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NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS USING CONVEXIFACTORS FOR MATHEMATICAL PROGRAMS WITH EQUILIBRIUM CONSTRAINTS

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Abstract. The main aim of this paper is to develop necessary Optimality conditions using Convexifactors for mathematical programs with equilibrium constraints (MPEC). For this purpose a nonsmooth version of the standard Guignard constraint qualification (GCQ) and strong stationarity are introduced in terms of convexifactors for MPEC. It is shown that Strong stationarity is the first order necessary optimality condition under nonsmooth version of the standard GCQ. Finally, notions of asymptotic pseudoconvexity and asymptotic quasiconvexity are used to establish the sufficient optimality conditions for MPEC.

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1. Introduction

In this paper, we study following, mathematical programs with equilibrium constraints (MPEC):

$$\begin{aligned}
\text{(MPEC)}\quad & \text{Minimize } f(x)\\
& \text{subject to } g(x) \leq 0,\ h(x) = 0,\\
& \qquad G(x) \geq 0,\ H(x) \geq 0,\ G(x)^t H(x) = 0,
\end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, $h : \mathbb{R}^n \to \mathbb{R}^p$, $G : \mathbb{R}^n \to \mathbb{R}^l$, $H : \mathbb{R}^n \to \mathbb{R}^l$.

Let $K := \{x \in \mathbb{R}^n : g(x) \leq 0, h(x) = 0, G(x) \geq 0, H(x) \geq 0, G(x)^t H(x) = 0\}$ denote the feasible set for MPEC.

Before proceeding further, we define the following sets in the context of MPECs. From the complimentarity term in MPEC, we have that for a feasible point $\bar{x} \in K$, either $G_i(\bar{x})$ or $H_i(\bar{x})$, $i = 1, 2, \ldots, l$ or both must be zero. To differentiate between these cases, we divide the indices of G and H into three sets which are as follows:

$$\begin{aligned}
A := A(\bar{x}) &:= \{i \in \{1, 2, \ldots, l\} : G_i(\bar{x}) = 0, H_i(\bar{x}) > 0\},\\
B := B(\bar{x}) &:= \{i \in \{1, 2, \ldots, l\} : G_i(\bar{x}) = 0, H_i(\bar{x}) = 0\},\\
D := D(\bar{x}) &:= \{i \in \{1, 2, \ldots, l\} : G_i(\bar{x}) > 0, H_i(\bar{x}) = 0\}.
\end{aligned}$$

Keywords. Mathematical programs with equilibrium constraints, Convexifactors, Guignard constraint qualification, Strong stationarity, Optimality conditions.

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B is called degenerate set.

Throughout this paper, we assume that B is a nonempty set. A partition of B is of the form (B_1, B_2) where $B = B_1 \cup B_2$ and $B_1 \cap B_2 = \phi$. We denote the set of all partitions of B by $P(B)$.

Now we recall a nonlinear program MPEC (B_1, B_2) as given by Ye [39], with respect to a partition (B_1, B_2) of B.

$$\begin{aligned}\text{MPEC}(B_1,B_2) \quad & \text{Minimize } f(x)\\ & \text{subject to } g(x) \leq 0,\ h(x) = 0,\\ & \qquad G_i(x) = 0,\ i \in A \cup B_2,\ H_i(x) = 0,\ i \in D \cup B_1,\\ & \qquad G_i(x) \geq 0,\ i \in B_1,\ H_i(x) \geq 0,\ i \in B_2.\end{aligned}$$

Obviously $\bar{x}$ is a local optimal solution of MPEC if and only if it is a local optimal solution of MPEC (B_1, B_2) for all partitions $(B_1, B_2) \in P(B)$.

Let $I := \{1, 2, \ldots, m\}$, $I' := \{1, 2, \ldots, p\}$ and $I(\bar{x}) := \{i \in I : g_i(\bar{x}) = 0\}$.

MPECs play an important role in modeling of many practical problems which appear in the field of engineering design, economics, game equilibria and transportation planning. For applications of MPEC one can see [5,31,33]. However, because of their structures, they are difficult to handle. For detailed study of these programs one is referred to the two monographs Luo *et al.* [25] and Outrata *et al.* [31].

We know that Karush–Kuhn–Tucker (KKT) type necessary optimality conditions are obtained using a constraint qualification (CQ). Unfortunately MPECs do not satisfy most of the common CQs such as Mangasarian–Fromovitz CQ (MFCQ), Slater CQ etc. at any feasible point. Hence, the usual KKT conditions cannot be viewed as first order optimality conditions for MPEC unless some strong assumptions are made. Therefore, it is the subject of intensive investigation to determine a suitable CQ for MPEC under which a local minimal solution of MPEC satisfies some first order optimality conditions. This has led to various stationarity concepts (or first order optimality conditions) and CQs suitable for MPEC. The most commonly discussed stationarity concepts in the literature are strong stationarity, M-stationarity and C-stationarity. Among these, strong stationarity is the strongest stationarity notion for MPEC. In [9], Flegel and Kanzow proved that strong stationarity holds under an MPEC variant of Linear independent CQ namely (MPEC–LICQ). Later on Flegel and Kanzow [10] showed that strong stationarity is a necessary optimality condition under GCQ assuming the functions to be continuously differentiable.

M-stationarity is the next strongest stationarity concept after strong stationarity. It was developed by many authors like Ye and Ye [40], Outrata [29, 30], Ye [38], Flegel and Kanzow [11], Movahedian and Nobakhtian [28], Kanzow and Schwartz [22], Guo and Lin [14]. Ye [39] gave M-stationarity condition for MPEC and proved optimality conditions under some MPEC generalized convexity assumptions. Movahedian and Nobakhtian [28] introduced M-stationarity concept for a feasible point of MPEC in terms of Clarke–Rockafellar subdifferential and proved optimality conditions for a nonsmooth locally Lipschitz MPEC.

For more study on these stationarity conditions one is referred to Pang and Fukushima [32], Scheel and Scholtes [34], Flegel and Kanzow [12], Flegel *et al.* [13], Movahedian and Nobakhtian [27], Henrion *et al.* [15] etc. Movahedian and Nobakhtian [27] introduced strong and M-stationarity notions and also CQs such as MPEC nonsmooth generalized Abadie CQ, MPEC GCQ and MPEC weak Abadie CQ (MPEC–WACQ). They showed that these stationarity concepts are first order optimality conditions under MPEC–WACQ. Recently, Ardali *et al.* [3] gave new types of MPEC stationarity conditions based on convexifactors for nonsmooth MPEC. Especially, they derived generalized strong (GS)-stationarity as the first order necessary optimality condition under a standard Abadie-type CQ. They also proved that generalized alternatively (GA)-stationarity is a necessary optimality condition under a weaker Abadie-type CQ namely MPEC ACQ.

But even Abadie constraint qualification (ACQ) which is considered to be the weakest CQ for standard nonlinear programs does not hold in general for MPEC. Pang and Fukushima [32] showed that it holds for MPEC under some restrictive assumptions.

In this paper, we consider GCQ, which is still weaker than ACQ to derive necessary and sufficient optimality conditions for nonsmooth MPEC. In order to develop KKT type necessary optimality conditions for MPEC, we have introduced nonsmooth versions of standard GCQ using the concept of convexifactors. Further, we have introduced strong stationarity concept in terms of convexifactors and proved that it is first order necessary optimality condition under a nonsmooth version of standard GCQ. Finally, we have defined the notions of asymptotic pseudoconvexity and asymptotic quasiconvexity in terms of convexifactors to prove sufficient optimality conditions for MPEC. Convexifactors are recent generalizations of the idea of subdifferential for scalar valued functions and were given by Demyanov [6]. They are always closed sets but not necessarily convex or compact (see [7, 18]) though the most well known subdifferentials such as Clarke, Michel–Penot etc. are always convex and compact. Due to these relaxations they can be applied to a large class of nonsmooth problems. Also in the form of convexifactor we get a smaller set and using it we obtain sharp optimality conditions. Thus, because of their immense importance, they have been explored by many researchers like Jeyakumar and Luc [18], Dutta and Chandra [7, 8], Li and Zhang [24], Babahadda and Gadhi [4], Suneja and Kohli [35–37], Kohli [23], Ardali *et al.* [3], Kabgani and Solemani-damaneh [19, 20], Kabgani *et al.* [21], Jayswal *et al.* [16, 17], Ahmad *et al.* [1] and others.

The paper is organized as follows. In Section 2, we give definitions of convexifactors and some basic results which are to be used in the proof of main results. We define notions of asymptotic pseudoconvex and asymptotic quasiconvex functions and introduce the nonsmooth versions of standard GCQ in terms of convexifactors in Section 3. In Section 4, we introduce strong stationarity concept and prove that it is necessary optimality condition for MPEC. Finally, sufficient optimality conditions under the assumption of asymptotic pseudoconvexity and asymptotic quasiconvexity on functions are established in Section 5.

2. Preliminaries

In this paper, we have focused on finite dimensional spaces. We begin by defining upper and lower Dini derivatives as follows:

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be an extended real valued function and let $x \in \mathbb{R}^n$ where $f(x)$ is finite. Then the upper and lower Dini derivatives of f at x in direction v are defined, respectively by

$$(f)_d^+(x, v) := \limsup_{t\to 0^+} \frac{f(x + tv) - f(x)}{t}$$

and

$$(f)_d^-(x, v) := \liminf_{t\to 0^+} \frac{f(x + tv) - f(x)}{t}.$$

Dini derivatives may be finite as well as infinite. In particular, if f is locally Lipschitz, both the upper and lower Dini derivatives are finite.

For any set $S \subset \mathbb{R}^n$, the closure, convex hull and the closed convex hull of S are denoted, respectively by $\operatorname{cl} S$, $\operatorname{conv} S$ and $\operatorname{clconv} S$.

We now give the definitions of convexifactors as given by Dutta and Chandra [7].

Definition 2.1. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be an extended real valued function and let $x \in \mathbb{R}^n$ where $f(x)$ is finite.

(i) f is said to admit an upper convexifactor (UCF) $\partial^u f(x)$ at x iff $\partial^u f(x) \subseteq \mathbb{R}^n$ is a closed set and

$$(f)_d^-(x, v) \leq \sup_{x^*\in\partial^u f(x)} \langle x^*, v\rangle, \quad \text{for all } v \in \mathbb{R}^n.$$

(ii) f is said to admit a lower convexifactor (LCF) $\partial_l f(x)$ at x iff $\partial_l f(x) \subseteq \mathbb{R}^n$ is a closed set and

$$(f)_d^+(x, v) \geq \inf_{x^*\in\partial_l f(x)} \langle x^*, v\rangle, \quad \text{for all } v \in \mathbb{R}^n.$$

(iii) f is said to admit a convexifactor (CF) $\partial^* f(x)$ at x iff $\partial^* f(x)$ is both (UCF) and (LCF) of f at x.
(iv) f is said to admit an upper semiregular convexifactor (USRCF) $\partial^{us} f(x)$ at x iff $\partial^{us} f(x) \subseteq \mathbb{R}^n$ is a closed set and

$$(f)_d^+(x, v) \leq \sup_{x^* \in \partial^{us} f(x)} \langle x^*, v\rangle, \quad \text{for all } v \in \mathbb{R}^n.$$

In particular, if equality holds in above, then, $\partial^{us} f(x)$ is called an upper regular convexifactor (URCF) of f at x.
(v) f is said to admit a lower semiregular convexifactor (LSRCF) $\partial_{ls} f(x)$ at x iff $\partial_{ls} f(x) \subseteq \mathbb{R}^n$ is a closed set and

$$(f)_d^-(x, v) \geq \inf_{x^* \in \partial_{ls} f(x)} \langle x^*, v\rangle, \quad \text{for all } v \in \mathbb{R}^n.$$

In particular, if equality holds in above, then, $\partial_{ls} f(x)$ is called a lower regular convexifactor (LRCF) of f at x.

Definition 2.2. Given a nonempty subset S of $\mathbb{R}^n$ the negative polar cone of S is defined by

$$S^- = \{v \in \mathbb{R}^n \,|\, \langle x, v\rangle \leq 0, \ \forall\, x \in S\}.$$

Now we give an important lemma which will be used in our main results.

Lemma 2.3. *Let B be a nonempty, convex and compact set and A be a convex cone. If*

$$\sup_{v \in B} \langle v, d\rangle \geq 0, \quad \textit{for all } d \in A^-,$$

then, $0 \in B + A$.

3. Generalized convexity and constraint qualification

3.1. ∂^u-asymptotic generalized convexity

We now define the notions of ∂^u-asymptotic generalized convex functions in terms of upper convexifactors on the lines of Luu [26] and Kabgani *et al.* [21]:

Definition 3.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Let f admits UCF $\partial^u f(\bar{x})$ at $\bar{x}$.

(i) f is said to be ∂^u-asymptotic pseudoconvex at $\bar{x}$ iff for all $x \in \mathbb{R}^n$

$$f(x) < f(\bar{x}) \quad \Rightarrow \quad \langle \xi^*, x - \bar{x}\rangle < 0 \quad \text{for all } \xi^* \in \operatorname{clconv} \partial^u f(\bar{x}).$$

(ii) f is said to be ∂^u-asymptotic quasiconvex at $\bar{x}$ iff for all $x \in \mathbb{R}^n$

$$f(x) \leq f(\bar{x}) \quad \Rightarrow \quad \langle \xi^*, x - \bar{x}\rangle \leq 0 \quad \text{for all } \xi^* \in \operatorname{clconv} \partial^u f(\bar{x}).$$

(iii) f is said to be ∂^u-asymptotic quasilinear at $\bar{x}$ iff f is ∂^u-asymptotic pseudoconvex and ∂^u-asymptotic quasiconvex at $\bar{x}$.

3.2. ∂^*-GCQ

Using the concept of upper convexifactors, now we introduce nonsmooth versions of standard GCQ, namely ∂^*-GCQ(B_1, B_2), $(B_1, B_2) \in P(B)$ for nonlinear program MPEC (B_1, B_2) and ∂^*-GCQ for MPEC.

Let $\bar{x} \in K$ be feasible for MPEC. We assume that the functions g_i, $i \in I(\bar{x})$, h_i, $i = 1, 2, \ldots, p$, $-G_i, G_i$, $i \in A$, $i \in B$, $-H_i$, H_i, $i \in D$, $i \in B$ admit upper convexifactors $\partial^u g_i(\bar{x})$, $i \in I(\bar{x})$, $\partial^u h_i(\bar{x})$, $i = 1, 2, \ldots, p$, $\partial^u(-G_i)(\bar{x})$, $\partial^u(G_i)(\bar{x})$, $i \in A$, $i \in B$, $\partial^u(-H_i)(\bar{x})$ and $\partial^u(H_i)(\bar{x})$, $i \in D$, $i \in B$, respectively at $\bar{x}$.

∂^*-GCQ(B_1, B_2) holds at $\bar{x}$ if

$$\Bigg(\bigcup_{i\in I(\bar{x})} \operatorname{conv} \partial^u g_i(\bar{x}) \bigcup \bigcup_{i=1}^{p} \operatorname{conv}\ \partial^u h_i(\bar{x}) \bigcup \bigcup_{i\in A\cup B_2} (\operatorname{conv} \partial^u(G_i)(\bar{x}) \cup \operatorname{conv} \partial^u(-G_i)(\bar{x}))$$
$$\bigcup \bigcup_{i\in D\cup B_1} (\operatorname{conv} \partial^u(H_i)(\bar{x}) \cup \operatorname{conv} \partial^u(-H_i)(\bar{x})) \bigcup \bigcup_{i\in B_1} \operatorname{conv} \partial^u(-G_i)(\bar{x}) \bigcup \bigcup_{i\in B_2} \operatorname{conv} \partial^u(-H_i)(\bar{x})\Bigg)^{-}$$
$$\subseteq \operatorname{clconv} T(K, \bar{x}).$$

∂^*-GCQ holds at $\bar{x}$ if

$$\Bigg(\bigcup_{i\in I(\bar{x})} \operatorname{conv} \partial^u g_i(\bar{x}) \bigcup \bigcup_{i=1}^{p} \operatorname{conv} \partial^u h_i(\bar{x}) \bigcup \bigcup_{i\in A} (\operatorname{conv} \partial^u(G_i)(\bar{x}) \cup \operatorname{conv} \partial^u(-G_i)(\bar{x}))$$
$$\bigcup \bigcup_{i\in D} (\operatorname{conv} \partial^u(H_i)(\bar{x}) \cup \operatorname{conv} \partial^u(-H_i)(\bar{x})) \bigcup \bigcup_{i\in B} \operatorname{conv} \partial^u(-G_i)(\bar{x}) \bigcup \bigcup_{i\in B} \operatorname{conv} \partial^u(-H_i)(\bar{x})\Bigg)^{-}$$
$$\subseteq \operatorname{clconv} T(K, \bar{x}).$$

In view of equation (15) in Ye [39], we can say that if ∂^*-GCQ(B_1, B_2) holds for all MPEC (B_1, B_2), $(B_1, B_2) \in P(B)$, then ∂^*-GCQ holds.

4. Necessary optimality conditions

In this section, we derive necessary optimality conditions for MPEC.

We begin by introducing ∂^*-strong stationarity concept in terms of upper convexifactors for MPEC.

Definition 4.1. A feasible point $\bar{x}$ of MPEC is said to be ∂^*-strong stationary point if there exist vectors $0 \neq (\lambda^g, \lambda^h, \lambda^G, \lambda^H, \mu^G, \mu^H) \in \mathbb{R}^{m+p+2l+2l}$ such that the following conditions hold:

$$0 \in \operatorname{cl}\Bigg[\operatorname{conv} \partial^u f(\bar{x}) + \Bigg\{ \sum_{i=1}^{m} \lambda_i^g \operatorname{conv} \partial^u g_i(\bar{x}) + \sum_{i=1}^{p} \lambda_i^h \operatorname{conv} \partial^u h_i(\bar{x})$$
$$+ \sum_{i=1}^{l} \left[\lambda_i^G \operatorname{conv} \partial^u(-G_i)(\bar{x}) + \lambda_i^H \operatorname{conv} \partial^u(-H_i)(\bar{x})\right]$$
$$+ \sum_{i=1}^{l} \left[\mu_i^G \operatorname{conv} \partial^u(G_i)(\bar{x}) + \mu_i^H \operatorname{conv} \partial^u(H_i)(\bar{x})\right] \Bigg\}\Bigg], \tag{4.1}$$

$\lambda_i^g \geq 0$, $\lambda_i^g g_i(\bar{x}) = 0$, $i \in I$, $\lambda_i^h \geq 0$, $i \in I'$, $\lambda_i^G = 0$, $i \in D$, $\lambda_i^G \geq 0$, $i \in B$, $i \in A$, $\lambda_i^H = 0$, $i \in A$, $\lambda_i^H \geq 0$, $i \in B$, $i \in D$, $\mu_i^G \geq 0$, $i \in A \cup B_2$, $\mu_i^G = 0$, $i \in B_1 \cup D$, $\mu_i^H \geq 0$, $i \in D \cup B_1$, $\mu_i^H = 0$, $i \in A \cup B_2$, with

$$\left[\sum_{i=1}^{m} \lambda_i^g + \sum_{i=1}^{p} \lambda_i^h + \sum_{i=1}^{l} \lambda_i^G + \sum_{i=1}^{l} \lambda_i^H + \sum_{i=1}^{l} \mu_i^G + \sum_{i=1}^{l} \mu_i^H\right] = 1.$$

Remark 4.2. If f, g_i, $i \in I$, h_i, $i \in I'$, $-G_i$, $-H_i$, G_i, H_i, $i := 1, 2, \ldots, l$ are differentiable and admit upper regular convexifactors $\partial^u f(\bar{x})$, $\partial^u g_i(\bar{x})$, $i \in I$, $\partial^u h_i(\bar{x})$, $i \in I'$, $\partial^u(-G_i)(\bar{x})$, $\partial^u(-H_i)(\bar{x})$, $\partial^u(G_i)(\bar{x})$ and $\partial^u(H_i)(\bar{x})$, $i := 1, 2, \ldots, l$, respectively at $\bar{x}$ and λ_i^G free, $i \in A$, λ_i^H free, $i \in D$ and $\mu_i^G = 0 = \mu_i^H$, $i := 1, 2, \ldots, l$ in the above stationarity condition, then it becomes close to the strongly stationarity condition given by Flegel and Kanzow [9, 10]. Also in the present form it is close to generalized strong stationarity condition given by Ardali *et al.* [3] and if λ_i^G free, $i \in A$, λ_i^H free, $i \in D$, $\mu_i^G = 0$, $i \in A$, $\mu_i^H = 0$, $i \in D$ in the above, then it becomes close to the strong stationarity condition given by Ye [39].

Before proceeding further, we give below an example to demonstrate the existence of above mentioned stationarity condition.

Example 4.3. Consider the problem

$$\text{Minimize } f(x,y) := \begin{cases} x+y, & x \geq 0, y \geq 0 \\ |x| + y^2, & x \geq 0,\ y < 0 \\ |x| + y^2, & x < 0,\ y \leq 0 \\ y, & x < 0,\ y > 0 \end{cases}$$

$$\text{subject to } g(x,y) := \begin{cases} |x| + |y|, & x > 0,\ y \in \mathbb{R} \\ \sqrt{-xy}, & x = 0,\ y < 0 \\ -\sqrt{-xy^2}, & x \leq 0,\ y \geq 0 \\ \sqrt{-x^2 y}, & x < 0,\ y < 0 \end{cases} \leq 0,$$

$$G(x,y) := y \geq 0, \quad H(x,y) := x^2 + |y| \geq 0, \quad G(x,y)^t H(x,y) := y(x^2 + |y|) = 0,$$

where $f, g, G, H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

It can be seen that f, g, $-G$, G, $-H$ and H admit UCFs $\partial^u f(0,0) := \{(1,1), (0,1), (-1,0), (1,0)\}$, $\partial^u g(0,0) := \{(x^*, 0) | x^* \geq 0\}$, $\partial^u(-G)(0,0) := \{(0,-1)\}$, $\partial^u(G)(0,0) := \{(0,1)\}$, $\partial^u(-H)(0,0) := \{(0,1), (0,-1)\}$ and $\partial^u(H)(0,0) := \{(0,-1), (0,1)\}$, respectively at $(0,0)$.

Then, there exist scalars $\lambda^g = \frac{1}{12}$, $\lambda^G = \frac{1}{12}$, $\lambda^H = \frac{1}{4}$, $\mu^G = \frac{1}{3}$, $\mu^H = \frac{1}{4}$, with $\lambda^g + \lambda^G + \lambda^H + \mu^G + \mu^H = 1$ such that

$$\begin{aligned}(0,0) \in \text{cl}\Big[\text{conv}\, \partial^u f(0,0) + \Big\{ & \lambda^g \text{conv}\, \partial^u g(0,0) + \lambda^G \text{conv}\, \partial^u(-G)(0,0) \\ & + \lambda^H \text{conv}\, \partial^u(-H)(0,0) + \mu^G \text{conv}\, \partial^u(G)(0,0) \\ & + \mu^H \text{conv}\, \partial^u(H)(0,0) \Big\}\Big].\end{aligned}$$

We now derive necessary optimality conditions for MPEC in the form of the following theorem.

Theorem 4.4. *Let $\bar{x}$ be a local optimal solution of MPEC. Assume that f is locally Lipschitz and admits a bounded upper semiregular convexifactor $\partial^{us} f(\bar{x})$ at $\bar{x}$. Let g_i, $i = 1, 2, \ldots, m$, h_i, $i = 1, 2, \ldots, p$, $-G_i$, G_i, $i \in A$, $i \in B$, $-H_i$, H_i, $i \in D$, $i \in B$ admit upper convexifactors $\partial^u g_i(\bar{x})$, $i = 1, 2, \ldots, m$, $\partial^u h_i(\bar{x})$, $i = 1, 2, \ldots, p$, $\partial^u(-G_i)(\bar{x})$, $\partial^u(G_i)(\bar{x})$, $i \in A$, $i \in B$, $\partial^u(-H_i)(\bar{x})$ and $\partial^u(H_i)(\bar{x})$, $i \in D$, $i \in B$, respectively at $\bar{x}$. Suppose that ∂^*-GCQ(B_1, B_2) holds for all $(B_1, B_2) \in P(B)$ and hence ∂^*-GCQ holds at $\bar{x}$. Then, $\bar{x}$ is a ∂^*-strong stationary point.*

Proof. Let $v \in \text{cl}(\text{conv}(T(K, \bar{x})))$ be arbitrary. Then there exists a sequence $v_k \in \text{conv}(T(K, \bar{x}))$ such that $v_k \to v$ as $k \to \infty$.

Since $v_k \in \text{conv}(T(K,\bar{x}))$, using Carathéodary's Theorem, we can find elements $v_k^s \in T(K,\bar{x})$ and $\lambda_k^s \geq 0$, with $\sum_{s=1}^{n+1} \lambda_k^s = 1$ such that

$$v_k = \sum_{s=1}^{n+1} \lambda_k^s v_k^s, \quad \text{for all } k \in N.$$

Then, on taking limit $k \to \infty$, above gives

$$v = \lim_{k\to\infty} \left(\sum_{s=1}^{n+1} \lambda_k^s v_k^s \right). \tag{4.2}$$

As $v_k^s \in T(K,\bar{x})$, by definition of tangent cone, we can find sequences $v_k^{s,j} \to v_k^s$ and $t_k^{s,j} \downarrow 0$ such that $\bar{x} + t_k^{s,j} v_k^{s,j} \in K$, for all $j \in N$.

Since $\bar{x}$ is a minimum of f over K, therefore, we have

$$\frac{f(\bar{x} + t_k^{s,j} v_k^{s,j}) - f(\bar{x})}{t_k^{s,j}} \geq 0, \quad \text{for sufficiently large } k, j \in N. \tag{4.3}$$

Now

$$\frac{f(\bar{x} + t_k^{s,j} v_k^{s,j}) - f(\bar{x})}{t_k^{s,j}} = \frac{f(\bar{x} + t_k^{s,j} v_k^{s,j}) - f(\bar{x} + t_k^{s,j} v_k^s)}{t_k^{s,j}} + \frac{f(\bar{x} + t_k^{s,j} v_k^s) - f(\bar{x})}{t_k^{s,j}}. \tag{4.4}$$

Since f is locally Lipschitz, therefore

$$\frac{f(\bar{x} + t_k^{s,j} v_k^{s,j}) - f(\bar{x} + t_k^{s,j} v_k^s)}{t_k^{s,j}} \to 0 \ \text{ as } \ k \to \infty.$$

Taking Limit supremum on both sides of equation (4.4) and using above and (4.3), we get

$$\limsup_{t_k^{s,j} \to 0^+} \frac{f(\bar{x} + t_k^{s,j} v_k^{s,j}) - f(\bar{x})}{t_k^{s,j}} = (f)_d^+(\bar{x}, v_k^s) \geq 0, \qquad s = 1, 2, \ldots, n+1.$$

That is,

$$(f)_d^+(\bar{x}, v_k^s) \geq 0, \ \text{ for all } \ v_k^s \in T(K,\bar{x}) \ \text{ as } \ v_k^s \text{ is arbitrary, } s = 1, 2, \ldots, n+1.$$

Thus, we have $\lambda_k^s (f)_d^+(\bar{x}, v_k^s) \geq 0$, as $\lambda_k^s \geq 0$, $s = 1, 2, \ldots, n+1$.

Now because Dini derivatives are positively homogeneous in direction (Ansari *et al.* [1], Thm. 2.8(a)), so we have from above

$$(f)_d^+(\bar{x}, \lambda_k^s v_k^s) \geq 0, \quad s = 1, 2, \ldots, n+1.$$

By upper semiregularity of $\partial^{us} f(\bar{x})$ at $\bar{x}$, it follows that

$$\sup_{\xi \in \partial^{us} f(\bar{x})} \langle \xi, \lambda_k^s v_k^s \rangle \geq 0, \ \text{ for all } v_k^s \in T(K,\bar{x}), \ s = 1, 2, \ldots, n+1.$$

Adding above for all $s = 1, 2, \ldots, n+1$, we get

$$\sup_{\xi \in \partial^{us} f(\bar{x})} \langle \xi, \sum_{s=1}^{n+1} \lambda_k^s v_k^s \rangle \geq 0.$$

Taking limit as $k \to \infty$ and using (4.2), we get

$$\sup_{\xi \in \partial^{us} f(\bar{x})} \langle \xi, v \rangle \geq 0, \quad \text{for all} \quad v \in \text{cl}(\text{conv}(T(K, \bar{x}))).$$

That is,

$$\sup_{\xi \in \text{clconv}\, \partial^{us} f(\bar{x})} \langle \xi, v \rangle \geq 0, \quad \text{for all} \quad v \in \text{cl}(\text{conv}(T(K, \bar{x}))).$$

Since ∂^*-GCQ(B_1, B_2) holds for every $(B_1, B_2) \in P(B)$ at $\bar{x}$, we have

$$\sup_{\xi \in \text{clconv}\, \partial^{us} f(\bar{x})} \langle \xi, v \rangle \geq 0, \quad \text{for all} \quad v \in A^-,$$

where A^- is the negative polar cone of A and A is defined by

$$\begin{aligned} A := \text{coneconv}\Bigg(& \bigcup_{i \in I(\bar{x})} \text{conv}\, \partial^u g_i(\bar{x}) \bigcup \bigcup_{i=1}^{p} \text{conv}\, \partial^u h_i(\bar{x}) \\ & \bigcup \bigcup_{i \in A \cup B_2} (\text{conv}\, \partial^u (G_i)(\bar{x}) \cup \text{conv}\, \partial^u (-G_i)(\bar{x})) \\ & \bigcup \bigcup_{i \in D \cup B_1} (\text{conv}\, \partial^u (H_i)(\bar{x}) \cup \text{conv}\, \partial^u (-H_i)(\bar{x})) \\ & \bigcup \bigcup_{i \in B_1} \text{conv}\, \partial^u (-G_i)(\bar{x}) \bigcup \bigcup_{i \in B_2} \text{conv}\, \partial^u (-H_i)(\bar{x}) \Bigg). \end{aligned}$$

Using Lemma 2.3, we get

$$\begin{aligned} 0 \in \text{clconv}\Bigg(\text{clconv}\, \partial^{us} f(\bar{x}) + \text{coneconv}\Bigg\{ & \bigcup_{i \in I(\bar{x})} \text{conv}\, \partial^u g_i(\bar{x}) \bigcup \bigcup_{i=1}^{p} \text{conv}\, \partial^u h_i(\bar{x}) \\ & \bigcup \bigcup_{i \in A \cup B_2} (\text{conv}\, \partial^u (G_i)(\bar{x}) \cup \text{conv}\, \partial^u (-G_i)(\bar{x})) \\ & \bigcup \bigcup_{i \in D \cup B_1} (\text{conv}\, \partial^u (H_i)(\bar{x}) \cup \text{conv}\, \partial^u (-H_i)(\bar{x})) \\ & \bigcup \bigcup_{i \in B_1} \text{conv}\, \partial^u (-G_i)(\bar{x}) \bigcup \bigcup_{i \in B_2} \text{conv}\, \partial^u (-H_i)(\bar{x}) \Bigg\} \Bigg). \end{aligned}$$

Using convex hull property of subsets S_1 and S_2 of $\mathbb{R}^n$, $\text{conv}\,(S_1 + S_2) = \text{conv}\, S_1 + \text{conv}\, S_2$, we have

$$\begin{aligned} 0 \in \text{cl}\Bigg(\text{convclconv} \partial^{us} f(\bar{x}) + \text{convconeconv}\Bigg\{ & \bigcup_{i \in I(\bar{x})} \text{conv}\, \partial^u g_i(\bar{x}) \bigcup \bigcup_{i=1}^{p} \text{conv}\, \partial^u h_i(\bar{x}) \\ & \bigcup \bigcup_{i \in A \cup B_2} (\text{conv}\, \partial^u (G_i)(\bar{x}) \cup \text{conv}\, \partial^u (-G_i)(\bar{x})) \\ & \bigcup \bigcup_{i \in D \cup B_1} (\text{conv}\, \partial^u (H_i)(\bar{x}) \cup \text{conv}\, \partial^u (-H_i)(\bar{x})) \\ & \bigcup \bigcup_{i \in B_1} \text{conv}\, \partial^u (-G_i)(\bar{x}) \bigcup \bigcup_{i \in B_2} \text{conv}\, \partial^u (-H_i)(\bar{x}) \Bigg\} \Bigg). \end{aligned}$$

Since $\text{convcl}S \subset \text{clconv}S$ for any subset S of $\mathbb{R}^n$, we have from above

$$0 \in \text{cl}\Bigg(\text{clconv}\,\partial^{us} f(\bar{x}) + \text{coneconv}\Bigg\{\bigcup_{i\in I(\bar{x})} \text{conv}\,\partial^u g_i(\bar{x}) \bigcup \bigcup_{i=1}^{p} \text{conv}\,\partial^u h_i(\bar{x}) \\ \bigcup \bigcup_{i\in A\cup B_2} (\text{conv}\,\partial^u (G_i)(\bar{x}) \cup \text{conv}\,\partial^u(-G_i)(\bar{x})) \\ \bigcup \bigcup_{i\in D\cup B_1} (\text{conv}\,\partial^u (H_i)(\bar{x}) \cup \text{conv}\,\partial^u(-H_i)(\bar{x})) \\ \bigcup \bigcup_{i\in B_1} \text{conv}\,\partial^u(-G_i)(\bar{x}) \bigcup \bigcup_{i\in B_2} \text{conv}\,\partial^u(-H_i)(\bar{x})\Bigg\}\Bigg).$$

Using closure property of subsets S_1 and S_2 of $\mathbb{R}^n$, $\text{cl}(S_1 + \text{cl}S_2) = \text{cl}(S_1 + S_2)$, we get

$$0 \in \text{cl}\Bigg(\text{conv}\,\partial^{us} f(\bar{x}) + \text{coneconv}\Bigg\{\bigcup_{i\in I(\bar{x})} \text{conv}\,\partial^u g_i(\bar{x}) \bigcup \bigcup_{i=1}^{p} \text{conv}\,\partial^u h_i(\bar{x}) \\ \bigcup \bigcup_{i\in A\cup B_2} (\text{conv}\,\partial^u (G_i)(\bar{x}) \cup \text{conv}\,\partial^u(-G_i)(\bar{x})) \\ \bigcup \bigcup_{i\in D\cup B_1} (\text{conv}\,\partial^u (H_i)(\bar{x}) \cup \text{conv}\,\partial^u(-H_i)(\bar{x})) \\ \bigcup \bigcup_{i\in B_1} \text{conv}\,\partial^u(-G_i)(\bar{x}) \bigcup \bigcup_{i\in B_2} \text{conv}\,\partial^u(-H_i)(\bar{x})\Bigg\}\Bigg),$$

which implies that there exists a sequence

$$x_k \in \Bigg(\text{conv}\,\partial^{us} f(\bar{x}) + \text{coneconv}\Bigg\{\bigcup_{i\in I(\bar{x})} \text{conv}\,\partial^u g_i(\bar{x}) \bigcup \bigcup_{i=1}^{p} \text{conv}\,\partial^u h_i(\bar{x}) \\ \bigcup \bigcup_{i\in A\cup B_2} (\text{conv}\,\partial^u (G_i)(\bar{x}) \cup \text{conv}\,\partial^u(-G_i)(\bar{x})) \\ \bigcup \bigcup_{i\in D\cup B_1} (\text{conv}\,\partial^u (H_i)(\bar{x}) \cup \text{conv}\,\partial^u(-H_i)(\bar{x})) \\ \bigcup \bigcup_{i\in B_1} \text{conv}\,\partial^u(-G_i)(\bar{x}) \bigcup \bigcup_{i\in B_2} \text{conv}\,\partial^u(-H_i)(\bar{x})\Bigg\}\Bigg),$$

such that $x_k \to 0$ as $k \to \infty$.

Since convexifactors are in general nonconvex sets, therefore, there exist sequences of scalars $\{\lambda_{ik}^g\}$, $\lambda_{ik}^g \geq 0$, $i \in I(\bar{x})$, $\{\lambda_{ik}^h\}$, $\lambda_{ik}^h \geq 0$, $i = 1, 2, \ldots, p$, $\{\lambda_{ik}^G\}$, $\lambda_{ik}^G \geq 0$, $i \in A \cup B_2 \cup B_1$, $\{\lambda_{ik}^H\}$, $\lambda_{ik}^H \geq 0$, $i \in B_2 \cup D \cup B_1$, $\{\mu_{ik}^G\}$, $\mu_{ik}^G \geq 0$, $i \in A \cup B_2$, $\{\mu_{ik}^H\}$, $\mu_{ik}^H \geq 0$, $i \in D \cup B_1$, with

$$\lim_{k\to\infty}\Bigg[\sum_{i\in I(\bar{x})} \lambda_{ik}^g + \sum_{i=1}^{p} \lambda_{ik}^h + \sum_{i\in A\cup B_2\cup B_1} \lambda_{ik}^G + \sum_{i\in B_2\cup D\cup B_1} \lambda_{ik}^H \\ + \sum_{i\in A\cup B_2} \mu_{ik}^G + \sum_{i\in D\cup B_1} \mu_{ik}^H\Bigg] := 1$$

such that

$$x_k \in \Bigg[\operatorname{conv} \partial^{us} f(\bar{x}) + \Bigg\{ \sum_{i \in I(\bar{x})} \lambda_{ik}^{g} \operatorname{conv} \partial^{u} g_i(\bar{x}) + \sum_{i=1}^{p} \lambda_{ik}^{h} \operatorname{conv} \partial^{u} h_i(\bar{x}) + \sum_{i \in A \cup B_2 \cup B_1} \lambda_{ik}^{G} \operatorname{conv} \partial^{u}(-G_i)(\bar{x}) + \sum_{i \in B_1 \cup D \cup B_2} \lambda_{ik}^{H} \operatorname{conv} \partial^{u}(-H_i)(\bar{x}) + \sum_{i \in A \cup B_2} \mu_{ik}^{G} \operatorname{conv} \partial^{u}(G_i)(\bar{x}) + \sum_{i \in D \cup B_1} \mu_{ik}^{H} \operatorname{conv} \partial^{u}(H_i)(\bar{x}) \Bigg\} \Bigg].$$

Since the sequences $\{\lambda_{ik}^g\}$, $i \in I(\bar{x})$, $\{\lambda_{ik}^h\}$, $i = 1, 2, \ldots, p$, $\{\lambda_{ik}^G\}$, $i \in A \cup B$, $\{\lambda_{ik}^H\}$, $i \in B \cup D$, $\{\mu_{ik}^G\}$, $i \in A \cup B_2$, $\{\mu_{ik}^H\}$, $i \in D \cup B_1$ are bounded, we may assume that the sequences $\lambda_{ik}^g \to \lambda_i^g$, $i \in I(\bar{x})$, $\lambda_{ik}^h \to \lambda_i^h$, $i = 1, 2, \ldots, p$, $\lambda_{ik}^G \to \lambda_i^G$, $i \in A \cup B$, $\lambda_{ik}^H \to \lambda_i^H$, $i \in B \cup D$, $\mu_{ik}^G \to \mu_i^G$, $i \in A \cup B_2$, $\mu_{ik}^H \to \mu_i^H$, $i \in D \cup B_1$ as $k \to \infty$.

Thus, we have

$$0 \in \operatorname{cl}\Bigg[\operatorname{conv} \partial^{us} f(\bar{x}) + \Bigg\{ \sum_{i \in I(\bar{x})} \lambda_{i}^{g} \operatorname{conv} \partial^{u} g_i(\bar{x}) + \sum_{i=1}^{p} \lambda_{i}^{h} \operatorname{conv} \partial^{u} h_i(\bar{x}) + \sum_{i \in A \cup B} \lambda_{i}^{G} \operatorname{conv} \partial^{u}(-G_i)(\bar{x}) + \sum_{i \in B \cup D} \lambda_{i}^{H} \operatorname{conv} \partial^{u}(-H_i)(\bar{x}) + \sum_{i \in A \cup B_2} \mu_{i}^{G} \operatorname{conv} \partial^{u}(G_i)(\bar{x}) + \sum_{i \in D \cup B_1} \mu_{i}^{H} \operatorname{conv} \partial^{u}(H_i)(\bar{x}) \Bigg\} \Bigg],$$

with

$$\Bigg[\sum_{i \in I(\bar{x})} \lambda_i^g + \sum_{i=1}^{p} \lambda_i^h + \sum_{i \in A \cup B} \lambda_i^G + \sum_{i \in B \cup D} \lambda_i^H + \sum_{i \in A \cup B_2} \mu_i^G + \sum_{i \in D \cup B_1} \mu_i^H \Bigg] := 1.$$

Now, $g_i(\bar{x}) := 0$ for $i \in I(\bar{x})$, therefore $\lambda_i^g g_i(\bar{x}) := 0$ for $i \in I(\bar{x})$.
For $i \notin I(\bar{x})$, $g_i(\bar{x}) < 0$, so let $\lambda_i^g = 0$, $i \notin I(\bar{x})$.
Thus, we have $\lambda_i^g g_i(\bar{x}) := 0$, $i \in I$.
Also $G_i(\bar{x}) = 0$ for $i \in A \cup B_2$.
For $i \notin A \cup B_2$, that is, for $i \in D \cup B_1$, $G_i(\bar{x}) > 0$. Let $\lambda_i^G = 0$, $i \in D \cup B_1$.
Similarly proceeding as above, since $H_i(\bar{x}) = 0$ for $i \in D \cup B_1$, we get $\lambda_i^H = 0$, $i \in A \cup B_2$.
Also by taking $\mu_i^G = 0$, $i \in B_1 \cup D$, $\mu_i^H = 0$, $i \in A \cup B_2$, we get

$$0 \in \operatorname{cl}\Bigg[\operatorname{conv} \partial^{us} f(\bar{x}) + \Bigg\{ \sum_{i=1}^{m} \lambda_{i}^{g} \operatorname{conv} \partial^{u} g_i(\bar{x}) + \sum_{i=1}^{p} \lambda_{i}^{h} \operatorname{conv} \partial^{u} h_i(\bar{x}) + \sum_{i=1}^{l} \lambda_{i}^{G} \operatorname{conv} \partial^{u}(-G_i)(\bar{x}) + \sum_{i=1}^{l} \lambda_{i}^{H} \operatorname{conv} \partial^{u}(-H_i)(\bar{x}) + \sum_{i=1}^{l} \mu_{i}^{G} \operatorname{conv} \partial^{u}(G_i)(\bar{x}) + \sum_{i=1}^{l} \mu_{i}^{H} \operatorname{conv} \partial^{u}(H_i)(\bar{x}) \Bigg\} \Bigg],$$

with

$$\Bigg[\sum_{i=1}^{m} \lambda_i^g + \sum_{i=1}^{p} \lambda_i^h + \sum_{i=1}^{l} \lambda_i^G + \sum_{i=1}^{l} \lambda_i^H + \sum_{i=1}^{l} \mu_i^G + \sum_{i=1}^{l} \mu_i^H \Bigg] = 1.$$

Hence, the theorem is proved. □

To illustrate above theorem, we provide the following example.

Example 4.5. Consider the problem

$$\text{Minimize } f(x,y) := \begin{cases} |x| + |y|, & x \geq 0, y \geq 0 \\ |y|, & x \geq 0, y < 0 \\ x^2 + |y|, & x < 0, y \in \mathbb{R} \end{cases}$$

$$\text{subject to } g(x,y) := \begin{cases} -\sqrt{x} - \sqrt{y}, & x \geq 0, y \geq 0 \\ \sqrt{-y}, & x \geq 0, y < 0 \\ \sqrt{-x} + \sqrt{-y}, & x < 0, y \leq 0 \\ \sqrt{-x}, & x < 0, y > 0 \end{cases} \leq 0,$$

$$G(x,y) := y \geq 0,\ H(x,y) := x^2 + |y| \geq 0,\ G(x,y)^t H(x,y) := y(x^2 + |y|) = 0,$$

where $f, g, G, H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.
It can be seen that f admits an USRCF $\partial^{us} f(0,0) := \{(1,1), (0,1), (0,-1)\}$,
g, $-G$, G, $-H$ and H admit UCFs $\partial^u g(0,0) := \{(x^*, y^*) \,|\, x^* \leq 0,\ y^* \leq 0\}$, $\partial^u(-G)(0,0) := \{(0,-1)\}$, $\partial^u(G)(0,0) := \{(0,1)\}$, $\partial^u(-H)(0,0) := \{(0,-1),(0,1)\}$ and $\partial^u(H)(0,0) := \{(0,1),(0,-1)\}$, respectively at $(0,0)$.
∂^*-GCQ holds at $(0,0)$.
Here $T(K,(0,0)) := \{(x,0) | x \geq 0\}$ and $K \subset \mathbb{R} \times \mathbb{R}$ is given by $K := \{(x,0) | x \geq 0\}$.
$(0,0)$ is an optimal solution of the problem.
Then, there exist scalars $\lambda^g = \frac{1}{6}$, $\lambda^G = \frac{1}{4}$, $\lambda^H = \frac{1}{6}$, $\mu^G = \frac{1}{4}$, $\mu^H = \frac{1}{6}$, with $\lambda^g + \lambda^G + \lambda^H + \mu^G + \mu^H = 1$ such that

$$\begin{aligned} &(0,0) \in \text{cl}[\text{conv}\, \partial^{us} f(0,0) + \{\lambda^g \text{conv}\, \partial^u g(0,0) + \lambda^G \text{conv}\, \partial^u(-G)(0,0) \\ &\qquad + \lambda^H \text{conv}\, \partial^u(-H)(0,0) + \mu^G \text{conv}\, \partial^u(G)(0,0) + \mu^H \text{conv}\, \partial^u(H)(0,0)\}], \\ &\lambda^g g(0,0) = 0. \end{aligned}$$

5. Sufficient optimality conditions

We begin this section by defining the following index sets.

$$\begin{aligned} &B_G^+ := \left\{i \in B : \mu_i^G = 0, \mu_i^H > 0\right\},\quad B_H^+ := \left\{i \in B : \mu_i^H = 0, \mu_i^G > 0\right\}, \\ &B^+ := \left\{i \in B : \mu_i^G > 0, \mu_i^H > 0\right\},\ A^+ := \left\{i \in A : \mu_i^G > 0\right\}, \\ &D^+ := \left\{i \in D : \mu_i^H > 0\right\}. \end{aligned}$$

The following theorem gives sufficient optimality conditions for MPEC.

Theorem 5.1. *Let $\bar{x}$ be a feasible solution of MPEC and ∂^*-strong stationarity condition holds at $\bar{x}$. Let f be ∂^u-asymptotic pseudoconvex, g_i, $i \in I(\bar{x})$, $-G_i$, $i \in A \cup B$, $-H_i$, $i \in B \cup D$ be ∂^u-asymptotic quasiconvex at $\bar{x}$ and h_i, $i = 1, 2, \ldots, p$ be ∂^u-asymptotic quasilinear at $\bar{x}$. If $A^+ \cup D^+ \cup B_G^+ \cup B_H^+ \cup B^+ = \emptyset$, then $\bar{x}$ is a global optimal solution of MPEC.*

Proof. Suppose that $\bar{x}$ is not an optimal solution of MPEC. Then there exists a feasible sequence $\{x_k\}$ of MPEC such that $f(x_k) < f(\bar{x})$. Since f is ∂^u-asymptotic pseudoconvex at $\bar{x}$, we have

$$\langle \xi, x_k - \bar{x} \rangle < 0 \quad \text{for all } \xi \in \text{clconv}\, \partial^u f(\bar{x}). \tag{5.1}$$

By (4.1), there exist $\xi_k \in \text{conv}\,\partial^u f(\bar{x})$, $\zeta_{ik} \in \text{conv}\,\partial^u g_i(\bar{x})$, $i = 1, 2, \ldots, m$, $\varsigma_{ik} \in \text{conv}\,\partial^u h_i(\bar{x})$, $i = 1, 2, \ldots, p$, $\gamma^*_{ik} \in \text{conv}\,\partial^u(-G_i)(\bar{x})$, $\gamma^{**}_{ik} \in \text{conv}\,\partial^u(G_i)(\bar{x})$, $\xi^*_{ik} \in \text{conv}\,\partial^u(-H_i)(\bar{x})$, $\xi^{**}_{ik} \in \text{conv}\,\partial^u(H_i)(\bar{x})$, $i = 1, 2, \ldots, l$ such that

$$\lim_{k\to\infty}\Bigg[\xi_k + \Bigg\{\sum_{i=1}^{m}\lambda_i^g\zeta_{ik} + \sum_{i=1}^{p}\lambda_i^h\varsigma_{ik} + \sum_{i=1}^{l}\lambda_i^G\gamma^*_{ik} + \sum_{i=1}^{l}\lambda_i^H\xi^*_{ik} + \sum_{i=1}^{l}\mu_i^G\gamma^{**}_{ik} + \sum_{i=1}^{l}\mu_i^H\xi^{**}_{ik}\Bigg\}\Bigg] = 0.$$

Thus, for any feasible solution $\{x_k\}$ of MPEC, we have

$$\lim_{k\to\infty}\Bigg[\langle\xi_k, x_k - \bar{x}\rangle + \Bigg\{\Bigg\langle\sum_{i=1}^{m}\lambda_i^g\zeta_{ik}, x_k - \bar{x}\Bigg\rangle + \Bigg\langle\sum_{i=1}^{p}\lambda_i^h\varsigma_{ik}, x_k - \bar{x}\Bigg\rangle + \Bigg\langle\sum_{i=1}^{l}\lambda_i^G\gamma^*_{ik}, x_k - \bar{x}\Bigg\rangle + \Bigg\langle\sum_{i=1}^{l}\lambda_i^H\xi^*_{ik}, x_k - \bar{x}\Bigg\rangle + \Bigg\langle\sum_{i=1}^{l}\mu_i^G\gamma^{**}_{ik}, x_k - \bar{x}\Bigg\rangle + \Bigg\langle\sum_{i=1}^{l}\mu_i^H\xi^{**}_{ik}, x_k - \bar{x}\Bigg\rangle\Bigg\}\Bigg] = 0. \tag{5.2}$$

Now, because $\xi \in \text{clconv}\,\partial^u f(\bar{x})$, there exists $\xi'_k \in \text{conv}\,\partial^u f(\bar{x})$ such that

$$\lim_{k\to\infty}\xi'_k = \xi. \tag{5.3}$$

Using above in (5.1), we get

$$\lim_{k\to\infty}\langle\xi'_k, x_k - \bar{x}\rangle < 0, \quad \text{for some } \xi'_k \in \text{conv}\,\partial^u f(\bar{x}).$$

Thus, for every $\xi \in \text{clconv}\,\partial^u f(\bar{x})$, there exists $\xi'_k \in \text{conv}\,\partial^u f(\bar{x})$ such that (5.3) holds.
In particular, it is true for $\xi \in \text{clconv}\,\partial^u f(\bar{x})$ for which there exists $\xi_k \in \text{conv}\,\partial^u f(\bar{x})$ such that

$$\lim_{k\to\infty}\xi_k = \xi.$$

Thus, we have

$$\lim_{k\to\infty}\langle\xi_k, x_k - \bar{x}\rangle < 0 \text{ as } \xi_k \in \text{conv}\,\partial^u f(\bar{x}). \tag{5.4}$$

Since x_k is feasible for MPEC, we have

$$g_i(x_k) \leq 0 = g_i(\bar{x}), \qquad i \in I(\bar{x}).$$

As $g_i, i \in I(\bar{x})$ is ∂^u-asymptotic quasiconvex at $\bar{x}$, we have

$$\langle\zeta_i, x_k - \bar{x}\rangle \leq 0 \text{ for all } \ \zeta_i \in \text{clconv}\,\partial^u g_i(\bar{x}),\ i \in I(\bar{x}),$$

which, on proceeding as earlier, gives

$$\lim_{k\to\infty}\langle\zeta_{ik}, x_k - \bar{x}\rangle \leq 0 \ \text{ as } \ \zeta_{ik} \in \text{conv}\,\partial^u g_i(\bar{x}),\ i \in I(\bar{x}).$$

Thus, we have

$$\lim_{k\to\infty}\Bigg\langle\sum_{i\in I(\bar{x})}\lambda_i^g\zeta_{ik}, x_k - \bar{x}\Bigg\rangle \leq 0 \text{ as } \lambda_i^g \geq 0,\ i \in I(\bar{x}).$$

Now, $g_i(\bar{x}) < 0$, $i \notin I(\bar{x})$, taking $\lambda_i^g = 0$ for $i \notin I(\bar{x})$, we get

$$\lim_{k\to\infty} \left\langle \sum_{i=1}^{m} \lambda_i^g \zeta_{ik}, x_k - \bar{x} \right\rangle \leq 0 \quad \text{as} \quad \zeta_{ik} \in \operatorname{conv} \partial^u g_i(\bar{x}). \tag{5.5}$$

Since,

$$h_i(x_k) = 0 = h_i(\bar{x}), \quad i = 1, 2, \ldots, p,$$

using ∂^u-asymptotic quasilinearity of h_i, $i = 1, 2, \ldots, p$ at $\bar{x}$, we have

$$\langle \varsigma_i, x_k - \bar{x} \rangle = 0 \text{ for all } \varsigma_i \in \operatorname{clconv} \partial^u h_i(\bar{x}),\ i = 1, 2, \ldots, p.$$

Thus,

$$\lim_{k\to\infty} \langle \varsigma_{ik}, x_k - \bar{x} \rangle = 0 \text{ as } \varsigma_{ik} \in \operatorname{conv} \partial^u h_i(\bar{x}),\ i = 1, 2, \ldots, p,$$

which gives

$$\lim_{k\to\infty} \left\langle \sum_{i=1}^{p} \lambda_i^h \varsigma_{ik}, x_k - \bar{x} \right\rangle = 0 \quad \text{as} \quad \lambda_i^h \geq 0,\ \varsigma_{ik} \in \operatorname{conv} \partial^u h_i(\bar{x}),\ i = 1, 2, \ldots, p. \tag{5.6}$$

Now, $-G_i(x_k) \leq 0 = -G_i(\bar{x})$, $i \in A \cup B$.
Since $-G_i$, $i \in A \cup B$ is ∂^u-asymptotic quasiconvex at $\bar{x}$, we have

$$\langle \gamma_i^*, x_k - \bar{x} \rangle \leq 0 \text{ for all } \gamma_i^* \in \operatorname{clconv} \partial^u(-G_i)(\bar{x}),\ i \in A \cup B.$$

Then as earlier, we have

$$\lim_{k\to\infty} \langle \gamma_{ik}^*, x_k - \bar{x} \rangle \leq 0 \text{ as } \gamma_{ik}^* \in \operatorname{conv} \partial^u(-G_i)(\bar{x}),\ i \in A \cup B. \tag{5.7}$$

Similarly, since $-H_i(x_k) \leq 0 = -H_i(\bar{x})$, $i \in B \cup D$, using ∂^u-asymptotic quasiconvexity of $-H_i$, $i \in B \cup D$ at $\bar{x}$ and proceeding as earlier, we get

$$\lim_{k\to\infty} \langle \xi_{ik}^*, x_k - \bar{x} \rangle \leq 0 \quad \text{as} \quad \xi_{ik}^* \in \operatorname{conv} \partial^u(-H_i)(\bar{x}),\ i \in B \cup D. \tag{5.8}$$

Since $A^+ \cup D^+ \cup B_G^+ \cup B_H^+ \cup B^+ = \emptyset$, multiplying (5.7) and (5.8) by $\lambda_i^G, i \in A \cup B$ and $\lambda_i^H, i \in B \cup D$, respectively and adding, we get

$$\lim_{k\to\infty} \left\langle \sum_{i=1}^{l} \lambda_i^G \gamma_{ik}^*, x_k - \bar{x} \right\rangle \leq 0 \quad \text{as} \quad \lambda_i^G = 0,\ i \in D \tag{5.9}$$

and

$$\lim_{k\to\infty} \left\langle \sum_{i=1}^{l} \lambda_i^H \xi_{ik}^*, x_k - \bar{x} \right\rangle \leq 0 \text{ as } \lambda_i^H = 0,\ i \in A. \tag{5.10}$$

Adding (5.4), (5.5), (5.6), (5.9) and (5.10), we get

$$\lim_{k\to\infty} \Bigg[\langle \xi_k, x_k - \bar{x} \rangle + \Bigg\{ \left\langle \sum_{i=1}^{m} \lambda_i^g \zeta_{ik}, x_k - \bar{x} \right\rangle + \left\langle \sum_{i=1}^{p} \lambda_i^h \varsigma_{ik}, x_k - \bar{x} \right\rangle + \left\langle \sum_{i=1}^{l} \lambda_i^G \gamma_{ik}^*, x_k - \bar{x} \right\rangle + \left\langle \sum_{i=1}^{l} \lambda_i^H \xi_{ik}^*, x_k - \bar{x} \right\rangle \Bigg\} \Bigg] < 0,$$

which contradicts (5.2).

Hence $\bar{x}$ is a global optimal solution of MPEC. □

Now we give an example to illustrate the above theorem.

Example 5.2. Consider the problem

$$\text{Minimize } f(x,y) := \begin{cases} \sqrt{x}+y^2, & x \geq 0, y \in \mathbb{R} \\ x^2, & x < 0, y \in \mathbb{R} \end{cases}$$

$$\text{subject to } g(x,y) := y^2 + y \leq 0,$$

$$G(x,y) := \begin{cases} x^2, & x \geq 0, y \in \mathbb{R} \\ x, & x < 0, y \in \mathbb{R} \end{cases} \geq 0,$$

$$H(x,y) := \begin{cases} \sqrt{x}+y^2, & x \geq 0, y \in \mathbb{R} \\ -\sqrt{-x}-y^2, & x < 0, y \in \mathbb{R} \end{cases} \geq 0,$$

$$G(x,y)^t H(x,y) := \begin{cases} x^2(\sqrt{x}+y^2), & x \geq 0, y \in \mathbb{R} \\ x(-\sqrt{-x}-y^2), & x < 0, y \in \mathbb{R} \end{cases} = 0,$$

where $f, g, G, H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

It can be seen that f is ∂^u-asymptotic pseudoconvex at $(0,0)$ with respect to $\partial^u f(0,0) := \{(x^* + \frac{1}{n}, y^* + \frac{1}{n}) | x^* \geq 0, y^* \leq 0, n \in N\} \cup \{(x^*, y^*) | x^* \geq 0, y^* \leq 0\}$.

g, $-G$ and $-H$ are ∂^u-asymptotic quasiconvex with respect to $\partial^u g(0,0) := \{(0, y^* + \frac{1}{n}) | y^* \geq 0, n \in N\} \cup \{(0, y^*) | y^* \geq 0\}$, $\partial^u(-G)(0,0) := \{(-1+\frac{1}{n}, 0), (\frac{1}{n}, 0) | n \in N\} \cup \{(-1,0)\}$ and $\partial^u(-H)(0,0) := \{(x^* + \frac{1}{n}, 0) | x^* \leq 0, n \in N\} \cup \{(x^*, 0) | x^* \leq 0\}$, respectively at $(0,0)$.

$A^+ \cup D^+ \cup B_G^+ \cup B_H^+ \cup B^+ = \emptyset$. Since ∂^*-strong stationarity condition holds at $(0,0)$, there exist scalars $\lambda^g = \frac{1}{2}$, $\lambda^G = \frac{1}{4}$, $\lambda^H = \frac{1}{4}$, with $\lambda^g + \lambda^G + \lambda^H = 1$ such that

$$\begin{aligned} (0,0) \in \text{cl}[&\text{conv}\, \partial^u f(0,0) + \{\lambda^g \text{conv}\, \partial^u g(0,0) + \lambda^G \text{conv}\, \partial^u(-G)(0,0) \\ &+ \lambda^H \text{conv}\, \partial^u(-H)(0,0)\}], \\ \lambda^g g(0,0) = 0&. \end{aligned}$$

$(0,0)$ is an optimal solution of the problem.

6. Conclusions

In this paper, using tool of nonsmooth analysis, convexifactors, a nonsmooth version of the standard GCQ and strong stationarity concept are introduced for MPEC. It is proved that strong stationarity is the first order necessary optimality condition under nonsmooth version of standard GCQ. Finally, under the assumption of asymptotic pseudoconvexity and asymptotic quasiconvexity on the functions, sufficient optimality conditions are established for MPEC.

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