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ANALYSIS OF STATE-DEPENDENT DISCRETE-TIME QUEUE WITH SYSTEM DISASTER

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Abstract. An explicit expression for time-dependent system size probabilities is obtained for the general state-dependent discrete-time queue with system disaster. Using generating function for the nth state transient probabilities, the underlying difference equation of system size probabilities are transformed into three-term recurrence relation which is then expressed as a continued fraction. The continued fractions are converted into formal power series which yield the time-dependent system size probabilities in closed form. Further, the busy period distribution is obtained for the considered model. As a special case, the system size probabilities and busy period distribution of Geo/Geo/1 queue are deduced. Finally, numerical illustrations are presented to visualize the system effect for various values of the parameters.

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1. Introduction

Since the last decade, considerable effort has been devoted to the study of continuous time queueing models more than the effort given to study their discrete time counterparts. Of late, the analysis of discrete time queueing models is gaining much attention in view of their applicability in the study of practical problems that arise in Airports traffic, power station, telecommunication systems, wireless sensor networks and computer networks, etc [27]. Its application areas include performance evaluation and the design of buffers for statistical multiplexors, traffic concentrators, switching modules and networks. However, the discrete-time models are more suitable for designing diverse productive processes than their continuous-time counterparts since the basic units in these systems are digital which corresponds with the machine cycle time. Discrete-time queueing approach is an effective scheme for reducing the power consumption of sensor nodes in wireless sensor networks (WSNs). The importance of discrete-time queue in WSN was extensively studied by Lee and Yang [23].

Recently there has been a rapid increase in the literature on queueing models with system disaster. Disaster is a kind of negative arrival and if a disaster enters a queueing system, it forces a customer who is in service to leave the system (ie., a customer in service and also in queue) and it has no effect on an empty system (see [33, 34]). System disaster can also be considered as a type of clearing mechanism which removes all workload

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in the system. Arrival of disaster destroys all the unfinished work and also breaks down the server. Disaster is also referred to as catastrophe [2, 3, 6, 8, 11, 20–22], mass exodus [9], queue flushing [28] and stochastic clearing systems [32].

Queues with negative customers, also called G-queues were first introduced by Gelenbe [14]. Negative arrivals have been interpreted as inhibiter and synchronization signals in neural and high speed communication network. Although many continuous-time queueing systems with system disaster were discussed in the past years, [4, 10, 16, 18, 25] their discrete-time counterparts received very little attention in the literature. Wang *et al.* [31] considered a single server discrete-time queue with correlated positive and negative customers arrival. Yechiali [34] discussed an M/M/1 queue in a multiphase random environment, where the system occasionally suffers a disastrous breakdown, causing all present jobs to be lost. Walraevens *et al.* [29] analyzed the transient delay in a discrete time FIFO buffer with batch arrival and they also calculated time dependent limiting steady state delay measures. Crescenzo *et al.* [12] considered a birth and death process with catastrophe and discussed the first occurrence of an "effective catastrophe" that occurs when the state of the system is not 0. In the recent years many authors have discussed queueing system with system disaster/catastrophe, see [7, 30] and references therein.

The model under consideration has broad applications in many fields. For example, consider a power station where generators supply power to different types of consumers based on their requirement. If there is an occurrence of a fault, it may lead to tripping of power to all the loads thereby leading to a massive blackout. The issue can be resolved only after rectification of the fault.

Another example: Everyday hundreds of thousands of transactions (passengers & packages) are fly all over the world for business and/or to maintain personal tie with their family and friends. The number of total passengers in an airport is measured in persons. This count includes any passenger who arrives at, departs from or is on a transit from that airport. In addition, total cargo handled is expressed in metric tons which include all the freight and mail that arrives at or departs from the airport. The number of total aircraft movements is measured in airplane-times and it includes the entire take-offs and landings of all kinds of aircraft in scheduled or charter conditions. Let us consider the complete shutdown of an airport due to a natural disaster. The arrival and departure of passengers or the passengers on transit and the metric tons of cargo at the airport will have an adverse impact. All activities in the airport become dormant until the airport recovers to its original status. The occurrence of this type of disasters is a kind of negative customer that breaks down the system completely. For a survey and comprehensive references for state-dependent queue with negative customer or disaster (see [1, 5, 19, 35]).

A classic real time example is the cancellation of flights from and to Chennai, India on 02nd December 2015 to 06th December 2015 due to torrential rains. Chennai had never witnessed before such a heavy downpour in almost 100 years of time. Airports authority of India could not operate the flights since the runway was completely inundated. The under carriage of aircraft was almost touching the water level. This impacted on an average of 320 landings and departures per day. This incident has become the motivating factor of this study.

In this paper, the time-dependent system size probabilities are obtained for the general state-dependent discrete-time queue in presence of system disaster in terms of the absence disaster. Moreover, the busy period distribution of discrete-time state-dependent queue with system disaster, and special cases such as transient probabilities, busy period distribution of single server discrete-time queue are deduced. Finally, numerical illustrations are presented to visualize the effect of various parameters on the transient system size probabilities, busy period distribution and also some performance measures.

2. Model description

Consider a state-dependent discrete-time queue with system disaster, where the time axis is divided into slots of equal length. All queueing activities can occur only at the slot boundaries. Let the time axis be marked by values $0, 1, 2, \ldots, m, \ldots$. Consider the time epoch m and assume that the arrivals and the disaster occur in

(m, m^+) and the departures occur in (m, m^-) where early arrival system (EAS) policy [17] is considered for the queueing system.

Let $\lambda_n,\ n = 0, 1, \ldots,$ and $\mu_n,\ n = 1, 2, \ldots,$ denote the arrival and service rates respectively, when the system is in state n. Assume that the arrivals (birth) occur according to a Bernoulli process with parameter $\lambda_n,\ n \geq 0$ and service completions (death) occur according to a geometric distribution with parameter $\mu_n,\ n \geq 0$ where $\mu_{n+1}^- = \mu_n$. It is also assumed that the probability of more than one arrival and/or departure during a given time slot is zero and the events in different slots are independent. When the system is not empty, disaster occurs according to a geometric distribution with rate ξ. A disaster event would make the system instantly empty, wherein all the customers in the system are lost. Simultaneously, the system becomes ready to accept new customers. Let

$$P_n(m) = \mathrm{P}(X_m = n | X_0 = 0),\ n = 0, 1, 2, \ldots, \tag{2.1}$$

with the initial condition $P_0(0) = 1$.

Equation (2.1) represents the conditional probability that there are n customers in the mth time epoch (m) where the system is empty at the initial time epoch. The following governing equations are

$$\begin{aligned} P_0(m+1) &= (1-\lambda_0)P_0(m) + (1-\lambda_1)\mu_1(1-\xi)P_1(m) + \lambda_0\mu_0(1-\xi)P_0(m) \\ &\quad + \lambda_0\xi P_0(m) + \xi(1 - P_0(m)) \end{aligned} \tag{2.2}$$

$$\begin{aligned} P_n(m+1) &= \lambda_{n-1}(1-\mu_{n-1})(1-\xi)P_{n-1}(m) + (1-\lambda_{n+1})\mu_{n+1}(1-\xi)P_{n+1}(m) \\ &\quad + [(1-\lambda_n)(1-\mu_n)(1-\xi) + \lambda_n\mu_n(1-\xi)]P_n(m),\ n = 1, 2, \ldots,\ m = 0, 1, 2, \ldots. \end{aligned} \tag{2.3}$$

with initial condition $P_n(0) = \delta_{0,n}$, the Kronecker delta.

Assume that $a_{2n} = \lambda_n(1-\mu_n)(1-\xi)$ and $a_{2n+1} = (1-\lambda_{n+1})\mu_{n+1}(1-\xi),\ n \geq 0$ and the reason for the notation may be known in the sequel.

Denote $\{\widehat{X}_m : m \geq 0\}$ be the state dependent discrete-time queue obtained from X_m by removing the possibility of disaster (*i.e.*, when $\xi = 0$).

Then the probabilities $P(\widehat{X}_m = n | \widehat{X}_0 = 0) = \widehat{P}_n(m),\ n = 0, 1, 2, \ldots,$ satisfy the system of difference equations obtained from (2.1) by setting $\xi = 0$.

If $G_z(n) = \displaystyle\sum_{m=0}^{\infty} P_n(m) z^m,\ |z| < 1$

then the above system of equations (2.2) and (2.3) can be reduced as follows:

$$\left[\frac{1}{z} - (1 - a_0 - \xi)\right] G_z(0) - a_1 G_z(1) = \frac{1}{z} + \frac{\xi}{1-z},$$

$$\left[\frac{1}{z} - (1 - \xi - a_{2n-1} - a_{2n})\right] G_z(n) = a_{2n-2} G_z(n-1) + a_{2n+1} G_z(n+1).$$

Let the transformation $s = \frac{1}{z} - 1$ and denoting $G_z(n)$ by $G_s(n)$, then above system reduces to

$$(s + a_0 + \xi)\, G_s(0) = (s+1)\left[1 + \frac{\xi}{s}\right] + a_1 G_s(1), \tag{2.4}$$

$$(s + \xi + a_{2n-1} + a_{2n})\, G_s(n) = a_{2n-2} G_s(n-1) + a_{2n+1} G_s(n+1),\ n \geq 1. \tag{2.5}$$

From (2.4) and (2.5), we obtain

$$G_s(0) = \frac{(s+1)\left(1 + \frac{\xi}{s}\right)}{s + a_0 + \xi - a_1\left[\frac{G_s(1)}{G_s(0)}\right]}, \tag{2.6}$$

and

$$\frac{G_s(n)}{G_s(n-1)} = \frac{a_{2n-2}}{s+\xi+a_{2n-1}+a_{2n}-a_{2n+1}\frac{G_s(n+1)}{G_s(n)}},\ n\geq 1.$$

The iteration of above equation turnout the following continued fraction:

$$\frac{G_s(n)}{G_s(n-1)} = \frac{a_{2n-2}}{s+\xi+a_{2n-1}+a_{2n}-\frac{a_{2n}a_{2n+1}}{s+\xi+a_{2n+1}+a_{2n+2}-\frac{a_{2n+2}a_{2n+3}}{s+\xi+a_{2n+3}+a_{2n+4}-}}}\ \cdots. \tag{2.7}$$

On iterating the above equation (2.7) along with (2.6) gives a Jacobi fraction (J-fraction) as follows:

$$G_s(0) = \frac{(s+1)\left(1+\frac{\xi}{s}\right)}{s+a_0+\xi-\frac{a_0a_1}{s+\xi+a_1+a_2-\frac{a_2a_3}{s+\xi+a_3+a_4-}}}\cdots. \tag{2.8}$$

In the following section, the time-dependent system size probabilities of general state-dependent discrete-time queue with system disaster in closed form are obtained by solving the system of difference equations (2.2) and (2.3). For the general models, the solutions are more complicated or in the implicit form, but the probabilities in explicit form are obtained from the infinite power series of continued fractions. These results are useful for the numerical evaluation of probabilities and other performance measures.

3. Transient analysis

The continued fraction given in (2.7) and (2.8) can be expressed as a formal power series in order to find the system size probabilities $P_n(m),\ n,m=0,1,2,\ldots$.

Theorem 3.1. *Let the expression given in (2.7) can be expanded as a formal power series of the form*

$$\frac{G_s(n)}{G_s(n-1)} = a_{2n-2}\sum_{r=0}^{\infty}(-1)^r B(r,n)\frac{1}{(s+\xi)^{r+1}}, \tag{3.1}$$

where

$$B(r,n) = \sum_{i_1=2n-1}^{2n} a_{i_1}\sum_{i_2=2n-1}^{i_1+1} a_{i_2}\ \ldots\sum_{i_r=2n-1}^{i_{r-1}+1} a_{i_r},\ \forall n\in N \tag{3.2}$$

with $B(0,n)=1$.

Proof. This proof is similar to its counterpart proof of Parthasarathy and Sudhesh [24]. ☐

Using Theorem 3.1, the transient probability of state-dependent discrete queue in presence of disaster when the system is empty is derived *i.e.*, $P_n(m)$ is expressed in terms of $\widehat{P_n}(m)$.

Theorem 3.2. *If* $G_s(0)=(s+1)\left(1+\frac{\xi}{s}\right)\sum_{r=0}^{\infty}(-1)^r\frac{A(r,0)}{(s+\xi)^{r+1}}$,
then $A(0,0)=1$ and for $r=1,2,3,\ldots$,

$$A(r,0) = \sum_{i_1=0}^{0} a_{i_1}\sum_{i_2=0}^{i_1+1} a_{i_2}\sum_{i_3=0}^{i_2+1} a_{i_3}\ \ldots\sum_{i_r=0}^{i_{r-1}+1} a_{i_r}. \tag{3.3}$$

Further for m = 0,1,2,. . . ,

$$P_0(m) = \widehat{P_0}(m)+\sum_{r=0}^{m-1}(-1)^r A(r,0)\sum_{j=0}^{m-(r+1)}(-1)^{j+1}\binom{r+j}{j+1}\xi^{j+1}\binom{r+j+2}{m-(r+j+1)}, \tag{3.4}$$

where $\widehat{P_0}(m) = \sum_{r=0}^{m}(-1)^r\binom{m}{r}A(r,0)$ *is a time-dependent probability in the absence of disaster.*

Proof. Assuming the power series expression of $G_s(0)$ as

$$G_s(0) = (s+1)\left(1+\frac{\xi}{s}\right)\sum_{r=0}^{\infty}(-1)^r\frac{A(r,0)}{(s+\xi)^{r+1}}\cdot \tag{3.5}$$

Using Theorem 2 of Parthasarathy and Sudhesh [24] and after some simple algebraic manipulation, we obtain

$$G_z(0) = \widehat{G_z}(0) + \sum_{r=0}^{\infty}(-1)^r A(r,0)\sum_{j=0}^{\infty}(-1)^{j+1}\binom{r+j}{j+1}\xi^{j+1}z^{r+j+1}(1-z)^{-(r+j+2)}.$$

The coefficient of z^m of the above equation gives the probability $P_0(m)$ which is given in (3.4). □

The following theorem gives expressions for $P_n(m),\ m=1,2,3,\ldots,\ n=0,1,2,\ldots$

Theorem 3.3. *For $n=0,1,2,\ldots,$ we have*

$$P_n(m) = \begin{cases} \widehat{P_n}(m) + L_{n-1}\sum\limits_{r=0}^{m-(n+1)}(-1)^r A(r,2n)\sum\limits_{j=0}^{m-(n+r+1)}(-1)^{j+1}\binom{r+j+n}{j+1}\xi^{j+1}\binom{r+j+n+2}{m-(r+j+n+1)} & \text{if } m>n\\ L_{n-1} & \text{if } m=n\\ 0 & \text{if } m<n, \end{cases} \tag{3.6}$$

where

$$\widehat{P_n}(m) = L_{n-1}\sum_{r=0}^{m-n}(-1)^r A(r,2n)\binom{m}{n+r},\ \text{if } m>n$$

and

$$A(r,n) = \sum_{i_1=0}^{n}a_{i_1}\sum_{i_2=0}^{i_1+1}a_{i_2}\sum_{i_3=0}^{i_2+1}a_{i_3}\ \ldots\sum_{i_r=0}^{i_{r-1}+1}a_{i_r},\ \forall n\in N, \tag{3.7}$$

and $L_{n-1} = a_0a_2a_4\ldots a_{2n-2},\ L_{-1}=1$ *with* $A(0,n)=1,\ \sum_{r=0}^{-n}(.)=0$ *and* $\binom{m}{-1}=0.$

Proof. Using Theorems 3.1 and 3.2, it is understood that the power series of $G_s(n)$ is as follows:

$$G_s(n) = L_{n-1}(s+1)\left(1+\frac{\xi}{s}\right)\sum_{r=0}^{\infty}(-1)^r\frac{A(r,2n)}{(s+\xi)^{r+n+1}} \tag{3.8}$$

satisfies equation (2.5).

Using Theorem 3 of Parthasarathy and Sudhesh [24] and after considerable simplifications,

$$G_z(n) = \widehat{G_z}(n) + L_{n-1}\sum_{r=0}^{\infty}(-1)^r A(r,2n)\sum_{j=0}^{\infty}(-1)^{j+1}\binom{r+n+j}{j+1}\xi^{j+1}\frac{z^{r+n+j+1}}{(1-z)^{r+n+j+2}}, \tag{3.9}$$

where

$$\widehat{G_z}(n) = L_{n-1}\sum_{r=0}^{\infty}(-1)^r A(r,2n)\sum_{i=0}^{\infty}\binom{n+r+i}{i}z^{n+r+i}.$$

The coefficients of z^m in (3.9) gives the probability $P_n(m)$ given in (3.6). □

3.1. Special case: Probability $P_m(n)$ of Geo/Geo/1 queue

Consider a single server discrete-time queue (Geo/Geo/1 queue) with system disaster in which the arrival and service rates are constants with respect to states, *i.e.*, rates are independent of states. The inter-arrival and service times follow geometrical distribution with parameters $\lambda\ (= a_{2n})$ and $\mu\ (= a_{2n+1}), n = 0, 1, 2, \ldots,$ respectively.

The power series coefficient $A(r, 0)$ is given by

$$A(r,0) = \frac{1}{\mu}\left(\frac{\lambda+\mu}{2}\right)^{r+1}\sum_{i=0}^{\left[\frac{r+1}{2}\right]}\frac{(-1)^i}{2(r-i)+1}\binom{2(r-i)+1}{r-i}\binom{r-i+1}{i}\left(\frac{\lambda-\mu}{\lambda+\mu}\right)^{2i}$$

and $A(0, 0) = 1$. The details of the proof for $A(r, 0)$ are given in Parthasarathy and Sudhesh [24].

Using (3.4), the probability $P_0(m)$ can be written as

$$P_0(m) = \sum_{r=0}^{m}(-1)^r\binom{m}{r}A(r,0) + \sum_{r=0}^{m-1}(-1)^r A(r,0)\sum_{j=0}^{m-(r+1)}(-1)^{j+1}\binom{r+j}{j+1}\xi^{j+1}\binom{r+j+2}{m-(r+j+1)}.$$

Thus, for $m = 0, 1, 2, \ldots,$

$$\begin{aligned}P_0(m) = 1 &+ \frac{1}{\mu}\sum_{r=1}^{m}(-1)^r\binom{m}{r}\left(\frac{\lambda+\mu}{2}\right)^{r+1}\sum_{i=0}^{\left[\frac{r+1}{2}\right]}\frac{(-1)^i}{2(r-i)+1}\binom{2(r-i)+1}{r-i}\binom{r-i+1}{i}\left(\frac{\lambda-\mu}{\lambda+\mu}\right)^{2i}\\ &+ \frac{1}{\mu}\sum_{r=1}^{m-1}(-1)^r\left(\frac{\lambda+\mu}{2}\right)^{r+1}\sum_{i=0}^{\left[\frac{r+1}{2}\right]}\frac{(-1)^i}{2(r-i)+1}\binom{2(r-i)+1}{r-i}\binom{r-i+1}{i}\left(\frac{\lambda-\mu}{\lambda+\mu}\right)^{2i}\\ &\times\sum_{j=0}^{m-(r+1)}(-1)^{j+1}\binom{r+j}{j+1}\xi^{j+1}\binom{r+j+2}{m-(r+j+1)}.\end{aligned} \tag{3.10}$$

Also, the continued fraction representation for $\frac{G_z(n)}{G_z(n-1)}$, $n = 1, 2, \ldots,$ is given by

$$\frac{G_z(n)}{G_z(n-1)} = \frac{\lambda}{s+\lambda+\mu+\xi-\frac{\lambda\mu}{s+\lambda+\mu+\xi-\frac{\lambda\mu}{s+\lambda+\mu+\xi-\ \cdots}}}.$$

It is well known that the above equation can be written as

$$\frac{G_z(n)}{G_z(n-1)} = \frac{\lambda}{s+\lambda+\mu+\xi}\sum_{r=0}^{\infty}C_r\left[\frac{\lambda\mu}{(s+\lambda+\mu+\xi)^2}\right]^r.$$

where

$$C_r = \frac{1}{r+1}\binom{2r}{r}, r\in Z^+,\ \text{a positive integer.}$$

is known as catalon number with $C_0 = 1$.

Replace s by $(1-z)/z$ in the above equation, then for n=1,2,..., we have

$$\frac{G_z(n)}{G_z(n-1)} = \frac{\lambda z}{1-(1-\lambda-\mu-\xi)z}\sum_{r=0}^{\infty}C_r\left[\frac{\lambda\mu z^2}{(1-(1-\lambda-\mu-\xi)z)^2}\right]^r.$$

Therefore,

$$G_z(n) = G_z(0)\left[\frac{\lambda z}{1-(1-\lambda-\mu-\xi)z}\right]^n \left[\sum_{r=0}^{\infty} C_r \left[\frac{\lambda\mu z^2}{(1-(1-\lambda-\mu-\xi)z)^2}\right]^r\right]^n.$$

It is well known that

$$\left[\sum_{k=0}^{\infty} C_r x^k\right]^n = \sum_{k=0}^{\infty} b(n,k)x^k,$$

where $b(n,k) = \dfrac{n}{2k+n}\dbinom{2k+n}{k}$ is the ballot number with $b(n,0)=1, \forall n \geq 1$ (see [15]).

Using the above result, we find

$$G_z(n) = \lambda^n G_z(0)\sum_{r=0}^{\infty} z^{r+n}\sum_{k=0}^{[r/2]} b(n,k)\binom{n+r-1}{r-2k}(\lambda\mu)^k(1-\lambda-\mu-\xi)^{r-2k}. \tag{3.11}$$

Substituting $G_z(0) = \sum_{m=0}^{\infty} z^m P_0(m)$ and comparing the coefficients of z^m on (3.11), for $m = n, n+1, n+2, \ldots$, gives

$$P_n(m) = \lambda^n \sum_{i=0}^{m-n}\sum_{k=0}^{[i/2]} b(n,k)\binom{n+i-1}{i-2k}(\lambda\mu)^k(1-\lambda-\mu-\xi)^{i-2k}P_0(m-n-i),$$

where $P_0(m-n-i)$ is found in (3.10).

In the following section, the busy period analysis for the state dependent queue with system disaster is discussed.

4. Busy period analysis of $\mathrm{Geo}_n/\mathrm{Geo}_n/1$ queue

Busy period analysis forms an integral part of the study of any queuing system and has been carried out for every queuing system discussed in the literature. The busy period analysis plays a vital role in the server point of view and is also essential for the efficient planning of the system resources. Let $T_{1,0}$ be the random variable denoting busy period starting with 1 job initially in the system. Assume that the arrival and service time follow Bernoulli process with parameter $\lambda_n, n = 0,1,2,\ldots$ and geometric distribution with parameter $\mu_n,\ n = 1,2,3,\ldots$, respectively. Then $q_n(m) = P(X_m = n,\ X_k \neq 0, k = 1,2,\ldots,m|X_0 = 1)$, where X_m denotes number of jobs at the mth time epoch.
Since 0 is an absorbing state and $(a_0 = 0)$, then the busy period distribution $q_0(m) = P(T_{1,0} \leq m)$ satisfies

$$q_0(m+1) = q_0(m) + (1-\lambda_1)\mu_1(1-\xi)q_1(m) + \xi(1-q_0(m)) \tag{4.1}$$

$$q_1(m+1) = [(1-\lambda_1)(1-\mu_1)(1-\xi) + \lambda_1\mu_1(1-\xi)]\, q_1(m) + (1-\lambda_2)\mu_2(1-\xi)q_2(m) \tag{4.2}$$

$$\begin{aligned} q_n(m+1) = {} & [(1-\lambda_n)(1-\mu_n)(1-\xi) + \lambda_n\mu_n(1-\xi)]\, q_n(m) \\ & +\lambda_{n-1}(1-\mu_{n-1})(1-\xi)q_{n-1}(m) + (1-\lambda_{n+1})\mu_{n+1}(1-\xi)q_{n+1}(m), \end{aligned} \tag{4.3}$$

with $q_1(0) = 1$.

Let us define the generating function

$$Q_z(n) = \sum_{m=0}^{\infty} q_n(m)z^m,\ n = 1,2,3,\ldots.$$

From (4.1) and (4.2), we get

$$Q_s(0) = (s+1)\frac{\xi}{s(s+\xi)} + \frac{a_1}{s+\xi}Q_s(1) \tag{4.4}$$

$$Q_s(1) = \frac{s+1}{s+\xi+a_1+a_2-a_3\frac{Q_s(2)}{Q_s(1)}}$$

and from (4.3), we arrive

$$\frac{Q_s(n)}{Q_s(n-1)} = \frac{a_{2n-2}}{s+\xi+a_{2n-1}+a_{2n}-a_{2n+1}\frac{Q_s(n+1)}{Q_s(n)}},\ n \geq 2.$$

Therefore,

$$Q_s(1) = \cfrac{s+1}{s+\xi+a_1+a_2-\cfrac{a_2a_3}{s+\xi+a_3+a_4-\cdots}}.$$

Using Theorem 3.1, we conclude that

$$Q_s(1) = (s+1)\sum_{r=0}^{\infty}(-1)^r\frac{B(r,1)}{(s+\xi)^{r+1}}.$$

Replace s by $\frac{1}{z}-1$ in the above equation, we get

$$Q_z(1) = \sum_{r=0}^{\infty}(-1)^rB(r,1)\sum_{k=0}^{\infty}\binom{r+k}{k}(1-\xi)^kz^{r+k}.$$

Equating the coefficients of z^m on both sides, we obtain

$$q_1(m) = \sum_{r=0}^{\infty}(-1)^rB(r,1)\binom{m}{m-r}(1-\xi)^{m-r}. \tag{4.5}$$

By using (4.4), we have

$$Q_z(0) = \frac{\xi z}{(1-z)[1-(1-\xi)z]} + a_1\sum_{r=0}^{\infty}(-1)^rB(r,1)\sum_{k=0}^{\infty}\binom{r+k+1}{k}(1-\xi)^kz^{r+k+1}.$$

The coefficients of z^m of the above equation yield

$$q_0(m) = \xi\sum_{i=0}^{m-1}(1-\xi)^{m-1-i} + a_1\sum_{r=0}^{m-1}(-1)^rB(r,1)\binom{m}{m-r-1}(1-\xi)^{m-r-1}. \tag{4.6}$$

The probability mass function of the busy period is

$$f_{m+1}(0) = P\left[T_{1,0} = m+1\right] = q_0(m+1) - q_0(m).$$

From (4.1),

$$f_{m+1}(0) = \xi\left(1-q_0(m)\right) + a_1q_1(m), \tag{4.7}$$

where $q_1(m)$ and $q_0(m)$ are expressed in (4.5) and (4.6).

4.1. Special case: Busy period distribution of Geo/Geo/1 queue

In this section, a busy period distribution of Geo/Geo/1 queue is obtained for the constant arrival and service rates by assuming $a_{2n} = \lambda$ and $a_{2n-1} = \mu$, $n = 1, 2, 3, \ldots$.
Then (4.2) and (4.3) yield,

$$Q_z(1) = \cfrac{1/z}{\frac{1}{z} - 1 + \xi + \lambda + \mu - \cfrac{\lambda\mu}{\frac{1}{z}-1+\xi+\lambda+\mu - \cfrac{\lambda\mu}{\frac{1}{z}-1+\xi+\lambda+\mu-\cdots}}}$$
$$= \frac{1}{z}\sum_{r=0}^{\infty} \frac{1}{r+1}\binom{2r}{r}\frac{(\lambda\mu)^r}{\left(\frac{1}{z} - 1 + \xi + \lambda + \mu\right)^{2r+1}}$$
$$Q_z(0) = \frac{\xi z}{(1-z)(1-(1-\xi)z)} + \frac{\mu}{(1-(1-\xi)z)}$$
$$\times \sum_{r=0}^{\infty}\left(\frac{1}{r+1}\right)\binom{2r}{r}\frac{(\lambda\mu)^r z^{2r+1}}{(1-(1-\lambda-\mu-\xi)z)^{2r+1}}.$$

The probability mass function of the busy period is given by

$$f_{m+1}(0) = P\left[T_{1,0} = m+1\right] = q_0(m+1) - q_0(m).$$

From (4.1),

$$f_{m+1}(0) = \xi\left(1 - q_0(m)\right) + \mu q_1(m), \tag{4.8}$$

where $q_1(m)$ and $q_0(m)$ are the coefficients of z^m in (4.5) and (4.6), respectively.

Thus,

$$q_0(m) = \xi\sum_{i=0}^{m-1}(1-\xi)^{m-1-i} + \sum_{r=0}^{\left[\frac{m-1}{2}\right]}\left(\frac{1}{r+1}\right)\binom{2r}{r}\lambda^r\mu^{r+1}$$
$$\times \sum_{j=0}^{m-(2r+1)}\binom{2r+j}{j}(1-\lambda-\mu-\xi)^j(1-\xi)^{m-(2r+1)-j}$$

and

$$q_1(m) = \sum_{r=0}^{\left[\frac{m}{2}\right]}\left(\frac{1}{r+1}\right)\binom{2r}{r}\binom{m}{2r}(\lambda\mu)^r(1-\lambda-\mu-\xi)^{m-2r}.$$

A substitution of the above equations in (4.8) gives a probability mass function of busy period which agrees with Sudhesh *et al.* [26].

Remark 4.1. When $\xi = 0$, the busy period distribution in the absence of disaster agrees with equation (39) of Parthasarathy and Sudhesh [24].

Remark 4.2. From the results (3.2) and (3.7), one can easily prove that $B(r, 1) = A(r, 2)$ when busy period occurs (*i.e.*, when $a_0 = \lambda_0 = 0$).

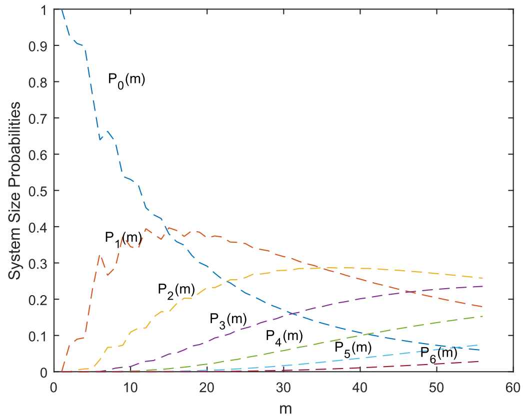

FIGURE 1. Probability curves for infinite model for the rates given in (5.1), while the disaster rate becomes non zero.

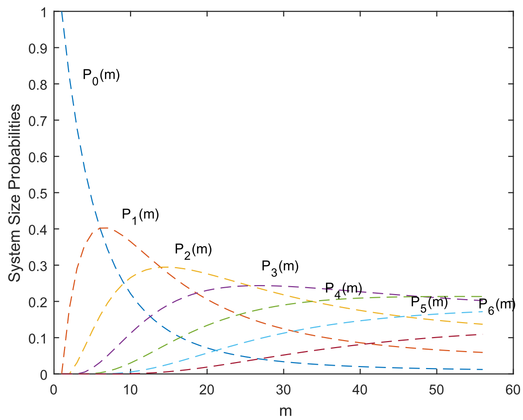

FIGURE 2. Probability curves for infinite model for the rates given in (5.1), while the disaster rate becomes zero

5. NUMERICAL ILLUSTRATIONS

In this section, numerical results of the system size probabilities, expected system size, variance and busy period distribution for various state-dependent arrival and service rates $\lambda_n,\ \mu_n$ and disaster rate ξ are discussed.

Figure 1, shows the plot for the system size probabilities $P_n(m)$, $(n = 0, 1, \dots)$ *vs.* time(m) for infinite capacity state-dependent queue in the presence of disaster with the arrival and service rates

$$\lambda_n = \begin{cases} \lambda & \text{if } n \leq s-1 \\ \left(\frac{s}{n+1}\right)^b \lambda & \text{if } n \geq s-1 \end{cases} \text{ and } \mu_n = \begin{cases} n\mu & \text{if } n \leq s \\ s\mu & \text{if } n \geq s \end{cases} \tag{5.1}$$

by assuming $\lambda = 0.18$, $\mu = 0.04$, $\xi = 0.6$, $s = 10$ and $b = 0.5$. It is assumed that initially the system is empty. It is observed that all the probability curves increase initially except the curve $P_0(m)$ and they tend to decrease gradually up to some time interval. Further, the probability distribution attains steady state as time increases. For clarity, only few probability curves are plotted.

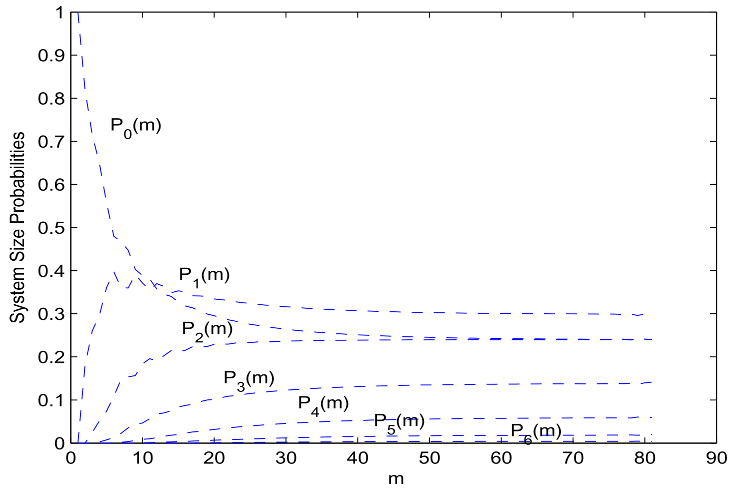

FIGURE 3. System size probability for different values of n with $\lambda = 0.03, \mu = 0.04, \xi = 0.09$ and $N = 11$.

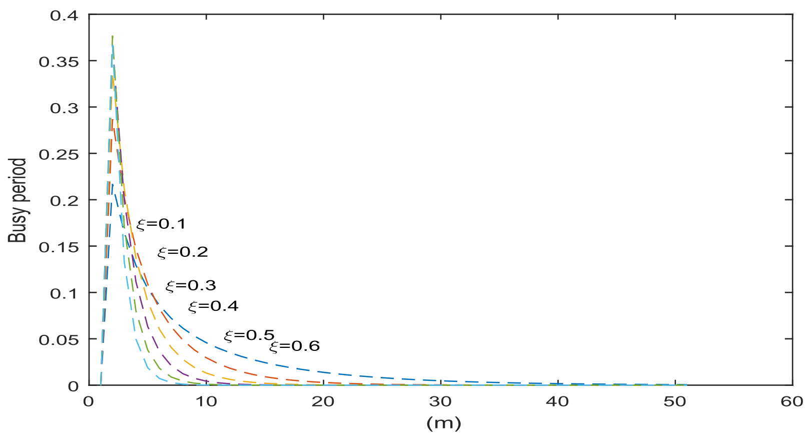

FIGURE 4. Busy period distribution for different values of ξ with $\lambda = 0.14, \ \mu = 0.18, b = 0.5$.

In our numerical study, we have confirmed that the disaster rate becomes zero ($i.e.\ \xi = 0$); then the probability curves are smooth, stable and looks similar to the example considered by Parthasarathy and Sudhesh [24] shown in Figure 2. Whereas the probability curves are fluctuating in the beginning time period, while the disaster rate becomes non-zero.

In Figure 3, the state-dependent probabilities are drawn for the following transition rates

$$\lambda_n = \frac{(N-n)(n+1)}{2(2n+1)}(\lambda+\mu) \text{ and } \mu_n = \frac{n(N+n+1)}{2(2n+1)}(\lambda+\mu),$$

where $\lambda = 0.03, \mu = 0.04, \ \xi = 0.09$, and $N = 11$. In this figure, curves are increasing (except $P_0(m)$) with increase in time during the initial time period, followed by a downward trend towards steady state.

The model considered with the non – linear rates mentioned above can represent a finite queueing system where new customers are discouraged from joining long queues. Also the customers in the system cooperate in

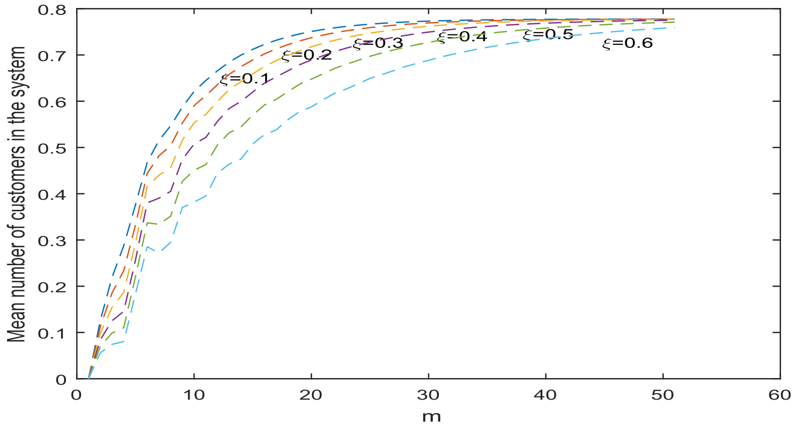

FIGURE 5. Mean system size probability for different values of ξ with $\lambda = 0.14, \mu = 0.18, b = 0.5$.

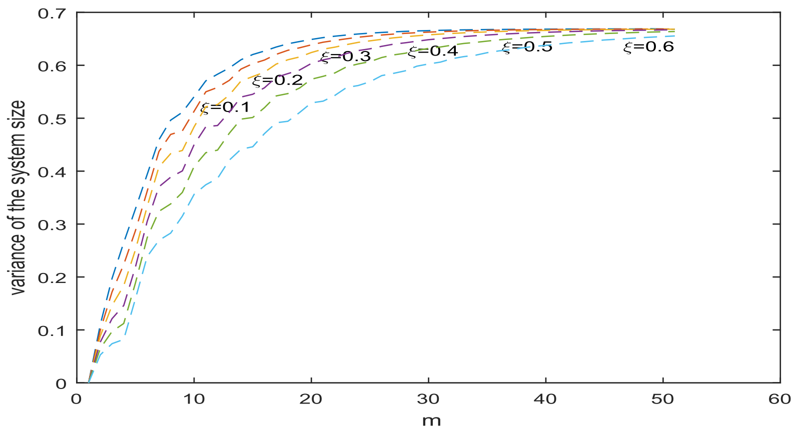

FIGURE 6. Variance of the system size for different values of ξ with $\lambda = 0.14, \mu = 0.18, b = 0.5$.

obtaining service, which results in a quadratic increase in the service rate. Such models find wide applications in computer networks and communication systems.

The busy-period probabilities $f_{m+1}(0)$ of Geo/Geo/1 queue for the transition rates are given in (5.1) with different disaster rates $\xi = 0.1, 0.2, 0.3, 0.4, 0.5$ and 0.6 are plotted in Figure 4.

Figures 5 and 6 depict the plot of the time-dependent expected system size and variance in discrete time slots for the transition rates given in (5.1) with varying values of ξ.

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