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FUZZY STOCHASTIC DATA ENVELOPMENT ANALYSIS WITH APPLICATION TO NATO ENLARGEMENT PROBLEM

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Abstract. Data Envelopment Analysis (DEA) is a widely used technique for measuring the relative efficiencies of Decision Making Units (DMUs) with multiple deterministic inputs and multiple outputs. However, in real-world problems, the observed values of the input and output data are often vague or random. Indeed, Decision Makers (DMs) may encounter a hybrid uncertain environment where fuzziness and randomness coexist in a problem. Hence, we formulate a new DEA model to deal with fuzzy stochastic DEA models. The contributions of the present study are fivefold: (1) We formulate a deterministic linear model according to the probability–possibility approach for solving input-oriented fuzzy stochastic DEA model, (2) In contrast to the existing approach, which is infeasible for some threshold values; the proposed approach is feasible for all threshold values, (3) We apply the cross-efficiency technique to increase the discrimination power of the proposed fuzzy stochastic DEA model and to rank the efficient DMUs, (4) We solve two numerical examples to illustrate the proposed approach and to describe the effects of threshold values on the efficiency results, and (5) We present a pilot study for the NATO enlargement problem to demonstrate the applicability of the proposed model.

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1. Introduction

Data Envelopment Analysis (DEA), initially introduced by Charnes *et al.* [3], is a well-known non-parametric methodology for computing the relative efficiency of a set of homogeneous units, named as Decision Making Units (DMU). DEA generalizes the intuitive single-input single-output ratio efficiency measurement into a multiple-input multiple-output model by using a ratio of the weighted sum of outputs to the weighted sum of inputs. It computes scalar efficiency scores with a range of zero to one that determine efficient level or position for each DMU under evaluation among all DMUs. A DMU is said to be efficient if its efficiency score is equal to one, otherwise it is said to be inefficient. Moreover, in real-life applications of DEA models the observed input and output data of the DMUs are often not known precisely. In such situations, being able to deal with vague and imprecise data may greatly contribute to the diffusion and application of DEA models. Two typical approaches namely probability-theoretic approach and fuzzy-theoretic approach can be used for such

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DEA models involving uncertainty. In what follows, an overview of such inexact optimization techniques were developed to tackle uncertainties in DEA models is provided.

To deal with imprecise data, the notions of fuzziness and randomness were introduced in DEA. Fuzzy sets can be used to represent ambiguous or imprecise information. On the other hand, the data can be obtained by statistics in some measurement errors and data entry errors characterized by random variables. However, in many practical situations, there is not a sufficient number of crisp statistic data. To handle such circumstances, a twofold uncertainty is needed. One approach to treat the uncertainty event is the measure based approach. Like the probability measure of a stochastic event, and the credibility measure of a fuzzy event based on a pair of dual fuzzy measures, *i.e.*, possibility and necessity. The same one in fuzzy random, the mean chance measure or simplified chance measure is presented. Exactly, that is the mean value, in the sense of probability of all the credibility values of the corresponding fuzzy events.

Hatami-Marbini *et al.* [11] classified the fuzzy DEA methods in the literature into five general groups, the tolerance approach [36, 42], the α-level based approach, the fuzzy ranking approach [10, 12], the possibility approach [20], and the fuzzy arithmetic approach [45]. Among these approaches, the α-level based approach is probably the most popular fuzzy DEA model in the literature. This approach generally tries to transform the FDEA model into a pair of parametric programs for each α-level. Kao and Liu [14], one of the most cited studies in the α-level approach's category, used Chen and Klein [4] method for ranking fuzzy numbers to convert the FDEA model to a pair of parametric mathematical programs for the given level of α. Saati *et al.* [34] proposed a fuzzy CCR model as a possibilistic programming problem and changed it into an interval programming problem by means of the α-level based approach. Parameshwaran *et al.* [29] proposed an integrated fuzzy analytic hierarchy process and DEA approach for the service performance measurement. Puri and Yadav [32] applied the suggested methodology by Saati *et al.* [34] to solve fuzzy DEA model with undesirable outputs. Shiraz *et al.* [15] proposed fuzzy free disposal hull models under possibility and credibility measures. Momeni *et al.* [25] used fuzzy DEA models to address the impreciseness and ambiguity associated with the input and output data in supply chain performance evaluation problems. Payan [31] evaluated the performance of DMUs with fuzzy data by using the common set of weights based on a linear program. Aghayi *et al.* [1] formulated a model to measure the efficiency of DMUs with interval inputs and ouputs based on common sets wieghts.

In order to evaluate the efficiency of DMUs with the deterministic inputs and the random outputs, Land *et al.* [19] extended the chance constrained DEA model. Olesen and Petersen [27] developed the chance constrained programming model for efficiency evaluation using a piecewise linear envelopment of confidence regions for observed stochastic multiple-input multiple-output combinations in DEA. Huang and Li [13] developed stochastic models in DEA by taking into account the possibility of random variations in input-output data. Cooper *et al.* [6], Li [21], and Bruni *et al.* [2] utilized joint chance constraints to extend the concept of stochastic efficiency. Cooper *et al.* [5] used chance-constrained programming for extending congestion DEA models. Tsionas and Papadakis [41] developed Bayesian inference techniques in chance-constrained DEA models. Udhayakumar *et al.* [44] used a genetic algorithm to solve the chance-constrained DEA models involving the concept of satisficing. Also some of the banking applications in relation to satisficing DEA can be found in Udhayakumar *et al.* [44] and Tsolas and Charles [43]. Farnoosh *et al.* [8] proposed chance-constrained FDH model with random input and random output. Wu *et al.* [47] proposed a stochastic DEA model by considering undesirable outputs with weak disposability. This model not only deals with the existence of random errors in the collected data, but also depicts the production rules uncovered by weak disposability of the undesirable outputs. Also, a comparison work between stochastic DEA and fuzzy DEA approaches have been introduced to evalutate the efficiency of Angolan banks by Wanke *et al.* [46]. A review of stochastic DEA models can be found in a recent work by Olesen and Petersen [28].

However, in the real-world problems decision makers may need to base decisions on information which are both fuzzily imprecise and probabilistically uncertain. Kwakernaak [17, 18] introduced the concept of fuzzy random variable, and then this idea enhanced by a number of researchers in the literature [9, 22, 24, 33]. Qin and Liu [33] developed a fuzzy random DEA (FRDEA) model where randomness and fuzziness exist simultaneously. The authors characterized the fuzzy random data with known possibility and probability distributions.

Tavana *et al.* [39] also introduced three different FDEA models consisting of probability–possibility, probability-necessity and probability–credibility constraints in which input and output data entailed fuzziness and randomness at the same time. Also, Tavana *et al.* [40] provided a chance-constrained DEA model with random fuzzy inputs and outputs with Poisson, uniform and normal distributions. After that, Tavana *et al.* [38] proposed DEA models with birandom input-output. Shiraz *et al.* [16] proposed fuzzy rough DEA models based on the expected value and possibility approaches. Paryab *et al.* [30] proposed DEA models using a bi-fuzzy data based possibility approach. However, there has been no attempt to study randomness and roughness simultaneously in DEA problems. Tavana *et al.* [37] also introduced a DEA model for problems characterized by random-rough variables. Nasseri *et al.* [26] proposed a new approach to consider the impact of undesirable outputs on the performance of DMUs in fuzzy stochastic environment. To deal with the uncertain environments, especially hybrid environments, the DEA model may disorder its structure when the uncertain parameter of input and output exist. For example, the method proposed by Tavana *et al.* [39] does not compute the efficiency scores of DMUs in the range of zero to one for input-oriented DEA models. Another shortcoming of this approach is the nonlinear (quadratic) form of the proposed DEA model. Hence, this study try to overcome the shortcomings of the existing approaches. To sum up with all the above aspects, the contributions of the present study are fivefold: (1) We formulate a deterministic linear model according to the probability- possibility approach for solving input-oriented fuzzy stochastic DEA model, (2) In contrast to the existing approach, which is infeasible for some threshold values; the proposed approach is feasible for all threshold values, (3) We apply the cross-efficiency technique to increase the discrimination power of the proposed fuzzy stochastic DEA model and to rank the efficient DMUs, (4) We solve two numerical examples to illustrate the proposed approach and to describe the effects of threshold values on the efficiency results, and (5) We present a pilot study for the NATO enlargement problem to demonstrate the applicability of the proposed model.

This paper is organized as follows: In the next section, some necessary concepts related to the fuzzy set theory and probability theory are reviewed. Section 3 presents our proposed CCR-DEA model. Section 4 gives the possibility–probablity approach based on chance constraint programming to solve the fuzzy stochastic DEA model. In Section 5, we solve two numerical examples to illustrate the proposed approach and to describe the effects of threshold values on the efficiency results. In Section 6, the results of the case conducted for the NATO enlargment problem to evaluate the efficiency of 18 countries are presented. Section 7 presents our conclusions and future research directions.

2. Preliminaries

In this section, we review some necessary concepts related to the fuzzy set theory and probability theory, which will be used in the rest of paper [7, 48, 49].

Definition 2.1. A fuzzy set $\tilde{A}$, defined on universal set X, is given by a set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))|x \in X\}$ where $\mu_{\tilde{A}}(x)$ gives the membership grade of the element x in the set $\tilde{A}$ and is called membership function.

Definition 2.2. A fuzzy set $\tilde{A}$, defined on universal set of real numbers R, is said to be a fuzzy number if its membership function has the following characteristics:

(1) $\tilde{A}$ is convex, *i.e.*, $\forall x, y \in R, \forall \lambda \in [0,1], \mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$.
(2) $\tilde{A}$ is normal, *i.e.*, $\exists \bar{x} \in R; \mu_{\tilde{A}}(\bar{x}) = 1$.
(3) $\mu_{\tilde{A}}$ is piecewise continuous.

Definition 2.3. A function $L : [0, \infty) \to [0, 1]$ (or $R : [0, \infty) \to [0, 1]$) is said to be reference function of fuzzy number if and only if $L(0) = 1$(or $R(0) = 1$) and L or R is non-increasing on $[0, \infty)$.

Definition 2.4 (Dubois and Prade [7]). A fuzzy number $\tilde{A} = (m, \alpha, \beta)_{LR}$ is said to be an LR fuzzy number, if its membership function is given by:

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\frac{m-x}{\alpha}), & \text{for} \quad x \leq m, \alpha > 0, \\ 1, & \text{for} \quad x = m, \\ R(\frac{x-n}{\beta}), & \text{for} \quad x \geq n, \beta > 0. \end{cases}$$

Remark 2.5. If $L(x) = R(x) = \max\{0, 1 - x\}$ then an LR fuzzy number $\tilde{A} = (m, \alpha, \beta)_{LR}$ is said to be a triangular fuzzy number and is denoted by $\tilde{A} = (m, \alpha, \beta)$.

Definition 2.6. Let $\tilde{A} = (m, \alpha, \beta)_{LR}$ be an LR fuzzy number and λ be a real number in the interval $[0, 1]$ then the crisp set, $\tilde{A}_\lambda = \{x \in R : \mu_{\tilde{A}}(x) \geq \lambda\} = [m - \alpha L^{-1}(\lambda), m + \beta R^{-1}(\lambda)]$ is said to be λ-cut of $\tilde{A}$.

Definition 2.7. Let $\tilde{A}_1 = (m_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{A}_2 = (m_2, \alpha_2, \beta_2)_{LR}$ be two LR fuzzy numbers and k be a non-zero real number. Then the exact formula for the extended addition and the scalar multiplication are defined as follows:

(i) $(m_1, \alpha_1, \beta_1)_{LR} + (m_2, \alpha_2, \beta_2)_{LR} = (m_1 + m_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)_{LR}$
(ii) $k > 0, k(m_1, \alpha_1, \beta_1)_{LR} = (km_1, k\alpha_1, k\beta_1)_{LR}$
(iii) $k < 0, k(m_1, \alpha_1, \beta_1)_{LR} = (km_1, -k\beta_1, -k\alpha_1)_{LR}$.

Definition 2.8 (Extension Principle). This principle allows the generalization of crisp mathematical concepts in fuzzy frameworks. For any function f, mapping points in set X to points in set Y, and any fuzzy set $A \in \tilde{P}(X)$ where $A = \mu_1(x_1) + \mu_2(x_2) + \ldots + \mu_n(x_n)$, this principle expresses $f(A) = f(\mu_1(x_1) + \mu_2(x_2) + \ldots + \mu_n(x_n)) = f(\mu_1(x_1)) + f(\mu_2(x_2)) + \ldots + f(\mu_n(x_n))$.

Definition 2.9. Let $(\Theta, P(\Theta), \text{Pos})$ be a possibility space where Θ is a non-empty set involving all possible events, and $P(\Theta)$ is the power set of Θ. For every $A \in P(\Theta)$, there is a non-negative number $\text{Pos}(A)$, so-called a possibility measure, satisfying the following axioms:

(i) $P(\emptyset) = 0, \quad P(\Theta) = 1,$
(ii) for every $A, B \in P(\Theta)$, $A \subseteq B$ implies $\text{Pos}(A) \leq \text{Pos}(B)$,
(iii) for every subset $\{A_w : w \in W\} \subseteq P(\Theta)$, $\text{Pos}(\cup_w A_w) = \text{Sup}_w \text{Pos}(A_w)$.

The elements of $P(\Theta)$ are also called *fuzzy events*.

Definition 2.10 (Liu and Liu [23]). Let ξ be a fuzzy variable on a possibility space $(\Theta, P(\Theta), \text{Pos})$. The possibility of the fuzzy event $\{\xi \geq r\}$, where r is any real number, is defined $\text{Pos}(\xi \geq r) = \underset{t \geq r}{\text{Sup}}\, \mu_\xi(t)$, where $\mu_\xi : \Re \to [0, 1]$ is the membership function of ξ.

Definition 2.11 (Liu and Liu [24]). Let (Ω, A, Pr) be a probability space where Ω is a sample space, A is the s-algebra of subsets of Ω (*i.e.*, the set of all possible potentially interesting events), and is a probability measure on Ω. A fuzzy random variable (FRV) is a function ξ from a probability space (Ω, A, Pr) to the set of fuzzy variables such that for every Borel set B of $\Re$, $\text{Pos}\{\xi(w), w \in B\}$ is a measurable function of ω.

Definition 2.12 (Liu and Liu [24]). A fuzzy random vector is a map from a sample space to a colleation of fuzzy vectors, $\xi = (\xi_1, \xi_2, \ldots, \xi_n) : \Omega \to F_v^n$, such that for any closed subset $F \in \Re^n$, $\text{Pos}\{\gamma \mid \xi(\omega, \gamma) \in F\}$ is a measurable fuction of $\omega \in \Omega$, *i.e.*, for any $t \in [0, 1]$, we have $\{\omega \in \Omega \,|\, \text{Pos}\{\gamma \,|\xi(\omega, \gamma) \in F\} \leq t\} \in A$. In the case of $n = 1$, ξ is called a fuzzy random variable.

Definition 2.13 (Fuzzy Random Arithmetic). Let ξ_1 and ξ_2 be two FRVs with the probability spaces $(\Omega_1, A_1, \text{Pr}_1)$ and $(\Omega_2, A_2, \text{Pr}_2)$, respectively. Then $\xi = \xi_2 + \xi_2$ is defined as $\xi(\omega_1, \omega_2) = \xi_1(\omega_1) + \xi_2(\omega_2)$ for any $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$, where $(\Omega_1 \times \Omega_2, A_1 \times A_2, \text{Pr}_1 \times \text{Pr}_2)$ is the corresponding probability space.

Definition 2.14. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ be a fuzzy random vector, and $f : \Re^n \to \Re$ be a continuous function. Then $f(\xi)$ will be a fuzzy random variable.

Definition 2.15. An LR fuzzy random variable will be denoted by $\xi(\omega)$, where $\omega \in \Omega$ and described by the following membership function:

$$\mu_{\xi(\omega)}(x) = \begin{cases} L\left(\frac{m(\omega)-x}{\alpha}\right), & x \leq m(\omega), \\ 1, & x = m(\omega), \\ R\left(\frac{x-m(\omega)}{\beta}\right), & x \geq m(\omega). \end{cases}$$

where $m(\omega)$ is the normally distributed random variable.

3. Proposed DEA-CCR model

Consider a set of n DMUs, where DMU_j has a production plan (x_j, y_j) and consumes m inputs $x_j = (x_{1j}, x_{2j}, \ldots, x_{mj})$ to produce s outputs $y_j = (y_{1j}, y_{2j}, \ldots, y_{sj})$. The technical efficiency of a given DMU_k under a constant return to scale (CRS) technology can be obtained by using the following problem, so-called input-oriented CCR primal model:

$$\begin{aligned} E_k = \text{Max} & \sum_{r=1}^{s} u_r y_{rk} \\ \text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{ik} = 1, \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{rj} \leq 0, j = 1, 2, \ldots, n, \\ & u_r \geq 0, v_i \geq 0, r = 1, 2, \ldots, s; i = 1, 2, \ldots, m. \end{aligned} \tag{3.1}$$

where u_r and v_i are the weights associated with the rth output and the ith input, respectively. The DMU_k is (technically) efficient if $E_k = 1$, otherwise if $0 < E_k < 1$, it is (technically) inefficient.

Substituting $\hat{x}_{ip} = v_i x_{ij}$ and $\hat{y}_{rp} = u_r y_{rj}$ into model (3.1), the following equivalent model is obtained:

$$\begin{aligned} & \theta_p^* = \max \sum_{r=1}^{s} \hat{y}_{rp} \\ & \text{s.t.} \\ & \left. \begin{aligned} & \sum_{i=1}^{m} \hat{x}_{ip} = 1, \\ & \sum_{r=1}^{s} \hat{y}_{rj} - \sum_{i=1}^{m} \hat{x}_{ij} \leq 0, \ \forall j \end{aligned} \right\} \quad (i) \\ & u_r y_{rj} \leq \hat{y}_{rj} \leq u_r y_{rj} \forall r, j \qquad (ii) \\ & v_i x_{ij} \leq \hat{x}_{ij} \leq v_i x_{ij} \forall i, j \qquad (iii) \\ & u_r, v_i \geq 0 \forall r, i \end{aligned} \tag{3.2}$$

This model allows us to provide a linear model in terms of uncertain data unlike the proposed model by Tavana *et al.* [39].

Remark 3.1. It is worthwhile to note that Saati *et al.* [34] used new variable substitutions for evaluating the fuzzy efficiency of DMUS with fuzzy data. Saati *et al.* [34] have used the variable substitutions on the interval DEA model derived from the fuzzy DEA model based on the concept of alpha cuts. That approach converts the non-linear programs into a linear one. We have used a similar variable substitution on a primary DEA model. In what follows, we extend the converted DEA model to a fuzzy stochastic environment. This approach not only leads to a linear model but also remains the basic properties of DEA models.

4. Fuzzy stochastic DEA-CCR model: a probability–possibility approach

The aim of this section is to propose a DEA-based method for evaluating the efficiencies of DMUs with fuzzy stochastic inputs and fuzzy stochastic outputs. To this end, consider n DMUs, each of which consumes m fuzzy stochastic inputs, denoted by $\tilde{\tilde{x}}_{ij} = \left(x_{ij}^m, x_{ij}^\alpha, x_{ij}^\beta\right)_{LR}, i = 1, \ldots, m, j = 1, \ldots, n$, and produces s fuzzy stochastic outputs, denoted by $\tilde{\tilde{y}}_{rj} = \left(y_{rj}^m, y_{rj}^\alpha, y_{rj}^\beta\right)_{LR}, r = 1, \ldots, s, j = 1, \ldots, n$. Let x_{ij}^m and y_{rj}^m, denoted by $x_{ij}^m \sim N\left(x_{ij}, \sigma_{ij}^2\right)$ and $y_{rj}^m \sim N\left(y_{rj}, \sigma_{rj}^2\right)$ be normally distributed. Therefore, $x_{ij}(y_{rj})$ and $\sigma_{ij}^2(\sigma_{rj}^2)$ are the mean and the variance of x_{ij}^m (y_{rj}^m) for DMU$_j$, respectively.

The chance-constrained programming (CCP) developed by Cooper *et al.* [5] is a stochastic optimization approach suitable for solving optimization problems with uncertain parameters. Building on CCP and possibility theory as the principal techniques, the following probability–possibility CCR model is proposed:

$$
\begin{aligned}
&\max \varphi\\
&\text{s.t.}\\
&\left.\begin{array}{l}
\varphi \le \sum_{r=1}^{s} \hat{y}_{rj}\\
\sum_{i=1}^{m} \hat{x}_{ij} = 1\\
\sum_{r=1}^{s} \hat{y}_{rj} - \sum_{i=1}^{m} \hat{x}_{ij} \le 0 \ \forall \text{j}
\end{array}\right\} \qquad (i)\\
&\Pr\left[\operatorname{Pos}\left(u_r y_{rj} \le \hat{y}_{rj} \le u_r y_{rj}\right) \ge \delta\right] \ge \gamma, \ \forall r, \text{j}, \qquad (ii)\\
&\Pr\left[\operatorname{Pos}\left(v_i x_{ij} \le \hat{x}_{ij} \le v_i x_{ij}\right) \ge \delta\right] \ge \gamma, \ \forall i, j, \qquad (iii)\\
&u_r, v_i \ge 0.
\end{aligned} \tag{4.1}
$$

where δ and $\gamma \in [0,1]$ in constraint (ii) and (iii) are the predetermined thresholds defined by the DM. Pos$[\cdot]$ and Pr$[\cdot]$ in model (4.1) denote the possibility and the probability of $[\cdot]$ event.

In addition, we presume that the fuzzy stochastic input $\tilde{\tilde{x}}_{ij}$ and the fuzzy stochastic output $\tilde{\tilde{y}}_{rj}$ are characterized, respectively, by the following two membership functions:

$$
\mu_{\tilde{\tilde{x}}_{ij}}(t) = \begin{cases} L\left(\frac{x_{ij}^m - t}{x_{ij}^\beta}\right), t \le x_{ij}^m,\\ R\left(\frac{t - x_{ij}^m}{x_{ij}^\beta}\right), t \ge x_{ij}^m. \end{cases} \tag{4.2}
$$

and

$$
\mu_{\tilde{\tilde{y}}_{rj}}(t) = \begin{cases} L\left(\frac{y_{rj}^m - t}{y_{rj}^\beta}\right), t \le y_{rj}^m,\\ R\left(\frac{t - y_{rj}^m}{y_{rj}^\beta}\right), t \ge y_{rj}^m. \end{cases} \tag{4.3}
$$

where $x_{ij}^m \sim N(x_{ij}, \sigma_{ij}^2)$ and $y_{rj}^m \sim N(y_{rj}, \sigma_{rj}^2)$.

In order to solve the probability–possibility constrained programming model (4.1), we convert the constraints in this model into their respective crisp equivalents. Thereby, Theorem 4.1 and Lemma 4.2 proposed, respectively, by Liu and Liu [24] and Sakawa [35] play a pivotal role in solving the fuzziness of proposed model (4.1).

Theorem 4.1. *Let ξ be a fuzzy random vector $g_j : \Re^n \to \Re$ are real-valued continuous functions $r = 1, \ldots, p$. Then the possibility* $\operatorname{Pos}\{g_j(\xi(w)) \le 0, j = 1, \ldots, n\}$ *is a random variable.*

Lemma 4.2. *Let $\bar{\lambda}_1$ and $\bar{\lambda}_2$ be two independent fuzzy numbers with continuous membership functions. For a given confidence level $\alpha \in [0,1]$,* $\operatorname{Pos}\left\{\bar{\lambda}_1 \ge \bar{\lambda}_2\right\} \ge \alpha$ *if and only if $\lambda_{1,\alpha}^R \ge \lambda_{2,\alpha}^R$, where $\lambda_{1,\alpha}^L$, $\lambda_{1,\alpha}^R$ and $\lambda_{2,\alpha}^L$, $\lambda_{2,\alpha}^R$ are the left and the right side extreme points of the α-level sets $\bar{\lambda}_1$ and $\bar{\lambda}_2$, respectively, and* $\operatorname{Pos}\{\bar{\lambda}_1 \ge \bar{\lambda}_2\}$ *present the degree of possibility.*

In what follows we show that the probability–possibility CCR model (4.1) can be equivalently transformed into a linear programing model.

The constraint (ii) in model (4.1), $\Pr\left[\text{Pos}\left(u_r y_{rj} \le \hat{y}_{rj} \le u_r y_{rj}\right) \ge \delta\right] \ge \gamma$, can be transformed into the following two constraints:

$$\Pr\left[\text{Pos}\left(u_r y_{rj} \le \hat{y}_{rj}\right) \ge \delta\right] \ge \gamma,$$
$$\Pr\left[\text{Pos}\left(\hat{y}_{rj} \le u_r y_{rj}\right) \ge \delta\right] \ge \gamma.$$

These constraints can be rewritten as the following constraints based on Lemma 4.2:

$$\Pr\left[\text{Pos}\left(y_{rj} \le \frac{\hat{y}_{rj}}{u_r}\right) \ge \delta\right] \ge \gamma \Leftrightarrow \Pr\left((y_{rj})^L_\delta \le \frac{\hat{y}_{rj}}{u_r}\right) \ge \gamma \Leftrightarrow \Pr\left(y^m_{rj} - L^{-1}(\delta) y^\alpha_{rj} \le \frac{\hat{y}_{rj}}{u_r}\right) \ge \gamma$$
$$\Pr\left[\text{Pos}\left(\frac{\hat{y}_{rj}}{u_r} \le y_{rj}\right) \ge \delta\right] \ge \gamma \Leftrightarrow \Pr\left(\frac{\hat{y}_{rj}}{u_r} \le (y_{rj})^R_\delta\right) \ge \gamma \Leftrightarrow \Pr\left(\frac{\hat{y}_{rj}}{u_r} \le y^m_{rj} + R^{-1}(\delta) y^\beta_{rj}\right) \ge \gamma.$$

In a similar way, constraint (iii) in model (4.1), $\Pr\left[\text{Pos}\left(v_i x_{ij} \le \hat{x}_{ij} \le v_i x_{ij}\right) \ge \delta\right] \ge \gamma$, can be rewritten as the following constraints:

$$\Pr\left[\text{Pos}\left(x_{ij} \le \frac{\hat{x}_{ij}}{v_i}\right) \ge \delta\right] \ge \gamma \Leftrightarrow \Pr\left((x_{ij})^L_\delta \le \frac{\hat{x}_{ij}}{v_i}\right) \ge \gamma \Leftrightarrow \Pr\left(x^m_{ij} - L^{-1}(\delta) x^\alpha_{rj} \le \frac{\hat{x}_{ij}}{v_i}\right) \ge \gamma$$
$$\Pr\left[\text{Pos}\left(\frac{\hat{x}_{ij}}{v_i} \le x_{ij}\right) \ge \delta\right] \ge \gamma \Leftrightarrow \Pr\left(\frac{\hat{x}_{ij}}{v_i} \le (x_{ij})^R_\delta\right) \ge \gamma \Leftrightarrow \Pr\left(\frac{\hat{x}_{ij}}{v_i} \le x^m_{ij} + R^{-1}(\delta) x^\beta_{ij}\right) \ge \gamma.$$

Therefore, model (4.1) can be reformulated as follows:

$$\begin{aligned}
&\max \varphi\\
&\text{s.t.}\\
&\varphi \le \sum_{r=1}^{s} \hat{y}_{rj}\\
&\sum_{i=1}^{m} \hat{x}_{ij} = 1\\
&\sum_{r=1}^{s} \hat{y}_{rj} - \sum_{i=1}^{m} \hat{x}_{ij} \le 0 \ \forall \text{j}\\
&\Pr\left(\tfrac{\hat{y}_{rj}}{u_r} \le y^m_{rj} + R^{-1}(\delta) y^\beta_{rj}\right) \ge \gamma, \forall r, j \qquad (i)\\
&\Pr\left(y^m_{rj} - L^{-1}(\delta) y^\alpha_{rj} \le \tfrac{\hat{y}_{rj}}{u_r}\right) \ge \gamma, \forall r, j \qquad (ii)\\
&\Pr\left(\tfrac{\hat{x}_{ij}}{v_i} \le x^m_{ij} + R^{-1}(\delta) x^\beta_{ij}\right) \ge \gamma, \forall i, j \qquad (iii)\\
&\Pr\left(x^m_{ij} - L^{-1}(\delta) x^\alpha_{rj} \le \tfrac{\hat{x}_{ij}}{v_i}\right) \ge \gamma, \ \forall i, j \qquad (iv)\\
&u_r, v_i \ge 0.
\end{aligned} \tag{4.4}$$

By the help of standardized normal distribution, (see, *e.g.*, [5]), model (4.4) can be transformed into a deterministic linear programming model. Consequently, let us consider constraint (i) in model (4.4) as $\Pr\left(\tilde{h} \ge 0\right) \ge \gamma$ where $\tilde{h} = y^m_{rj} + R^{-1}(\delta) y^\beta_{rj} - \frac{\hat{y}_{rj}}{u_r}$. Due to the normal distribution of y^m_{rp} , $\tilde{h}$ also has normal distribution with the following mean and variance:

$$E(\tilde{h}) = E\left[y^m_{rj} + R^{-1}(\delta) y^\beta_{rj} - \frac{\hat{y}_{rj}}{u_r}\right] = y_{rj} + R^{-1}(\delta) y^\beta_{rj} - \frac{\hat{y}_{rj}}{u_r}$$
$$\text{Var}(\tilde{h}) = \text{Var}\left(y^m_{rj} + R^{-1}(\delta) y^\beta_{rj} - \frac{\hat{y}_{rj}}{u_r}\right) = \text{Var}\left(y^m_{rj}\right) = \sigma^2_{rj}.$$

By standardizing the normal distribution, $\Pr\left(\tilde{h} \geq 0\right) \geq \gamma$ is converted to $\Pr\left(z \geq \frac{-E(\tilde{h})}{\sqrt{\operatorname{var}(\tilde{h})}}\right) \geq \gamma$ where $z = \frac{h-E(\tilde{h})}{\sqrt{\operatorname{var}(\tilde{h})}}$ is the standard normal random variable with zero mean and unit variance. The corresponding cumulative distribution function is $\Phi\left(\frac{-E(\tilde{h})}{\sqrt{\operatorname{Var}(\tilde{h})}}\right) \leq 1-\gamma$ and it is equal to $\frac{\hat{y}_{rj}}{u_r} - y_{rj} - R^{-1}(\delta)y_{rj}^{\beta} \leq \sigma_{rj}\Phi_{1-\gamma}^{-1}$, where $\Phi_{1-\gamma}^{-1}$ is the inverse of Φ at the level of $1-\gamma$. Finally, the deterministic version of constraint (i) in model (4.4) will be as follows:

$$\hat{y}_{rj} \leq u_r(y_{rj} + R^{-1}(\delta)y_{rj}^{\beta} + \sigma_{rj}\Phi_{1-\gamma}^{-1}), \forall r, j.$$

A similar procedure adopted for constraints (ii), (iii) and (iv) in model (4.4) results in the following constraints:

$$(ii) : u_r(y_{rj} - L^{-1}(\delta)y_{rj}^{\alpha} - \sigma_{rj}\Phi_{1-\gamma}^{-1}) \leq \hat{y}_{rj}, \forall r, j$$

$$(iii) : \hat{x}_{ij} \leq v_i(x_{ij} + R^{-1}(\delta)x_{ij}^{\beta} + \sigma_{ij}\Phi_{1-\gamma}^{-1}), \forall i, j$$

$$(iv) : v_i(x_{ij} - L^{-1}(\delta)x_{ij}^{\alpha} - \sigma_{ij}\Phi_{1-\gamma}^{-1}) \leq \hat{x}_{ij}, \forall i, j.$$

As a consequence, the deterministic equivalent for model (4.1) can be set as follows:

$$\begin{aligned}
&E_k(\delta,\gamma) = \max\ \varphi \\
&\text{s.t.} \\
&\varphi \leq \sum_{r=1}^{s} \hat{y}_{rj} \\
&\sum_{i=1}^{m} \hat{x}_{ij} = 1 \\
&\sum_{r=1}^{s} \hat{y}_{rj} - \sum_{i=1}^{m} \hat{x}_{ij} \leq 0\ \forall j \\
&u_r(y_{rj} - L^{-1}(\delta)y_{rj}^{\alpha} - \sigma_{rj}\Phi_{1-\gamma}^{-1}) \leq \hat{y}_{rj} \leq u_r(y_{rj} + R^{-1}(\delta)y_{rj}^{\beta} + \sigma_{rj}\Phi_{1-\gamma}^{-1}), \forall r, j \\
&v_i(x_{ij} - L^{-1}(\delta)x_{ij}^{\alpha} - \sigma_{ij}\Phi_{1-\gamma}^{-1}) \leq \hat{x}_{ij} \leq v_i(x_{ij} + R^{-1}(\delta)x_{ij}^{\beta} + \sigma_{ij}\Phi_{1-\gamma}^{-1}), \forall i, j \\
&u_r, v_i \geq 0.
\end{aligned} \tag{4.5}$$

The above model is obviously a linear program. It should be noted that the deterministic model obtained by Tavana *et al.* [39] is a non-linear program.

The following theorem shows that the objective function of model (4.5), $E_k(\delta,\gamma)$, is monotonously decreasing related to the each of $\delta$and γ level.

Theorem 4.3. *If $E_k(\delta,\gamma)$ is the optimum objective function value of model (4.5) then $E_k(\delta_1,\gamma) \geq E_k(\delta_2,\gamma) and E_k(\delta,\gamma_1) \geq E_k(\delta,\gamma_2)$ where $\delta_1 \leq \delta_2 and \gamma_1 \leq \gamma_2$.*

Proof. Denote the feasible space of model (4.5) by $S_{\delta,\gamma}$. We need to prove that $S_{\delta_2,\gamma_2} \subseteq S_{\delta_1,\gamma_1}$. To this, consider the following constraint of model (4.5)

$$v_i(x_{ij} - L^{-1}(\delta)x_{ij}^{\alpha} - \sigma_{ij}\Phi_{1-\gamma}^{-1}) \leq \hat{x}_{ij} \leq v_i(x_{ij} + R^{-1}(\delta)x_{ij}^{\beta} + \sigma_{ij}\Phi_{1-\gamma}^{-1}). \tag{4.6}$$

Let $\Phi^{-1}(\gamma) = \Phi_{\gamma}^{-1}$. As $\Phi^{-1}(1-\gamma)$, $L^{-1}(\delta)$ and $R^{-1}(\delta)$ are decreasing function, the functions $-\Phi^{-1}(1-\gamma), -L^{-1}(\delta)$ and $-R^{-1}(\delta)$ will be increasing. It is concluded that

$$\begin{aligned}
&\left[x_{ij} - L^{-1}(\delta_2)x_{ij}^{\alpha} - \sigma_{ij}\Phi^{-1}(1-\gamma_2), x_{ij} + R^{-1}(\delta_2)x_{ij}^{\beta} + \sigma_{ij}\Phi^{-1}(1-\gamma_2)\right] \subseteq \\
&\qquad \left[x_{ij} - L^{-1}(\delta_1)x_{ij}^{\alpha} - \sigma_{ij}\Phi^{-1}(1-\gamma_1), x_{ij} + R^{-1}(\delta_1)x_{ij}^{\beta} + \sigma_{ij}\Phi^{-1}(1-\gamma_1)\right].
\end{aligned}$$

In a similar way, we can conclud that

$$\Big[y_{rj} - L^{-1}(\delta_2)y^{\alpha}_{rj} - \sigma_{rj}\Phi^{-1}(1-\gamma_2), y_{rj} + R^{-1}(\delta_2)y^{\beta}_{rj} + \sigma_{rj}\Phi^{-1}(1-\gamma_2)\Big] \subseteq \Big[y_{rj} - L^{-1}(\delta_1)y^{\alpha}_{rj} - \sigma_{rj}\Phi^{-1}(1-\gamma_1), y_{rj} + R^{-1}(\delta_1)y^{\beta}_{rj} + \sigma_{rj}\Phi^{-1}(1-\gamma_1)\Big]$$

This completes the proof. □

We present the following defiition to define the efficiecy of each DMU.

Definition 4.4. For the given level $\delta$and γ, we define $\mathrm{E}^T_k(\delta,\gamma) = \mathrm{E}_k(\delta, \frac{\gamma}{2})$ as efficiency score of DMU_k in fuzzy random DEA model (4.5).

The corresponding model with $\mathrm{E}^T_k(\delta,\gamma)$ is as follows:

$$\begin{aligned}
&E^T_k(\delta,\gamma) = \max\ \varphi\\
&\text{s.t.}\\
&\varphi \le \sum_{r=1}^{s} \hat{y}_{rj}\\
&\sum_{i=1}^{m} \hat{x}_{ij} = 1\\
&\sum_{r=1}^{s} \hat{y}_{rj} - \sum_{i=1}^{m} \hat{x}_{ij} \le 0\ \forall j\\
&u_r(y_{rj} - L^{-1}(\delta)y^{\alpha}_{rj} - \sigma_{rj}\Phi^{-1}_{1-\frac{\gamma}{2}}) \le \hat{y}_{rj} \le u_r(y_{rj} + R^{-1}(\delta)y^{\beta}_{rj} + \sigma_{rj}\Phi^{-1}_{1-\frac{\gamma}{2}}), \forall r, j\\
&v_i(x_{ij} - L^{-1}(\delta)x^{\alpha}_{ij} - \sigma_{ij}\Phi^{-1}_{1-\frac{\gamma}{2}}) \le \hat{x}_{ij} \le v_i(x_{ij} + R^{-1}(\delta)x^{\beta}_{ij} + \sigma_{ij}\Phi^{-1}_{1-\frac{\gamma}{2}}), \forall i, j\\
&u_r, v_i \ge 0.
\end{aligned} \tag{4.7}$$

Theorem 4.5. *Consider $E^{\mathrm{T}}_k(\delta,\gamma)$ as the optimum objective function value of model (4.7) for DMU_k, then*

(a) *$E^T_k(\delta_1,\gamma) \ge E^T_k(\delta_2,\gamma)$ and $E^T_k(\delta,\gamma_1) \ge E^T_k(\delta,\gamma_2)$ where $\delta_1 \le \delta_2$ and $\gamma_1 \le \gamma_2$.*
(b) *$0 < E^T_j(\delta,\gamma) \le 1,(j = 1,2,\dots,n)$. Also, theres exists at lease one $k \in \{1,2,\dots,n\}$ suhc that $E^T_k(\delta,\gamma) = 1$.*
(c) *model (4.7) is feasible for any δ and γ.*

Proof. (a) It is straightforward using Theorem 4.3 and Definition 4.4.

(b) Obviously, it is followed immediately from the first, second and third constraints of model (4.6) that $E^T_j(\delta,\gamma) \le 1$. In what follows, we introduce such DMU_k with $E^{\mathrm{T}}_k(\delta,\gamma) = 1$. According to part (a), $E^{\mathrm{T}}_k(\delta,\gamma)$ is decreasing with respect to both δ and γ threshold, and so $E^{\mathrm{T}}_k(\delta,\gamma) \ge E^{\mathrm{T}}_k(1,1)$. Let $\delta = 1$ and $\gamma = 1$, then $L^{-1}(1) = R^{-1}(1) = 0$ and $\Phi^{-1}(0.5) = 0$. Hence, we have $\hat{x}_{ij} = v_i x_{ij}$, $\hat{y}_{rj} = u_r y_{rj}$ in model (4.7). Therefore, the corresponding model with $E^{\mathrm{T}}_k(1,1)$ will be as follows:

$$\begin{aligned}
&E^{\mathrm{T}}_k(1,1) = \max \sum_{r=1}^{s} u_r y_{rp}\\
&\text{s.t.}\\
&\sum_{i=1}^{m} v_i x_{ip} = 1,\\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0,\ j = 1,\dots,n\\
&u_r, v_i \ge 0.
\end{aligned} \tag{4.8}$$

As seen the above model is same with the traditional CCR-DEA model given in (3.1). So, $E^{\mathrm{T}}_k(1,1)$ would be positve as the objective function value of a traditional CCR-DEA model and then $E^{\mathrm{T}}_k(\delta,\gamma) \ge E^{\mathrm{T}}_k(1,1) > 0$. On the other hand, for such DMU_k, $E^{\mathrm{T}}_k(1,1)$ would be equal to 1. Hence, the relation $1 \ge E^{\mathrm{T}}_k(\delta,\gamma) \ge E^{\mathrm{T}}_k(1,1) = 1$ completes the proof of part (b).

TABLE 1. The data for Example 5.1.

	A_1	A_2	A_3	A_4
I_1	(N(9,1),1)	(N(5,1),1)	(N(4,1),1)	(N(10,1),1)
I_2	N(8,1)	N(4,1)	N(4,1)	N(7,1)
O_1	N(2.5,1)	N(2,1)	N(5,1)	N(3.5,1)

(c) Denote the feasible space of model (4.7) by $S^T_{\delta,\gamma}$. According to the proof of Theorem 4.3, $S^T_{1,1} \subseteq S^T_{\delta,\gamma}$. Therefore, it is sufficient to show that the feasible space $S^T_{1,1}$ is nonempty. According to the proof of part (b), $E_k^{\mathrm{T}}(1,1)$ is given by model (4.8) and this model is alwasy feasible as the traditional CCR-DEA model. This completes the proof of part (c).

□

5. Numerical examples

In this section, two numerical examples are solved to illustrate the proposed approach and the obtained results are discussed. In the first example, we compare the efficiency results of the proposed method with those obtained from the approach proposed by Tavana *et al.* [39]. The second example describes the effects of threshold values on the efficiency results.

Example 5.1. Consider four DMUs; A_1, A_2, A_3 and A_4 that consume two inputs and produce one output. The input I_1 is a normally random variable fuzzified with symmetric triangular fuzzy values. The input I_2 and the output O_1 are random variables with normal distribution. The data is given in Table 1.

Regarding model (4.7), the following linear programming problem is solved to evaluate unit A_1:

$$\begin{aligned}
&E_k^T(\delta,\gamma) = \max\ \varphi\\
&\text{s.t.}\\
&\varphi \le \hat{y}_{11}\\
&\hat{x}_{11} + \hat{x}_{21} = 1\\
&\hat{y}_{11} - \hat{x}_{11} - \hat{x}_{21} \le 0, \hat{y}_{12} - \hat{x}_{12} - \hat{x}_{22} \le 0\\
&\hat{y}_{13} - \hat{x}_{13} - \hat{x}_{23} \le 0, \hat{y}_{14} - \hat{x}_{14} - \hat{x}_{24} \le 0\\
&u_r(y_{rj} - 0.25y^{\alpha}_{rj} - 0.32\sigma_{rj}^{)} \le \hat{y}_{rj} \le u_r(y_{rj} + 0.25y^{\beta}_{rj} + 0.32\sigma_{rj}^{)}, \forall r, j\\
&v_i(x_{ij} - 0.25x^{\alpha}_{ij} - 0.32\sigma_{ij}^{)} \le \hat{x}_{ij} \le v_i(x_{ij} + 0.25x^{\beta}_{ij} + 0.32\sigma_{ij}^{)}, \forall i, j\\
&u_r, v_i \ge 0.
\end{aligned} \tag{5.1}$$

In a similar way, according to Tavana *et al.*'s approach the following model is solved in order to evaluate the efficiency of unit A_1:

$$\begin{aligned}
&\max \bar{\varphi}\\
&\text{s.t.}\\
&\bar{\varphi} - u_1(2.5) + u_1(0) - (-1.96)\bar{\theta}^O_p \le 0\\
&v_1\,(9 - 0.5(1)) + v_2\,(8 - 0.5(1)) + (-1.96)\bar{\theta}^I_p \ge 1\\
&v_1\,(9 - 0.5(1)) + v_2\,(8 - 0.5(1)) - (-1.96)\bar{\theta}^I_p \le 1\\
&u_1\,(2.5 + 0.5(0)) - v_1\,(9 - 0.5(1)) - v_2\,(8 - 0.5(1)) - (-1.96)\bar{\lambda}_1 \le 0\\
&u_1\,(2 + 0.5(0)) - v_1\,(5 - 0.5(1)) - v_2\,(4 - 0.5(1)) - (-1.96)\bar{\lambda}_2 \le 0\\
&u_1\,(5 + 0.5(0)) - v_1\,(4 - 0.5(1)) - v_2\,(4 - 0.5(1)) - (-1.96)\bar{\lambda}_3 \le 0
\end{aligned} \tag{5.2}$$

$$
\begin{aligned}
&u_1\,(3.5+0.5(0)) - v_1\,(10-0.5(1)) - v_2\,(7-0.5(1)) - (-1.96)\bar{\lambda}_4 \le 0\\
&\left(\theta_p^O\right)^2 = u_1^2(1)\\
&\left(\theta_p^I\right)^2 = v_1^2(1) + v_2^2(1)\\
&\bar{\lambda}_j^2 = u_1^2(1) + v_1^2(1) + v_2^2(1), \forall j\\
&u_r, v_i, \bar{\theta}_p^O, \bar{\theta}_p^I, \bar{\lambda}_j \ge 0.
\end{aligned}
$$

model (5.2) is simplified as follows:

$$
\begin{aligned}
&\max \bar{\varphi}\\
&\text{s.t.}\\
&\bar{\varphi} - 2.5u_1 + 1.96\bar{\theta}_p^O \le 0\\
&8.5v_1 + 7.5v_2 - 1.96\bar{\theta}_p^I \ge 1\\
&8.5v_1 + 7.5v_2 + 1.96\bar{\theta}_p^I \le 1\\
&2.5u_1 - 8.5v_1 - 7.5v_2 + 1.96\bar{\lambda}_1 \le 0\\
&2u_1 - 4.5v_1 - 3.5v_2 + 1.96\bar{\lambda}_2 \le 0\\
&5u_1 - 3.5v_1 - 3.5v_2 + 1.96\bar{\lambda}_3 \le 0\\
&3.5u_1 - 9.5v_1 - 6.5v_2 + 1.96\bar{\lambda}_4 \le 0\\
&\left(\theta_p^O\right)^2 = u_1^2\\
&\left(\theta_p^I\right)^2 = v_1^2 + v_2^2\\
&\bar{\lambda}_j^2 = u_1^2 + v_1^2 + v_2^2, \forall j\\
&u_r, v_i, \bar{\theta}_p^O, \bar{\theta}_p^I, \bar{\lambda}_j \ge 0.
\end{aligned} \tag{5.3}
$$

We use the same threshold values for both models (5.1) and (5.2) to evaluate the efficiency of DMU A_1. By considering $\delta = 0.75$ and $\gamma = 0.75$ in model (5.1), we obtain $E_1^T(\delta,\gamma) = 0.71$. Here it is to be noted that model (5.3) is infeasible for the same threshold values according to probability–credibility approach proposed by Tavana *et al.*'s [39]. In fact, it is obvious that the constraints $8.5v_1+7.5v_2-1.96\bar{\theta}_p^I \ge 1$, $8.5v_1+7.5v_2+1.96\bar{\theta}_p^I \le 1$ and $\left(\theta_p^I\right)^2 = v_1^2+v_2^2$ are not satisfied simultaneously.

In sum, there are two important reasons for using our proposed method compared with Tavana *et al.*'s approach. First, in contrast to Tavana *et al.*'s approach, our proposed model is always feasible for all threshold values. Second, we solve a linear model in order to evaluate the efficiency of DMUs with fuzzy stochastic data as compared with model proposed by Tavana *et al.*'s [39] approach. Hence, from a computation point of view the proposed method is preferable to the Tavana *et al.*'s approach for solving the fuzzy stochastic DEA model.

Example 5.2. Consider four DMUs; A_1, A_2, A_3 and A_4 with the information given in Table 2.

Now, we analyze the effect of different values of γ and δ on the efficiency results. We recall that γ and δ are the predetermined thresholds defined by the decision maker. Now, assume that the DM considers $\gamma = 0.25$

TABLE 2. The data for Example 5.2.

	A_1	A_2	A_3	A_4
I_1	(N(9,1),1)	(N(5,1),1)	(N(4,1),1)	(N(10,1),1)
I_2	8	4	4	7
O_1	N(2.5,1)	N(2,1)	N(5,1)	N(3.5,1)

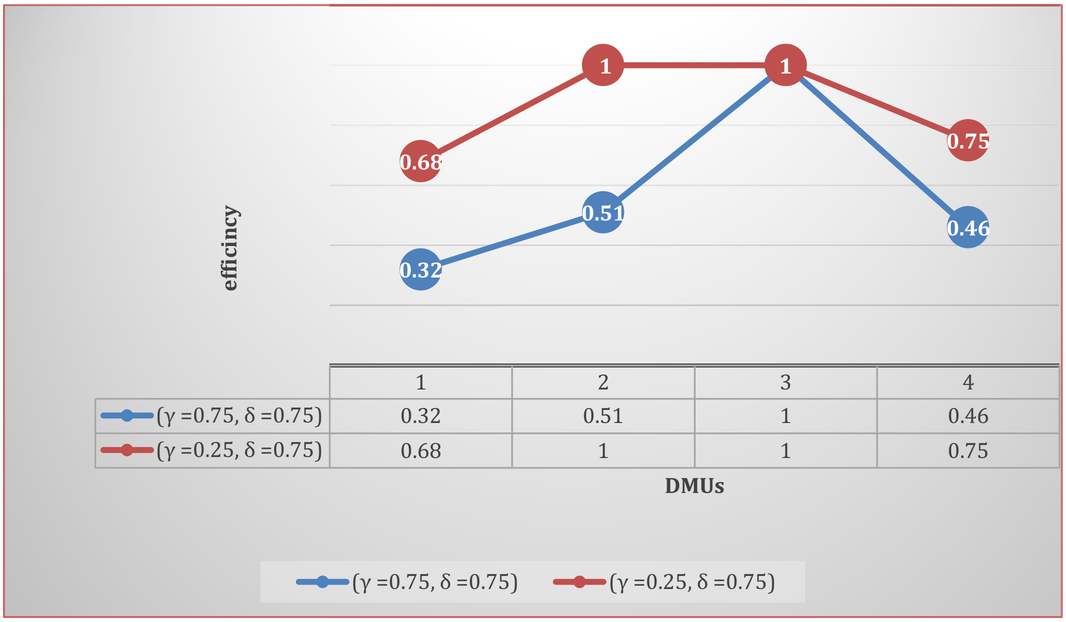

FIGURE 1. The efficiency results for different threshold values.

and $\delta = 0.75$ for evaluating DUMs of Table 1. Based on these threshold values, we obtain $E_1^T(\delta,\gamma) = 0.68$, $E_2^T(\delta,\gamma) = 1$, $E_3^T(\delta,\gamma) = 1$, and $E_4^T(\delta,\gamma) = 0.75$, respectively for DMUs A_1, A_2, A_3 and A_4. Figure 1 illustrates the efficiency of the DMUs in two thresholds ($\gamma = 0.75$, $\delta = 0.75$) and ($\gamma = 0.25$, $\delta = 0.75$) to investigate the influence of thresholds in efficiency scores. As we see only DMU A_3 is provided as the efficient unit for $\gamma = 0.75$ and $\delta = 0.75$, while two DMUs A_2 and A_3 are recognized as the efficient units for $\gamma = 0.25$ and $\delta = 0.75$. Hence, different threshold values according the DM's view, effect on both the efficiency scores of DMUs, and the situation of efficient units.

6. Case study

NATO's open door policy on enlargement invites European countries that are in a position to advance the principles of the North Atlantic Treaty and contribute to security in the Euro-Atlantic area, to join the alliance. Deciding ideal candidates for the expansion of NATO can be complicated. The integration of nonmembers in NATO is made in four steps indicating an increasing level of cooperation: Partnership for Peace (PFP), Individual Partnership Action Plan (IPAP), Intensified Dialogue (ID) and Membership Action Plan (MAP). The first level, PFP, was created in 1993 to create a dialog with neutral European and former Warsaw Block member states. The second level, IPAP, was installed in 2002 for eight countries within PFP potentially eligible for NATO membership.

The third level of integration, ID, is currently initiated for two countries prior vital and a unanimous vote protects the integrity of the alliance and prevents tension among the member countries to the final candidacy. Finally, the MAP is currently in action for three countries for which membership is under negotiation. Decisions on enlargement are ultimately made by NATO and its members; however, the North Atlantic Council is NATO's principal decision- making body and is responsible for inviting new members to join the alliance. Decisions to invite new members come as a result of a unanimous vote by current member countries in the final stage. Relationships between members of the alliance are vital and a unanimous vote protects the integrity of the

TABLE 3. The countries selected for this study.

DMU	Non-EU country
1	Armenia
2	Austria
3	Azerbaijan
4	Belarus
5	Bosnia and Herzegovina
6	Finland
7	FYR Macedonia
8	Georgia
9	Ireland
10	Kazakhstan
11	Malta
12	Moldova
13	Montenegro
14	Russia
15	Serbia
16	Sweden
17	Switzerland
18	Ukraine

TABLE 4. The fuzzy random input and output data.

DMU	Input 1	Input 2	Output 1	Output 2
1	(N(14.1,1),0.28,0.28)	(N(29.13,1),1.16,1.16)	(N(17.17,1),0.68,0.68)	(N(9.7,1),0.58,0.58)
2	(N(55.87,1),1.11,1.11)	(N(59.1,1),2.36.,2.36)	(N(301,1),12.04,12.04)	(N(45.12,1),2.70,2.70)
3	(N(15.26,1),0.30.0.30)	(N(6.7,1),0.26,0.26)	(N(64.66,1),2.58,2.58)	(N(10.45,1),0.62,0.62)
4	(N(20.16,1),0.40,0.40)	(N(3.57,1),0.14,0.14)	(N(103.5,1),4.14,4.14)	(N(20.05,1),1.20,1.20)
5	(N(48.29,1),0.96,0.96)	(N(34,1),1.36,1.36)	(N(14.78,1),0.59,0.59)	(N(48,1),2.88,2.88)
6	(N(30.87,1),0.61, 0.61)	(N(35.9,1),1.43,1.43)	(N(158.4,1),6.33,6.33)	(N(32.92,1),1.97,1.97)
7	(N(14.33,1),0.28, 0.28)	(N(30.8,1),1.23,1.23)	(N(17.35,1),0.69,0.69)	(N(14.46,1),0.86,0.86)
8	(N(21.95,1),0.43,0.43)	(N(26.37,1),1.05,1.05)	(N(20.6,1),0.82,0.82)	(N(17.86,1),1.07,1.07)
9	(N(48.26,1),0.96,0.96)	(N(30.9,1),1.23,1.23)	(N(181.6,1),7.26,7.26)	(N(42.65,1),2.55,2.55)
10	(N(25.06,1),0.50, 0.50)	(N(15.7,1),0.62,0.62)	(N(168.2,1),6.72,6.72)	(N(14.02,1),0.84,0.84)
11	(N(49.81,1),0.99, 0.99)	(N(39.73,1),1.58,1.58)	(N(9.4,1),0.37,0.37)	(N(37.07,1),2.22,2.22)
12	(N(28.87,1),0.57, 0.57)	(N(23.3,1),0.93,0.93)	(N(12.76,1),0.51,0.51)	(N(18.76,1),1.12,1.12)
13	(N(727.86,1),14.55, 14.55)	(N(43,1),1.72,1.72)	(N(5.92,1),0.03,0.03)	(N(108.28,1),6.49,6.49)
14	(N(12.49,1),0.24, 0.24)	(N(5.9,1),0.23,0.231)	(N(2097,1),83.88,83.88)	(N(14.26,1),0.85,0.85)
15	(N(12.68,1),0.25, 0.25)	(N(37,1),1.48,1.48)	(N(77.28,1),3.09,3.09)	(N(12.42,1),0.74,0.74)
16	(N(68.98,1),1.37, 1.37)	(N(41.7,1),1.66,1.66)	(N(338.5,1),13.54,13.54)	(N(73.59,1),4.41,4.41)
17	(N(56.67,1),1.13, 1.13)	(N(44.2,1),1.76,1.76)	(N(263.2,1),10.52,10.52)	(N(49.67,1),2.98,2.98)
18	(N(26.87,1),0.53, 0.53)	(N(11.7,1),0.46,0.46)	(N(314.8,1),12.59,12.59)	(N(13.41,1),0.80,0.80)

alliance and prevents tension among the member countries. Each decision to expand is made individually on a case by case basis and is a result of an agreement that the invited country will add to the security and stability of the alliance. The determination must also allow the alliance to preserve the ability to perform its main function of defense. Countries outside of the alliance are not given a voice in these decisions nor should countries be excluded for consideration due to membership of other groups or organizations.

Table 5. The stochastic fuzzy efficiency scores.

	($\gamma = 0.25$, $\delta = 0.5$)	($\gamma = 0.25$, $\delta = 0.75$)	($\gamma = 0.5$, $\delta = 0.5$)	($\gamma = 0.5$, $\delta = 0.25$)	($\gamma = 0.75$, $\delta = 0.25$)
1	0.8770	0.8434	0.79961	0.8318	0.7532
2	0.8874	0.8526	0.8590	0.8940	0.8480
3	0.9392	0.9006	0.8279	0.8628	0.7679
4	1.0000	1.0000	1.0000	1.0000	1.0000
5	1.0000	1.0000	1.0000	1.0000	1.0000
6	1.0000	1.0000	1.0000	1.0000	1.0000
7	1.0000	1.0000	1.0000	1.0000	1.0000
8	0.9577	0.9204	0.9017	0.9382	0.8711
9	0.9749	0.9367	0.9425	0.9809	0.9292
10	0.6692	0.6419	0.6240	0.6498	0.6006
11	0.8235	0.7913	0.7951	0.8275	0.7832
12	0.7526	0.7234	0.7133	0.7422	0.6926
13	0.7198	0.6856	0.6231	0.6545	0.5820
14	1.0000	1.0000	1.0000	1.0000	1.0000
15	1.0000	1.0000	1.0000	1.0000	1.0000
16	1.0000	1.0000	1.0000	1.0000	1.0000
17	0.9601	0.9224	0.9307	0.9687	0.9197
18	0.6444	0.6184	0.5820	0.6065	0.5506

Table 6. Average Cross Efficiency of efficient DMUs.

DMUs	($\gamma = 0.25$, $\delta = 0.5$)	($\gamma = 0.25$, $\delta = 0.75$)	($\gamma = 0.5$, $\delta = 0.5$)	($\gamma = 0.5$, $\delta = 0.25$)	($\gamma = 0.75$, $\delta = 0.25$)
DMU 4	1.0612	1.004	0.9882	1.0152	0.9698
DMU 5	0.8488	0.8299	0.8341	0.8488	0.8197
DMU 6	0.8364	0.8279	0.8426	0.8505	0.8326
DMU 7	0.7030	0.7070	0.7320	0.7283	0.7310
DMU 14	1.0982	1.0660	1.0540	1.0753	1.0348
DMU 15	0.6418	0.6511	0.6789	0.6694	0.6827
DMU 16	0.9344	0.9108	0.9120	0.9299	0.8954

Table 7. Complete ranking of the DMUs.

δ	γ	Complete ranking of the DMUs
0.5	0.25	14>4>16>5>6>7>15>9>8>17>3>2>1>11>12>13>10>18
0.75	0.25	14>4>16>5>6>7>15>9>17>8>3>2>1>11>12>13>10>18
0.5	0.5	14>4>16>6>5>7>15>9>17>8>2>3>1>11>12>13>10>18
0.25	0.5	14>4>16>6>5>7>15>9>17>8>2>3>1>11>12>13>10>18
0.25	0.75	14>4>16>6>5>7>15>9>17>8>2>3>1>11>12>13>10>18

The specification of the model follows the publicly announced criteria related to economic and social stability, as well as the absence of conflicts with existing or future members or partners of the alliance. Consequently, in this study, revenue or Gross Domestic Product (GDP) (output 1), and Budget (Revenues) (output 2) are considered as the output variables while budget expenditures (Input 1) and public debt (Input 2) are considered as the input variables. Table 3 shows the list of the country.

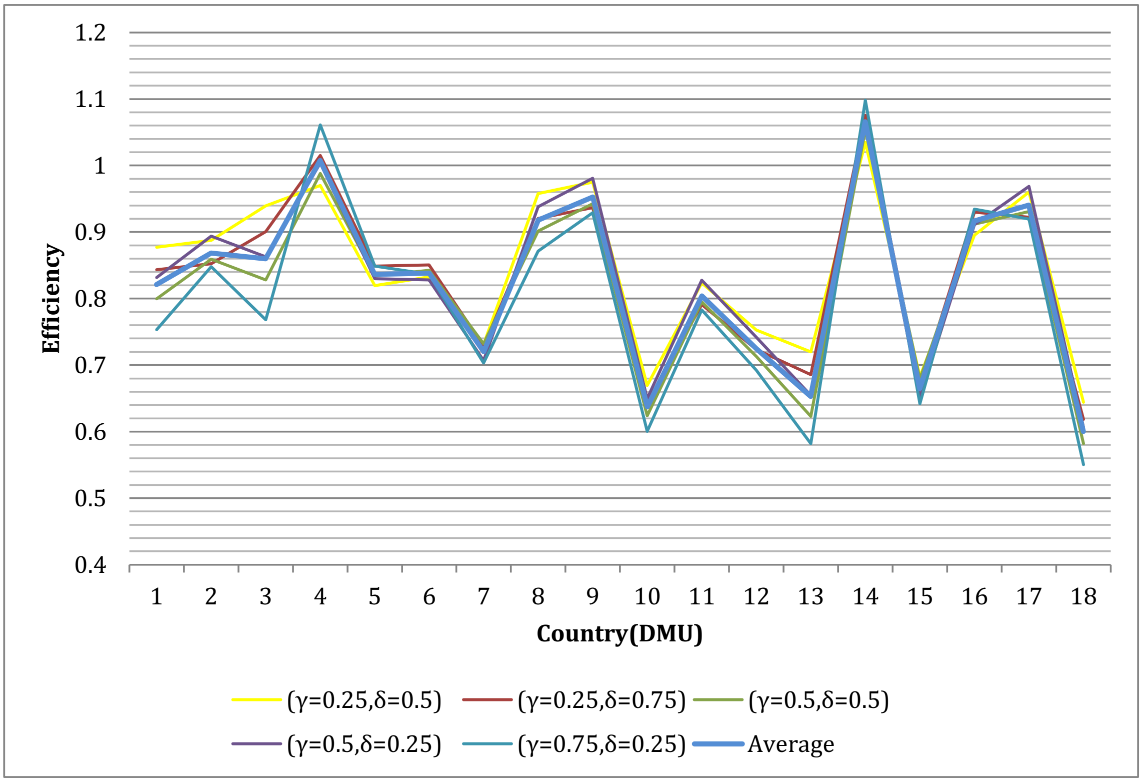

FIGURE 2. The final result.

In Table 4, we present the two fuzzy random inputs and the three fuzzy random outputs, where all the inputs and outputs data are triangular fuzzy random numbers. These data are denoted by $(m,\ \alpha,\ \beta)$ where m is the center value and α and β are the left and right tails, respectively.

Table 5 shows the evaluating results by model (4.6) when we set the predetermined minimum probability level δ and the predetermined acceptable level of possibility γ in five different threshold levels of ($\gamma = 0.25$, $\delta = 0.5$), ($\gamma = 0.25$, $\delta = 0.75$), ($\gamma = 0.5$, $\delta = 0.5$), ($\gamma = 0.5$, $\delta = 0.25$) and ($\gamma = 0.75$, $\delta = 0.25$). With the variation in the satisfaction levels δ and γ, DMUs 4, 5, 6, 7, 14, 15 and 16 are efficient at five given levels. Generally from Table 5, we can see the applicability of Theorem 4.3 when the efficiency scores of the DMUs decrease by increasing the level δ from ($\gamma = 0.25$, $\delta = 0.5$) to ($\gamma = 0.25$, $\delta = 0.75$) and the level γ from ($\gamma = 0.25$, $\delta = 0.5$) to ($\gamma = 0.25$, $\delta = 0.75$).

Table 6 presents the Average Cross Efficiency (ACE), denoted by $\bar{E}_k^*(\delta, \gamma)$, for each efficient DMU at levels stated above. Also, these ACE scores are used to obtain a complete ranking of DMUs.

Table 7 shows the complete ranking of DMUs. As seen from Table 7, this ranking is similar except for some relocation in surrounding DMUs. The first three ranks of the countries belong to Russia, Belarus and Sweden, respectively, in each level. Also, the last three ranks of the countries belong to Montenegro, Kazakhstan and Ukraine, respectively, in each level. Another point obtained from Table 6 shows that the influence of the variations of stochastic level γ is more than fuzzy level δ on $E_k^T(\delta, \gamma)$. Indeed, with the same increasing in each of levels $\delta$and γ, the objective value decreases further by increasing the stochastic level γ, and so the number of efficient DMUs is fall down in this case. Figure 2 illustrates the efficiency scores of the activities. In this figure the ACE values for efficient DMUs are used.

7. Conclusions

In this paper, we formulated a DEA model in fuzzy random environment. A methodology in chance constraint programming adopted to solve such DEA model. Unlike the proposed model by Tavana *et al.* [39], our proposed approach not only leads to a linear program, but also it gives efficiency scores with the range of zero to one for DMUs similar to traditional input-oriented DEA models. Also, a case study for NATO enlargement problem illustrated how a complex socio-economic problem with multiple resources and multiple fuzzy stochastic consequences can be addressed to inform decision-making body about the decision. For future study, a new measure in fuzzy stochastic programming can also be planned in chance constraint programming.

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