RAIRO-Oper. Res. 53 (2019) 1721–1728
https://doi.org/10.1051/ro/2018074

RAIRO Operations Research
www.rairo-ro.org

DOMINATION INTEGRITY OF SOME GRAPH CLASSES

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Abstract. The stability of a communication network has a great importance in network design. There are several vulnerability measures used to determine the resistance of network to the disruption in this sense. Domination theory provides a model to measure the vulnerability of a graph network. A new vulnerability measure of domination integrity was introduced by Sundareswaran in his Ph.D. thesis (Parameters of vulnerability in graphs (2010)) and defined as $\mathrm{DI}(G) = \min\{\,|S|+m(G-S) : S \in V(G)\}$ where $m(G-S)$ denotes the order of a largest component of graph $G-S$ and S is a dominating set of G. The domination integrity of an undirected connected graph is such a measure that works on the whole graph and also the remaining components of graph after any break down. Here we determine the domination integrity of wheel graph $W_{1,n}$, Ladder graph L_n, $S_{m,n}$, Friendship graph F_n, Thorn graph of P_n and C_n which are commonly used graph models in network design.

Mathematics Subject Classification. 05C12, 05C35, 05C69, 68R10.

Received March 6, 2017. Accepted September 4, 2018.

1. Introduction

A communication network should be constructed as stable as possible for the failures on links or nodes that can cause large damage for communication. Vulnerability measures such as binding number, toughness, scattering number, integrity, tenacity and rupture degree are the measures, based on the whole vertex set of a graph model, commonly used to determine the stability of a communication network. On the other hand, in analysis of the vulnerability of a communication network to disruption, the most important two questions are (i) what is the number of elements that are not functioning and (ii) what is the size of the largest remaining component, in which mutual communication still exists. Integrity is one of the well-known vulnerability measures that try to find answers to these questions. The integrity of a graph is introduced by Barefoot *et al.* [2] and it is defined as follows.

Definition 1.1. The integrity of a graph G is denoted by $I(G)$ and defined by $I(G) = \min\{|S| + m(G-S) : S \subset V(G)\}$ where $m(G-S)$ is the order of a maximum component of $G-S$ [2].

Definition 1.2. An I-set of G is any subset S of $V(G)$ for which $I(G) = |S| + m(G-S)$ [2].

Bagga *et al.* [1] have presented many results concerning integrity in a survey article. Goddard and Swart [5] have investigated the interrelations between integrity and other graph parameters. Mamut and Vumar [10] have discussed the integrity of middle graphs. Dundar and Aytac [4] have discussed the integrity of total graphs.

Keywords. Integrity, domination, domination integrity.

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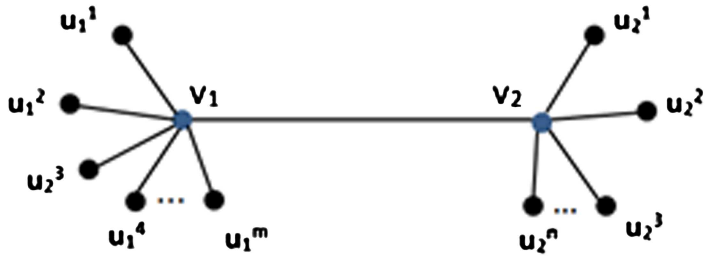

FIGURE 1. Double star graph $S_{m,n}$.

Depending on network models new vulnerability measures take a great role in any failure not only on nodes but also on links which have special properties. For example, domination is a famous concept in network design, there are many types such as connected domination, independent domination, double domination etc.

Definition 1.3. A subset S of $V(G)$ is called dominating set if for every $v \in V-S$, there exist a $u \in S$ such that v is adjacent to u [8].

Definition 1.4. The minimum cardinality of a minimal dominating set in G is called the domination number of G denoted as $\gamma(G)$ and the corresponding minimal dominating set is called a γ-set of G [8].

By taking both integrity and domination concepts into consideration a new vulnerability measure of domination integrity was defined by Sundareswaran Swaminathan in [12,13] as follows.

Definition 1.5. The domination integrity of a connected graph G is denoted by DI(G) and defined as DI$(G) = \min\{|S| + m(G-S) \ : S$ is a dominating set$\}$ where $m(G-S)$ is the order of a maximum component of $G-S$. The definition above is given by Sundareswaran in his Ph.D. thesis [13].

Later on, many new results concerning domination integrity were found by Sundareswaran and Swaminathan in [14,15]. Vaidya and Kothari [17] have discussed domination integrity in the context of some graph operations like duplication of an edge by vertex and duplication of vertex by an edge. The domination integrity of splitting graph of path P_n and cycle C_n was investigated by the same authors in [18]. Vaidya and Shah [19] determined the domination integrity of total graphs of path P_n, cycle C_n and star $K_{1,n}$ and also determined the domination integrity of square graph of path in [20]. Computational complexity of domination integrity in graphs is studied by Sundareswaran and Swaminathan [16].

Definition 1.6. For $n \geq 3$, the *wheel* $W_{1,n}$ is defined to be a graph $K_1 + C_n$ [3].

Definition 1.7. Let $p_1, p_2, \ldots, p_n$ be non-negative integers. The *thorn graph* of G, with parameters $p_1, p_2, \ldots, p_n$, is obtained by attaching p_i new vertices of degree one to the vertex u_i of the graph G, $i = 1, 2, \ldots, n$. The thorn graph of G will be denoted by G^*, or if the respective parameters need to be specified, by $G^*(p_1, p_2, \ldots, p_n)$ [6].

Definition 1.8. The *double star graph* $S_{m,n}$ is obtained by attaching $K_{1,m}$ and $K_{1,n}$ at the end vertices of path P_2 [3] (Fig. 1).

Definition 1.9. A *friendship graph* F_n is a graph which consists of n triangles with a common vertex [11] (Fig. 2).

Definition 1.10. The *n-ladder graph* L_n can be defined as $P_2 \times P_n$, where P_n is a path graph [9] (Figs. 3 and 4).

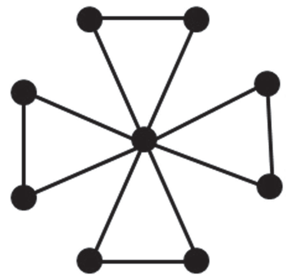

FIGURE 2. Friendship graph F_4.

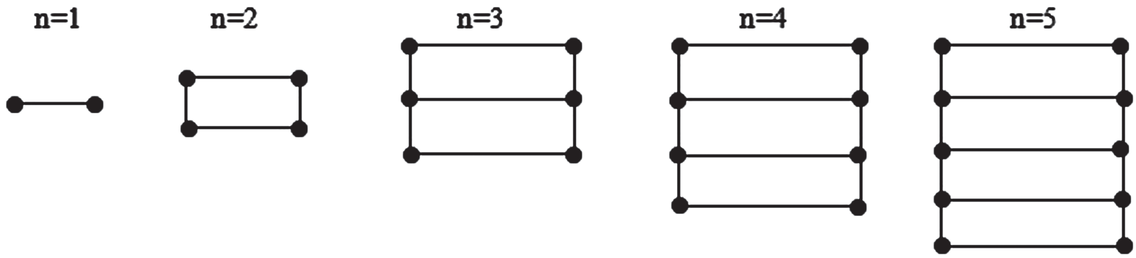

FIGURE 3. Ladder graphs L_1, L_2, L_3, L_4, L_5.

2. MAIN RESULTS

In this paper we present some new results concerning this parameter on graph structures.

Proposition 2.1 ([13]).

$$(i)\ \mathrm{DI}\,(P_n) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor + 1; & n = 2,3,4,5,6,7 \\ \left\lceil \frac{n}{3} \right\rceil + 2; & n \geq 8 \end{cases}$$

$$(ii)\ \mathrm{DI}\,(C_n) = \begin{cases} 3; & n = 3,4 \\ \left\lceil \frac{n}{3} \right\rceil + 2; & n \geq 5 \end{cases}$$

$$(iii)\ \mathrm{DI}\,(K_{m,n}) = \min\{m,n\} + 1$$

Theorem 2.2. *For $n \geq 3$, $\mathrm{DI}\,(W_{1,n}) = \lceil 2\sqrt{n} \rceil$.*

Proof. Let v be central vertex of $W_{1,n}$. Every $W_{1,n}$ contains a cycle C_n whose order is n and each vertex on this cycle is connected to this central vertex v. Remove a domination set $S = \{v\} \cup X$ from $W_{1,n}$ where $X \subset V(C_n)$ and $|X| = r$. Since $w(W_{1,n} - S) \leq r$ then $m\,(W_{1,n} - S) \geq \frac{(n+1)-(r+1)}{r}$. (When $|S| = r$ vertices are removed from any connected graph G then $m\,(G - S) \geq \frac{p-r}{w(G-S)}$. (p is the number of vertices in G and $w\,(G - S)$ is the number of components in $G - S$). Hence;

$$\begin{aligned} \mathrm{DI}\,(W_{1,n}) &= \min_{S \subset V(W_{1,n})} \{|S| + m\,(W_{1,n} - S)\colon S \text{ is a dominating set}\} \\ &\geq \min_{r} \left\{ r+1+\frac{n-r}{r} \right\}. \end{aligned}$$

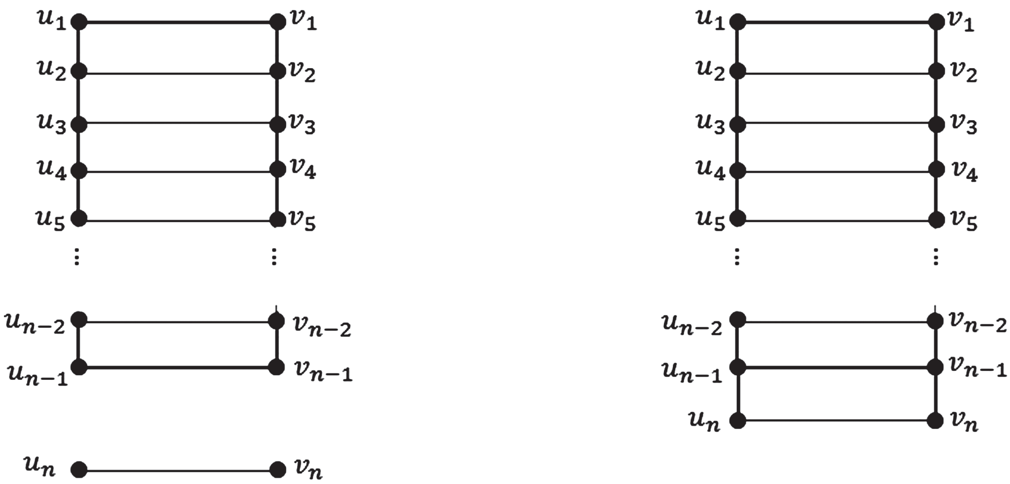

FIGURE 4. Construction of Ladder graph L_n.

To find the minimum value of the function $f(r) = r{+}1{+}\frac{n-r}{r} = \frac{r^2+n}{r}$, we derivate the function:

$$f^{'}(r) = \frac{2r.r - (r^2+n)}{r^2} = 0$$
$$r^2 - n = 0 \Rightarrow r = \mp\sqrt{n}$$

Since the number of vertices cannot be a negative value then $r = \sqrt{n}$.
$f(\sqrt{n}) = \sqrt{n}{+}1{+}\frac{n-\sqrt{n}}{\sqrt{n}} = \sqrt{n}{+}1{+}\sqrt{n} - 1 = 2\sqrt{n}$ is obtained. Thus, $\mathrm{DI}(W_{1,n}) \geq 2\sqrt{n}$.

If $S \subset V(W_{1,n})$ dominating set is constructed by adding a central vertex v to I-set of C_n, then $m(W_{1,n} - S) = \frac{n-r}{r}$ and by this construction we obtain $\mathrm{DI}(W_{1,n}) = 2\sqrt{n}$. Since integrity is integer valued [1], we round domination integrity up to get an upper bound. Thus $\mathrm{DI}(W_{1,n}) = \lceil 2\sqrt{n} \rceil$. □

Theorem 2.3. *Let $l_1, l_2, \ldots, l_n$ be positive integers, P_n be a path with n vertices and C_n be a cycle with n vertices then for thorn graph of P_n, ${P_n}^*$ and thorn graph of C_n, ${C_n}^*$ following results are obtained.*

$$(i)\ \mathrm{DI}({P_n}^*(l_1, l_2, \ldots, l_n)) = n + 1$$

$$(ii)\ \mathrm{DI}({C_n}^*(l_1, l_2, \ldots, l_n)) = n + 1$$

Proof. (i) Let P_n be a path with n vertices and $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. In the thorn graph of P_n, with parameters $l_1, l_2, \ldots, l_n$, if l_i vertices of which is attached to the vertex v_i $(i = 1, 2, \ldots, n)$, are added to the dominating set of ${P_n}^*(l_1, l_2, \ldots, l_n)$, this is contrary to the requirement of minimum value in Definition 1.5. Therefore, $S = \{v_1, v_2, \ldots, v_n\}$ is taken as the dominating set of ${P_n}^*(l_1, l_2, \ldots, l_n)$. ${P_n}^*(l_1, l_2, \ldots, l_n) - S$ consists of isolated vertices. So $m({P_n}^*(l_1, l_2, \ldots, l_n) - S) = 1$. In that case, $\mathrm{DI}({P_n}^*(l_1, l_2, \ldots, l_n)) = |S| + 1 = n + 1$.

(ii) Let C_n be a cycle with n vertices and $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. In the thorn graph of C_n, with parameters $l_1, l_2, \ldots, l_n$, $S = \{v_1, v_2, \ldots, v_n\}$ is taken as the dominating set of ${C_n}^*(l_1, l_2, \ldots, l_n)$ because of the same reason in *proof (i)* of Theorem 2.3. ${C_n}^*(l_1, l_2, \ldots, l_n) - S$ consists of isolated vertices. So $m({C_n}^*(l_1, l_2, \ldots, l_n) - S) = 1$. Hence, $\mathrm{DI}({C_n}^*(l_1, l_2, \ldots, l_n)) = |S| + 1 = n + 1$.

□

Theorem 2.4. *The domination integrity of the double star graph $S_{m,n}$ is 3.*

Proof. Let P_2 be a path, $V(P_2) = \{v_1, v_2\}$ and m vertices be attached to v_1, n vertices be attached to v_2 for obtaining $S_{m,n}$. $m+n$ vertices which are attached at the end vertices of P_2, have one degree. If these vertices are added to the dominating set of $S_{m,n}$, it is contrary to the requirement of minimum value in Definition 1.5. Hence, $S = \{v_1, v_2\}$ is taken as the minimum dominating set of $S_{m,n}$. $S_{m,n} - S$ consists of isolated vertices so $m(S_{m,n} - S) = 1$. If X is any dominating set of $S_{m,n}$, then $|X| + m(S_{m,n} - X) \geq |S| + 1$. Therefore, $\mathrm{DI}(S_{m,n}) = 3$. □

Theorem 2.5. *The domination integrity of friendship graph F_n is 3 (Fig. 2).*

Proof. Let F_n be a friendship graph and v be the central vertex of F_n. $S = \{v\}$ is the minimum dominating set of F_n. The remaining graph $F_n - S$ consists of n components which each of them is P_2 so the maximum size of the remaining components is $m(F_n - S) = 2$. If X is any dominating set of F_n, then $|X| + m(F_n - X) \geq |S| + 2$. Hence, $\mathrm{DI}(F_n) = 3$. □

Theorem 2.6. *For* $1 \leq n \leq 8$, $\mathrm{DI}(L_n) = n + 1$.

Proof. Let $V(L_n) = \{u_i, v_i \colon 1 \leq i \leq n\}$ where $1 \leq n \leq 8$.

Case 1: $n = 1$. It is clear that L_1 is the path graph P_2. So from the Proposition 2.1(i), $\mathrm{DI}(L_1) = \mathrm{DI}(P_2) = 2$.

Case 2: $n = 2$. It is clear that L_2 is the cycle graph C_4. So from the Proposition 2.1(ii), $\mathrm{DI}(L_2) = \mathrm{DI}(C_4) = 3$.

Case 3: $n = 3$. Let's use a dominating set $S_1 = \{u_1, v_2, u_3\} \subset V(L_3)$ where $|S_1| = 3$ and $m(L_3 - S_1) = 1$. Then we have $|S_1| + m(L_3 - S_1) = 4$. Let's take another dominating set $S_2 = \{u_2, v_2\} \subset V(L_3)$ where $|S_2| = 2$ and $m(L_3 - S_2) = 2$. Then we have $|S_2| + m(L_3 - S_2) = 4$. Since there is no such a dominating set $X \subset V(L_3)$ which satisfies $|X| + m(L_3 - X) < 4$ then $\mathrm{DI}(L_3) = 4$.

Case 4: $n = 4$. Let's use a dominating set $S_1 = \{u_1, v_2, u_3, v_4\} \subset V(L_4)$ where $|S_1| = 4$ and $m(L_4 - S_1) = 1$. Then we have $|S_1| + m(L_4 - S_1) = 5$. Let's take another dominating set $S_2 = \{u_2, v_2, u_4, v_4\} \subset V(L_4)$ so $|S_2| = 4$ and $m(L_4 - S_2) = 2$. We obtain $|S_2| + m(L_4 - S_2) = 6 > 5$. For another dominating set $X \subset V(L_4)$ let $|X| < 4$. Thus we have $m(L_4 - X) \geq 3$ and $|X| + m(L_3 - X) > 5$. Hence $\mathrm{DI}(L_4) = 5$.

Case 5: $n = 5$. Let's use a dominating set $S_1 = \{u_1, v_2, u_3, v_4, u_5\} \subset V(L_5)$ where $|S_1| = 5$ and $m(L_5 - S_1) = 1$. Then we have $|S_1| + m(L_5 - S_1) = 6$. Let's take another dominating set $S_2 = \{u_2, v_2, u_4, v_4\} \subset V(L_5)$ where $|S_2| = 4$ and $m(L_5 - S_2) = 2$. So we obtain $|S_2| + m(L_5 - S_2) = 6$. Let $|X| \leq 3$ for the dominating set $X \subset V(L_5)$. Since value of $m(L_5 - X)$ will be increased then $|X| + m(L_5 - X) > 6$ holds. Hence $\mathrm{DI}(L_5) = 6$.

Case 6: $n = 6$. Let's use a dominating set $S_1 = \{u_1, v_2, u_3, v_4, u_5, v_6\} \subset V(L_6)$ where $|S_1| = 6$ and $m(L_6 - S_1) = 1$. Then we have $|S_1| + m(L_6 - S_1) = 7$. Let's take another dominating set $S_2 = \{u_2, v_2, u_4, v_4, u_6, v_6\} \subset V(L_6)$ where $|S_2| = 6$ ve $m(L_6 - S_2) = 2$. So we have $|S_2| + m(L_6 - S_2) = 8 > 7$ dir. Let $|X| \leq 5$ for the dominating set $X \subset V(L_6)$. Since the value of $m(L_6 - X)$ will be increased then $|X| + m(L_6 - X) > 7$ holds. Hence $\mathrm{DI}(L_6) = 7$.

Case 7: $n = 7$. Let's use a dominating set $S_1 = \{u_1, v_2, u_3, v_4, u_5, v_6, u_7\} \subset V(L_7)$ where $|S_1| = 7$ and $m(L_7 - S_1) = 1$. So we obtain $|S_1| + m(L_7 - S_1) = 8$. Let's take another dominating set $S_2 = \{u_2, v_2, u_4, v_4, u_6, v_6\} \subset V(L_7)$ where $|S_2| = 6$ and $m(L_7 - S_2) = 2$. Then we have $|S_2| + m(L_7 - S_2) = 8$. Let $|X| \leq 5$ for the dominating set $X \subset V(L_7)$. Since the value of $m(L_7 - X)$ will be increased then $|X| + m(L_7 - X) > 8$ holds. Hence $\mathrm{DI}(L_7) = 8$.

Case 8: $n = 8$. Let's use a dominating set $S_1 = \{u_1, v_2, u_3, v_4, u_5, v_6, u_7, v_8\} \subset V(L_8)$ where $|S_1| = 8$ and $m(L_8 - S_1) = 1$. Then we have $|S_1| + m(L_8 - S_1) = 9$. Let's take another dominating set $S_2 = \{u_2, v_2, u_4, v_4, u_6, v_6, u_8, v_8\} \subset V(L_8)$ where $|S_2| = 8$ and $m(L_8 - S_2) = 2$. Then we obtain $|S_2| + m(L_8 - S_2) = 10 > 9$. For the dominating set $X \subset V(L_8)$ let $|X| \leq 7$. Since the value of $m(L_8 - X)$ will be increased then $|X| + m(L_8 - X) > 9$ holds. Hence $\mathrm{DI}(L_8) = 9$.

Now, let's have a look for the value of $\mathrm{DI}(L_{\mathrm{n}})$ where $n \geq 9$.

□

TABLE 1. Observations for $D(\gamma - set)$ and dominating set S.

n	$\|V(G)\| = 2n$	$\|D\|$	$m(G-D)$	$\|D\| + m(G-D)$	$\|S\|$	$m(G-S)$	$\|S\| + m(G-S)$
9	18	5	13	18	6	4	10
10	20	6	12	18	7	4	11
11	22	6	16	22	8	4	12
12	24	7	15	22	8	4	12
13	26	7	19	26	9	4	13
14	28	8	18	26	10	4	14
15	30	8	22	30	10	4	14
16	32	9	21	30	11	4	15
17	34	9	25	34	12	4	16
18	36	10	24	34	12	4	16
19	38	10	28	38	13	4	17
20	40	11	27	38	14	4	18
21	42	11	31	42	14	4	18
22	44	12	30	42	15	4	19
23	46	12	34	46	16	4	20
24	48	13	33	46	16	4	20
25	50	13	37	50	17	4	21

Theorem 2.7. *For ladder graph L_n where $n \geq 9$ domination integrity value of L_n is defined as follows.*

$$\mathrm{DI}\left(L_n\right) = \begin{cases} 2\left\lceil \frac{n}{3} \right\rceil + 4; & n \equiv 0 \text{ and } n \equiv 2 \pmod 3 \\ 2\left\lceil \frac{n-1}{3} \right\rceil + 5; & n \equiv 1 \pmod 3 \end{cases}$$

Proof. In a ladder graph a vertex can dominate 4 vertices with itself at most. That's why proof is done by applying mod3 on n vertices where $n \geq 9$. Let $V\left(L_n\right) = \{u_k, v_k \colon 1 \leq k\}$ and $S \subset V\left(L_n\right)$ to be a dominating set defined as in 3 cases:

(1) If $\boldsymbol{n} \equiv \mathbf{0}\,(\mathbf{mod3})$ then $S = \left\{u_{3k-1},\, v_{3k-1} : 1 \leq k \leq \frac{n}{3}\right\}$ and $|S| = 2\left\lceil \frac{n}{3} \right\rceil$. In this case, $u_{3k-2}, u_{3k} \in N(u_{3k-1})$ and $v_{3k-2}, v_{3k} \in N(v_{3k-1})$ then S is a dominating set of L_n and we obtain $m\left(L_n - S\right) = 4$.

(2) If $\boldsymbol{n} \equiv \mathbf{1}\,(\mathbf{mod3})$ then S can be chosen as $S = \left\{u_{3k-1},\, v_{3k-1},\, v_n : 1 \leq k \leq \frac{n-1}{3}\right\}$ or $S = \left\{v_{3k-1},\, u_{3k-1},\, u_n : 1 \leq k \leq \frac{n-1}{3}\right\}$. So $|S| = 2\left\lceil \frac{n-1}{3} \right\rceil + 1$. Since $u_{3k-2},\, u_{3k} \in N(u_{3k-1})$, $v_{3k-2}, v_{3k} \in N(v_{3k-1})$ and $u_n \in N(v_n)$ or $v_n \in N(u_n)$ then S is a dominating set for L_n and we obtain $m\left(L_n - S\right) = 4$

(3) If $\boldsymbol{n} \equiv \mathbf{2}\,(\mathbf{mod3})$ then let's choose S as $S = \left\{u_{3k-1},\ v_{3k-1} : 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil\right\}$ and we have $|S| = 2\left\lceil \frac{n}{3} \right\rceil$. So for any k value in $1 \leq k < \left\lceil \frac{n}{3} \right\rceil$ we select vertices as $u_{3k-2}, u_{3k} \in N(u_{3k-1})$ and $v_{3k-2}, v_{3k} \in N(v_{3k-1})$. For $k = \left\lceil \frac{n}{3} \right\rceil$ since $u_{3k-2} \in N(u_{3k-1})$ and $v_{3k-2} \in N(v_{3k-1})$, S is a dominating set for L_n and we obtain $m\left(L_n - S\right) = 4$.

Table 1 shows the relation between DI based on any S dominating and minimum D dominating sets of L_n where D is a $\gamma -$set. The values obtained by $|D| + m(L_n - S)$ and $|S| + m\left(L_n - S\right)$ are also shown in Table 1. It is observed that $\mathrm{DI}(L_n)$ gets its minimum value when S is chosen as dominating set of L_n.

As seen from Table 1, $|S| + m\left(L_n - S\right) < |D| + m(L_n - S)$, this inequality holds for the dominating sets $D(\gamma - set)$ and S. Then, $\mathrm{DI}\left(L_n\right) \leq |S| + m\left(L_n - S\right) < |D| + m\left(L_n - S\right)$ is obtained. Let's choose another dominating set S_i of L_n which satisfies the inequality $|D| \leq |S_i| < |S|$. Since value of $m\left(L_n - S_i\right)$ will increase then we have $|S| + m\left(L_n - S\right) < |S_i| + m(L_n - S_i)$. Let $S_j \subset V(L_n)$ be a dominating set with minimum cardinality which satisfies $m\left(L_n - S_j\right) = j$ ($1 \leq j \leq 3$).

Thus we have $|S| + m\left(L_n - S\right) < |S_j| + m(L_n - S_j)$ and for some values of n we obtain equalities instead of inequalities.

TABLE 2. DI results when values of $m(G-S)$ are 1, 2 and 3, respectively.

n	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
$\|S_1\|$	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
$m(G-S_1)$	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
$\|S_1\|+m(G-S_1)$	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
$\|S_2\|$	8	10	10	12	12	14	14	16	16	18	18	20	20	22	22	24	24
$m(G-S_2)$	**2**	**2**	**2**	**2**	**2**	**2**	**2**	**2**	**2**	**2**	**2**	**2**	**2**	**2**	**2**	**2**	**2**
$\|S_2\|+m(G-S_2)$	10	12	12	14	14	16	16	18	18	20	20	22	22	24	24	26	26
$\|S_3\|$	7	8	9	10	11	11	12	13	14	15	15	16	17	18	19	19	20
$m(G-S_3)$	**3**	**3**	**3**	**3**	**3**	**3**	**3**	**3**	**3**	**3**	**3**	**3**	**3**	**3**	**3**	**3**	**3**
$\|S_3\|+m(G-S_3)$	10	11	12	13	14	14	15	16	17	18	18	19	20	21	22	22	23

Table 2 shows the results of DI when dominating sets are chosen to obtain value of $m\,(G-S)$ to be at most 1, 2 or 3. The reason why dominating set are chosen to have these $m\,(G-S)$ values is explained as follows. Let $S_t \subset V(L_n)$ be a minimal dominating set which satisfies $m\,(L_n - S_t) \geq 5$. In this case, it is clear that $|S| + m\,(L_n - S) < |S_t| + m(L_n - S_t)$. So dominating sets are not chosen in a way that where $m\,(L_n - S) > 4$.

From the investigations above, it is clear that DI(L_n) gets its minimum value when $m\,(L_n - S)$ is 4 at most. So, domination integrity of L_n when $n \geq 9$ is given as;

$$\mathrm{DI}\,(L_n) = \begin{cases} 2\left\lceil \frac{n}{3} \right\rceil + 4; & n \equiv 0 \text{ and } n \equiv 2 \pmod 3 \\ 2\left\lceil \frac{n-1}{3} \right\rceil + 5; & n \equiv 1 \pmod 3. \end{cases}$$

□

3. Concluding results

Domination is an important vulnerability concept in the design of communication networks. Integrity is also another well-known vulnerability measure in the stability comparison of two given network models. Sundareswaran and Swaminathan have defined domination integrity by taking into these two important concepts domination and integrity. In this study we investigate domination integrity of $W_{1,n}$, Ladder L_n, double star $S_{m,n}$, Friendship F_n, Thorn graphs of P_n and C_n which are commonly used network models whose domination integrity values have not been studied.

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