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SECOND-ORDER SENSITIVITY ANALYSIS FOR PARAMETRIC EQUILIBRIUM PROBLEMS IN SET-VALUED OPTIMIZATION

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Abstract. In the paper, we first establish relationships between second-order contingent derivatives of a given set-valued map and that of the weak perturbation map. Then, these results are applied to sensitivity analysis for parametric equilibrium problems in set-valued optimization.

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1. Introduction

For optimization-related problems, after having their optimal solutions/optimal value, we often face many questions, such as: how do these solutions represent by the influence of other elements? How do you know that the numerical results are acceptable? What are extrapolation limits of the obtained results? . . . Answers to these questions are given by stability and sensitivity analyses. In mathematics, stability is understood as studies of various continuity properties of solution maps/optimal-valued maps of concerned problems, while sensitivity means studies of derivatives of the above-mentioned maps, see [12].

In the paper, we focus on sensitivity analysis in set-valued optimization. Many results have been obtained in this topic. In [35, 36], the behavior of perturbation maps in terms of contingent derivatives was established. The TP-derivative was introduced in [31] and employed in some conditions in [35]. In [32], Shi discussed some properties of perturbation maps in convex vector optimization problems. In [34], Sun and Li proposed sensitivity analysis for a general class of generalized vector quasi-equilibrium problems. Similar results for parametric weak vector equilibrium problems were obtained in [19]. In [8], Chuong and Yao studied generalized Clarke epiderivatives of the efficient value map for parametric vector optimization . In [34], sensitivity analysis for parametric vector optimization via lower Studniarski derivative was given. Inspired by these results, developments in sensitivity analysis have been increasing recently, see [3, 7, 10, 22, 23, 30].

From the above-mentioned results, the concept of derivatives is very essential to study sensitivity analysis. For the differentiability in set-valued optimization, there have been many kinds of generalized derivatives introduced with their applications in optimality conditions and duality. To get more information in optimality conditions, derivatives of higher orders have been proposed. They have been employed in some results on sensitivity analysis recently, see [3, 10, 34, 37].

Keywords. Sensitivity analysis, weak perturbation map, directional metric subregularity, contingent derivative, set-valued map.

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It is well known that the equilibrium problem plays an important role in optimization. Several types of this problem, such as the variational inequality problem, the saddle point problem, the complementarity problem, etc., have been investigated, see [6, 11, 14, 17, 20]. In detail, the lower semicontinuity of the solution maps of the parametric weak equilibrium problem was studied in [6, 14, 17]. The existence of the generalized equilibrium problem was obtained in [11, 20]. However, to the best of our knowledge, there are very few results dealing with higher-order sensitivity analysis for parametric equilibrium problems in set-valued optimization. This motivates us to study second-order sensitivity analysis for parametric set-valued equilibrium problems in terms of the weak efficiency in the paper.

The structure of the paper is as follows. In Section 2, we introduce some main notations and definitions used in the sequel. Section 3 is devoted to study relationships between second-order contingent derivatives of a set-valued map and that of the weak perturbation map. These results are applied to sensitivity analysis for parametric equilibrium problems in Section 4.

2. Preliminaries

Let X, Y be normed spaces, C be a closed convex pointed cone in Y. For a nonempty subset $A \subseteq Y$, $\mathrm{int}A$ and $\mathrm{cl}A$ stand for the interior and closure of A, respectively. $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_+$ are used for sets of the natural numbers, real numbers, and nonnegative real numbers, respectively. A nonempty convex subset B of C is called a base of C if $0 \notin \mathrm{cl}B$ and $C = \mathrm{cone}B := \{rb | b \in B, r \geq 0\}$. If $\mathrm{int}\, C \neq \emptyset$, $a_0 \in A$ is said to be a weak efficient point of A ($a_0 \in \mathrm{WMin}_C\, A$) if, see [21],

$$(A - \{a_0\}) \cap (-\mathrm{int}\, C) = \emptyset.$$

For a given set-valued map $F : X \to 2^Y$, the domain, image and graph of F are defined by

$$\begin{aligned} \mathrm{dom}\, F &:= \{x \in X | F(x) \neq \emptyset\}, \qquad && \mathrm{im}\, F := \{y \in Y | y \in F(x)\}, \\ \mathrm{gr}\, F &:= \{(x, y) \in X \times Y | y \in F(x)\}, && \text{respectively.} \end{aligned}$$

The profile map of F is the map $(F + C)(x) := F(x) + C$.

Recall that F is called to be metric regular at $(x_0, y_0) \in \mathrm{gr}\, F$ if there exist $\alpha, \lambda > 0$ such that for all $x \in B_X(x_0, \lambda)$, $y \in B_Y(y_0, \lambda)$,

$$d(x, F^{-1}(y)) \leq \alpha d(y, F(x)), \tag{2.1}$$

where $B_X(x, r)$ stands for the open ball in X centered at x with radius $r > 0$.

Remark 2.1. When F is a set-valued map, the inclusion $y \in F(x)$ can be considered as a constraint system (including inequality and equality conditions on x) with respect to y as a parameter. The solution set of $y \in F(x)$, denoted by $F^{-1}(y)$, is nonempty iff $y \in \mathrm{im}\, F$. An important question is raised: how is the behavior of $F^{-1}(y)$ corresponding to perturbation in y? It means that we need to learn about: bounds of the solution map $F^{-1}(.)$ (y as variable) under perturbations, and how large a perturbation in y can be to maintain good behavior of the solution map $F^{-1}(.)$. The best answer for these questions is metric regularity.

Indeed, (2.1) gives us an estimate of the distance from a candidate x to the solution set with respect to y. It is bounded above by a multiple α of the distance between y and $F(x)$, which measures the residual when $y \notin F(x)$.

If we fix $y = y_0$ in (2.1), the metric regularity reduces to the metric subregularity. Let S be a nonempty subset in X, F is metric subregular at (x_0, y_0) with respect to S if there exist $\alpha, \lambda > 0$ such that for all $x \in B_X(x_0, \lambda) \cap S$,

$$d(x, F^{-1}(y_0) \cap S) \leq \alpha d(y_0, F(x)).$$

The metric regularity (metric subregularity) property of F is equivalent to the Aubin property (the calmness, respectively) of the inverse map $F^{-1} : Y \to 2^X$, see [5]. The reader is referred to books [5, 24, 25, 29] and papers [9, 13, 15, 16] for more properties and applications of metric (sub)regularity.

In the paper, we propose another concept of metric subregularity as follows. Let X, Y, Z be normed spaces, $F : X \times Y \to 2^Z$, $((x_0, y_0), z_0) \in \operatorname{gr} F$ and $(u_1, v_1) \in X \times Y$.

Definition 2.2. The map F is said to be directionally metric subregular of order 2 at $((x_0, y_0), z_0)$ in direction $(u, v) \in X \times Y$ with respect to a subset $S \subseteq X \times Y$ and (u_1, v_1) if there exist $\alpha, \lambda > 0$ such that for all $t \in (0, \lambda)$, $u' \in B_X(u, \lambda)$, $v' \in B_Y(v, \lambda)$ with $(x_0 + tu_1 + t^2u', y_0 + tv_1 + t^2v') \in S$,

$$d((x_0 + tu_1 + t^2u', y_0 + tv_1 + t^2v'), F^{-1}(z_0) \cap S) \leq \alpha d(z_0, F(x_0 + tu_1 + t^2u', y_0 + tv_1 + t^2v')).$$

It is obvious to see that if F is metric subregular at $((x_0, y_0), z_0)$ with respect to a subset S, then F is directionally metric subregular of order 2 at $((x_0, y_0), z_0)$ in direction (u, v) with respect to S and (u_1, v_1), for all $(u, v), (u_1, v_1) \in X \times Y$. Thus, Definition 2.2 can be considered as an extension of the metric subregularity.

Definition 2.3 ([5])**.** Let $S \subseteq X$, $x_0 \in \operatorname{cl} S$.

(i) The first-order contingent (adjacent) cone of S at x_0 is defined by

$$T_S(x_0) := \{x \in X | \exists t_n \to 0^+, \exists x_n \to x, x_0 + t_n x_n \in S\}$$
$$(T_S^\flat(x_0) := \{x \in X | \forall t_n \to 0^+, \exists x_n \to x, x_0 + t_n x_n \in S\}, \text{ respectively}).$$

(ii) For $x_1 \in X$, the second-order contingent (adjacent) set of S at x_0 with respect to x_1 is defined by

$$T_S^2(x_0, x_1) := \{x \in X | \exists t_n \to 0^+, \exists x_n \to x, x_0 + t_n x_1 + t_n^2 x_n \in S\}$$
$$(T_S^{2(\flat)}(x_0, x_1) := \{x \in X | \forall t_n \to 0^+, \exists x_n \to x, x_0 + t_n x_1 + t_n^2 x_n \in S\}, \text{ respectively}).$$

Definition 2.4 ([5])**.** Let $F : X \to 2^Y$, $(x_0, y_0) \in \operatorname{gr} F$.

(i) The first-order contingent (adjacent) derivative of F at (x_0, y_0) is the set-valued map $DF(x_0, y_0) : X \to 2^Y$ $(D^\flat F(x_0, y_0) : X \to 2^Y)$ defined by

$$\operatorname{gr}(DF(x_0, y_0)) := T_{\operatorname{gr} F}(x_0, y_0)$$
$$(\operatorname{gr}(D^\flat F(x_0, y_0)) := T^\flat_{\operatorname{gr} F}(x_0, y_0), \text{ respectively}).$$

(ii) For $(u_1, v_1) \in X \times Y$, the second-order contingent (adjacent) derivative of F at (x_0, y_0) with respect to (u_1, v_1) is the set-valued map $D^2F(x_0, y_0, u_1, v_1) : X \to 2^Y$ $(D^{2(\flat)}F(x_0, y_0, u_1, v_1) : X \to 2^Y)$ defined by

$$\operatorname{gr}(D^2F(x_0, y_0, u_1, v_1)) := T^2_{\operatorname{gr} F}(x_0, y_0, u_1, v_1)$$
$$(\operatorname{gr}(D^{2(\flat)}F(x_0, y_0, u_1, v_1)) := T^{2(\flat)}_{\operatorname{gr} F}(x_0, y_0, u_1, v_1), \text{ respectively}).$$

Definition 2.4 can be expressed in terms of sequences equivalently as follows

$$\begin{aligned}
DF(x_0, y_0)(u) &= \{v \in Y | \exists t_n \to 0^+,\ \exists (u_n, v_n) \to (u, v),\ y_0 + t_n v_n \in F(x_0 + t_n u_n)\}.\\
D^\flat F(x_0, y_0)(u) &= \{v \in Y | \forall t_n \to 0^+,\ \exists (u_n, v_n) \to (u, v),\ y_0 + t_n v_n \in F(x_0 + t_n u_n)\}.\\
D^2F(x_0, y_0, u_1, v_1)(u) &= \{v \in Y | \exists t_n \to 0^+,\ \exists (u_n, v_n) \to (u, v),\ y_0 + t_n v_1 + t_n^2 v_n \in F(x_0 + t_n u_1 + t_n^2 u_n)\}.\\
D^{2(\flat)}F(x_0, y_0, u_1, v_1)(u) &= \{v \in Y | \forall t_n \to 0^+,\ \exists (u_n, v_n) \to (u, v),\ y_0 + t_n v_1 + t_n^2 v_n \in F(x_0 + t_n u_1 + t_n^2 u_n)\}.
\end{aligned}$$

Definition 2.5. Let $F : X \to 2^Y$, $(x_0, y_0) \in \operatorname{gr} F$, and $(u_1, v_1) \in X \times Y$, the second-order lower contingent derivative of F at (x_0, y_0) with respect to (u_1, v_1) is the set-valued map $D_l^2 F(x_0, y_0, u_1, v_1) : X \to 2^Y$ defined by

$$D_l^2 F(x_0, y_0, u_1, v_1)(u) := \{v \in Y | \forall t_n \to 0^+, \forall u_n \to u, \exists v_n \to v, y_0 + t_n v_1 + t_n^2 v_n \in F(x_0 + t_n u_1 + t_n^2 u_n)\}.$$

We can check that $D_l^2F(x_0,y_0,u_1,v_1)(u) \subseteq D^{2(\flat)}F(x_0,y_0,u_1,v_1)(u) \subseteq D^2F(x_0,y_0,u_1,v_1)(u)$.

Definition 2.6. Let $F : X \to 2^Y$, $(x_0,y_0) \in \operatorname{gr} F$, and $(u_1,v_1) \in X \times Y$.

(i) The map F is said to have the second-order proto-contingent derivative at (x_0,y_0) with respect to (u_1,v_1) if for all $u \in X$,

$$D^2F(x_0,y_0,u_1,v_1)(u) = D^{2(\flat)}F(x_0,y_0,u_1,v_1)(u).$$

(ii) The map F is said to have the second-order semi-contingent derivative at (x_0,y_0) with respect to (u_1,v_1) if for all $u \in X$,

$$D^2F(x_0,y_0,u_1,v_1)(u) = D_l^2F(x_0,y_0,u_1,v_1)(u).$$

The terminology "proto-contingent derivative" is used in the paper according to the idea of [26] and [28]. If F has the second-order semi-contingent derivative then it has the second-order proto-contingent derivative.

3. Second-order contingent derivatives of weak perturbation maps

Suppose that X,Y are normed spaces, $C \subseteq Y$ is a closed convex pointed cone, and $F : X \to 2^Y$. We first establish relationships between the second-order contingent derivative of F and that of $F + C$.

Definition 3.1. Let $F : X \to 2^Y$, $(x_0,y_0) \in \operatorname{gr}F$, and $(u_1,v_1) \in \operatorname{gr}DF(x_0,y_0)$. The second-order radial-contingent derivative of F is defined by, for $u \in X$,

$$\begin{aligned} D_r^{''}F(x_0,y_0,u_1,v_1)(u) := \{&v \in Y | \exists t_n \to 0^+, \exists s_n > 0, \exists (u_n,v_n) \to (u,v),\\ &y_0 + t_nv_1 + s_nv_n \in F(x_0 + t_nu_1 + s_nu_n)\}. \end{aligned}$$

Proposition 3.2. *Let $F : X \to 2^Y$, $(x_0,y_0) \in \operatorname{gr}F$, and $(u_1,v_1) \in \operatorname{gr}DF(x_0,y_0)$. Then, for $u \in X$,*

$$D^2F(x_0,y_0,u_1,v_1)(u) + C \subseteq D^2(F + C)(x_0,y_0,u_1,v_1)(u). \tag{3.1}$$

If C has a compact base and the condition

$$D_r^{''}F(x_0,y_0,u_1,v_1)(0) \cap (-C) = \{0\} \tag{3.2}$$

is fulfilled, then (3.1) becomes an equality.

Proof. Let $v \in D^2F(x_0,y_0,u_1,v_1)(u) + C$, *i.e.*, there exist $w \in D^2F(x_0,y_0,u_1,v_1)(u)$ and $c \in C$ such that $v = w + c$. For w, there are $t_n \to 0^+$, $(u_n,w_n) \to (u,w)$ such that

$$y_0 + t_nv_1 + t_n^2w_n \in F(x_0 + t_nu_1 + t_n^2u_n),$$

which implies that

$$y_0 + t_nv_1 + t_n^2(w_n + c) \in F(x_0 + t_nu_1 + t_n^2u_n) + C.$$

Thus, $v = w + c \in D^2(F + C)(x_0,y_0,u_1,v_1)(u)$.

For the converse conclusion, let $v \in D^2(F + C)(x_0,y_0,u_1,v_1)(u)$, then there exist $t_n \to 0^+$, $(u_n,v_n) \to (u,v)$, and $c_n \in C$ such that

$$y_0 + t_nv_1 + t_n^2v_n - c_n \in F(x_0 + t_nu_1 + t_n^2u_n).$$

Since C has a compact base, we may assume that $c_n = \alpha_nb_n$ with $\alpha_n > 0$ and $b_n \to b(b \in C \setminus \{0\})$, which implies that

$$y_0 + t_nv_1 + t_n^2v_n - \alpha_nb_n \in F(x_0 + t_nu_1 + t_n^2u_n). \tag{3.3}$$

We consider two cases as follows:

Case 1: $\alpha_n/t_n^2 \to +\infty$. From (3.3), one has

$$y_0 + t_n v_1 + \alpha_n \left(\frac{t_n^2}{\alpha_n} v_n - b_n \right) \in F\left(x_0 + t_n u_1 + \alpha_n \left(\frac{t_n^2}{\alpha_n} u_n \right) \right),$$

i.e., $-b \in D_r^{''} F(x_0, y_0, u_1, v_1)(0)$, which contradicts (3.2).

Case 2: α_n/t_n^2 is bounded and we suppose that $\alpha_n/t_n^2 \to r \geq 0$. It follows from (3.3) that

$$y_0 + t_n v_1 + t_n^2 \left(v_n - \frac{\alpha_n}{t_n^2} b_n \right) \in F(x_0 + t_n u_1 + t_n^2 u_n).$$

Thus, $v - rb \in D^2 F(x_0, y_0, u_1, v_1)(u)$, *i.e.*, $v \in D^2 F(x_0, y_0, u_1, v_1)(u) + C$. □

The following example illustrates that the condition (3.2) is necessary for the inverse conclusion of (3.1).

Example 3.3. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, and $C = \mathbb{R}_+$. We consider the following set-valued map

$$F(x_1, x_2) := \begin{cases} \{(x_1 + x_2)/2, -1\}, & \text{if } x_1 = x_2 \geq 0, \\ \{-1/2\}, & \text{otherwise.} \end{cases}$$

Let $(x_0, y_0) = ((0,0), 0)$ and $(u_1, v_1) = ((1,1), 1) \in \text{gr}DF(x_0, y_0)$, by calculating, we get

$$(F + C)(x_1, x_2) = \begin{cases} \{y \in Y | y \geq -1\}, & \text{if } x_1 = x_2 \geq 0, \\ \{y \in Y | y \geq -1/2\}, & \text{otherwise,} \end{cases}$$

and

$$D^2 F(x_0, y_0, u_1, v_1)(x_1, x_2) = \begin{cases} \{(x_1 + x_2)/2\}, & \text{if } x_1 = x_2, \\ \emptyset, & \text{otherwise,} \end{cases}$$

$$D^2(F + C)(x_0, y_0, u_1, v_1)(x_1, x_2) = \mathbb{R}.$$

Thus, $D^2 F(x_0, y_0, u_1, v_1)(x_1, x_2) + C \subsetneq D^2(F+C)(x_0, y_0, u_1, v_1)(x_1, x_2)$. The reason is that the condition (3.2) does not hold. Indeed, by taking $t_n := 1/n$, $s_n := 1$, $u_n := (-1/n, 0) \to (0,0)$, and $v_n := -1/2 - 1/n \to -1/2$, we can check that

$$y_0 + t_n v_1 + s_n v_n \in F(x_0 + t_n u_1 + s_n u_n),$$

i.e., $-1/2 \in D_r^{''} F(x_0, y_0, u_1, v_1)(0) \cap (-C)$.

Lemma 3.4. *Let $A \subseteq Y$ be a nonempty subset and $\hat{C} \subseteq \text{int}C \cup \{0\}$ be a closed convex pointed cone. Then,*

$$\text{WMin}_C A = \text{WMin}_C (A + \hat{C}).$$

Proof. "⊆:" Let $y_0 \in \text{WMin}_C A$, suppose that $y_0 \notin \text{WMin}_C(A + \hat{C})$, *i.e.*, there exist $y \in A$ and $c \in \hat{C}$ such that $y + c - y_0 \in -\text{int}C$. This implies that $y - y_0 \in -\text{int}C - c \subseteq -\text{int}C$, which contradicts the weak efficiency of y_0.

"⊇:" Let $y_0 \in \text{WMin}_C(A + \hat{C})$, *i.e.*, $y_0 \in A + \hat{C}$ and $(A + \hat{C} - \{y_0\}) \cap (-\text{int}C) = \emptyset$. It follows from the closeness of $\hat{C}$ that $0 \in \hat{C}$, so

$$(A - \{y_0\}) \cap (-\text{int}C) = \emptyset. \tag{3.4}$$

We just need to prove that $y_0 \in A$. Suppose to the contrary, *i.e.*, $y_0 \notin A$, then there exists $y \in A$ $(y \neq y_0)$ such that $y_0 \in y + \hat{C}$ (since $y_0 \in A + \hat{C}$). Consequently, one gets

$$y - y_0 \in (A - \{y_0\}) \cap (-\hat{C}) \subseteq (A - \{y_0\}) \cap (-\text{int}C),$$

which contradicts (3.4). □

We cannot replace $\hat{C}$ by C in Lemma 3.4 as indicated by the following example

Example 3.5. Let $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and $A = \{(0, 0)\}$. Then,

$$\text{WMin}_C A(= \{(0, 0)\}) \subsetneq \text{WMin}_C(A + C)(= \mathbb{R}^2_+ \setminus \text{int}\mathbb{R}^2_+).$$

From Proposition 3.2 and Lemma 3.4, we obtain a proposition as follows.

Proposition 3.6. *Let $F : X \to 2^Y$, $(x_0, y_0) \in \text{gr}F$, $(u_1, v_1) \in \text{gr}DF(x_0, y_0)$, and $\hat{C} \subseteq \text{int}C \cup \{0\}$ be a closed convex pointed cone. If $\hat{C}$ has a compact base and the condition (3.2) is fulfilled with respect to $\hat{C}$, then for $u \in X$,*

$$\text{WMin}_C D^2F(x_0, y_0, u_1, v_1)(u) = \text{WMin}_C D^2(F + \hat{C})(x_0, y_0, u_1, v_1)(u).$$

Example 3.7. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$ and $F : X \to 2^Y$ be defined by $F(x) := \{(y_1, y_2) \in Y | y_1 + y_2 \geq 0\}$. With $\hat{C} := \{(y_1, y_2) \in Y | y_2 \leq 2y_1\} \cap \{(y_1, y_2) \in Y | y_2 \geq (1/2)y_1\}$, $\hat{C}$ is a closed convex pointed cone having a compact base and $\hat{C} \subseteq \text{int}C \cup \{0\}$.

Let $x_0 = 0, y_0 = (0, 0)$, $u_1 = 2$ and $v_1 = (1, 1)$, we now check that (3.2) is fulfilled with respect to $\hat{C}$. In fact, let $w = (w_1, w_2) \in D_r^{''}F(x_0, y_0, u_1, v_1)(0)$, then there are $t_n \to 0^+$, $s_n > 0$, $u_n \to 0$ and $(w_n^1, w_n^2) \to (w_1, w_2)$ such that

$$(0, 0) + t_n(1, 1) + s_n(w_n^1, w_n^2) \in F(0 + 2t_n + s_n u_n), \tag{3.5}$$

i.e.,

$$2t_n + s_n(w_n^1 + w_n^2) \geq 2t_n + s_n u_n.$$

Thus, $w \in \{(w_1, w_2) \in Y | w_1 + w_2 \geq 0\}$, which implies that $D_r^{''}F(x_0, y_0, u_1, v_1)(0) \subseteq \{(w_1, w_2) \in Y | w_1 + w_2 \geq 0\}$. For the converse inclusion, let $w \in \{(w_1, w_2) \in Y | w_1 + w_2 \geq 0\}$ and by taking $t_n = 1/n$, $s_n = 1$, $u_n = 0 \to 0$, $(w_n^1, w_n^2) = w \to w$, then (3.5) is satisfied, *i.e.*, $w \in D_r^{''}F(x_0, y_0, u_1, v_1)(0)$. Hence, we have

$$D_r^{''}F(x_0, y_0, u_1, v_1)(0) = \{(w_1, w_2) \in Y | w_1 + w_2 \geq 0\}.$$

Consequently, one gets

$$D_r^{''}F(x_0, y_0, u_1, v_1)(0) \cap (-\hat{C}) = \{0\}.$$

Therefore, Proposition 3.6 is fulfilled. Indeed, by a direct calculation, we obtain

$$\begin{aligned}\text{WMin}_C D^2F(x_0, y_0, u_1, v_1)(u) &= \text{WMin}_C D^2(F + \hat{C})(x_0, y_0, u_1, v_1)(u) \\ &= \{(w_1, w_2) \in Y | w_1 + w_2 = u\}.\end{aligned}$$

Using the above results, relationships between the second-order contingent derivative of F and that of the weak perturbation map, denoted by $W(x) := \text{WMin}_C F(x)$, are implied. Recall that the map F is said to have the weak domination property near x_0 with respect to $\hat{C}$, where $\hat{C} \subseteq \text{int}C \cup \{0\}$ is a closed convex pointed cone, if there exists a neighborhood V of x_0 such that for all $x \in V$,

$$F(x) \subseteq \text{WMin}_C F(x) + \hat{C},$$

equivalently,

$$F(x) \subseteq W(x) + \hat{C}.$$

Lemma 3.8. *Let $(x_0, y_0) \in \text{gr}W$ and $(u_1, v_1) \in \text{gr}DF(x_0, y_0)$. If F has the weak domination property near x_0 with respect to $\hat{C}$, then for $u \in X$,*

$$D^2(F + \hat{C})(x_0, y_0, u_1, v_1)(u) = D^2(W + \hat{C})(x_0, y_0, u_1, v_1)(u). \tag{3.6}$$

Proof. We first prove that $F(x)+\hat{C}=W(x)+\hat{C}$ for all $x\in V$, where V is a neighborhood of x_0. Indeed, one has

$$\begin{aligned}F(x)+\hat{C}&\subseteq W(x)+\hat{C}\ \text{(the weak domination property)}\\&=\text{WMin}_C F(x)+\hat{C}\\&\subseteq F(x)+\hat{C}.\end{aligned}$$

Let $v\in D^2(F+\hat{C})(x_0,y_0,u_1,v_1)(u)$, *i.e.*, there exist $t_n\to 0^+$, $(u_n,v_n)\to(u,v)$ such that

$$y_0+t_nv_1+t_n^2v_n\in(F+\hat{C})(x_0+t_nu_1+t_n^2u_n)=F(x_0+t_nu_1+t_n^2u_n)+\hat{C}.$$

For n large enough, $x_0+t_nu_1+t_n^2u_n\in V$ then, equivalently, we get

$$y_0+t_nv_1+t_n^2v_n\in W(x_0+t_nu_1+t_n^2u_n)+\hat{C},$$

i.e., $v\in D^2(W+\hat{C})(x_0,y_0,u_1,v_1)(u)$. ☐

Theorem 3.9. *Let $(x_0,y_0)\in \text{gr}W$, $(u_1,v_1)\in \text{gr}DW(x_0,y_0)$, and $\hat{C}\subseteq \text{int}C\cup\{0\}$ be a closed convex pointed cone. If F has the weak domination property near x_0 with respect to $\hat{C}$, $\hat{C}$ has a compact base, and the condition (3.2) is fulfilled with respect to $\hat{C}$, then for $u\in X$,*

$$\text{WMin}_C D^2F(x_0,y_0,u_1,v_1)(u)=\text{WMin}_C D^2W(x_0,y_0,u_1,v_1)(u).$$

Proof. Since $D_r^{''}W(x_0,y_0,u_1,v_1)(u)\subseteq D_r^{''}F(x_0,y_0,u_1,v_1)(u)$, it follows from the condition (3.2) that

$$D_r^{''}W(x_0,y_0,u_1,v_1)(0)\cap(-\hat{C})=\{0\}.$$

Hence, from Proposition 3.6 and Lemma 3.8, one obtains

$$\begin{aligned}\text{WMin}_C D^2F(x_0,y_0,u_1,v_1)(u)&=\text{WMin}_C D^2(F+\hat{C})(x_0,y_0,u_1,v_1)(u)\\&=\text{WMin}_C D^2(W+\hat{C})(x_0,y_0,u_1,v_1)(u)\\&=\text{WMin}_C D^2W(x_0,y_0,u_1,v_1)(u).\end{aligned}$$

☐

The following example illustrates Theorem 3.9.

Example 3.10. Let $X=\mathbb{R}^2$, $Y=\mathbb{R}$, $C=\mathbb{R}_+$, and $F(x_1,x_2):=\{y\in Y|y\geq x_1+x_2\}$. Let $(x_0,y_0)=((0,0),0)$ and $(u_1,v_1)=((1,1),2)\in\text{gr}DF(x_0,y_0)$, by calculating, we get

$$W(x_1,x_2)=\text{WMin}_C F(x_1,x_2)=\{x_1+x_2\},$$

and

$$\begin{aligned}D^2F(x_0,y_0,u_1,v_1)(x_1,x_2)&=\{y\in Y|y\geq x_1+x_2\},\\D^2W(x_0,y_0,u_1,v_1)(x_1,x_2)&=\{x_1+x_2\}.\end{aligned}$$

It is easy to check that all conditions of Theorem 3.9 are fulfilled. Indeed, we now show that (3.2) is satisfied. Let $v\in D_r^{''}F(x_0,y_0,u_1,v_1)(0)$, by Definition 3.1, there exist $t_n\to 0^+$, $s_n>0$, $(u_n^1,u_n^2)\to(0,0)$, $v_n\to v$ such that

$$0+2t_n+s_nv_n\in F(0+t_n+s_nu_n^1,0+t_n+s_nu_n^2),\tag{3.7}$$

i.e.,

$$2t_n+s_nv_n\geq 2t_n+s_n(u_n^1+u_n^2),$$

which implies that $v_n \geq u_n^1 + u_n^2$. Thus, $v \geq 0$, *i.e.*, $D_r^{''}F(x_0, y_0, u_1, v_1)(0) \subseteq \mathbb{R}_+$.

For the converse inclusion, let $v \in \mathbb{R}_+$, there exist $t_n = 1/n$, $s_n = 1$, $u_n^1 = u_n^2 = 1/n$, $v_n = v + u_n^1 + u_n^2$ such that (3.7) is satisfied, *i.e.*, $v \in D_r^{''}F(x_0, y_0, u_1, v_1)(0)$. Hence, $D_r^{''}F(x_0, y_0, u_1, v_1)(0) = \mathbb{R}_+(= C)$, which means that (3.2) is fulfilled. Therefore, by Theorem 3.9, we have

$$\text{WMin}_C D^2F(x_0, y_0, u_1, v_1)(x_1, x_2) = \text{WMin}_C D^2W(x_0, y_0, u_1, v_1)(x_1, x_2) = \{x_1 + x_2\}.$$

Theorem 3.11. *Let $(x_0, y_0) \in \text{gr}W$, $(u_1, v_1) \in \text{gr}DW(x_0, y_0)$, and $\hat{C} \subseteq \text{int}C \cup \{0\}$ be a closed convex pointed cone. If all conditions of Theorem 3.9 are satisfied, then for $u \in X$,*

$$\text{WMin}_C D^2F(x_0, y_0, u_1, v_1)(u) \subseteq D^2W(x_0, y_0, u_1, v_1)(u). \tag{3.8}$$

If, additionally, F has the second-order proto-contingent derivative at (x_0, y_0) and the map $g : (X \times Y)^2 \to \mathbb{R}_+$ defined by $g(\beta_1, \gamma_1, \beta_2, \gamma_2) := ||\beta_1 - \beta_2||$ is directionally metric subregular of order 2 at $((x_0, y_0, x_0, y_0), 0)$ in the direction $(u, \overline{v}, u, \hat{v})$ with respect to $\text{gr}\, W \times \text{gr}\, F$ and (u_1, v_1), for all $(u, \overline{v}) \in \text{gr}D^2W(x_0, y_0, u_1, v_1)$ and $(u, \hat{v}) \in \text{gr}D^2F(x_0, y_0, u_1, v_1)$, then (3.8) becomes an equality.

Proof. The inclusion (3.8) can be implied directly from Theorem 3.9. For the converse inclusion, let $\overline{v} \in D^2W(x_0, y_0, u_1, v_1)(u)$, then there exist $t_n \to 0^+$, $(\overline{u}_n, \overline{v}_n) \to (u, \overline{v})$ such that

$$y_0 + t_n v_1 + t_n^2 \overline{v}_n \in W(u_0 + t_n u_1 + t_n^2 \overline{u}_n).$$

If $\overline{v} \notin \text{WMin}_C\, D^2F(x_0, y_0, u_1, v_1)(u)$, then there exists $\hat{v} \in D^2F(x_0, y_0, u_1, v_1)(u)$ with $\hat{v} - \overline{v} \in -\text{int}\, C$. Because F has the second-order proto-contingent derivative at (x_0, y_0), with t_n above, there is $(\hat{u}_n, \hat{v}_n) \to (u, \hat{v})$ satisfying

$$y_0 + t_n v_1 + t_n^2 \hat{v}_n \in F(x_0 + t_n u_1 + t_n^2 \hat{u}_n).$$

It follows from the directionally metric subregularity that there exist $\alpha > 0$ and $\lambda > 0$ such that for every $t \in (0, \lambda)$ and $(u_1^{'}, v_1^{'}, u_2^{'}, v_2^{'}) \in B_{X \times Y}((u, \overline{v}), \lambda) \times B_{X \times Y}((u, \hat{v}), \lambda)$ with

$$(x_0 + tu_1 + t^2u_1^{'},\ y_0 + tv_1 + t^2v_1^{'},\ x_0 + tu_1 + t^2u_2^{'},\ y_0 + tv_1 + t^2v_2^{'}) \in \text{gr}\, W \times \text{gr}\, F,$$

and

$$d\left((x_0 + tu_1 + t^2u_1^{'},\ y_0 + tv_1 + t^2v_1^{'},\ x_0 + tu_1 + t^2u_2^{'},\ y_0 + tv_1 + t^2v_2^{'}),\, g^{-1}(0) \cap (\text{gr}\, W \times \text{gr}\, F)\right)$$

$$\leq \alpha d\left(0, g\left(x_0 + tu_1 + t^2u_1^{'}, y_0 + tv_1 + t^2v_1^{'}, x_0 + tu_1 + t^2u_2^{'}, y_0 + tv_1 + t^2v_2^{'}\right)\right). \tag{3.9}$$

It is easy to see that $t_n \in (0, \lambda)$ and $(\overline{u}_n, \overline{v}_n, \hat{u}_n, \hat{v}_n) \in B_{X \times Y}((u, \overline{v}), \lambda) \times B_{X \times Y}((u, \hat{v}), \lambda)$ for n large enough. Thus, from (3.9), for n large enough, there exists $(\overline{x}_n, \overline{y}_n,\ \hat{x}_n, \hat{y}_n) \in \text{gr}\, W \times \text{gr}\, F$ with $\overline{x}_n = \hat{x}_n$ such that

$$||\left(x_0 + t_n u_1 + t_n^2 \overline{u}_n,\ y_0 + t_n v_1 + t_n^2 \overline{v}_n,\ x_0 + t_n u_1 + t_n^2 \hat{u}_n,\ y_0 + t_n v_1 + t_n^2 \hat{v}_n\right) - (\overline{x}_n, \overline{y}_n, \hat{x}_n, \hat{y}_n)\, ||$$

$$< \alpha {t_n}^2 ||\overline{u}_n - \hat{u}_n|| + t_n^3,$$

which implies

$$\left\|x_0 + t_n u_1 + t_n^2 \overline{u}_n - \overline{x}_n\right\| < \alpha t_n^2 ||\overline{u}_n - \hat{u}_n|| + t_n^3,$$
$$\left\|y_0 + t_n v_1 + t_n^2 \overline{v}_n - \overline{y}_n\right\| < \alpha t_n^2 ||\overline{u}_n - \hat{u}_n|| + t_n^3,$$
$$\left\|y_0 + t_n v_1 + t_n^2 \hat{v}_n - \hat{y}_n\right\| < \alpha t_n^2 ||\overline{u}_n - \hat{u}_n|| + t_n^3.$$

Consequently,

$$\left\|\frac{\overline{x}_n - x_0 - t_n u_1}{t_n^2} - \overline{u}_n\right\| < \alpha ||\overline{u}_n - \hat{u}_n|| + t_n,$$

$$\left\| \frac{\overline{y}_n - y_0 - t_n v_1}{t_n^2} - \overline{v}_n \right\| < \alpha ||\overline{u}_n - \hat{u}_n|| + t_n,$$

$$\left\| \frac{\hat{y}_n - y_0 - t_n v_1}{t_n^2} - \hat{v}_n \right\| < \alpha ||\overline{u}_n - \hat{u}_n|| + t_n.$$

By setting $\overline{v}_n^1 := \dfrac{\overline{y}_n - y_0 - t_n v_1}{t_n^2}$, $\hat{v}_n^1 := \dfrac{\hat{y}_n - y_0 - t_n v_1}{t_n^2}$ and $u_n := \dfrac{\overline{x}_n - x_0 - t_n u_1}{t_n^2}$, then $\overline{v}_n^1 \to \overline{v}$, $\hat{v}_n^1 \to \hat{v}$, $u_n \to u$ and

$$y_0 + t_n v_1 + t_n^2 \overline{v}_n^1 = \overline{y}_n \in W(\overline{x}_n) = \text{WMin}_C F(x_0 + t_n u_1 + t_n^2 u_n),$$
$$y_0 + t_n v_1 + t_n^2 \hat{v}_n^1 = \hat{y}_n \in F(\hat{x}_n) = F(\overline{x}_n) = F(x_0 + t_n u_1 + t_n^2 u_n).$$

Hence, for n large enough, one has

$$(y_0 + t_n v_1 + t_n^2 \hat{v}_n^1) - (y_0 + t_n v_1 + t_n^2 \overline{v}_n^1) = t_n^2 (\hat{v}_n^1 - \overline{v}_n^1) \in -\text{int}\, C,$$

which contradicts the fact that $y_0 + t_n v_1 + t_n^2 \overline{v}_n^1 \in \text{WMin}_C\, F(u_0 + t_n u_1 + t_n^2 u_n)$. □

Remark 3.12. The converse conclusion of (3.8) was studied in some existing results using other kinds of generalized derivatives. For example, Anh and Khanh assumed that F has the semi-variational set of order m in Proposition 4.1 of [3]. In Proposition 5.2 of [10], the authors proposed an assumption on the mth-order semi-contingent-type derivative to get the converse inclusion of (3.8) in terms of contingent-type derivatives. In [3, 10], these conditions are called "proto-variational set" and "proto-contingent-type derivative", respectively. However, by Penot's idea in [26], we use the terminology "semi-variational set" and "semi-contingent-type derivative" here to compare them with our results conveniently. In [34], the authors employed lower Studniarski derivatives in their main results.

The above-mentioned conditions require the existence of some kinds of lower derivatives, but it is quite strict. Moreover, "semi-variational set" and "semi-contingent-type derivative" mean "semi-contingent derivative" in the paper. By Definition 2.6, we try to use the "proto-contingent derivative" property to obtain the converse conclusion of (3.8). With a weakened assumption, we need to have some supplementary conditions. One of them is introduced in Theorem 3.11, and it is not difficult to check this condition (see Example 3.13).

From the above observation, we propose some open questions as follows: are there other conditions to get the converse inclusion of (3.8)? If yes, how are relationships between them?

The following example shows a case where Theorem 3.11 can be employed, while the results mentioned in Remark 3.12 cannot.

Example 3.13. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$ and $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(x_1, x_2) := \begin{cases} \emptyset, & \text{if } x_1, x_2 \in \left\{ \dfrac{1}{n^3} | n \in \mathbb{N} \right\}, \\ \{(x_1 + x_2)/2\}, & \text{otherwise.} \end{cases}$$

It is obvious that $W(x_1, x_2) = \text{WMin}_C F(x_1, x_2) = F(x_1, x_2)$. Let $(x_0, y_0) = ((0, 0), 0)$ and $(u_1, v_1) = ((1, 1), 1) \in \text{gr} DW(x_0, y_0)$. We can check that F has the second-order proto-contingent derivative at $((0, 0), 0)$ with respect to (u_1, v_1) and

$$D^2 F(x_0, y_0, u_1, v_1)(x_1, x_2) = \{(x_1 + x_2)/2\},$$

but conditions mentioned in Proposition 4.1 of [3] and Proposition 5.2 of [10] do not hold, and the second-order lower Studniarski derivative in Theorem 4.2 of [34] does not exist. Thus, these results do not work in this case.

However, Theorem 3.11 is useful for this example. Indeed, all conditions of Theorem 3.11 are satisfied. For instance, we just check that the directionally metric subregularity of order 2 of the map g in Theorem 3.11 is fulfilled. In this example, the map $g : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}_+$ is given by

$$g(x_1, x_2, w_1, y_1, y_2, w_2) := ||(x_1, x_2) - (y_1, y_2)||.$$

By Definition 2.2, let $(u_1, u_2, (u_1 + u_2)/2) \in \operatorname{gr} D^2W(x_0, y_0, u_1, v_1)$, $(u_1', u_2', (u_1' + u_2')/2) \in \operatorname{gr} D^2F(x_0, y_0, u_1, v_1)$ and $\lambda > 0$, it is enough to show that there exists $\alpha > 0$ such that for all $t \in (0, \lambda)$, $(\hat{u}_1, \hat{u}_2, \hat{v}) \in B_{\mathbb{R}^3}((u_1, u_2, (u_1 + u_2)/2), \lambda)$, $(\overline{u}_1, \overline{u}_2, \overline{v}) \in B_{\mathbb{R}^3}((u_1', u_2', (u_1' + u_2')/2), \lambda)$ with

$$(0 + t + t^2\hat{u}_1, 0 + t + t^2\hat{u}_2, 0 + t + t^2\hat{v}, 0 + t + t^2\overline{u}_1, 0 + t + t^2\overline{u}_2, 0 + t + t^2\overline{v}) \in \operatorname{gr} W \times \operatorname{gr} F,$$

then

$$\begin{aligned} &d((t + t^2\hat{u}_1, t + t^2\hat{u}_2, t + t^2\hat{v}, t + t^2\overline{u}_1, t + t^2\overline{u}_2, t + t^2\overline{v}), g^{-1}(0) \cap (\operatorname{gr} W \times \operatorname{gr} F)) \\ &\quad \le \alpha d(0, g(t + t^2\hat{u}_1, t + t^2\hat{u}_2, t + t^2\hat{v}, t + t^2\overline{u}_1, t + t^2\overline{u}_2, t + t^2\overline{v})). \end{aligned}$$

Since $(t + t^2\hat{u}_1, t + t^2\hat{u}_2, t + t^2\hat{v}, t + t^2\overline{u}_1, t + t^2\overline{u}_2, t + t^2\overline{v}) \in \operatorname{gr} W \times \operatorname{gr} F$, we get $\hat{v} = (\hat{u}_1 + \hat{u}_2)/2$ and $\overline{v} = (\overline{u}_1 + \overline{u}_2)/2$. Thus, we need to find α such that

$$\begin{aligned} &\inf_{\substack{(x,y)\in\mathbb{R}^2,\\ v\in F(x,y),\\ w\in G(x,y)}} \left\{||(t + t^2\hat{u}_1, t + t^2\hat{u}_2) - (x, y)|| + ||(t + t^2\overline{u}_1, t + t^2\overline{u}_2) - (x, y)|| + |t + t^2\hat{v} - v| + |t + t^2\overline{v} - w|\right\} \\ &\qquad \le \alpha t^2||(\hat{u}_1, \hat{u}_2) - (\overline{u}_1, \overline{u}_2)||. \end{aligned} \tag{3.10}$$

We can see that

$$\begin{aligned} &\inf_{\substack{(x,y)\in\mathbb{R}^2,\\ v\in F(x,y),\\ w\in G(x,y)}} \left\{||(t + t^2\hat{u}_1, t + t^2\hat{u}_2) - (x, y)|| + ||(t + t^2\overline{u}_1, t + t^2\overline{u}_2) - (x, y)|| + |t + t^2\hat{v} - v| + |t + t^2\overline{v} - w|\right\} \\ &\qquad = \inf_{(x,y)\in\mathbb{R}^2}\{||(t + t^2\hat{u}_1, t + t^2\hat{u}_2) - (x, y)|| + ||(t + t^2\overline{u}_1, t + t^2\overline{u}_2) - (x, y)|| \\ &\qquad\quad + |t + t^2(\hat{u}_1 + \hat{u}_2)/2 - (x + y)/2| + |t + t^2(\overline{u}_1 + \overline{u}_2)/2 - (x + y)/2|\}. \end{aligned}$$

By setting

$$x := \frac{t + t^2\hat{u}_1 + t + t^2\overline{u}_1}{2} = t + \frac{t^2}{2}(\hat{u}_1 + \overline{u}_1)$$

and

$$y := \frac{t + t^2\hat{u}_2 + t + t^2\overline{u}_2}{2} = t + \frac{t^2}{2}(\hat{u}_2 + \overline{u}_2),$$

then

$$\begin{aligned} ||(t + t^2\hat{u}_1, t + t^2\hat{u}_2) - (x, y)|| &= \frac{t_n^2}{2}||(\hat{u}_1, \hat{u}_2) - (\overline{u}_1, \overline{u}_2)||, \\ |t + t^2(\hat{u}_1 + \hat{u}_2)/2 - (x + y)/2| &= \frac{t_n^2}{4}|(\hat{u}_1 - \overline{u}_1) + (\hat{u}_2 - \overline{u}_2)| \\ &\le \frac{t_n^2}{4}\sqrt{((\hat{u}_1 - \overline{u}_1)^2 + (\hat{u}_2 - \overline{u}_2)^2)(1^2 + 1^2)} \\ &\le (\sqrt{2}/4)t^2||(\hat{u}_1, \hat{u}_2) - (\overline{u}_1, \overline{u}_2)||. \end{aligned}$$

Similarly, we get

$$\begin{aligned} ||(t + t^2\overline{u}_1, t + t^2\overline{u}_2) - (x, y)|| &= \frac{t_n^2}{2}||(\hat{u}_1, \hat{u}_2) - (\overline{u}_1, \overline{u}_2)||, \\ |t + t^2(\overline{u}_1 + \overline{u}_2)/2 - (x + y)/2| &\le (\sqrt{2}/4)t^2||(\hat{u}_1, \hat{u}_2) - (\overline{u}_1, \overline{u}_2)||, \end{aligned}$$

which implies that

$$\inf_{\substack{(x,y)\in\mathbb{R}^2,\\ v\in F(x,y),\\ w\in G(x,y)}} \left\{||(t+t^2\hat{u}_1, t+t^2\hat{u}_2)-(x,y)|| + ||(t+t^2\overline{u}_1, t+t^2\overline{u}_2)-(x,y)|| + |t+t^2\hat{v}-v| + |t+t^2\overline{v}-w|\right\}$$
$$\leq (1+\sqrt{2}/2)t^2||(\hat{u}_1,\hat{u}_2)-(\overline{u}_1,\overline{u}_2)||.$$

Thus, (3.10) is true for any $\alpha \geq 1+\sqrt{2}/2$. It means that there exist $\lambda > 0$ and $\alpha \geq 1+\sqrt{2}/2$ for which Definition 2.2 is satisfied with respect to the map g in this example, *i.e.*, g is directionally metric subregular of order 2.

Therefore, it follows from Theorem 3.11 that

$$D^2W(x_0,y_0,u_1,v_1)(x_1,x_2) \subseteq \mathrm{WMin}_C D^2F(x_0,y_0,u_1,v_1)(x_1,x_2).$$

4. Sensitivity analysis for parametric equilibrium problems

Let X, P, Y be normed spaces and $C \subseteq Y$ be a closed convex pointed cone. We consider the following parametric equilibrium problem (PEP): find $x \in K(p)$ such that

$$F(x,y,p) \cap (-\mathrm{int}C) = \emptyset, \quad \forall y \in K(p),$$

where $F : X \times X \times P \to 2^Y$ and $K : P \to 2^X$. For each $p \in P$, the solution map of (PEP) is denoted by

$$S(p) := \{x \in K(p) | F(x,y,p) \cap (-\mathrm{int}C) = \emptyset, \ \forall y \in K(p)\}.$$

The map S can be rewritten by for $x \in X$ and $p \in P$,

$$S(p) = \{x \in K(p) | 0 \in W(p,x)\},$$

where $W(p,x) := \mathrm{WMin}_C G(p,x)$ and $G(p,x) := \bigcup_{y\in K(p)} (F(x,y,p) \cup \{0\})$.

Definition 4.1. ([27]) The map S is said to be Robinson metric regular around $(p_0,x_0) \in \mathrm{gr}S$ if there exist $\mu > 0$, $\gamma > 0$, and neighborhoods U of p_0, V of x_0 such that

$$d(x,S(p)) \leq \mu d(0,W(p,x)), \quad \text{whenever } p \in U,\ x \in V,\ d(0,W(p,x)) < \gamma.$$

We now propose an extension of Definition 4.1 as follows

Definition 4.2. Let $(p_0,x_0) \in \mathrm{gr}S$ and $(x_1,p_1) \in X \times P$. The map S is said to be directionally Robinson metric regular of order 2 along K around (p_0,x_0) in the direction (x,p) with respect to (x_1,p_1) if there exist $\mu > 0$, $\gamma > 0$, $\lambda > 0$ such that

$$d(x_0+tx_1+t^2x', S(p_0+tp_1+t^2p')) \leq \mu d(0, W(p_0+tp_1+t^2p', x_0+tx_1+t^2x')),$$

whenever $t \in (0,\lambda)$, $p' \in B_P(p,\lambda)$, $x' \in B_X(x,\lambda)$,

$$x_0+tx_1+t^2x' \in K(p_0+tp_1+t^2p')$$

and

$$d(0, W(p_0+tp_1+t^2p', x_0+tx_1+t^2x')) < \gamma.$$

It is obvious to see that if S is Robinson metric regular around (p_0,x_0) then S is directionally Robinson metric regular of order 2 around (p_0,x_0) in the direction (x,p) with respect to (x_1,p_1), for all $(x,p),(x_1,p_1) \in X \times P$.

Proposition 4.3. *Let $(p_0, x_0) \in \mathrm{gr}S$ and $(p_1, x_1) \in \mathrm{gr}DS(p_0, x_0)$. Then, for $p \in P$,*

$$D^2S(p_0, x_0, p_1, x_1)(p) \subseteq \{x \in D^2K(p_0, x_0, p_1, x_1)(p) | 0 \in D^2W(p_0, x_0, 0, p_1, x_1, 0)(p, x)\}. \quad (4.1)$$

If, additionally, S is directionally Robinson metric regular of order 2 along K around (p_0, x_0) in the direction $(x, p) \in M$ with respect to (p_1, x_1), where $M := \{(p, x)|(p, x, 0) \in \mathrm{gr}D^2W(p_0, x_0, 0, p_1, x_1, 0)\}$, K has the second-order proto contingent derivative at (p_0, x_0) with respect to (p_1, x_1), and the map $g : (P \times X) \times (P \times X \times Y) \to \mathbb{R}_+$ defined by $g(\beta_1, \gamma_1, \beta_2, \gamma_2, \delta) := ||\beta_1 - \beta_2|| + ||\gamma_1 - \gamma_2||$ is directionally metric subregular of order 2 at $((p_0, x_0, p_0, x_0, 0), 0)$ in the direction $(p, \overline{x}, p, \hat{x}, 0)$ with respect to $\mathrm{gr}\, K \times \mathrm{gr}\, W$ and (p_1, x_1), for all $(p, \overline{x}) \in \mathrm{gr}D^2K(p_0, x_0, p_1, x_1)$ and $(p, \hat{x}, 0) \in \mathrm{gr}D^2W(p_0, x_0, 0, p_1, x_1, 0)$, then (4.1) becomes an equality.

Proof. Let $x \in D^2S(p_0, x_0, p_1, x_1)(p)$, then there exist $t_n \to 0^+$, $(p_n, x_n) \to (p, x)$ such that

$$x_0 + t_nx_1 + t_n^2x_n \in S(p_0 + t_np_1 + t_n^2p_n).$$

By the definition of S, we get

$$x_0 + t_nx_1 + t_n^2x_n \in K(p_0 + t_np_1 + t_n^2p_n)$$

and

$$0 \in W(p_0 + t_np_1 + t_n^2p_n, x_0 + t_nx_1 + t_n^2x_n),$$

which implies that $x \in D^2K(p_0, x_0, p_1, x_1)(p)$ and $0 \in D^2W(p_0, x_0, 0, p_1, w_1, 0)(p, x)$.

For the converse of (4.1), let $x \in D^2K(p_0, x_0, p_1, x_1)(p)$ such that $0 \in D^2W(p_0, x_0, 0, p_1, w_1, 0)(p, x)$. For $x \in D^2K(p_0, x_0, p_1, x_1)(p)$, there are $t_n \to 0^+$, $(p_n, x_n) \to (p, x)$ such that

$$x_0 + t_nx_1 + t_n^2x_n \in K(p_0 + t_np_1 + t_n^2p_n).$$

Due to the second-order proto-contingent derivative property of K, with t_n above, there exists$(\hat{p}_n, \hat{x}_n) \to (p, x)$ and $z_n \to 0$ such that

$$t_n^2z_n \in W(p_0 + t_np_1 + t_n^2\hat{p}_n, x_0 + t_nx_1 + t_n^2\hat{x}_n).$$

From the directionally metric subregularity of g and the proof similar to that of Theorem 3.2, we get $(u_n, v_n) \to (p, x)$ and $w_n \to 0$ satisfying

$$x_0 + t_nx_1 + t_n^2v_n \in K(p_0 + t_np_1 + t_n^2u_n)$$

and

$$t_n^2w_n \in W(p_0 + t_np_1 + t_n^2u_n, x_0 + t_nx_1 + t_n^2v_n).$$

Since S is directionally Robinson metric regular of order 2 along K around (p_0, x_0), there exist $\lambda > 0$, $\mu > 0$, $\gamma > 0$ such that for n large enough, we get $x_n \in B_X(x, \lambda)$, $p_n \in B_P(p, \lambda)$,

$$d(0, W(p_0 + t_np_1 + t_n^2u_n, x_0 + t_nx_1 + t_n^2v_n)) \leq t_n^2||w_n|| < \gamma,$$

and

$$d(x_0 + t_nx_1 + t_n^2v_n, S(p_0 + t_np_1 + t_n^2u_n)) \leq \mu d(0, W(p_0 + t_np_1 + t_n^2u_n, x_0 + t_nx_1 + t_n^2v_n)) \leq \mu t_n^2||w_n||.$$

Thus, for n large enough, there exists $y_n \in S(p_0 + t_np_1 + t_n^2u_n)$ such that

$$\left\|x_0 + t_nx_1 + t_n^2v_n - y_n\right\| < \mu t_n^2||w_n|| + t_n^3,$$

which implies that

$$\left\|\frac{y_n - x_0 - t_nx_1}{t_n^2} - v_n\right\| < \mu||w_n|| + t_n.$$

Taking $n \to +\infty$, then $\hat{v}_n := (y_n - x_0 - t_nx_1)/t_n^2 \to x$, *i.e.*, we get $x \in D^2S(p_0, x_0, p_1, x_1)(p)$. ☐

The following definition is necessary for our next result.

Definition 4.4. Let $F : X \to 2^Y$ and $(x_0, y_0) \in \text{gr}F$. The mth-order Studniarski derivative of F is defined by, for $x \in X$,

$$D_S^m F(x_0, y_0)(u) := \{v \in Y | \exists t_n \to 0^+, \exists (u_n, v_n) \to (u, v), y_0 + t_n^m v_n \in F(x_0 + t_n u_n)\}.$$

The reader is referred to [1,2,4,33,34] for more properties and applications of this derivative. In the paper, we employ the second-order Studniarski derivative to obtain the following proposition.

Proposition 4.5. *Let X be finite dimensional, $(p_0, x_0) \in \text{gr}G$ and $(p_1, x_1) \in \text{gr}DG(p_0, x_0)$. Suppose that $\text{gr}F$ is closed, $\text{gr}K$ is compact, and for each $y_0 \in \Omega(0)$, where $\Omega(0) := \{y \in K(p_0) | 0 \in F(x_0, y, p_0)\}$,*

$$D_S^2 K(p_0, y_0)(0) = \{0\}. \tag{4.2}$$

Then, for $(p, x) \in P \times X$, one gets

$$D^2 G(p_0, x_0, 0, p_1, x_1, 0)(p, x) \subseteq \bigcup_{y_0 \in \Omega(0)} \bigcup_{y \in D^2 K(p_0, y_0, p_1, 0)(p)} \big(D^2 F(x_0, y_0, p_0, 0, x_1, 0, p_1, 0)(x, y, p) \cup \{0\}\big). \tag{4.3}$$

If, additionally, F has the second-order proto-contingent derivative at (x_0, y_0, p_0) and the map $g : (X \times X \times P \times Y) \times (P \times X) \to \mathbb{R}_+$ defined by $g((\beta_1, \gamma_1, \rho_1, \upsilon_1), (\rho_2, \gamma_2)) := ||\rho_1 - \rho_2|| + ||\gamma_1 - \gamma_2||$ is directionally metric subregular of order 2 at $((x_0, y_0, p_0, 0, p_0, y_0), 0)$ in the direction $(x, y, p, v, \hat{p}, \hat{y})$ with respect to $\text{gr}\, F \times \text{gr}\, K$ and $(x_1, 0, p_1, 0, p_1, 0)$, for all $(x, y, p, v) \in \text{gr}D^2 F(x_0, y_0, p_0, 0, x_1, 0, p_1, 0)$ and $(\hat{p}, \hat{y}) \in \text{gr}D^2 K(p_0, y_0, p_1, 0)$, then (4.3) becomes an equality.

Proof. Let $v \in D^2 G(p_0, x_0, p_1, x_1)(p, x)$. If $v = 0$, it is trivial. We assume that $v \neq 0$, then there exist $t_n \to 0^+$, $(p_n, x_n, v_n) \to (p, x, v)$ such that

$$0 + t_n.0 + t_n^2 v_n \in G(p_0 + t_n p_1 + t_n^2 p_n, x_0 + t_n x_1 + t_n^2 x_n).$$

By the definition of G, there exists $y_n \in K(p_0 + t_n p_1 + t_n^2 p_n)$ such that

$$t_n^2 v_n \in F(x_0 + t_n x_1 + t_n^2 x_n, y_n, p_0 + t_n p_1 + t_n^2 p_n),$$

Since $\text{gr}K$ is compact, $\{y_n\}$ has a subsequence converging to $y_0 \in K(p_0)$. It follows from the closeness of $\text{gr}F$ that $0 \in F(x_0, y_0, p_0)$, *i.e.*, $y_0 \in \Omega(0)$.

We now prove that $\{(y_n - y_0)/t_n^2\}$ is bounded. Suppose to the contrary, *i.e.*, $||y_n - y_0||/t_n^2 \to +\infty$, then one has

$$\begin{aligned} y_0 + ||y_n - y_0|| \frac{y_n - y_0}{||y_n - y_0||} &= y_n \in K\left(p_0 + \sqrt{||y_n - y_0||} \frac{t_n}{\sqrt{||y_n - y_0||}} p_1 + ||y_n - y_0|| \frac{t_n^2}{||y_n - y_0||} p_n\right) \\ &= K\left(p_0 + \sqrt{||y_n - y_0||}\left(\frac{t_n}{\sqrt{||y_n - y_0||}} p_1 + \sqrt{||y_n - y_0||} \frac{t_n^2}{||y_n - y_0||} p_n\right)\right). \end{aligned}$$

Since X is finite dimensional, we assume that $(y_n - y_0)/||y_n - y_0||$ has a subsequence converging to $\hat{y}$ with $||\hat{y}|| = 1$. It is easy to see that

$$\frac{t_n}{\sqrt{||y_n - y_0||}} p_1 + \sqrt{||y_n - y_0||} \frac{t_n^2}{||y_n - y_0||} p_n \to 0,$$

so with $s_n := \sqrt{||y_n - y_0||} \to 0^+$, we obtain $\hat{y} \in D_S^2 K(p_0, y_0)(0)$, which contradicts (4.2). Thus, $\{(y_n - y_0)/t_n^2\}$ is bounded.

By setting $\overline{y}_n := (y_n - y_0)/t_n^2$, without loss of generality, we suppose that $\overline{y}_n \to \overline{y}$. Consequently, one gets

$$y_0 + t_n.0 + t_n^2\overline{y}_n \in K(p_0 + t_np_1 + t_n^2p_n)$$

and

$$t_n^2v_n \in F(x_0 + t_nx_1 + t_n^2x_n, y_0 + t_n.0 + t_n^2\overline{y}_n, p_0 + t_np_1 + t_n^2p_n),$$

which implies that $v \in D^2F(x_0, y_0, p_0, 0, x_1, 0, p_1, 0)(x, \overline{y}, p)$ and $\overline{y} \in D^2K(p_0, y_0, p_1, 0)(p)$.

For the converse of (4.3), let $v \in \bigcup_{y_0\in\Omega(0)} \bigcup_{y\in D^2K(p_0,y_0,p_1,0)(p)} \big(D^2F(x_0, y_0, p_0, 0, x_1, 0, p_1, 0)(x, y, p) \cup \{0\}\big)$. If $v = 0$, then by the definition of G, for any $t_n \to 0^+$, $(p_n, x_n) \to (p, x)$, we have

$$0 \in G(p_0 + t_np_1 + t_n^2p_n, x_0 + t_nx_1 + t_n^2x_n),$$

i.e., $0 \in D^2G(p_0, x_0, 0, p_1, x_1, 0)$. If $v \neq 0$, there exist $y_0 \in \Omega(0)$ and $y \in D^2K(p_0, y_0, p_1, 0)(p)$ such that $v \in D^2F(x_0, y_0, p_0, 0, x_1, 0, p_1, 0)(x, y, p)$. Thus, there are $t_n \to 0^+$, $(p_n, y_n) \to (p, y)$ such that

$$y_0 + t_n.0 + t_n^2y_n \in K(p_0 + t_np_1 + t_n^2p_n).$$

Since F has the second-order proto-contingent derivative, with t_n above, there exist $(\hat{x}_n, \hat{y}_n, \hat{p}_n, \hat{v}_n) \to (x, y, p, v)$ such that

$$t_n^2\hat{v}_n \in F(x_0 + t_nx_1 + t_n^2\hat{x_n}, y_0 + t_n.0 + t_n^2\hat{y}_n, p_0 + t_np_1 + t_n^2\hat{p}_n).$$

From the directionally metric subregularity of g and the proof similar to that of Theorem 3.11, there are $\overline{y}_n \to y$, $\overline{p}_n \to p$ with

$$y_0 + t_n.0 + t_n^2\overline{y}_n \in K(p_0 + t_np_1 + t_n^2\overline{p}_n)$$

and

$$t_n^2\hat{v}_n \in F(x_0 + t_nx_1 + t_n^2\hat{x_n}, y_0 + t_n.0 + t_n^2\overline{y}_n, p_0 + t_np_1 + t_n^2\overline{p}_n),$$

which implies that

$$t_n^2\hat{v} \in G(p_0 + t_np_1 + t_n^2\overline{p}_n, x_0 + t_nx_1 + t_n^2\hat{x_n}),$$

i.e., $v \in D^2G(p_0, x_0, 0, p_1, x_1, 0)(p, x)$. □

From Theorem 3.11, Propositions 4.3 and 4.5, we obtain sensitivity analysis for (PEP) as follows.

Theorem 4.6. *Let* $(p_0, x_0) \in \mathrm{gr}S$ *and* $(p_1, x_1) \in \mathrm{gr}DS(p_0, x_0)$. *Suppose that all conditions of Theorem 3.11 and Propositions 4.3, 4.5 are fulfilled for* (PEP). *Then, for* $p \in P$,

$$D^2S(p_0, x_0, p_1, x_1)(p) = \{x \in D^2K(p_0, x_0, p_1, x_1)(p)|$$
$$D^2F(x_0, y_0, p_0, 0, x_1, 0, p_1, 0)(x, y, p) \cap (-\mathrm{int}C) = \emptyset, \forall y_0 \in \Omega(0), \forall y \in D^2K(p_0, y_0, p_1, 0)(p)\}.$$

Theorem 4.6 can be considered as an extension of Theorem 3.1 in [19] from smooth cases with the first order to set-valued cases with the second order.

5. Conclusions

In the paper, we study a topic related to equilibrium problems. More precisely, we establish second-order sensitivity analysis for set-valued parametric equilibrium problems. Several examples are given to illustrate our results.

For possible developments of this paper, since several theoretical models in optimization can be expressed as special cases of equilibrium problems, such as constrained set-valued optimization problems, cone saddle point problems, variational inequalities (see [18]), we can learn about applications of the obtained results in the paper to these particular cases. Furthermore, finding answers for open questions in Remark 3.12 may be a promising study.

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