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FACET-INDUCING INEQUALITIES WITH ACYCLIC SUPPORTS FOR THE CATERPILLAR-PACKING POLYTOPE

Javier Marenco[1,2,*]

Abstract. A *caterpillar* is a connected graph such that the removal of all its vertices with degree 1 results in a path. Given a graph G, a *caterpillar-packing of* G is a set of vertex-disjoint (not necessarily induced) subgraphs of G such that each subgraph is a caterpillar. In this work we consider the set of caterpillar-packings of a graph, which corresponds to feasible solutions of the *2-schemes strip cutting problem with a sequencing constraint* (2-SSCPsc) presented by Rinaldi and Franz (*Eur. J. Oper. Res.* **183** (2007) 1371–1384). Facet-preserving procedures have been shown to be quite effective at explaining the facet-inducing inequalities of the associated polytope, so in this work we continue this issue by exploring such procedures for valid inequalities with acyclic supports. In particular, the obtained results are applicable when the underlying graph is a tree.

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1. Introduction

A *caterpillar* is a (possibly empty) connected graph such that the removal of all its vertices with degree 1 results in a path, see Figure 1(a). Given a graph $G = (V, E)$, the subgraph induced by an edge subset $F \subseteq E$ is $G_F = (V_F, F)$, where $V_F \subseteq V$ is the set of endpoints of the edges in F. A *caterpillar-packing of* G is a set $\mathcal{F} = \{F_1, \dots, F_k\}$ such that (a) $F_i \subseteq E$ and G_{F_i} is a caterpillar, for $i = 1, \dots, k$, and (b) $V_{F_i} \cap V_{F_j} = \emptyset$, for $i \neq j$. In other words, a caterpillar-packing of G is a set of vertex-disjoint (not necessarily induced) subgraphs of G such that each subgraph is a caterpillar.

Caterpillar-based structures in graphs have been tackled with integer programming techniques in previous works. The *minimum spanning caterpillar problem* asks for a spanning caterpillar minimizing a linear cost function that assigns different costs to leaf edges and edges from the central path. Integer programming and heuristic approaches for this problem are presented in [14,15]. Theoretical developments concerning its approximability include [4,5]. This structure is related to ring-stars, where the central path of the spanning caterpillar is replaced by a cycle. Both integer programming (see [7,8]) and heuristic (see [2,3]) approaches have been pursued for the

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[1] Sciences Institute, National University of General Sarmiento, J. M. Gutiérrez 1150, Malvinas Argentinas, 1613 Provincia de Buenos Aires, Argentina.

[2] Computer Science Dept., FCEyN, University of Buenos Aires. Int. Güiraldes y Av. Cantilo, Pabellón I, Planta Baja, 1428 Buenos Aires, Argentina.

[*] Corresponding author: jmarenco@ungs.edu.ar; jmarenco@dc.uba.ar

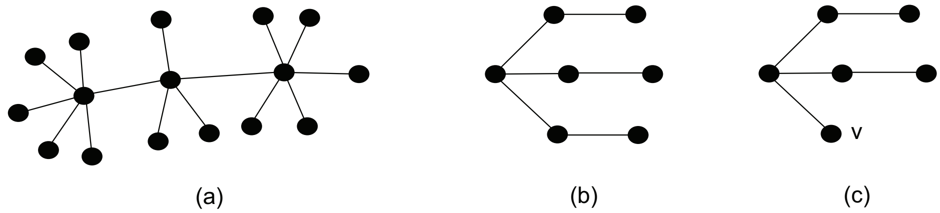

FIGURE 1. *Panel a*: a caterpillar. *Panel b*: the bipartite claw. *Panel c*: the incomplete bipartite claw, with v its dangling vertex.

minimum ring-star problem. The natural generalization of this problem asking for more than one cycle is called the *m-ring-star problem* and has been studied in [1, 12, 16, 17].

The interest in caterpillar-packings comes from an integer programming approach to the *2-schemes strip cutting problem with a sequencing constraint* (2-SSCPsc), a problem that arises in the context of corrugated cardboard machines [13]. The 2-SSCPsc is essentially a unidimensional cutting stock problem with at most two orders per pattern, with the additional *sequencing constraint* that requires every order to appear in consecutive patterns. This last constraint makes the problem quite difficult in practice. If we define the *schemes graph* to be $SG = (\mathcal{O}, S)$, where $\mathcal{O}$ is the set of orders and $S = \{ij$: there exists a feasible pattern with orders i and $j\}$, then the two-order patterns present in any feasible solution induce a caterpillar-packing of SG [13]. This observation motivated the introduction in [10] of an integer programming model aiming to exploit this structure.

If $e \in E$, we define $\mathbf{u}_e \in \mathbb{R}^{|E|}$ to be the unit vector associated with the edge e, *i.e.*, $(\mathbf{u}_e)_{e'} = 1$ if $e' = e$ and $(\mathbf{u}_e)_{e'} = 0$ otherwise. If $F \subseteq E$, we define $\mathbf{u}_F = \sum_{e \in F} \mathbf{u}_e$ to be its *characteristic vector*. We define the *caterpillar-packing polytope* associated to the graph G to be

$$CPP(G) = \text{conv}\{\mathbf{u}_F : F \text{ is the edge set } \cup \mathcal{F} \text{ of a caterpillar-packing } \mathcal{F} \text{ of } G\}.$$

Families of valid inequalities for $CPP(G)$ may be incorporated to a cutting-plane-based procedure for the 2-SSCPsc, thus motivating the study of this polytope. In [11] such a polyhedral study was started, showing that $CPP(G)$ has remarkable properties, key among them the existence of a straightforward lifting lemma and the existence of facet-preserving procedures, which take as input a valid (respectively, facet-inducing) inequality and produce an enlarged inequality that is also valid (respectively, facet-inducing if additional constraints hold).

In this work we are interested in facet-inducing inequalities with acyclic supports for $CPP(G)$, that could be used when G is a tree or a forest. Only the first facet-preserving procedure presented in [11] can be applied when G is a tree, since the remaining procedures generate cycles in the support of the obtained inequality. Hence, we must consider new such procedures, generating inequalities with acyclic supports. Preliminary computational experience suggests that the facet-preserving procedures presented in this work might give complete descriptions of $CPP(G)$ when G is a tree, and this fact would provide a crucial insight on the computational complexity of the 2-SSCPsc over trees. The well-known equivalence between optimization and separation [6] provides the theoretical background for such a study.

This paper is organized as follows. Section 2 presents the caterpillar-packing polytope in detail and provides some properties of this polytope. Section 3 presents a new family of facet-inducing inequalities. Section 4 presents the new facet-preserving procedures and proves them correct. Finally, Section 5 states some conclusions and open problems.

2. FORMULATION AND BASIC PROPERTIES

We introduce in this section a natural integer programming formulation for $CPP(G)$, based on the following classical result. The *bipartite claw* is the graph depicted in Figure 1(b).

Theorem 2.1 ([9]). *A connected graph H is a caterpillar if and only if H does not contain any cycle and any bipartite claw.*

We assume $E \neq \emptyset$ throughout this work. We introduce a binary variable x_e for each $e \in E$, such that $x_e = 1$ if and only if the solution includes the edge e. We denote by $\mathcal{C}(G)$ the set of all (not necessarily induced) cycles in G, and by $\mathcal{B}(G)$ the set of all (not necessarily induced) bipartite claws of G, in both cases regarded as sets of edges. In this setting, $CPP(G)$ is the convex hull of the points $x \in \{0,1\}^{|E|}$ satisfying the following constraints:

$$\sum_{e \in C} x_e \leq |C| - 1 \quad \forall\, C \in \mathcal{C}(G), \tag{2.1}$$

$$\sum_{e \in B} x_e \leq 5 \qquad \forall\, B \in \mathcal{B}(G). \tag{2.2}$$

The *cycle constraints* (2.1) ask feasible solutions not to contain any cycle, whereas the *bipartite claw constraints* (2.2) forbid bipartite claws. Hence, Theorem 2.1 ensures that integer points in $CPP(G)$ represent caterpillar-packings of G. Note that the definition of a caterpillar-packing does not ask every vertex to be included in some caterpillar (*e.g.*, the empty set of edges is a caterpillar-packing), so we do not have constraints asking for such conditions.

It is easy to verify that $CPP(G)$ is full-dimensional and that any facet-inducing inequality different from $x_e \geq 0$ for any $e \in E$ has non-negative coefficients. Also, the bipartite claw constraints (2.2) define facets of $CPP(G)$, even if the bipartite claw is not induced in G. On the other hand, the *cycle constraints* are not facet-inducing in general [11].

The particular structure of caterpillars implies the following lifting and projection lemmas. If $W \subseteq V$, we denote by $E(W) = \{ij \in E : i, j \in W\}$ the set of edges induced by W. We denote by G_W the subgraph of G induced by the vertex set W, *i.e.*, $G_W = (W, E(W))$. Finally, for $E' \subseteq E$, we define $\pi_{E'}$ to be the projection of π onto the space of the variables associated with the edges in E'.

Lifting Lemma [11]. *Let $W \subseteq V$ and let $\pi x \leq \pi_0$ be a valid inequality such that $\pi_e = 0$ for $e \notin E(W)$. If $\pi_{E(W)} x \leq \pi_0$ is facet-inducing for $CPP(G_W)$, then $\pi x \leq \pi_0$ is facet-inducing for $CPP(G)$.*
Projection Lemma [11]. *Let $\pi x \leq \pi_0$ be a facet-inducing inequality for $CPP(G)$. Let $e \in E$ with $\pi_e = 0$ and define $G' = (V, E')$, where $E' = E \backslash \{e\}$. Then, $\pi_{E'} x \leq \pi_0$ is facet-inducing for $CPP(G')$.*

3. A family of facet-inducing inequalities

We present in this section a new family of facet-inducing inequalities, which will be one of the base cases for the facet-preserving procedures introduced in the next section.

We define a *central claw* $C \subseteq E$ to be the structure depicted in Figure 2, *i.e.*, $C = \{u_1u_2\} \cup T \subseteq E$, where $T = \{u_1v_1, u_1v_2, u_2v_3, u_2v_4\} \cup \{v_iw_i\}_{i=1}^4$. We define

$$2x_{u_1u_2} + \sum_{e \in T} x_e \;\leq\; 8 \tag{3.1}$$

to be the *central claw inequality* associated with C. Figure 2 shows a picture of such a valid inequality, by representing variables with coefficient 1 with single-line edges, and variables with coefficient 2 with double-line edges. Note that the central claw $C \subseteq E$ need not be induced in G (*i.e.*, there may exist an edge $ij \in E$, $i, j \in V_C$, such that $ij \notin C$). Also note that this inequality cannot be obtained from a bipartite claw constraint with the procedures presented in [11] and in this manuscript.

For $i \in V$, we define $\delta(i) = \{ij : ij \in E \text{ for some } j \in V\}$. For $A \subseteq E$ and $x \in CPP(G)$, we define $x(A) = \sum_{e \in A} x_e$.

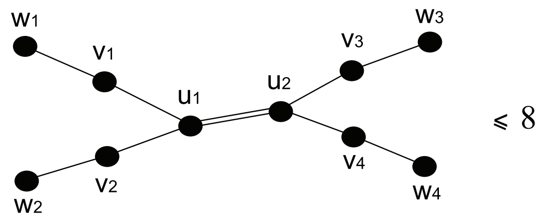

FIGURE 2. The central claw inequality (3.1).

An *incomplete bipartite claw* is the graph H depicted in Figure 1(c), and we say that v is the *dangling vertex* of H. If $x = \mathbf{u}_S \in CPP(G)$, we say that v is a *dangling vertex in* x if S contains an incomplete bipartite claw H such that v is the dangling vertex of H.

If $\pi x \leq \pi_0$ is a valid inequality, we call $E_\pi = \{e \in E : \pi_e \neq 0\}$ the *support* of π. We denote by G_π the subgraph of G given by the edges in E_π, *i.e.*, $G_\pi = (V_\pi, E_\pi)$, where $V_\pi = \{i \in V : ij \in E_\pi$ for some $j \in V\}$. For $u \in V_\pi$, we denote by $d_\pi(u)$ the degree of u in G_π. We define $\mathbf{0} \in \mathbb{R}^{|E|}$ to be the all-zeros vector with $|E|$ entries.

For $e \in E$, the valid inequalities $x_e \geq 0$ and $x_e \leq 1$ are called the *trivial* inequalities associated with e. An inequality is *nontrivial* if it differs from $x_e \geq 0$ and $x_e \leq 1$, for any $e \in E$.

Theorem 3.1. *The central claw inequality (3.1) is valid and facet-inducing for $CPP(G)$.*

Proof. Let $x \in CPP(G) \cap \mathbb{Z}^{|E|}$ be a feasible solution. If $x_{u_1u_2} = 0$ then $\sum_{e \in T} x_e \leq 8$ implies that (3.1) is satisfied, so assume $x_{u_1u_2} = 1$. In this case, we cannot have $x_e = 1$ for every $e \in T$, since x would then contain the bipartite claw $B = \{v_2u_1, u_1u_2, u_2v_3, v_3w_3, u_2v_4, v_4w_4\}$. We also cannot have $x_e = 1$ for every $e \in T \backslash \{u_1v_1\}$ or $x_e = 1$ for every $e \in T \backslash \{v_1w_1\}$, since in these cases the bipartite claw B again appears in x. A similar analysis shows that, for any $e' \in T$, the configuration $x_e = 1$ for every $e \in T \backslash \{e'\}$ and $x_{e'} = 0$ is not feasible. We conclude that $x(T) \leq 6$, so (3.1) is satisfied.

Now for facetness. By the lifting lemma, we may assume that G is the subgraph induced by the vertices $\{u_1, u_2\} \cup \{v_i, w_i\}_{i=1}^4$. Call F the face of $CPP(G)$ defined by (3.1), and assume $\lambda x = \lambda_0$ for every $x \in F$. We shall prove that λ is a multiple of the coefficient vector of (3.1), thus showing that F is a facet of $CPP(G)$.

Consider the solution $\bar{x} = \mathbf{u}_{u_1u_2} + \mathbf{u}_{T \backslash \{v_1w_1, v_3w_3\}}$ and the solution $\bar{x}' = \mathbf{u}_{u_1u_2} + \mathbf{u}_{T \backslash \{u_1v_1, v_3w_3\}}$. Both points are feasible and satisfy (3.1) with inequality, hence $\lambda \bar{x} = \lambda_0 = \lambda \bar{x}'$. Since $\bar{x}$ and $\bar{x}'$ differ in the variables $x_{v_1w_1}$ and $x_{u_1v_1}$, then $\lambda_{v_1w_1} = \lambda_{u_1v_1}$. By a similar procedure with the edges in T, we can verify that $\lambda_{v_iw_i} = \lambda_{u_jv_i}$ for $i = 1, \ldots, 4$ and $j \in \{1, 2\}$ such that $u_jv_i \in E$.

Consider now the solution $\bar{x}'' = \mathbf{u}_{u_1u_2} + \mathbf{u}_{T \backslash \{v_2w_2, v_3w_3\}}$. Again, $\bar{x}''$ is feasible and satisfies (3.1) with inequality. The points $\bar{x}$ and $\bar{x}''$ only differ in the variables $x_{v_1w_1}$ and $x_{v_2w_2}$, hence $\lambda_{v_1w_1} = \lambda_{v_2w_2}$. By a similar procedure with the edges $\{v_iw_i\}_{i=1}^4$, we get $\lambda_{v_iw_i} = \lambda_{v_jw_j}$ for $i, j = 1, \ldots, 4$. By combining the previous observations, we conclude that $\lambda_e = \lambda_{e'}$ for $e, e' \in T$.

Finally, consider the solution $\tilde{x} = \mathbf{u}_T$. Again, $\tilde{x}$ is feasible and satisfies (3.1) with equality. The existence of the solutions $\bar{x}$ and $\tilde{x}$ implies $\lambda_{u_1u_2} = \lambda_{v_1w_1} + \lambda_{v_3w_3}$. Since $\lambda_e = \lambda_{e'}$ for $e, e' \in T$, we conclude $\lambda_{u_1u_2} = 2\lambda_e$ for any $e \in T$.

We now show that $\lambda_e = 0$ for $e \in E \backslash C$. For every $e \in \delta(w_3) \backslash C$, it is not hard to verify that $\bar{x}' + \mathbf{u}_e \in CPP(G)$, hence we have $\bar{x}' \in F$ and $\bar{x}' + \mathbf{u}_e \in F$, implying $\lambda_e = 0$. By repeating this argument with $\{w_i\}_{i \neq 3}$, we conclude that $\lambda_e = 0$ for $e \in \delta(w_i) \backslash C$ and $i = 1, \ldots, 4$.

Finally, for each $e \in \delta(v_1) \backslash C$, construct the solution $\hat{x} = \mathbf{u}_{u_1u_2} + \mathbf{u}_{T \backslash \{u_1v_1, u_2v_2\}}$. Again, we have $\hat{x} \in F$ and $\hat{x} + \mathbf{u}_e \in F$, implying $\lambda_e = 0$. By repeating this argument with $\{v_i\}_{i \neq 1}$, we get $\lambda_e = 0$ for $e \in \delta(v_i) \backslash C$ and $i = 1, \ldots, 4$.

We conclude that λ is a multiple of the coefficient vector of (3.1) which, therefore, defines a facet of $CPP(G)$. □

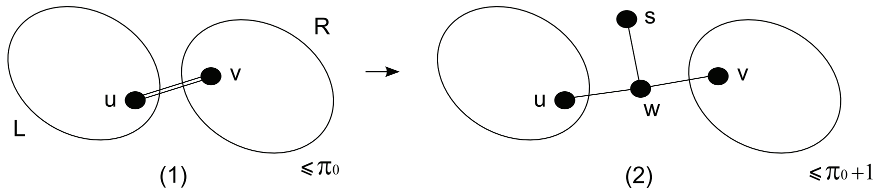

FIGURE 3. The construction specified by Procedure 1.

It must be noted that the central claw inequalities can be separated in polynomial time by exhaustive enumeration. The structure obtained from a central claw by splitting u_1u_2, thus generating an extra vertex z connected to u_1 and u_2, gives the following valid inequality by Theorem 4 of [11]:

$$\sum_{e\in T} x_e + 2(x_{u_1z} + x_{zu_2}) \ \leq \ 10. \tag{3.2}$$

However, in spite of Theorem 3.1, there is a feasible solution x in the face defined by the central claw inequality (3.1) such that u_2 is a dangling vertex in x, which means that Theorem 4 of [11] cannot be applied to guarantee facetness. Indeed, (3.2) is not facet-inducing since it can be obtained as the result of the addition of two bipartite claw constraints, both including u_1z and zu_2.

4. FACET-PRESERVING PROCEDURES WITH ACYCLIC SUPPORTS

The polytope $CPP(G)$ admits many families of facet-inducing inequalities involving different graph structures. Interestingly, many of these families of facets can be explained in terms of facet-preserving procedures. Each such procedure takes as input a facet-inducing inequality and produces a sligthly modified inequality that is also facet-inducing. This section explores two such procedures for $CPP(G)$ that do not introduce cycles in the obtained inequality, hence are applicable when G is a tree.

4.1. First facet-preserving procedure

The first facet-preserving procedure takes as input a valid inequality with at least one variable with coefficient 2, and turns the associated edge into three new edges in a modified graph G'. The new inequality is valid for $CPP(G')$ and, if the original inequality is facet-inducing for $CPP(G)$ and additional technical hypotheses hold, then it is also facet-inducing for $CPP(G')$.

Procedure 1. Let $\pi x \leq \pi_0$ be a nontrivial valid inequality. Let $uv \in E_\pi$ with $\pi_{uv} = 2$. Call $L \subseteq V$ the connected component of $G_\pi \backslash \{uv\}$ including u, and call $R \subseteq V$ the connected component of $G_\pi \backslash \{uv\}$ including v (see Fig. 3). Let $G' = (V', E')$ be the graph defined by $V' = V \cup \{w, s\}$ and $E' = E\backslash\{uv\} \cup \{uw, wv, ws\}$. In this setting, the procedure constructs the inequality

$$\sum_{e\in E\backslash\{uv\}} \pi_e x_e + (x_{uw} + x_{wv} + x_{ws}) \ \leq \ \pi_0 + 1. \tag{4.1}$$

Theorem 4.1. *Assume the hypotheses of Procedure 1 hold. If*

(a) *L and R are disjoint,*
(b) *there exists $t \in L$ with $ut \in E$ and $\pi_{ut} = 1$,*
(c) *there exists $t \in R$ with $vt \in E$ and $\pi_{vt} = 1$,*
(d) *at most one edge $ut \in E_\pi$ with $t \in L$ has $\pi_{ut} \geq 2$, and in this case every edge $tp \in E_\pi$ with $p \neq u$ has $\pi_{tp} = 1$,*

(e) *at most one edge $vt \in E_\pi$ with $t \in R$ has $\pi_{vt} \geq 2$, and in this case every edge $tp \in E_\pi$ with $p \neq v$ has $\pi_{tp} = 1$,*

then (4.1) is valid for $CPP(G')$.

Proof. Denote (4.1) by $\hat{\pi}x \leq \hat{\pi}_0$. Let $\bar{x} \in CPP(G') \cap \{0,1\}^{|E'|}$, and define $T = \{uw, vw, ws\}$.

If $\bar{\boldsymbol{x}}\boldsymbol{(T)} \leq \mathbf{1}$, construct $\bar{x}' \in CPP(G)$ by $\bar{x}'_{uv} = 0$, $\bar{x}'_e = \bar{x}_e$ for $e \in E_{\hat{\pi}} \backslash T$, and $\bar{x}'_e = 0$ for $e \in E \backslash E_{\hat{\pi}}$. Let A be the edge set of the solution represented by $\bar{x}$. Since the edge set of the solution represented by $\bar{x}'$ is $A \backslash T$, then $\bar{x}' \in CPP(G)$, hence $\pi\bar{x}' \leq \pi_0$. This implies

$$\hat{\pi}\bar{x} \;=\; \pi\bar{x}' + \bar{x}(T) \;\leq\; \pi_0 + 1 \;=\; \hat{\pi}_0.$$

If $\bar{\boldsymbol{x}}_{\boldsymbol{uw}} = \bar{\boldsymbol{x}}_{\boldsymbol{vw}} = \mathbf{1}$ (implying $\bar{x}(T) \geq 2$), then construct $\bar{x}' \in \mathbb{Z}^{|E|}$ by $\bar{x}'_{uv} = 1$, $\bar{x}'_e = \bar{x}_e$ for $e \in E_{\hat{\pi}} \backslash T$, and $\bar{x}'_e = 0$ for $e \in E \backslash E_{\hat{\pi}}$. By contradiction, if $\bar{x}'$ has a cycle (resp. bipartite claw) B, then $\bar{x}$ contains the cycle (resp. bipartite claw) $B \backslash \{uv\} \cup \{uw, vw\}$, hence $\bar{x}' \in CPP(G)$. This implies $\pi\bar{x}' \leq \pi_0$, and

$$\hat{\pi}\bar{x} \;=\; \pi\bar{x}' \underbrace{-2\bar{x}'_{uv} + \bar{x}(T)}_{\leq 1} \;\leq\; \pi_0 + 1 \;=\; \hat{\pi}_0.$$

Finally, if $\bar{\boldsymbol{x}}\boldsymbol{(T)} = \mathbf{2}$ **but** $\bar{\boldsymbol{x}}_{\boldsymbol{uw}} \neq \bar{\boldsymbol{x}}_{\boldsymbol{vw}}$, we either have $\bar{x}_{uw} = \bar{x}_{ws} = 1$ or $\bar{x}_{vw} = \bar{x}_{ws} = 1$. Construct $\bar{x}'$ as in the previous paragraph. Since $\bar{x}(T) = 2$ and $\bar{x}'_{uv} = 1$, we have $\hat{\pi}\bar{x} = \pi\bar{x}' - 2\bar{x}'_{uv} + \bar{x}(T) = \pi\bar{x}'$. If $\bar{x}' \in CPP(G)$, then $\pi\bar{x}' \leq \pi_0$, hence

$$\hat{\pi}\bar{x} \;=\; \pi\bar{x}' - 2\bar{x}'_{uv} + \bar{x}(T) \;=\; \pi\bar{x}' \;\leq\; \pi_0 \;\leq\; \hat{\pi}_0.$$

So assume $\bar{x}' \notin CPP(G)$, hence $\bar{x}'$ contains a bipartite claw B ($\bar{x}'$ cannot contain a cycle since $\bar{x}_L$ and $\bar{x}_R$ do not contain any cycle, and the hypothesis (a) holds). We must have $uv \in B$, since otherwise B is included within L or R, a contradiction since $\bar{x}$ is feasible. This implies that B has one of the configurations shown in Figure 4, or their symmetrical counterparts with L. Assume w.l.o.g. that B has one of the configurations presented in Figure 4. In both cases we have $\bar{x}_{vw} = 0$, since otherwise $B \backslash \{uv\} \cup \{vw\}$ in case (i) and $B \backslash \{uv, ut\} \cup \{vw, ws\}$ in case (ii) is a bipartite claw in $\bar{x}$, a contradiction. We now identify an edge $\bar{e}$ such that $\bar{x}' - \mathbf{u}_{\bar{e}} \in CPP(G)$ and $\pi_{\bar{e}} = 1$, as follows.

- In case (i), we cannot have $\bar{x}_{vq} = 1$ for $q \in R$, $q \neq p$, since in this case the bipartite claw $B \backslash \{uv\} \cup \{vq\}$ belongs to R. Hence, $\bar{x}' - \mathbf{u}_{vp} \in CPP(G)$. If $\pi_{vp} = 1$ then we set $\bar{e} := vp$. On the other hand, if $\pi_{vp} \geq 2$ then the hypothesis (e) ensures that $\pi_{pr} = 1$ for every $pr \in E_\pi$ with $r \neq v$, so we take $\bar{e}$ to be any such edge within B.
- In case (ii), the hypotheses (c) and (e) ensure that there exists $p \in R$ with $\pi_{vp} = 1$. We have $\bar{x}' - \mathbf{u}_{vp} \in CPP(G)$ since otherwise $\bar{x}$ contains a bipartite claw $B' \neq B$ with $uv \in B'$ and having at least four edges in $E(R)$ (see Fig. 5), which translates into $\bar{x}$, a contradiction. We take, therefore, $\bar{e} := vp$.

In all cases, we have $\bar{x}' - \mathbf{u}_{\bar{e}} \in CPP(G)$ and $\pi_{\bar{e}} = 1$. This implies $\pi(\bar{x}' - \mathbf{u}_{\bar{e}}) \leq \pi_0$, hence

$$\hat{\pi}\bar{x} \;=\; \pi\bar{x}' \;=\; \pi(\bar{x}' - \mathbf{u}_{\bar{e}}) + \pi_{\bar{e}} \;\leq\; \pi_0 + 1 \;=\; \hat{\pi}_0.$$

Since $\bar{x}$ is an arbitrary feasible solution in $CPP(G')$, we conclude that $\hat{\pi}x \leq \hat{\pi}_0$ is a valid inequality for this polytope. ☐

Theorem 4.2. *Assume the hypotheses of Procedure 1 and the conditions (a)–(e) hold. If (i) $\pi x \leq \pi_0$ defines a facet F of $CPP(G)$, (ii) there exists $x \in F$ with $x_{uv} = 0$ such that u is not a dangling vertex in x, and (iii) there exists $x \in F$ with $x_{uv} = 0$ such that v is not a dangling vertex in x, then (4.1) induces a facet of $CPP(G')$.*

Proof. Denote (4.1) by $\hat{\pi}x \leq \hat{\pi}_0$. By the lifting lemma, we may assume $G = G_\pi$. Since $CPP(G)$ is full-dimensional, let $x^1, \ldots, x^{|E|}$ be affinely independent points such that $\pi x^i = \pi_0$ for $i = 1, \ldots, |E|$. For $i = 1, \ldots, |E|$, construct $\bar{x}^i$ from x^i as follows.

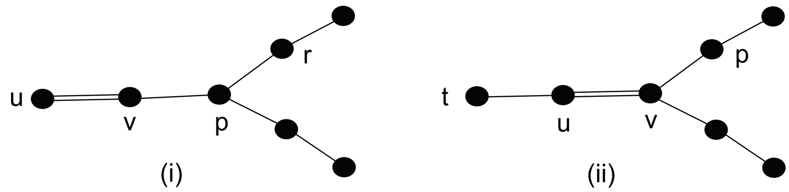

FIGURE 4. Possible configurations for B in the proof of Theorem 4.1.

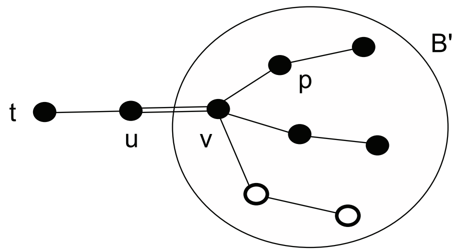

FIGURE 5. Appearance of a bipartite claw B' in case (ii) of the proof of Theorem 4.1.

- If $x^i_{uv} = 0$, then set $\bar{x}^i_e = x^i_e$ for $e \in E \backslash \{uv\}$, $\bar{x}^i_{uw} = \bar{x}^i_{vw} = 0$, and $\bar{x}^i_{ws} = 1$.
- If $x^i_{uv} = 1$, then set $\bar{x}^i_e = x^i_e$ for $e \in E \backslash \{uv\}$ and $\bar{x}^i_{uw} = \bar{x}^i_{vw} = \bar{x}^i_{ws} = 1$.

In both cases we get that $\bar{x}^i$ is a feasible solution and $\hat{\pi}\bar{x}^i = \hat{\pi}_0$ since $\pi x^i = \pi_0$. Furthermore, since the projection of $\bar{x}^i$ onto the variables associated with $E \backslash \{uv\} \cup \{uw\}$ coincides with x^i, then $\bar{x}^i$ is affinely independent with $\{\bar{x}^j\}_{j<i}$.

Let $t \in \{1, \ldots, |E|\}$ such that $x^t_{uv} = 0$ and such that u is not a dangling vertex in x^t (the hypothesis (ii) ensures that such a point exists), and construct the point $\tilde{x}^t$ by setting $\tilde{x}^t_e = x^t_e$ for $e \in E \backslash \{uv\}$, $\tilde{x}^t_{vw} = \tilde{x}^t_{ws} = 0$, and $\tilde{x}^t_{uw} = 1$. This point is feasible, has $\hat{\pi}\tilde{x}^t = \pi x^t + \hat{\pi}_{uw} = \pi_0 + 1 = \hat{\pi}_0$, and is affinely independent with $\{\bar{x}^1, \ldots, \bar{x}^{|E|}\}$ since $\tilde{x}^t_{uw} \neq \tilde{x}^t_{vw}$ but $\bar{x}^i_{uw} = \bar{x}^i_{vw}$ for $i = 1, \ldots, |E|$.

Finally, let $k \in \{1, \ldots, |E|\}$ such that $x^k_{uv} = 0$ and such that v is not a dangling vertex in x^k (the hypothesis (iii) ensures that such a point exists), and construct the point $\hat{x}^k$ by setting $\hat{x}^k_e = x^k_e$ for $e \in E \backslash \{uv\}$, $\hat{x}^k_{uw} = \hat{x}^k_{ws} = 0$, and $\hat{x}^k_{vw} = 1$. Again, this point is feasible, has $\hat{\pi}\hat{x}^k = \hat{\pi}_0$, and is affinely independent with $S = \{\bar{x}^1, \ldots, \bar{x}^{|E|}\} \cup \{\tilde{x}^t\}$ since $x_{uw} = 1 - x_{ws} + x_{vw}$ for $x \in S$ but $\hat{x}^k_{uw} \neq 1 - \hat{x}^k_{ws} + \hat{x}^k_{vw}$.

We have, therefore, constructed $|E|+2 = |E'|$ affinely independent points satisfying (4.1) with equality which, therefore, defines a facet of $CPP(G')$. □

A central claw satisfies the conditions of Theorem 4.2. Consequently, if Procedure 1 is applied to the facet-inducing inequality (3.1), then the resulting inequality

$$\sum_{e \in T} x_e + (x_{u_1 z} + x_{z u_2} + x_{zs}) \ \leq \ 9 \qquad (4.2)$$

is also facet-inducing.

4.2. Second facet-preserving procedure

We now introduce the second facet-preserving procedure. This procedure takes as input a valid inequality having a claw in its support, and adds an edge to the support while changing at the same time two coefficients from the original inequality. Again, the new inequality is valid and, if some additional technical conditions

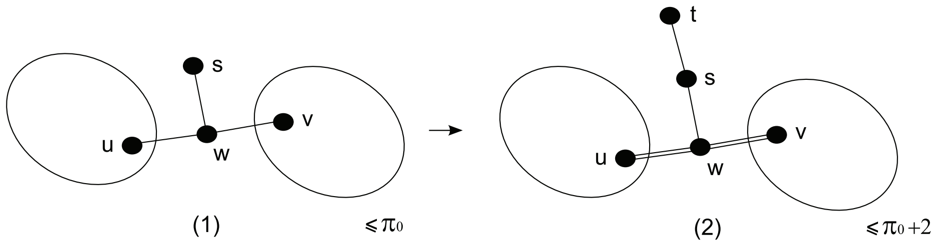

FIGURE 6. The construction specified by Procedure 2.

hold, this procedure also preserves facetness.

Procedure 2. Let $\pi x \leq \pi_0$ be a nontrivial valid inequality. Let $uw, vw, ws \in E_\pi$ with $\pi_{uw} = \pi_{vw} = \pi_{ws} = 1$, $d_\pi(w) = 3$, and $d_\pi(s) = 1$ (see Fig. 6). Let $t \in V \backslash V_\pi$ such that $st \in E$. In this setting, the procedure constructs the inequality

$$\sum_{e \in E} \pi_e x_e + (x_{uw} + x_{wv} + x_{st}) \ \leq \ \pi_0 + 2. \tag{4.3}$$

Theorem 4.3. *Assume the hypotheses of Procedure 2 hold. If every point $x \in CPP(G)$ with $x_{uw} = x_{vw} = x_{ws} = 1$ has $\pi x \leq \pi_0 - 1$, then (4.3) is valid for $CPP(G)$.*

Proof. Denote (4.3) by $\hat{\pi} x \leq \hat{\pi}_0$. Let $\bar{x} \in CPP(G) \cap \mathbb{Z}^{|E|}$ (which, therefore, has $\pi \bar{x} \leq \pi_0$) and define $T = \{uw, vw, ws, st\}$. Assume w.l.o.g. $\bar{x}_e = 0$ for $e \notin E_{\hat{\pi}}$. If $\bar{x}_{uw} + \bar{x}_{wv} + \bar{x}_{st} \leq 2$, we have $\hat{\pi}\bar{x} = \pi\bar{x} + (\bar{x}_{uw} + \bar{x}_{wv} + \bar{x}_{st}) \leq \pi_0 + 2 = \hat{\pi}_0$. So, assume $\bar{x}_{uw} = \bar{x}_{wv} = \bar{x}_{st} = 1$. If $\bar{x}_{ws} = 0$ then $\bar{x}' := \bar{x} - \mathbf{u}_{st} + \mathbf{u}_{ws}$ is also feasible, since w has exactly two incident edges in the solution induced by $\bar{x}$ and s has no incident edges in the solution induced by $\bar{x} - \mathbf{u}_{st}$ (since $\bar{x}_e = 0$ for $e \notin E_{\hat{\pi}}$). The fact that $\bar{x}'$ is feasible shows that $\bar{x}$ cannot satisfy $\pi\bar{x} \leq \pi_0$ with equality (if this was the case, then $\pi\bar{x}' = \pi\bar{x} + \pi_{ws} = \pi_0 + 1$, a contradiction). Since $\pi\bar{x}' \leq \pi_0$, we have

$$\begin{aligned} \hat{\pi}\bar{x} &= \pi\bar{x} + (\bar{x}_{uw} + \bar{x}_{wv} + \bar{x}_{st}) \\ &= \pi\bar{x}' + (\bar{x}_{uw} + \bar{x}_{wv}) \\ &\leq \pi_0 + 2 \ = \ \hat{\pi}_0. \end{aligned}$$

On the other hand, if $\bar{x}_{ws} = 1$ then the hypothesis ensures that $\pi\bar{x} \leq \pi_0 - 1$, hence

$$\hat{\pi}\bar{x} \ = \ \pi\bar{x} + (\bar{x}_{uw} + \bar{x}_{wv} + \bar{x}_{st}) \ \leq \ (\pi_0 - 1) + 3 \ = \ \hat{\pi}_0. \tag{4.4}$$

In all cases, we get $\hat{\pi}\bar{x} \leq \hat{\pi}_0$. Since $\bar{x}$ is an arbitrary feasible solution, we conclude that (4.3) is a valid inequality. □

The hypothesis in Theorem 4.3 asking for every point $x \in CPP(G)$ with $x_{uw} = x_{vw} = x_{ws} = 1$ to have $\pi x \leq \pi_0 - 1$ is quite unsatisfactory, since it involves a nontrivial check on $\pi x \leq \pi_0$. Unfortunately, additional hypotheses are indeed required in order to ensure the validity of the inequality constructed by the procedure, as the counterexample in Figure 7 shows. The first inequality $\pi x \leq 11$ in the figure is valid but does not satisfy this hypothesis. The second inequality $\hat{\pi} x \leq 13$ in the figure is the inequality obtained by applying Procedure 2 to $\pi x \leq 11$, but it is not valid since the point $\bar{x} = \mathbf{u}_S$ for $S = E_{\hat{\pi}} \backslash \{e_1, e_2\}$ has $\hat{\pi}\bar{x} = 14$.

Theorem 4.4. *Assume the hypotheses of Procedure 2 and Theorem 4.3 hold. If (i) $\pi x \leq \pi_0$ defines a facet F of $CPP(G)$, (ii) there exist $x^1, \ldots, x^{|E|} \in F$ affinely independent such that no $i \in \{1, \ldots, |E|\}$ has $x^i_{uw} = x^i_{vw} = 0$ and $x^i_{ws} = 1$ and (iii) some $i \in \{1, \ldots, |E|\}$ has $x^i_{uw} = x^i_{vw} = x^i_{ws} = 1$, then (4.3) induces a facet of $CPP(G)$.*

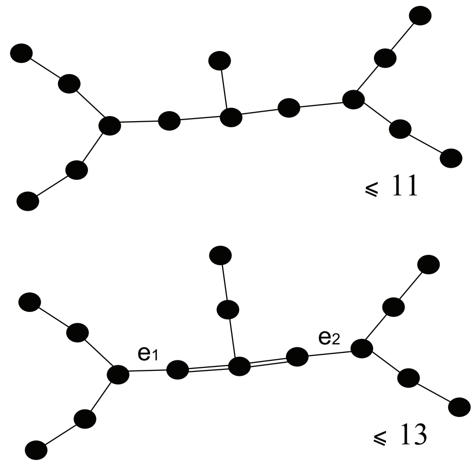

FIGURE 7. Counterexample to the validity of the inequality generated by Procedure 2 when the hypothesis in Theorem 4.3 does not hold.

Proof. Denote (4.3) by $\hat{\pi}x \leq \hat{\pi}_0$. By the lifting lemma, we may assume $G = G_{\hat{\pi}}$. Let $x^1, \ldots, x^{|E|}$ be affinely independent points satisfying the hypotheses (ii) and (iii). By the projection lemma, $\pi x \leq \pi_0$ is facet-defining for $CPP(G) \cap \{x \in \{0,1\}^{|E|} : x_{st} = 0\}$. Let $E' = E \backslash \{st\}$. Since $x^i \in \{0,1\}^{|E|}$ for $i = 1, \ldots, |E|$, then the projection of $\{x^1, \ldots, x^{|E|}\}$ onto all the variables with the exception of x_{st} contains $|E'| = |E| - 1$ affinely independent points, say $x^1, \ldots, x^{|E'|}$. Assume w.l.o.g. that x^1 satisfies the hypothesis (iii), *i.e.*, $x^1_{uw} = x^1_{vw} = x^1_{ws} = 1$.

No $i \in \{1, \ldots, |E'|\}$ has $x^i_{uw} = x^i_{vw} = 1$ and $x^i_{ws} = 0$, since in this case the vertex w only has two incident edges in the solution induced by x, hence the point $x^i + \mathbf{u}_{ws}$ is feasible and $\pi(x^i + \mathbf{u}_{ws}) > \pi_0$, a contradiction since $\pi x \leq \pi_0$ is valid. Also note that no $i \in \{1, \ldots, |E'|\}$ has $x^i_{uw} = x^i_{vw} = x^i_{ws} = 0$, since again the point $x^i + \mathbf{u}_{ws}$ is feasible and $\pi(x^i + \mathbf{u}_{ws}) > \pi_0$, a contradiction.

Let $T = \{uw, vw, ws\}$. For $i = 1, \ldots, |E'|$, construct $\bar{x}^i$ from x^i as follows.

- If $x^i(T) = 3$, then take $\bar{x}^i_e = x^i_e$ for $e \in E \backslash \{st\}$ and $\bar{x}^i_{st} = 0$. We have $\hat{\pi}\bar{x}^i = \pi x^i + (x^i_{uw} + x^i_{vw}) = \pi_0 + 2 = \hat{\pi}_0$.
- If $x^i(T) \leq 2$, then take $\bar{x}^i_e = x^i_e$ for $e \in E \backslash \{st\}$ and $\bar{x}^i_{st} = 1$. By the previous observations, we cannot have $x^i(T) = 0$ and the hypothesis (ii) implies that $x^i_{uw} + x^i_{vw} \geq 1$. We have already verified that in this case we cannot have $x^i_{uw} = x^i_{vw} = 1$, hence $x^i_{uw} + x^i_{vw} = 1$. We conclude that

$$\hat{\pi}\bar{x}^i \;=\; \pi x^i + (\underbrace{\bar{x}^i_{uw} + \bar{x}^i_{vw}}_{=1} + \underbrace{\bar{x}^i_{ws}}_{=1}) \;=\; \pi_0 + 2 \;=\; \hat{\pi}_0.$$

The projection of $\bar{x}^i$ and x^i onto all the variables with the exception of x_{st} coincide, hence $\bar{x}^i$ is affinely independent w.r.t. $\{\bar{x}^j\}_{j<i}$.

Finally, construct $\tilde{x}^1$ by $\tilde{x}^1_e = x^1_e$ for $e \in E \backslash \{ws, st\}$, $\tilde{x}^1_{ws} = 0$, and $\tilde{x}^1_{st} = 1$. We have

$$\begin{aligned}\hat{\pi}\tilde{x}^1 &= (\pi x^1 - \pi_{ws}) + (\tilde{x}^1_{uw} + \tilde{x}^1_{vw} + \tilde{x}^1_{st}) \\ &= (\pi_0 - 1) + 3 \;=\; \hat{\pi}_0.\end{aligned}$$

The point $\tilde{x}^1$ is affinely independent w.r.t. the points $\bar{x}^1, \ldots, \bar{x}^{|E'|}$ since $x^i_{st} + x^i_{uw} + x^i_{vw} = 2$ for $i = 1, \ldots, |E'|$ but $\tilde{x}^1_{st} + \tilde{x}^1_{uw} + \tilde{x}^1_{vw} = 3$.

This way, we construct the set $\{\bar{x}^1, \ldots, \bar{x}^{|E'|}\} \cup \{\tilde{x}^1\}$ of $|E|$ affinely independent points satisfying $\hat{\pi}x \leq \hat{\pi}_0$ with equality which, therefore, defines a facet of $CPP(G)$. □

The hypothesis of Theorem 4.3 in Theorem 4.4 can be replaced by the condition that (4.3) is valid. Procedure 2 can be applied to the inequality (4.2), which in turn was generated by Procedure 1, in order to obtain a new inequality. The hypotheses of the previous theorems hold in this case, hence the resulting inequality is valid and facet-inducing.

5. Concluding remarks

In this work we have studied facet-preserving procedures that generate facet-inducing inequalities with acyclic supports from existing strong valid inequalities. Similar procedures presented in previous works create cycles in the support of the constructed valid inequalities, so they cannot be applied to $CPP(G)$ when G is a tree.

The theorems guaranteeing that the obtained inequalities are valid involve hypotheses that may be quite difficult to check in practice, hence we do not expect the procedures presented in this work to be useful in practice (contrary to the practical effectiveness of previously-presented procedures). In particular, the hypothesis (b)–(e) in Theorem 4.1 may complicate the search for violated inequalities generated by Procedure 1, and the hypothesis in Theorem 4.3 asking every feasible point x with $x_{uw} = x_{wv} = x_{ws} = 1$ to satisfy $\pi x \leq \pi_0 - 1$ may be intractable to check in practice. We leave as an open problem the search for procedures with hypotheses more amenable to a practical treatment.

The main interest in these results is theoretical, and stems from the fact that a complete description of $CPP(G)$ with constraints separable in polynomial time would automatically yield a proof of the computational tractability of the 2-SSCPsc over trees. Computational experiments with polyhedral software on small instances suggest that the results presented in this work might provide such complete descriptions for small instances, and it would be interesting to explore whether such complete descriptions are possible for arbitrarily-sized instances.

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References

[1] R. Baldacci, M. Dell'Amico and J. Salazar González, The capacitated m-ring-star problem. *Oper. Res.* **55** (2007) 1147–1162.
[2] H. Calvete, C. Galé and J. Iranzo, An efficient evolutionary algorithm for the ring star problem. *Eur. J. Oper. Res.* **231** (2013) 22–33.
[3] T. Dias, G. de Sousa, E. Macambira, L. Cabral and M. Fampa, An efficient heuristic for the ring star problem. In: *Proceedings of the 5th International Workshop on Experimental Algorithms WEA 2006, Menorca, Spain*. In Vol. 4007 of *Lect. Note Comput. Sci.* (2006) 24–35.
[4] M. Dinneen and M. Khosravani, A linear time algorithm for the minimum spanning caterpillar problem for bounded treewidth graphs. In: *Proceedings of the 17th International Colloquium on Structural Information and Communication Complexity, Sirince*. In Vol. 6058 of *Lect. Notes Comput. Sci.* (2010) 237–246.
[5] M. Dinneen and M. Khosravani, Hardness of approximation and integer programming frameworks for searching for caterpillar trees. In: *Proceedings of the Seventeenth Computing on The Australasian Theory Symposium, Perth, Australia* (2011) 145-150.
[6] M. Grötschel, L. Lovász and A. Schrijver, Geometric Algorithms and Combinatorial Optimization. Springer-Verlag, Berlin (1993).
[7] S. Kedad-Sidhoum and V. Nguyen, An exact algorithm for solving the ring star problem. *Optimization* **59** (2010) 125–140.
[8] M. Labbé, G. Laporte, I. Martín and J. González, The ring star problem: polyhedral analysis and exact algorithm. *Networks* **43** (2004) 177–189.
[9] C. Lekkerkerker and J. Boland, Representation of a finite graph by a set of intervals on the real line. *Fund. Math.* **51** (1962) 45–64.
[10] S. Lucero, J. Marenco and F. Martínez, An integer programming approach for the 2-schemes strip cutting problem with a sequencing constraint. *Optim. Eng.* **16** (2015) 605–632.
[11] J. Marenco, The caterpillar-packing polytope. *Discrete Appl. Math.* **245** (2018) 4–15.
[12] Z. Naji-Azimi, M. Salari and P. Toth, A heuristic procedure for the capacitated m-ring-star problem. *Eur. J. Oper. Res.* **207** (2010) 1227–1234.
[13] F. Rinaldi and A. Franz, A two-dimensional strip cutting problem with sequencing constraint. *Eur. J. Oper. Res.* **183** (2007) 1371–1384.
[14] L. Simonetti, Y. Frota and C.C. de Souza, An exact method for the minimum caterpillar spanning problem. In: *Proceedings of the 8th Cologne-Twente Workshop on Graphs and Combinatorial Optimization, Paris, France* (2009) 48-51.

[15] L. Simonetti, Y. Frota and C.C. de Souza, Upper and lower bounding procedures for the minimum caterpillar spanning problem. *Eletron. Notes Discrete Math.* **35** (2009) 83–88.

[16] K. Sundar and S. Rathinam, Multiple depot ring star problem: a polyhedral study and exact algorithm. *J. Global Optim.* **67** (2017) 527–551.

[17] Z. Zhang, H. Qin and A. Lim, A memetic algorithm for the capacitated m-ring-star problem. *Appl. Intell.* **40** (2014) 305–321.