stock level and, in the second, it is taken as product functions of stock level and price. Both models have been studied through some numerical examples and analyzed the sensitivity of the main parameters. From sensitivity analysis, it has been observed that parameters of the Model-II are more sensitive than those of the Model-I.

There are many future scope of research by extending our model. The extension can be done by considering the effect of deterioration, relaxing zero inventory level at the terminal point of the total inventory cycle. The model can be extended by the inclusion of more members in the model with coordination strategy.

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