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AN OPTIMIZATION TECHNIQUE FOR NATIONAL INCOME DETERMINATION MODEL WITH STABILITY ANALYSIS OF DIFFERENTIAL EQUATION IN DISCRETE AND CONTINUOUS PROCESS UNDER THE UNCERTAIN ENVIRONMENT

Biswajit Sarkar[1], Sankar Prasad Mondal[2], Sun Hur[1,*], Ali Ahmadian[3], Soheil Salahshour[4], Rekha Guchhait[1,5] and Muhammad Waqas Iqbal[1]

Abstract. The paper represents a variation of the national income determination model with discrete and continuous process in fuzzy environment, a significant implication in economics planning, by means of fuzzy assumptions. This model is re-recognized and deliberated with fuzzy numbers to estimate its uncertain parameters whose values are not precisely known. Exhibition of imprecise solutions of the concerned model is carried out by using the proposed two methods: generalized Hukuhara difference and generalized Hukuhara derivative (gH-derivative) approaches. Moreover, the stability analysis of the model in two different systems in fuzzy environment is illustrated. Additionally, different illustrative examples for optimization of national income determination model are undertaken with the constructive graph and table for convenience for clarity of the projected approaches.

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1. Introduction

The thought of national income moves a very significant figure in economic theory. National income is the entire flow of goods and services of a nation during a year. In another words, national income is the total money cost of all goods and services emanated in a country during a year, after excluding wear and tear and decadence of plant and machinery used in the production of goods and services. By national income of a country, we average the sum-total of incomes attained by the citizens of the country during a given stage of time due to their participation in the production process. It is in general weighed for a year. It should be marked that national income is not the sum of all incomes which gather due to sharing in the production method are accessory in

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[1] Department of Industrial & Management Engineering, Hanyang University, 155 88 Ansan, Gyeonggi-do, South Korea.
[2] Department of Natural Science, Maulana Abul Kalam Azad University of Technology, Haringhata, Nadia 741249, West Bengal, India.
[3] Institute for Mathematical Research (INSPEM), University Putra Malaysia, 434 00 UPM Serdang, Selangor, Malaysia.
[4] Department of Mathematics, Mobarakeh Branch, Islamic Azad University, Mobarakeh, Iran.
[5] Department of Mathematics & Statistics, Banasthali Vidyapith, 304 002 Rajasthan, India.
*Corresponding author: hursun@hanyang.ac.kr

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the national income total. Zadeh [67] published a new direction of fuzzy concept within literature. In modeling with real natural phenomena with uncertain discrete and continuous system difference and differential equation play an important role.

1.1. The main reason for taking the economical parameter as an imprecise parameter

For modeling real world problem in economical sciences, all of the parameters are not necessarily a fixed quantity but it may frequently differ due to various reasons. Thus, the parameters used in biological model are sensible and it can be considered as imprecise number on the place of crisp number. To handle such type of impreciseness many researchers take different approaches like fuzzy approach, stochastic approach, and interval approach etc. In the fuzzy set theory cases the imprecise parameters can be treated by fuzzy numbers whereas in stochastic approach the random variables are used with known probability distributions. The concept of membership and non-membership function comes for intuitionistic fuzzy approach. The concept of interval number and interval valued function comes when we take interval approach. Moreover, it is very tough to suitable imprecise biological parameters and use it for modeling. In this paper we take some parameter as fuzzy number and try to investigate the behavior of the model.

1.2. Review on fuzzy difference equation

In recent times the study of difference equations and their system has grown an immense interest of researchers. A difference equation is in fact an equation which specifies the transform of the variable between two periods. A difference equation is said to be a fuzzy difference equation if (i) initial data is fuzzy number, (ii) coefficients are fuzzy number, (iii) initial data and coefficients are both fuzzy numbers. There exists different research where fuzzy difference equation is studied. Some of the published articles are recommended to understand the theory well as [15, 17–19, 26, 34, 39–41, 43–46, 60, 63, 64, 68].

1.3. Review on Fuzzy differential equation

The application of differential equations has been widely explored in various fields like engineering, economics, biology, and physics. For constructing different type of problems in real life situation the fuzzy set theory plays an important role. The applicability of non-sharp or imprecise concept is very useful for explored different sectorfor its applicability. A differential equation may be called fuzzy differential equation if: (1) only the coefficient (or coefficients) of the differential equation is fuzzy valued number (2) only the initial value (or values) or boundary value (or values) are fuzzy valued number (3) the forcing term is fuzzy valued function (4) All the condition (1)–(3) or its combinations are present on the differential equation.

There exist two types of strategy for solving the FDEs, which are

(a) Ignoring the definition of fuzzy derivative
(b) Based on the definition of fuzzy derivative.

Now we look on some different procedure and concepts of derivation in the following table:

Fuzzy differential equation	
Ignoring the definition of fuzzy derivative	
Name of the theory	Previous authors' work
Fuzzy differential inclusion	Baidosov [8], Hllermeier [24]
Zadeh's Extension principle	Oberguggenberger and Pittschmann [42], Buckley and Feuring [12]

Based on the definition of fuzzy derivative	
Name of the theory	Some references
Dubois–Prade derivative	Dubois and Prade [20]
Puri–Ralescu derivative	Puri and Ralescu [54]
Goetschel–Voxman derivative	Goetschel and Voxman [22]
Friedman–Ming–Kandel derivative	Friedman *et al.* [21]
Seikkala derivative	Seikkala [5, 13, 58]
SGH derivative	Bede and Gal [10, 13, 62]
Same-order and reverse-order derivative	Yue and Guangyuan [66]
π-derivative	Chalco-Cano *et al.* [13, 14]
gH derivative	Stefanini and Bede [2, 3, 6, 25, 30, 31, 61, 62]
g derivative	Bede and Stefanini [1, 7, 11, 62]
H_2 derivative	Mazandarani and Najariyan [32]
Interactive derivative	Barros and Pedro [16]
gr derivative	Mazandarani [33]

Recently there exists very little of the literature where fuzzy differential equation is applied in real life problem, see [4, 23, 27–29, 35–38, 40, 41, 48–50, 52, 53, 55–57, 59][6].

1.4. Motivation

Mathematical modeling is very important for any research methodologies. Specially the economics theory has grown much attention nowadays. National income determination is very important for researchers. It is important to determine the behavior of a system with uncertainty. Moreover, modeling the systems for discrete and continuous cases with uncertainty has got more attention. Thus, for solving these problems, one may use the concept of fuzzy difference and fuzzy differential equation.

1.5. Noveltie

Some of the developments mentioned above have already been accomplished. Innovations of this article are mentioned below:

(a) National income determination model is considered in discrete and continuous system in fuzzy environment.
(b) Fuzzy difference equation concept is applied here.
(c) Fuzzy differential equation concept is also applied here.
(d) Stability criterion of fuzzy difference equation is described.
(e) Stability criterion of fuzzy differential equation is described.
(f) Numerical results are taken and the solution and stability is determined to show the importance of the proposed model.

1.6. Structure of the paper

The construction of the article is as follows: in first section we set up the prior work on fuzzy set theory, fuzzy differential and difference equation, notation, and assumptions of the model. Second section is full of preliminary concept. Stability analysis of difference equation and differential equation are addressed in Section 3. The algorithm for solving a fuzzy model is described in Section 4. National income determination model with discrete process in fuzzy environment is described in Section 5. In Section 6, we have described a national income determination model with continuous process in fuzzy environment. The conclusion is made in Section 7.

[6]Basically the appliance of gH difference and derivative is shown in this article. Anyone can take different approaches for derivative of fuzzy function. The results are not same if we use different derivative approaches. Considering generalized Hukuhara difference and differentiability we transformed the associated fuzzy differential equation into system of ordinary differential equation system and which it is nothing but two differential equations with some parametric form of a fuzzy number (Similar thing may happen for fuzzy difference equation).

1.7. Notation

Notation and abbreviations are used within this paper as follows:

$\tilde{\mathcal{F}}$	fuzzy set which is definite by a pair $\left(\mathcal{U}, \mu_{\tilde{\mathcal{F}}}(y)\right)$
$\mathcal{U}$	universal nonempty set
b	autonomous
I^*	investment per period ($/period)
a	marginal propensity to consume
$\mu_{\tilde{\mathcal{F}}}(\mathscr{Y})$	gradation of the membership function of $\mathscr{Y}$ in $\tilde{\mathcal{F}}$
$\tilde{\mathcal{F}}_i$	fuzzy number in triangular, where $i = 1, 2, 3$
$\mathscr{P}, \mathscr{Q}$	two fuzzy functions
$\mathcal{G} : [a, b] \to \Re_{\mathcal{F}}$	parametric representations of a fuzzy valued function
U^*	equilibrium solution
ϑ_i	real and distinct Eigen values of the coefficients matrix, where $i = 1, 2$
ϑ^*	real and equal Eigen value of the coefficients matrix
$\gamma \pm i\delta$	complex conjugate Eigen values of the coefficients matrix
t	time period (yr)
C_t	consumption in period t (units/yr)
Y	national income ($/yr)
τ	positive adjustment coefficient
gH	general Hukuhara
$\mathcal{G}$	generalized Hukuhara differentiable at t_0
$\mathfrak{I}$	an interval of real number set
$\mathfrak{A}, \mathfrak{B}$	positive numbers
$\widetilde{\mathcal{F}_\alpha}$	α-cut of $\tilde{\mathcal{F}}$

1.8. Assumptions

1. A simple two-sector model is introduced with structural equations.
2. The national income determination model is considered in both discrete and continuous system under fuzzy environment.
3. Fuzzy difference and differential equation's concept are applied here.
4. Stability criterion of both the fuzzy difference and fuzzy differential equations is defined.
5. Generalized Hukuhara difference and differentiability are considered to convert associated fuzzy differential equation into system of ordinary differential equation.
6. Initial income is taken as fuzzy *i.e.*, $Y_0 = \tilde{Y}_0$.
7. The flow of money from firms to households is given as payment for the factors of production.
8. The portion side is the total flow of money received by the firms, either in the form of investment or as payment for goods bought by households.
9. The gH-difference and gH-derivative approach are utilized.
10. Uncertain parameters are estimated, whose values are not precisely known.

2. Preliminaries

Definition 2.1 (Fuzzy set)**.**
Let $\tilde{\mathcal{F}}$ be a fuzzy set which is definite by a pair $(\mathcal{U}, \mu_{\tilde{\mathcal{F}}}(y))$, where $\mathcal{U}$ be auniversal nonempty set and

$$\mu_{\tilde{\mathcal{F}}}(y) : \mathcal{U} \to [0, 1].$$

For every $\mathscr{Y} \in \mathcal{U}$, $\mu_{\tilde{\mathcal{F}}}(\mathscr{Y})$ is the gradation of the membership function of $\mathscr{Y}$ in $\tilde{\mathcal{F}}$.

Definition 2.2 (Fuzzy number in triangular form).

A fuzzy number in triangular form represented by $\tilde{\mathcal{F}} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ and the presentation can be illustrated as membership function as below:

$$\mu_{\tilde{\mathcal{F}}}(y) = \begin{cases} 0, & y \leq \mathcal{F}_1 \\ \frac{y-\mathcal{F}_1}{\mathcal{F}_2-\mathcal{F}_1}, & \mathcal{F}_1 \leq y \leq \mathcal{F}_2 \\ 1, & y = \mathcal{F}_2 \\ \frac{\mathcal{F}_3-y}{\mathcal{F}_3-\mathcal{F}_2}, & \mathcal{F}_2 \leq y \leq \mathcal{F}_3 \\ 0, & y \geq \mathcal{F}_3. \end{cases}$$

Definition 2.3 (Parametric form of a fuzzy number $\tilde{\mathcal{F}}$).

The α-cut of $\tilde{\mathcal{F}} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is given by

$$\mathcal{F}_\alpha = [\mathcal{F}_1 + \alpha(\mathcal{F}_2 - \mathcal{F}_1), \mathcal{F}_3 - \alpha(\mathcal{F}_3 - \mathcal{F}_2)], \quad \forall \alpha \in [0,1].$$

Definition 2.4 (Generalized Hukuhara difference for two fuzzy valued function).

The generalized Hukuhara difference of two fuzzy functions $p, q \in \Re_{\mathcal{F}}$ are defined by the formulae

$$p \ominus_g q = r \Leftrightarrow \begin{cases} (i)\, p = q \oplus r \\ \text{or } (ii)\, p = q \oplus (-1) r \end{cases}$$

where one can take $[r]_\alpha = [r_1(\alpha), r_1(\alpha)]$, then $r_1(\alpha) = \min\{p_1(\alpha) - q_1(\alpha), p_2(\alpha) - q_2(\alpha)\}$ and $r_2(\alpha) = \max\{p_1(\alpha) - q_1(\alpha), p_2(\alpha) - q_2(\alpha)\}$.

Here, the parametric representations of a fuzzy valued function $\mathcal{G} : [a,b] \to \Re_{\mathcal{F}}$ is expressed by

$$[\mathcal{G}(t)]_\alpha = [\mathcal{G}_1(t,\alpha), \mathcal{G}_2(t,\alpha)], \quad t \in [a,b], \alpha \in [0,1].$$

Definition 2.5 (Generalized Hukuhara derivative on a fuzzy valued function).

The generalized Hukuhara derivative of a fuzzy valued function $\mathcal{G} : (a,b) \to \Re_{\mathcal{F}}$ at t_0 is defined as

$$\mathcal{G}'(t_0) = \lim_{h \to 0} \frac{\mathcal{G}(t_0 + h) \ominus_g \mathcal{G}(\ominus_g)}{h}. \tag{2.1}$$

If $\mathcal{G}'(t_0) \in \Re_{\mathcal{F}}$ satisfying the equation (2.1) exists finitely, we can say that $\mathcal{G}$ is generalized Hukuhara differentiable at t_0.

Also we can say that $\mathcal{G}(t)$ is (i)-gH differentiable at t_0 if

$$[\mathcal{G}'(t_0)]_\alpha = [\mathcal{G}_1'(t_0,\alpha), \mathcal{G}_2'(t_0,\alpha)] \tag{2.2}$$

and $\mathcal{G}(t)$ is (i)-gH differentiable at t_0 if

$$[\mathcal{G}'(t_0)]_\alpha = [\mathcal{G}_2'(t_0,\alpha), \mathcal{G}_1'(t_0,\alpha)]. \tag{2.3}$$

Definition 2.6 (Strong and Weak solution of fuzzy differential equation and fuzzy difference equation).

Take the first order linear fuzzy differential equation of type $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x(t))$ with $x(t_0) = x_0 \cdot (k$ or (and) x_0 be fuzzy number(s)) or, consider the first order linear difference equation of type $x_{n+1} = f(x_n, x_{n-1})$ with $x_{n=0} = x_0$.

Now if the solution (possible solution) of the above differential or difference equation be $\tilde{x}(t)$ and its parametric form or α-cut be $x(t,\alpha) = [x_1(t,\alpha), x_2(t,\alpha)]$.

The condition:

(i) $x_1(t,\alpha) \leq x_2(t,\alpha)$ and
(ii) $x_1(t,\alpha)$ is increasing and $x_2(t,\alpha)$ is decreasing as the condition that α goes from 0 to 1, then $\tilde{x}(t)$ is called strong solution otherwise $\tilde{x}(t)$ is called weak solution. For the case of weak solution the α-cut of the solution is given by

$$x(t,\alpha) = [\min\{x_1(t,\alpha), x_2(t,\alpha)\}, \max\{x_1(t,\alpha), x_2(t,\alpha)\}].$$

Theorem 2.7. *If we take $\mathcal{I}$ is an interval of real number set and consider $\hbar : \mathcal{I} \times \mathcal{I} \rightarrow \mathcal{I}$ be the continuous function.*

Consider the difference equation

$$\mathcal{X}_{t+1} = \hbar(\mathcal{X}_t, x_{t-1}),\ t = 0, 1, \ldots, \tag{2.4}$$

with the initial data are $t_{-1}, t_0 \in \mathcal{I}$ and $\hbar$ obeys the following criteria as mentioned below:

(a) *There exist positive number $\mathfrak{A}$ and $\mathfrak{B}$ with $\mathfrak{A} < \mathfrak{B}$ so that $\mathfrak{A} \leq \hbar(\mathcal{X}, \mathcal{Y}) \leq \mathfrak{B}$ for all $\mathcal{X}, \mathcal{Y} \in [\mathfrak{A}, \mathfrak{B}]$.*
(b) *$\hbar(\mathcal{X}, \mathcal{Y})$ is increasing function in $\mathcal{X} \in [\mathfrak{A}, \mathfrak{B}]$ for each and every $\mathcal{Y} \in [\mathfrak{A}, \mathfrak{B}]$, and $\hbar(x, y)$ is decreasing in $\mathcal{Y} \in [\mathfrak{A}, \mathfrak{B}]$ for each and every $\mathcal{X} \in [\mathfrak{A}, \mathfrak{B}]$.*
(c) *(a) has no solutions in the prime period two in $[\mathfrak{A}, \mathfrak{B}]$.*

Then it can be said that there exists exactly one equilibrium solution $\tilde{\mathcal{X}}$ of (2.4) which lies in the interval $[\mathfrak{A}, \mathfrak{B}]$. In addition, each and every solution of (2.4) with initial data $\mathcal{X}_{-1}, \mathcal{X}_0 \in [\mathfrak{A}, \mathfrak{B}]$ converges to $\tilde{\mathcal{X}}$.

Theorem 2.8 (The existence and uniqueness of the solution)**.**

Consider the fuzzy initial value problem as

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t, x(t)), t \in [t_0, T] \tag{2.5}$$

with initial condition $x(t_0) = a_0$, where $f : I \times E \rightarrow E$ is a continuous fuzzy mapping and a_0 or the coefficient of the differential equation or both are fuzzy number. The interval may be like $[0,T]$ for some $T > 0$ or $I = [0, \infty)$.

If $f : I \times E \rightarrow E$ is a continuous fuzzy function such that there exists $k > 0$ such that $(f(t,y), f(t,z)) \leq kD(y,z)$, $\forall t \in I, y, z \in E$. Then (2.5) has two different solutions namely (i)-gH differentiable solution and (ii)-gH differentiable solution on I.

Proof. The study [51] provides the proof with the fuzzy initial condition. The proof for fuzzy environment is given below:

First, the characterization theorem is found when the function has (i)-gH fferentiable solution.

The problem has a generalized interval valued fuzzy number whose parametric representations are like

$$(\tilde{a}_0)_\alpha = [a_l(\alpha), a_r(\alpha)].$$

Also consider that

$$(x(t))_\alpha = [x_l(t,\alpha), x_r(t,\alpha)]$$

and since it is (i)-gH differentiable,

$$\left[\frac{x(t)}{\mathrm{d}t}\right]_\alpha = \left[\frac{x_l(t,\alpha)}{\mathrm{d}t}, \frac{x_r(t,\alpha)}{\mathrm{d}t}\right].$$

Now, after taking parametric form of (2.5), it becomes

$$\frac{x_l(t,\alpha)}{\mathrm{d}t} = F(t, x_l(t,\alpha), x_r(t,\alpha))$$
$$\frac{x_r(t,\alpha)}{\mathrm{d}t} = G(t, x_l(t,\alpha), x_r(t,\alpha))$$

with initial condition $x_l(t_0, \alpha) = a_l(\alpha)$, $x_r(t_0, \alpha) = a_r(\alpha)$, where $[F, G]$ is the parametric form of the fuzzy valued function $f(t, x(t))$.

Similarly, for (ii)-gH differentiable case,

$$\left[\frac{x(t)}{\mathrm{d}t}\right]_\alpha = \left[\frac{x_r(t,\alpha)}{\mathrm{d}t}, \frac{x_l(t,\alpha)}{\mathrm{d}t}\right].$$

Now, after taking the parametric form of (2.5), it becomes

$$\frac{x_r(t,\alpha)}{\mathrm{d}t} = F(t, x_l(t,\alpha), x_r(t,\alpha))$$
$$\frac{x_l(t,\alpha)}{\mathrm{d}t} = G(t, x_l(t,\alpha), x_r(t,\alpha)),$$

with initial condition $x_l(t_0, \alpha) = a_l(\alpha)$, $x_r(t_0, \alpha) = a_r(\alpha)$, where $[F, G]$ is the parametric form of the fuzzy valued function $f(t, x(t))$. □

Remarks. The corresponding transformed system is a crisp system of differential equation. The existences of the solution can be easily proved. The existence and uniqueness condition for fuzzy differential equation is already provide in the existing literature [23, 47, 65].

3. Stability analysis of the difference equation and differential equation

This section contains stability analysis for both difference and differential equation.

3.1. Stability analysis of linear difference equation

An autonomous linear discrete equation (which is also namely difference equation) of the given form (see Ref. [9]) is considered as follows:

$$U_t = \mathcal{a}U_{t-1} + \mathcal{b}(\mathcal{a} \neq 0).$$

If U^* is the equilibrium solution of the model, then $U_t = U_{t-1} = U^*$ (there is no change from generation $n-1$ to generation n)

$$i.e.,\ \mathcal{a}U^* + \mathcal{b} = U^*$$
$$i.e.,\ U^* = \frac{\mathcal{b}}{1-\mathcal{a}}.$$

The equilibrium point *i.e.*, U^* is said to be stable if all the solutions of the above difference equation approaches to $\frac{\mathcal{b}}{1-\mathcal{a}}$ as n becomes large. The equilibrium point is unstable if all solutions (if exist) diverge from u^* to $\pm\infty$. The stabilities of the equilibrium solutions depend on the constant $\mathcal{a}$. It can be said that it is stable if $|\mathcal{a}| < 1$ and unstable if $|\mathcal{a}| > 1$. We reach an ambiguous result if $\mathcal{a} = \pm 1$.

3.1.1. *Stability analysis of system linear homogeneous difference equation*

The system linear difference equations in general form are considered as follows:

$$U_{n+1} = \mathcal{a}U_n + \mathcal{b}V_n$$
$$V_{n+1} = cU_n + dV_n.$$

The equations can be written as in the matrix equation form as,

$$\begin{bmatrix} U_{n+1} \\ V_{n+1} \end{bmatrix} = \begin{bmatrix} \mathcal{a} & \mathcal{b} \\ c & d \end{bmatrix} \begin{bmatrix} U_n \\ V_n \end{bmatrix}.$$

One can find that $(0, 0)$ is the equilibrium point of the corresponding homogeneous system.

Theorem 3.1. *Let ϑ_1 and ϑ_2 be two real and distinct eigenvalues of the coefficients matrix of the above system. Then the equilibrium point $(0,0)$ is*

(a) *Stable when $|\vartheta_1| < 1$ and $|\vartheta_2| < 1$*
(b) *Unstable when $|\vartheta_1| > 1$ and $|\vartheta_2| > 1$*
(c) *Saddle when $|\vartheta_1| < 1$, $|\vartheta_2| > 1$ or $|\vartheta_1| > 1$, $|\vartheta_2| < 1$.*

Proof. It is omitted due to the simplicity. □

Theorem 3.2. *Let us consider that $\vartheta_1 = \vartheta_2 = \vartheta^*$ be real and equal eigenvalues of the coefficients matrix then the equilibrium point $(0,0)$ is*

(a) *Stable when $|\vartheta^*| < 1$*
(b) *Unstable when $|\vartheta^*| > 1$.*

Proof. It is omitted due to the simplicity of the proof. □

Theorem 3.3. *Take $\gamma + i\delta$ and $\gamma - i\delta$ be the complex conjugate eigenvalues of the coefficients matrix the equilibrium point $(0,0)$ is*

(a) *Stable when $|\gamma \pm i\delta| < 1$*
(b) *Unstable when $|\gamma \pm i\delta| > 1$.*

Proof. It is omitted due to the simplicity of the theorem. □

3.1.2. Stability analysis of system linear non homogeneous difference equation

The system of linear difference equations is as follows:

$$U_{n+1} = aU_n + bV_n + G$$
$$V_{n+1} = cU_n + dV_n + H.$$

Considering a transformation such that it changes to

$$\mathcal{X}_{n+1} = a\mathcal{X}_n + b\mathcal{Y}_n$$
$$\mathcal{X}_{n+1} = c\mathcal{X}_n + d\mathcal{Y}_n.$$

Then proceeding similar way as previous section we can check the stability criterion.

3.2. Stability analysis of linear system of differential equation

The system of linear differential equation of the form is taken as

$$\frac{\mathrm{d}\mathcal{X}(t)}{\mathrm{d}t} = \mathcal{A}\mathcal{X}(t) + \mathcal{B}(t)$$

where $\mathcal{X}(t) = (u_1 u_2 u_3 \dots u_n)^T$ and $\mathcal{A} = (a_{ij})_{n\times n}$ $(i,j = 1,2,3,\dots,n)$ and $\mathcal{B}(t) = (g_1 g_2 g_3 \dots g_n)^T$.

The above system can be written in the form

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = a_{11}u_1 + a_{12}u_2 + \dots + a_{1n}u_n + g_1$$
$$\frac{\mathrm{d}u_2}{\mathrm{d}t} = a_{21}u_1 + a_{22}u_2 + \dots + a_{2n}u_n + g_2$$
$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = a_{n1}u_1 + a_{n_2}u_2 + \dots + a_{nn}u_n + g_n.$$

The above system is called system of linear homogeneous differential equation if $\mathcal{B}(t)$ is the matrix whose all elements are zero and non-homogeneous if the element of $\mathcal{B}(t)$ are not all zero.

Now we try to find the stability criterion of linear system of homogeneous and non-homogeneous differential equation.

3.2.1. Stability analysis of system of linear homogeneous differential equation

If the above $\mathcal{g}_1, \mathcal{g}_2, \ldots, \mathcal{g}_n = 0$, then the above system is written as

$$\begin{bmatrix} \frac{\mathrm{d}u_1}{\mathrm{d}t} \\ \frac{\mathrm{d}u_2}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}u_n}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

$$i.e., \ \frac{\mathrm{d}\mathcal{X}(t)}{\mathrm{d}t} = \mathcal{A}\mathcal{X}(t)$$

where $\mathcal{X} = \begin{pmatrix} u_1 & u_2 & \ldots & u_n \end{pmatrix}^T$ and $\mathcal{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$.

Now we can find the eigenvalues of the matrix $\mathcal{A}$. The stability criterion depends on the nature of the eigenvalues.

Theorem 3.4 (Routh–Hurwitz criteria)**.**

Routh–Hurwitz criteria is if the characteristic equation of the given matrix is like

(a) *For quadratic equation* $\vartheta^2 + a_1\vartheta + a_2 = 0$*, the stability criteria are* $a_1 > 0, a_2 > 0$
(b) *For cubic equation* $\vartheta^3 + a_1\vartheta^2 + a_2\vartheta + a_3 = 0$*, the stability criteria are* $a_1 > 0, a_2 > 0$*,* $a_3 > 0, a_1a_2 - a_3 > 0$
(c) *For biquadratic equation* $\vartheta^4 + a_1\vartheta^3 + a_2\vartheta^2 + a_3\vartheta + a_4 = 0$*, the stability criteria are* $a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0, a_1a_2 - a_3 > 0, a_1a_2a_3 - a_3^2 - a_1^2a_4 > 0$.

It should be noted that there are some theorems on the basis of the stability of nth order differential equation. To construct the theory in this paper, we use the above three theorems.

3.2.2. Stability analysis of system linear nonhomogeneous differential equation

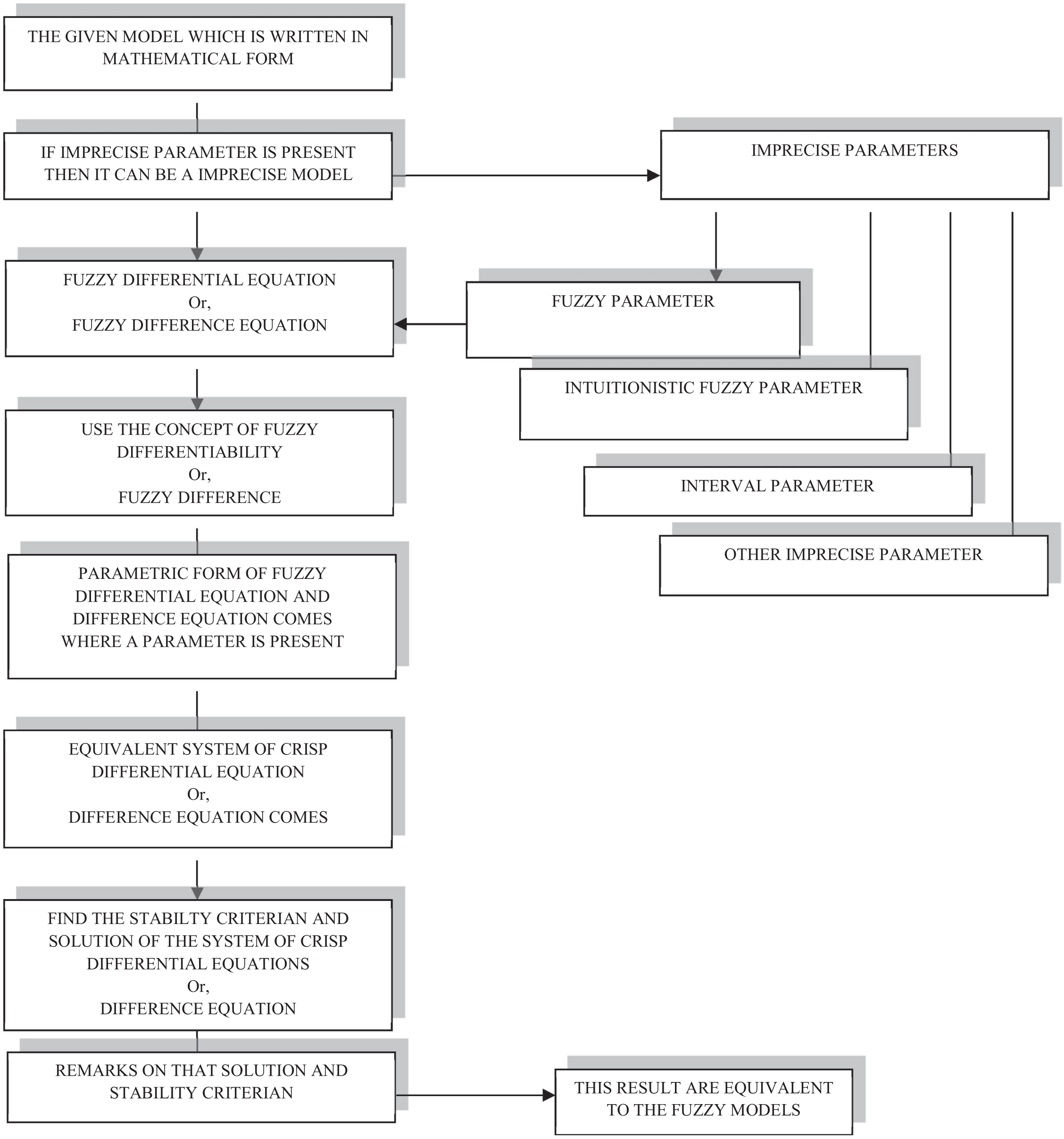
Now if the system of differential equation is nonhomogeneous then we can perform a transformation such that $u_1 = u_1' + \epsilon_1$, $u_2 = u_2' + \epsilon_2$ and $u_n = u_n' + \epsilon_n$ the nonhomogeneous system of linear equation becomes

$$\begin{bmatrix} \frac{\mathrm{d}u_1'}{\mathrm{d}t} \\ \frac{\mathrm{d}u_2'}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}u_n'}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix}.$$

The above system of differential equation is system of homogeneous differential equation. Therefore, the stability analysis is same as previous case.

4. Flowchart for solving mathematical problem associated with differential equation and difference equation in fuzzy environment

Now it is very important challenge how a mathematical model is solved when it is defined in imprecise environment and how can we conclude some remarks for futuristic study. A crisp model is very easy to handle as compared to the model with uncertainties. There are some concepts by which we can convert an imprecise system to a corresponding system of crisp system. In this section one can make a flow chart for the readers for understanding the conversion of imprecise to crisp system.



5. National income determination model with discrete process in fuzzy environment

The basic concept for modeling discrete system is that the present situation is not dependent on the previous situation. Only the initial data may be taken. Now our main motive is to explore that how national income determination model is in discrete system, what is its solution with fuzzy initial condition and the model is stable for a specific situation.

5.1. Model formulation

This section contains crisp problem, numerical experiments, and analysis of the model.

5.1.1. Crisp problem

We have introduced a simple two-sector model with structural equations

$$Y = C + I$$
$$C = aY + b$$
$$I = I^*$$

where b and I^* denote autonomous and investment and a is the marginal propensity to consume, which in the range $0 < a < 1$. In writing down the equations in this form, we are implicitly assuming that only one-time period is involved, that consumption depends on national income within the involved time period, which consumes instantaneously. In practice, there is a time lag between consumption and national income. Consumption, C_t, in period t depends on national income, Y_{t-1}, in the previous period, $t-1$. The corresponding consumption function is given by

$$C_t = aY + b.$$

If one can assume that investment is the same in all time periods, then

$$I = I^*.$$

Finally, if the flow of money is the balance in each time period, one has

$$Y_t = C_t + I_t.$$

Substituting the expressions for C_t and I_t into above equation, it can be found

$$Y_t = aY_{t-1} + b + I^*.$$

which one can recognize as a difference equation of the standard from give in this section. This equation can be solved and the time path is analyzed.

5.1.2. Fuzzy problem

Now, if the decision maker calculates the national income after certain times where he/she does not know the exact actual income at previous time then how is his/her approach. The easiest thing is to make the initial income as fuzzy and find the solution.

The above problem is treated as a fuzzy problem if any one of a, b, I^* or the initial condition Y_0 is fuzzy number.

This paper considers the initial condition as fuzzy number *i.e.*, $Y_0 = \tilde{Y}_0$.

Solution. The problem becomes
$Y_t = aY_{t-1} + b + I^*$ with $Y_0 = \tilde{Y}_0$.
Taking α-cut of the equation, one can find

$$Y_{t,1}(\alpha) = aY_{t-1,1}(\alpha) + b + I^*$$

and

$$Y_{t,2}(\alpha) = aY_{t-1,2}(\alpha) + b + I^*$$

with initial condition $Y_{t=0,1}(\alpha) = Y_{01}(\alpha)$ and $Y_{t=0,2}(\alpha) = Y_{02}(\alpha)$

The general solution is written as

$$Y_{t,1}(\alpha) = Y_{01}(\alpha)\, a^t + \frac{1-a^t}{1-a}(b + I^*)$$

and

$$Y_{t,2}(\alpha) = Y_{02}(\alpha)\, a^t + \frac{1-a^t}{1-a}(b + I^*).$$

5.1.3. Stability analysis of fuzzy problem

The fuzzy problem is written as

$$\begin{bmatrix} Y_{t,1}(\alpha) \\ Y_{t,2}(\alpha) \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} Y_{t-1,1}(\alpha) \\ Y_{t-1,2}(\alpha) \end{bmatrix} + \begin{bmatrix} b + I^* \\ b + I^* \end{bmatrix}.$$

Now after giving transformation $Y_{t,1}(\alpha) = Y'_{t,1}(\alpha) + \epsilon_1$ and $Y_{t,2}(\alpha) = Y'_{t,2}(\alpha) + \epsilon_2$, one can find

$$\begin{bmatrix} Y'_{t,1}(\alpha) \\ Y'_{t,2}(\alpha) \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} Y'_{t-1,1}(\alpha) \\ Y'_{t-2,2}(\alpha) \end{bmatrix}.$$

Eigenvalues of coefficient matrix is $\lambda = a, a$.

The real and equal eigenvalue of the coefficients matrix is a then the equilibrium point $(0, 0)$ is

(i) Stable if $|a| < 1$
(ii) Unstable if $|a| > 1$.

5.2. Numerical illustration

Example 5.1. Solve the problem when $a = 0.8$, $b = 100$, $I^* = 200$ ($/yr) and the initial condition is $\tilde{Y}_0 = (1500, 1700, 1800)$ ($/yr)

Solution. The solution is written as

$$Y_{t,1}(\alpha) = 1500 + 200\alpha(0.8)^t$$

and

$$Y_{t,2}(\alpha) = 1500 + (300 - 100\alpha)(0.8)^t.$$

Remarks. Figure 1 and Table 1 are for $Y_{t,1}(\alpha)$ and $Y_{t,2}(\alpha)$ when $t = 2, 3, 5$, and 8, respectively. For each t, $Y_{t,1}(\alpha)$ is increasing function and $Y_{t,2}(\alpha)$ is a decreasing function as α from 0 to 1. Hence, one can conclude that the solution is a strong solution.

Stability situation. The problem is stable as $|a = 0.8| < 1$ (using the result of Sect. 3.1.1).

Example 5.2. Solve the problem when $a = 0.9$, $b = 250$, $I^* = 350$ ($/yr) and the initial condition is $\tilde{Y}_0 = (6400, 6500, 6800)$ ($/yr).

Solution. The solutions are written as

$$Y_{t,1}(\alpha) = 600 + (5800 + 100\alpha)(0.9)^t$$

and

$$Y_{t,2}(\alpha) = 600 + (6200 - 300\alpha)(0.9)^t.$$

Remarks. Table 2 and Figure 2 are for $Y_{t,1}(\alpha)$ and $Y_{t,2}(\alpha)$ where $t = 4, 8, 10$, and 12, respectively. For each t, $Y_{t,1}(\alpha)$ is increasing function and $Y_{t,2}(\alpha)$ is a decreasing function as α from 0 to 1. The results must go to a strong solution.

Stability situation. The problem is stable as $|a = 0.9| < 1$ (using the result of Sect. 3.1.1).

TABLE 1. Solution for different values of t in Example 5.1.

$t=2$ (yr)			$t=3$ (yr)		
α	$Y_{t,1}(\alpha)$ (\$/yr)	$Y_{t,2}(\alpha)$ (\$/yr)	α	$Y_{t,1}(\alpha)$ (\$ /yr)	$Y_{t,2}(\alpha)$ (\$/yr)
0	1500.00	1692.00	0	1500.00	1653.60
0.1	1512.80	1685.60	0.1	1510.24	1648.48
0.2	1525.60	1679.20	0.2	1520.48	1643.36
0.3	1538.40	1672.80	0.3	1530.72	1638.24
0.4	1551.20	1666.40	0.4	1540.96	1633.12
0.5	1564.00	1660.00	0.5	1551.20	1628.00
0.6	1576.80	1653.60	0.6	1561.44	1622.88
0.7	1589.60	1647.20	0.7	1571.68	1617.76
0.8	1602.40	1640.80	0.8	1581.92	1612.64
0.9	1615.20	1634.40	0.9	1592.16	1607.52
1	1628.00	1628.00	1	1602.40	1602.40
$t=5$ (yr)			$t=8$ (yr)		
0	1500.00	1598.30	0	1500.00	1550.33
0.1	1506.55	1595.02	0.1	1503.35	1548.65
0.2	1513.10	1591.75	0.2	1506.71	1546.97
0.3	1519.66	1588.47	0.3	1510.06	1545.29
0.4	1526.21	1585.19	0.4	1513.42	1543.62
0.5	1532.76	1581.92	0.5	1516.77	1541.94
0.6	1539.32	1578.64	0.6	1520.13	1540.26
0.7	1545.87	1575.36	0.7	1523.48	1538.58
0.8	1552.42	1572.08	0.8	1526.84	1536.90
0.9	1558.98	1568.81	0.9	1530.19	1535.23
1	1565.53	1565.53	1	1533.55	1533.55

5.3. Analysis of the model

Clearly, the national income determination model is considered with a discrete process. In this process, the model turned into difference equation. Now when one considers it in fuzzy environment it differs in behavior due to different behavior between two fuzzy functions. One can convert the difference equation into two systems of difference equations and solve them to check whether it obeys the fuzzy rules or not.

6. National income determination model with continuous process in fuzzy environment

The basic concept for modeling in continuous system is that the present situation depends on the previous situation. Now the main motive is to determine how national income determination model works in continuous system, what is its solution with fuzzy initial condition and the model stability in a specific situation.

6.1. Model formulation

This section contains crisp problem, fuzzy problem, and stability analysis of fuzzy problem.

6.1.1. Crisp problem

The defining equations of the usual two-sector model are

$$Y = C + I \tag{6.1}$$

$$C = aY + b \tag{6.2}$$

$$I = I^*. \tag{6.3}$$

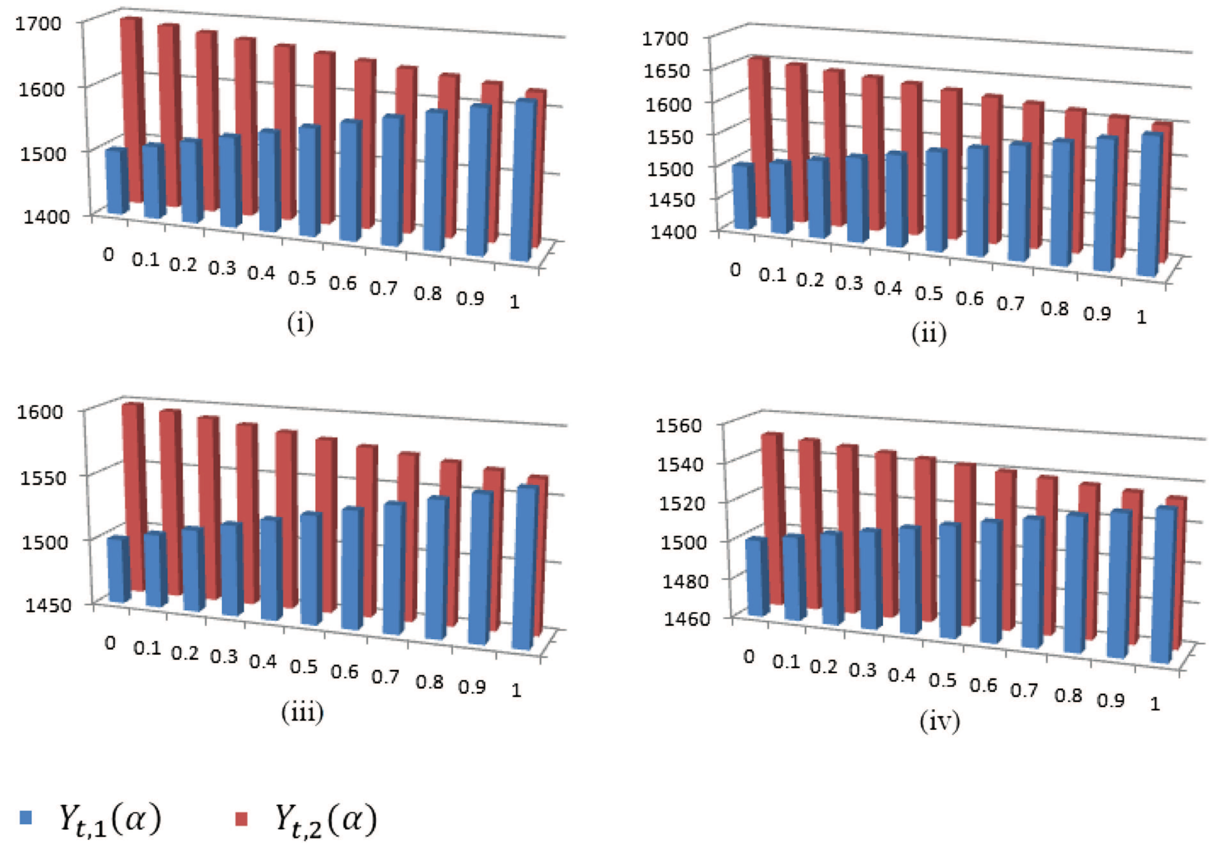


FIGURE 1. Figure of $Y_{t,1}(\alpha)$ and $Y_{t,2}(\alpha)$ for different values of t in Example 5.1.

The first of these is simply a statement that the economy is already in the balance. The left-hand portion of the equation (6.1) is the flow of money from firms to households given as payment for the factors of production. The portion side is the total flow of money received by the firms, either in the form of investment, or as payment for goods bought by households. In practice, the equilibrium values are not immediately attained and one may need to make an alternative assumption about how Y is proportional to the excess expenditure, $C+I-Y$: that is

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \tau\left(C + I - Y\right). \tag{6.4}$$

For some positive adjustment coefficient τ, this makes sense because

(x) if $C + I > Y$, it gives $\mathrm{d}Y/\mathrm{d}t > 0$ and thus Y rises in order to achieve a balance expenditure and income.
(y) if $C + I = Y$, it gives the $\mathrm{d}Y/\mathrm{d}t = 0$ and thus Y is held constant at the equilibrium level.
(z) if $C + I < Y$, it gives $\mathrm{d}Y/\mathrm{d}t < 0$ and thus Y falls in order to achieve a balance between expenditure and income.

Thus usual relations (6.2) and (6.3) can be substituted into the new equation (6.4) to obtain

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \tau\left(aY + b + I^{*} - Y\right) \tag{6.5}$$

$$= \tau\left(a - 1\right)Y + \alpha\left(b + I^{*}\right) \tag{6.6}$$

TABLE 2. Solution for different values of t in Example 5.2.

$t = 4$ (yr)			$t = 8$ (yr)		
α	$Y_{t,1}(\alpha)$ (\$/yr)	$Y_{t,2}(\alpha)$ (\$/yr)	α	$Y_{t,1}(\alpha)$ (\$/yr)	$Y_{t,2}(\alpha)$ (\$/yr)
0	4405.38	4667.82	0	3096.70	3268.89
0.1	4411.94	4648.13	0.1	3101.01	3255.98
0.2	4418.50	4628.45	0.2	3105.31	3243.06
0.3	4425.06	4608.77	0.3	3109.62	3230.15
0.4	4431.62	4589.08	0.4	3113.92	3217.24
0.5	4438.18	4569.40	0.5	3118.23	3204.32
0.6	4444.74	4549.72	0.6	3122.53	3191.41
0.7	4451.30	4530.03	0.7	3126.84	3178.49
0.8	4457.86	4510.35	0.8	3131.14	3165.58
0.9	4464.42	4490.67	0.9	3135.45	3152.67
1	4470.99	4470.99	1	3139.75	3139.75
$t = 10$ (yr)			$t = 12$ (yr)		
0	2622.33	2761.80	0	2238.09	2351.06
0.1	2625.82	2751.34	0.1	2240.91	2342.59
0.2	2629.30	2740.88	0.2	2243.73	2334.11
0.3	2632.79	2730.42	0.3	2246.56	2325.64
0.4	2636.28	2719.96	0.4	2249.38	2317.17
0.5	2639.76	2709.50	0.5	2252.21	2308.69
0.6	2643.25	2699.04	0.6	2255.03	2300.22
0.7	2646.74	2688.58	0.7	2257.86	2291.75
0.8	2650.22	2678.12	0.8	2260.68	2283.28
0.9	2653.71	2667.66	0.9	2263.51	2274.80
1	2657.20	2657.20	1	2266.33	2266.33

which are recognized as a differential equation of the standard form given in this section[7].

6.1.2. Fuzzy problem

Now if the decision maker calculates the national income after certain times where he does not know the exact actual income at previous time then how his/her approach varies. The easiest thing is to make the initial income as fuzzy and find the solution.

The above problem is treated as a fuzzy problem if any one of a, b, τ, I^* or the initial condition Y_0 is fuzzy number.

This paper considers the initial condition as fuzzy number *i.e.*, $Y_0 = \tilde{Y}_0$.

Solution. The problem becomes
$\frac{\mathrm{d}Y}{\mathrm{d}t} = \tau\left(aY + b + I^* - Y\right)$ with $Y_0 = \tilde{Y}_0$.

Case 1. $Y(t)$ is (i)-gH differentiable

The above problem reduces to

$$\frac{\mathrm{d}Y_1(t,\alpha)}{\mathrm{d}t} = a\tau Y_1\left(t,\alpha\right) - \tau Y_2\left(t,\alpha\right) + \alpha(b + I^*)$$

and

$$\frac{\mathrm{d}Y_2(t,\alpha)}{\mathrm{d}t} = a\tau Y_2\left(t,\alpha\right) - \tau Y_1\left(t,\alpha\right) + \alpha(b + I^*)$$

with initial condition $Y_{t=0,1}\left(\alpha\right) = Y_{01}(\alpha)$ and $Y_{t=0,2}\left(\alpha\right) = Y_{02}(\alpha)$.

[7]Here (6.5) and (6.6) are equal in crisp sense but not on fuzzy sense.

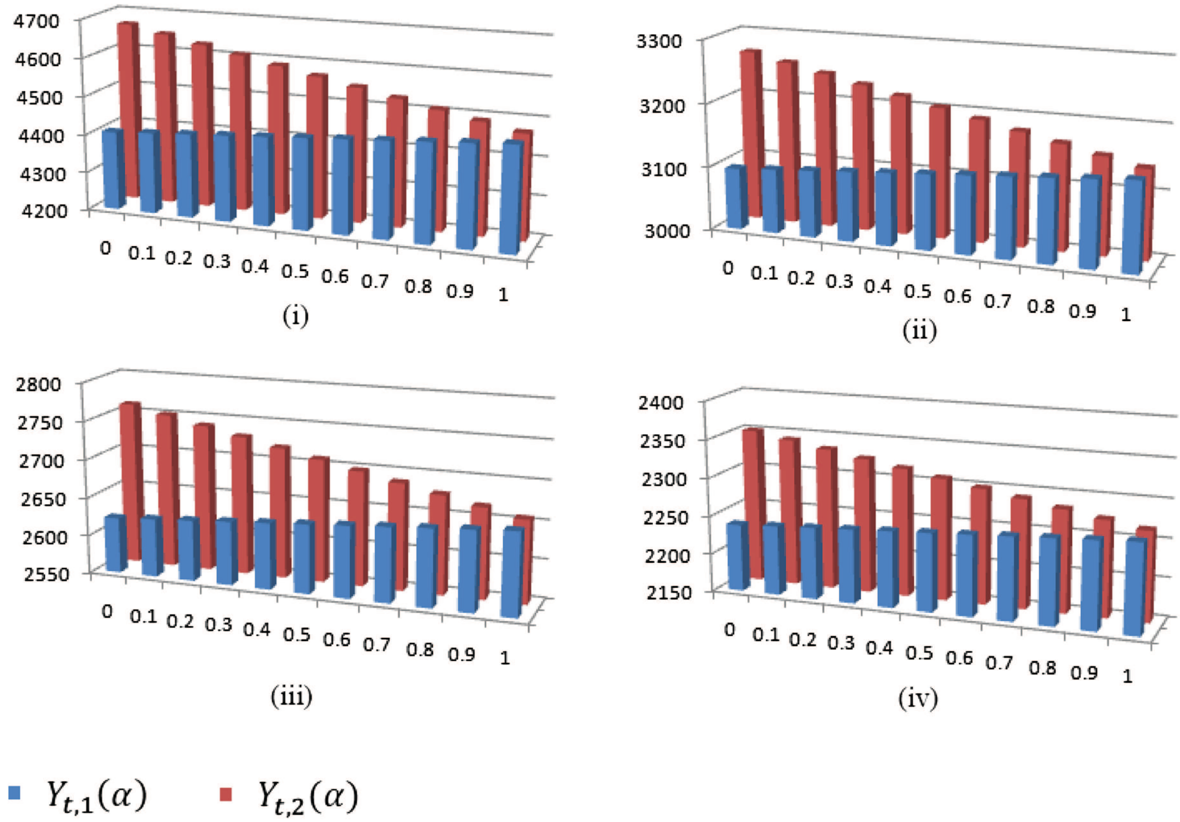


FIGURE 2. Figure of $Y_{t,1}(\alpha)$ and $Y_{t,2}(\alpha)$ for different values of t in Example 5.2.

Case 2. $Y(t)$ is (ii)-gH differentiable

The above problem reduces to

$$\frac{\mathrm{d}Y_1(t,\alpha)}{\mathrm{d}t} = a\tau Y_2(t,\alpha) - \tau Y_1(t,\alpha) + \tau(b+I^*)$$

and

$$\frac{\mathrm{d}Y_2(t,\alpha)}{\mathrm{d}t} = a\tau Y_1(t,\alpha) - \tau Y_2(t,\alpha) + \tau(b+I^*)$$

with initial condition $Y_{t=0,1}(\alpha) = Y_{01}(\alpha)$ and $Y_{t=0,2}(\alpha) = Y_{02}(\alpha)$.

6.1.3. Stability analysis of fuzzy problem

Case 1. The problem is written as

$$\begin{bmatrix} \dfrac{\mathrm{d}Y_1(t,\alpha)}{\mathrm{d}t} \\ \dfrac{\mathrm{d}Y_2(t,\alpha)}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} a\tau & -\tau \\ -\tau & a\tau \end{bmatrix} \begin{bmatrix} Y_1(t,\alpha) \\ Y_2(t,\alpha) \end{bmatrix} + \begin{bmatrix} \alpha(b+I^*) \\ \alpha(b+I^*) \end{bmatrix}.$$

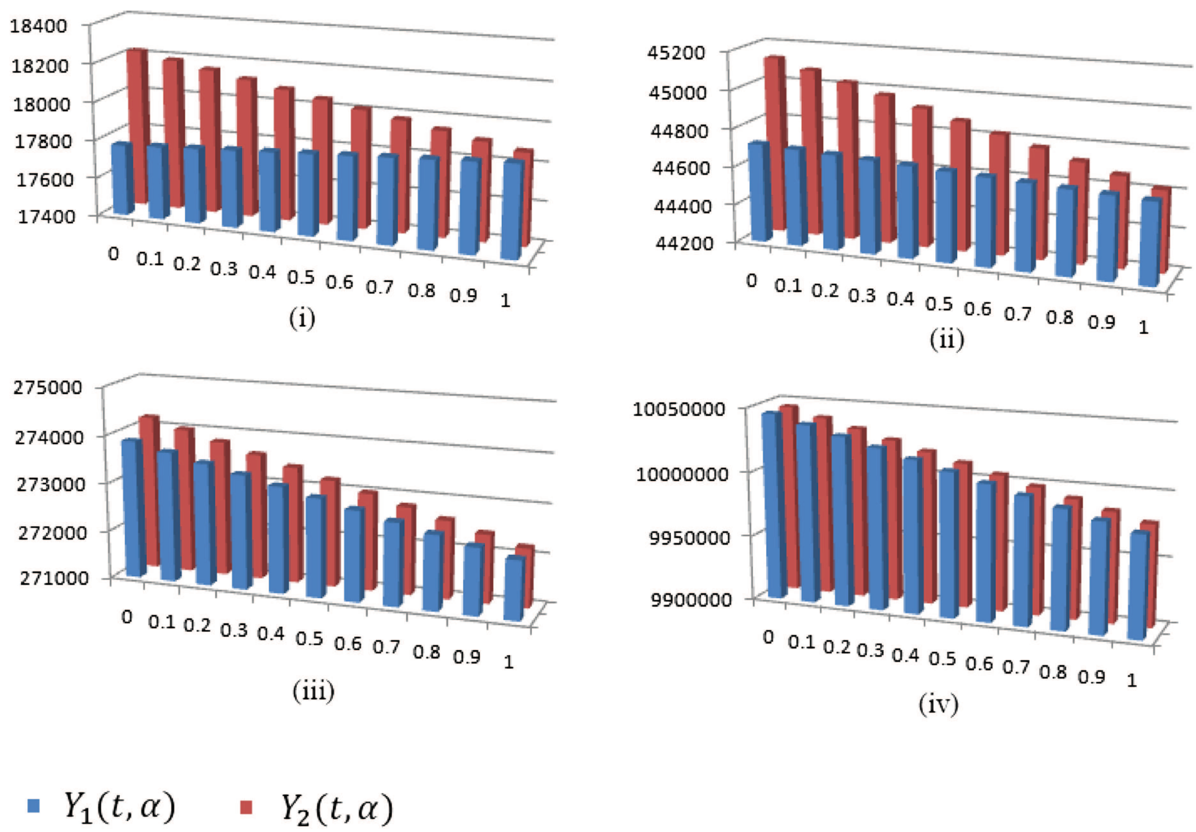


FIGURE 3. Figure of $Y_1(t,\alpha)$ and $Y_2(t,\alpha)$ for different values of t in Example 6.1.

After giving the transformation $Y_1(t,\alpha) = Y_1'(t,\alpha) + \epsilon_3$ and $Y_2(t,\alpha) = Y_2'(t,\alpha) + \epsilon_4$ such that the above differential equation becomes

$$\begin{bmatrix} \dfrac{\mathrm{d}Y_1'(t,\alpha)}{\mathrm{d}t} \\ \dfrac{\mathrm{d}Y_2'(t,\alpha)}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} a\tau & -\tau \\ -\tau & a\tau \end{bmatrix} \begin{bmatrix} Y_1'(t,\alpha) \\ Y_2'(t,\alpha) \end{bmatrix}.$$

The characteristic equation of the above system is

$$\lambda^2 - 2a\tau\lambda + \tau^2\left(a^2 - 1\right) = 0.$$

The above system is stable if $a\tau < 0$ and $\tau^2\left(a^2-1\right) > 0$.

Case 2. The problem is written as

$$\begin{bmatrix} \dfrac{\mathrm{d}Y_1(t,\alpha)}{\mathrm{d}t} \\ \dfrac{\mathrm{d}Y_2(t,\alpha)}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} -\tau & a\tau \\ a\tau & -\tau \end{bmatrix} \begin{bmatrix} Y_1(t,\alpha) \\ Y_2(t,\alpha) \end{bmatrix} + \begin{bmatrix} \tau(b+I^*) \\ \tau(b+I^*) \end{bmatrix}.$$

TABLE 3. Solution for different values of t in Example 6.1 when $Y(t)$ is (i)-gH differentiable.

	$t=1$ (yr)			$t=2$ (yr)	
α	$Y_1(t,\alpha)$ (\$/yr)	$Y_2(t,\alpha)$ (\$/yr)	α	$Y_1(t,\alpha)$ (\$/yr)	$Y_2(t,\alpha)$ (\$/yr)
0	17770.81	18223.23	0	44717.67	45127.03
0.1	17781.13	18188.31	0.1	44707.89	45076.32
0.2	17791.45	18153.39	0.2	44698.11	45025.60
0.3	17801.78	18118.47	0.3	44688.33	44974.88
0.4	17812.10	18083.55	0.4	44678.55	44924.17
0.5	17822.42	18048.63	0.5	44668.77	44873.45
0.6	17832.75	18013.71	0.6	44658.99	44822.73
0.7	17843.07	17978.80	0.7	44649.21	44772.02
0.8	17853.39	17943.88	0.8	44639.43	44721.36
0.9	17863.72	17908.96	0.9	44629.65	44670.59
1	17874.04	17874.04	1	44619.87	44619.87
	$t=4$ (yr)			$t=8$ (yr)	
0	273869.17	274204.33	0	10045168.40	10045393.06
0.1	273702.94	274004.58	0.1	10038482.48	10038684.67
0.2	273536.71	273804.84	0.2	10031796.55	10031976.29
0.3	273370.47	273605.09	0.3	10025110.63	10025267.90
0.4	273204.24	273405.34	0.4	10018424.71	10018559.51
0.5	273038.01	273205.59	0.5	10011738.79	10011851.13
0.6	272871.77	273005.84	0.6	10005052.87	10005142.74
0.7	272705.54	272806.09	0.7	9998366.95	9998434.35
0.8	272539.31	272606.34	0.8	9991681.03	9991725.96
0.9	272373.07	272406.59	0.9	9984995.11	9985017.58
1	272206.84	272206.84	1	9978309.19	9978309.19

After giving the transformation $Y_1(t,\alpha) = Y_1'(t,\alpha) + \epsilon_5$ and $Y_2(t,\alpha) = Y_2'(t,\alpha) + \epsilon_6$ such that the above differential equation becomes

$$\begin{bmatrix} \dfrac{\mathrm{d}Y_1'(t,\alpha)}{\mathrm{d}t} \\ \dfrac{\mathrm{d}Y_2'(t,\alpha)}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} -\tau & a\tau \\ a\tau & -\tau \end{bmatrix} \begin{bmatrix} Y_1'(t,\alpha) \\ Y_2'(t,\alpha) \end{bmatrix}.$$

The characteristic equation of the above system is

$$\lambda^2 + 2\tau\lambda + \tau^2\left(1-a^2\right) = 0.$$

The above system is stable if $\tau > 0$ and $\left(1-a^2\right) > 0$.

6.2. Numerical illustration

Example 6.1. Solve the problem when $\tau = 0.5a = 0.8$, $b = 400$, $I^* = 600$ (\$/yr) and the initial condition is $\tilde{Y}_0 = (6800, 7000, 7300)$ (\$/yr).

Solution. The solution is written as

Case 1. When $Y(t)$ is (i)-gH differentiable

The differential equation is

$$\frac{\mathrm{d}Y_1(t,\alpha)}{\mathrm{d}t} = 0.4Y_1(t,\alpha) - 0.5Y_2(t,\alpha) + 500$$

and

$$\frac{\mathrm{d}Y_2(t,\alpha)}{\mathrm{d}t} = 0.4Y_2(t,\alpha) - 0.5Y_1(t,\alpha) + 500$$

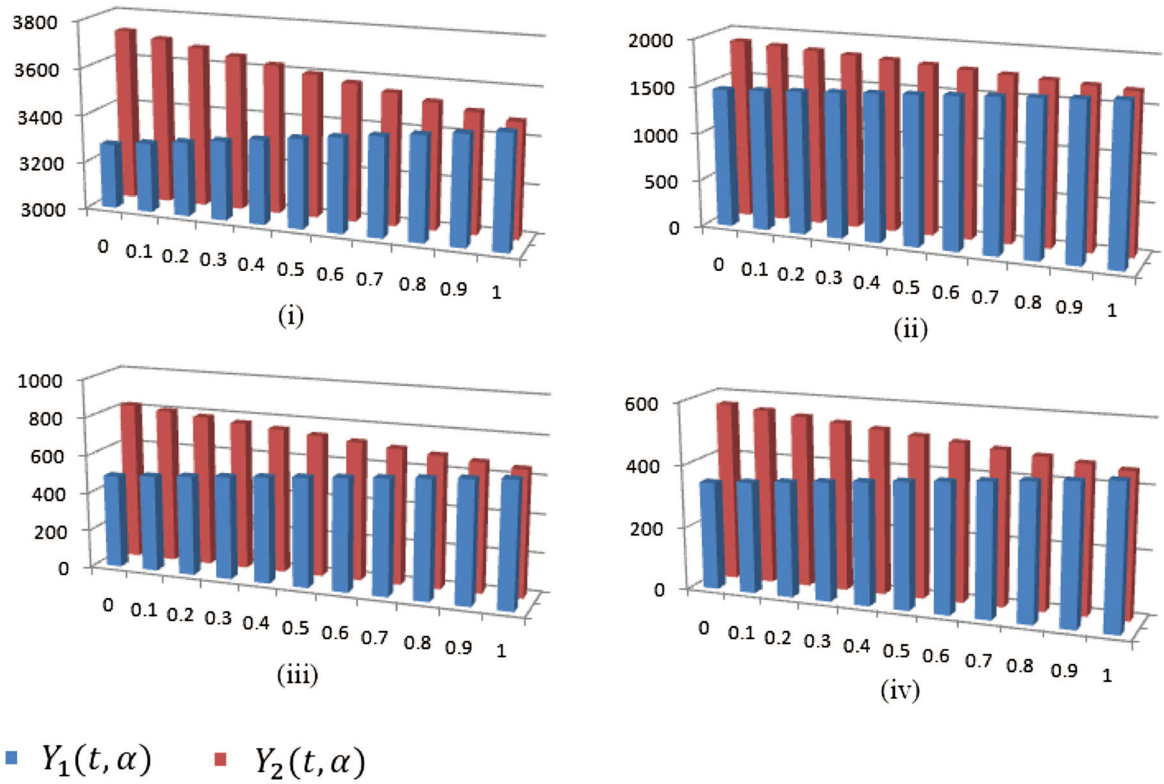


FIGURE 4. Figure of $Y_1(t,\alpha)$ and $Y_2(t,\alpha)$ for different values of t in Example 6.1 when $Y(t)$ in (ii)-gH differentiable.

with initial condition $Y_{t=0,1}(\alpha) = Y_{01}(\alpha) = 6800 + 200\alpha$ and $Y_{t=0,2}(\alpha) = Y_{02}(\alpha) = 7300 - 300\alpha$.

Therefore, the solutions are

$$Y_1(t,\alpha) = -450 + (250\alpha - 250)\,\mathrm{e}^{-0.1t} + (7500 - 50\alpha)\mathrm{e}^{0.9t}$$

and

$$Y_2(t,\alpha) = -450 + (250 - 250\alpha)\,\mathrm{e}^{-0.1t} + (7500 - 50\alpha)\mathrm{e}^{0.9t}.$$

Remarks. Figure 3 and Table 3 are for $Y_1(t,\alpha)$ and $Y_2(t,\alpha)$ when $t = 1, 2, 4$ and 8, respectively. For $t = 1, 2$, $Y_1(t,\alpha)$ is increasing and $Y_2(t,\alpha)$ is a decreasing function as α moves from 0 to 1. Hence, the solution goes to strong solution for $t = 1, 2$. For $t = 4$ and 8, $Y_1(t,\alpha)$ and $Y_2(t,\alpha)$ are both decreasing function as α moves from 0 to 1. Hence, the solution is a weak solution for $t = 4$ and 8.

Stability situation. The problem is unstable as the characteristic equation is of type $\lambda^2 - 0.8\lambda - 0.9 = 0$ (using Thm. 3.4).

Case 2. When $Y(t)$ is (ii)-gH differentiable

The differential equation is

$$\frac{\mathrm{d}Y_2(t,\alpha)}{\mathrm{d}t} = 0.4Y_1(t,\alpha) - 0.5Y_2(t,\alpha) + 500$$

and

$$\frac{\mathrm{d}Y_1(t,\alpha)}{\mathrm{d}t} = 0.4Y_2(t,\alpha) - 0.5Y_1(t,\alpha) + 500$$

TABLE 4. Solution for different values of t in Example 6.1 when *Y(t)* is (ii)-gH differentiable.

$t = 1$ (yr)			$t = 2$ (yr)		
α	$Y_1(t,\alpha)$ (\$/yr)	$Y_2(t,\alpha)$ (\$/yr)	α	$Y_1(t,\alpha)$ (\$/yr)	$Y_2(t,\alpha)$ (\$/yr)
0	3273.06	3725.48	0	1463.53	1894.42
0.1	3293.65	3700.82	0.1	1485.32	1873.12
0.2	3314.23	3676.17	0.2	1507.12	1851.83
0.3	3334.82	3651.52	0.3	1528.91	1830.54
0.4	3355.41	3626.86	0.4	1550.71	1809.24
0.5	3376.00	3602.21	0.5	1572.50	1787.95
0.6	3396.59	3577.55	0.6	1594.29	1766.65
0.7	3417.17	3552.90	0.7	1616.09	1745.36
0.8	3437.76	3528.25	0.8	1637.88	1724.06
0.9	3458.35	3503.59	0.9	1659.68	1702.77
1	3478.94	3478.94	1	1681.47	1681.47
$t = 4$ (yr)			$t = 8$ (yr)		
0	487.34	822.50	0	343.26	567.93
0.1	503.96	805.61	0.1	354.49	556.69
0.2	520.59	788.71	0.2	365.72	545.45
0.3	537.21	771.82	0.3	376.95	534.22
0.4	553.83	754.92	0.4	388.18	522.98
0.5	570.45	738.03	0.5	399.41	511.74
0.6	587.07	721.14	0.6	410.64	500.50
0.7	603.69	704.24	0.7	421.87	489.27
0.8	620.31	687.35	0.8	433.10	478.03
0.9	636.94	670.45	0.9	444.33	466.79
1	653.56	653.56	1	455.56	455.56

with initial condition $Y_{t=0,1}(\alpha) = Y_{01}(\alpha) = 6800 + 200\alpha$ and $Y_{t=0,2}(\alpha) = Y_{02}(\alpha) = 7300 - 300\alpha$.

Therefore, the solution is

$$Y_1(t,\alpha) = 450 + (250\alpha - 250)\,\mathrm{e}^{-0.1t} + (7500 - 50\alpha)\mathrm{e}^{-0.9t}$$

and

$$Y_2(t,\alpha) = 450 + (250 - 250\alpha)\,\mathrm{e}^{-0.1t} + (7500 - 50\alpha)\mathrm{e}^{-0.9t}.$$

Remarks. Figure 4 and Table 4 are for $Y_1(t,\alpha)$ and $Y_2(t,\alpha)$ when $t = 1, 2, 4$, and 8, respectively. Clearly for $t = 1, 2, 4$ and 8, $Y_1(t,\alpha)$ is increasing function and $Y_2(t,\alpha)$ is a decreasing function as α moves from 0 to 1. Hence the solution represents a strong solution for $t = 1, 2, 4$, and 8.

Stability situation. The problem is stable as the characteristic equation is of type $\lambda^2 + \lambda + 0.09 = 0$ (using the Thm. 3.4).

Example 6.2. Solve the problem when $\tau = 0.1$, $a = 0.9$, $b = 100$, $I^* = 300$ (\$/yr) and the initial condition is $\tilde{Y}_0 = (1750, 2000, 2200)$ (\$/yr).

Solution. The solution is written as

Case 1. $Y(t)$ is (i)-gH differentiable

The differential equation is

$$\frac{\mathrm{d}Y_1(t,\alpha)}{\mathrm{d}t} = 0.9Y_1(t,\alpha) - 0.1Y_2(t,\alpha) + 40$$

and

$$\frac{\mathrm{d}Y_2(t,\alpha)}{\mathrm{d}t} = 0.9Y_2(t,\alpha) - 0.1Y_1(t,\alpha) + 40$$

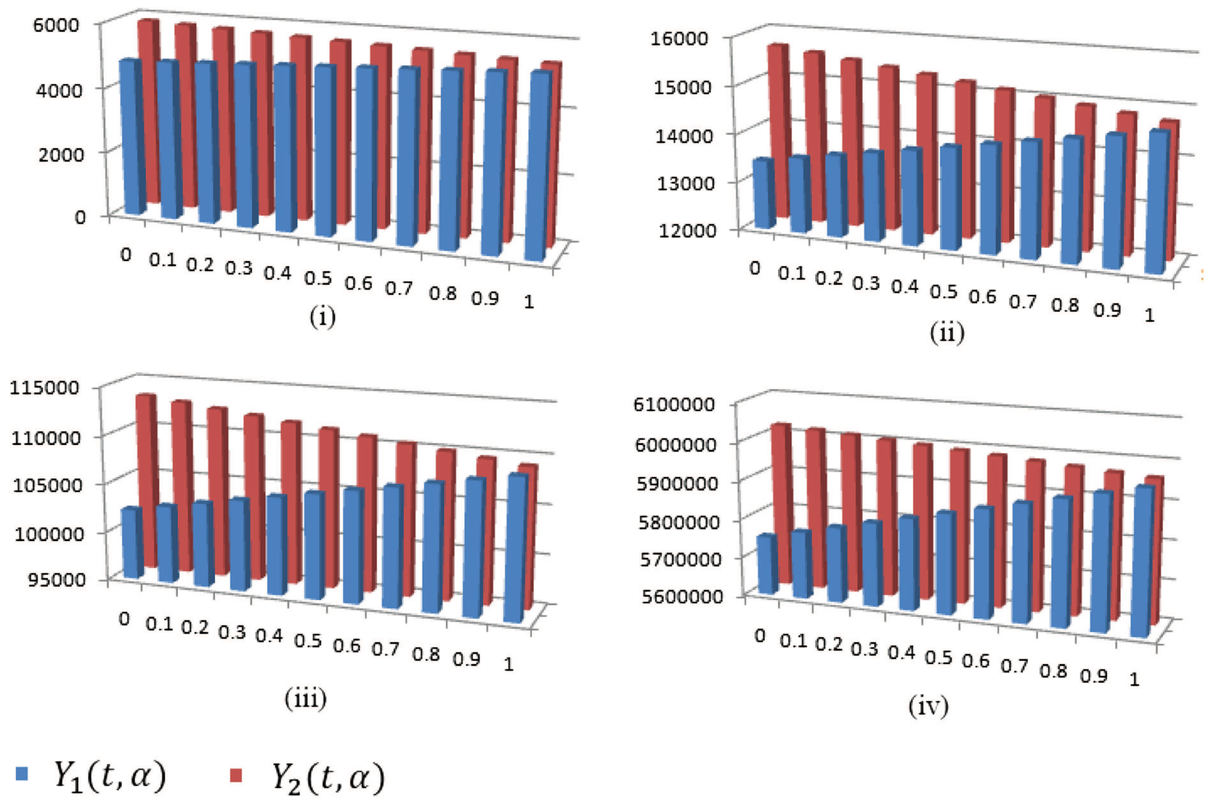


FIGURE 5. Figure of $Y_1(t,\alpha)$ and $Y_2(t,\alpha)$ for different values of t in Example 6.2 for Case 1.

with initial condition $Y_{t=0,1}(\alpha) = Y_{01}(\alpha) = 1750 + 250\alpha$ and $Y_{t=0,2}(\alpha) = Y_{02}(\alpha) = 2200 - 200\alpha$.

Therefore, the solution is

$$Y_1(t,\alpha) = -40 + (-225 + 225\alpha)\,\mathrm{e}^{0.8t} + (2015 + 25\alpha)\mathrm{e}^{t}$$

and

$$Y_2(t,\alpha) = -40 + (225 - 225\alpha)\,\mathrm{e}^{0.8t} + (2015 + 25\alpha)\mathrm{e}^{t}.$$

Remarks. Figure 4 and Table 4 are for $Y_1(t,\alpha)$ and $Y_2(t,\alpha)$ when $t = 1, 2, 4$, and 8, respectively. Clearly for $t = 1, 2, 4$ and 8, $Y_1(t,\alpha)$ is increasing function and $Y_2(t,\alpha)$ is a decreasing function as α moves from 0 to 1. Hence the solution is recommend for a strong solution for $t = 1, 2, 4$, and 8.

Stability situation. The system is unstable as the characteristic equation is of type $\lambda^2 - 1.8\lambda + 0.80 = 0$ (using the Thm. 3.4).

Case 2. $Y(t)$ is (ii)-gH differentiable

The differential equations are

$$\frac{\mathrm{d}Y_2(t,\alpha)}{\mathrm{d}t} = 0.9Y_1(t,\alpha) - 0.1Y_2(t,\alpha) + 40$$

and

$$\frac{\mathrm{d}Y_1(t,\alpha)}{\mathrm{d}t} = 0.9Y_2(t,\alpha) - 0.1Y_1(t,\alpha) + 40$$

Table 5. Solution for different values of t in Example 6.2 when $Y(t)$ is (i)-gH differentiable.

$t = 1$ (yr)			$t = 2$ (yr)		
α	$Y_1(t,\alpha)$ (yr)	$Y_2(t,\alpha)$ (yr)	α	$Y_1(t,\alpha)$ (yr)	$Y_2(t,\alpha)$ (yr)
0	4827.85	5829.35	0	13438.95	15667.81
0.1	4884.73	5786.07	0.1	13568.86	15574.84
0.2	4941.60	5742.79	0.2	13698.78	15481.87
0.3	4998.47	5699.51	0.3	13828.70	15388.90
0.4	5055.34	5656.23	0.4	13958.61	15295.93
0.5	5112.21	5612.95	0.5	14088.53	15202.96
0.6	5169.08	5569.67	0.6	14218.44	15109.99
0.7	5225.95	5526.40	0.7	14348.36	15017.02
0.8	5282.82	5483.12	0.8	14478.28	14924.05
0.9	5339.69	5439.84	0.9	14608.19	14831.08
1	5396.56	5396.56	1	14738.11	14738.11
$t = 4$ (\$/yr)			$t = 8$ (\$/yr)		
0	102271.52	113311.16	0	5751936.89	6022767.15
0.1	102960.00	112895.67	0.1	5772930.79	6016678.03
0.2	103648.48	112480.19	0.2	5793924.70	6010588.92
0.3	104336.95	112064.70	0.3	5814918.61	6004499.80
0.4	105025.43	111649.21	0.4	5835912.52	5998410.68
0.5	105713.91	111233.73	0.5	5856906.43	5992321.56
0.6	106402.39	110818.24	0.6	5877900.38	5986232.44
0.7	107090.86	110402.75	0.7	5898894.21	5980143.32
0.8	107779.34	109987.27	0.8	5919888.15	5974054.21
0.9	108467.82	109571.78	0.9	5940882.06	5967965.09
1	109156.30	109156.30	1	5961875.97	5961875.97

with initial condition $Y_{t=0,1}(\alpha) = Y_{01}(\alpha) = 1750 + 250\alpha$ and $Y_{t=0,2}(\alpha) = Y_{02}(\alpha) = 2200 - 200\alpha$.

Therefore, the solutions are

$$Y_1(t,\alpha) = 40 + (-225 + 225\alpha)\,\mathrm{e}^{0.8t} + (2015 + 25\alpha)\mathrm{e}^{-t}$$

and

$$Y_2(t,\alpha) = 40 + (225 - 225\alpha)\,\mathrm{e}^{0.8t} + (2015 + 25\alpha)\mathrm{e}^{-t}.$$

Remarks. Figure 6 and Table 6 are for $Y_1(t,\alpha)$ and $Y_2(t,\alpha)$ when $t = 1, 2, 4$, and 8, respectively. Clearly, for $t = 1, 2, 4$ and 8, $Y_1(t,\alpha)$ is increasing function and $Y_2(t,\alpha)$ is a decreasing function as α moves from 0 to 1. Hence the solution is of strong solution for $t = 1, 2, 4$, and 8.

Stability situation. The system is unstable as the characteristic equation is of type $\lambda^2 + 0.2\lambda - 0.80$ (using the Thm. 3.4).

6.3. Analysis of the model

Clearly, one can conclude that when we take national income determination problem is taken as continuous process, it turns into a continuous differential equation. But when it is modeled with fuzzy then it becomes a fuzzy differential equation. The solution procedures for fuzzy and crisp differential equations are not the same. One should know the derivative of fuzzy function *i.e.*, fuzzy differentiability concept. Using well known fuzzy differentiability concepts, one can convert the fuzzy differential equation to parametric two crisp differential equations and solve them. Lastly, one can check whether the solution also follows the fuzzy rules or not.

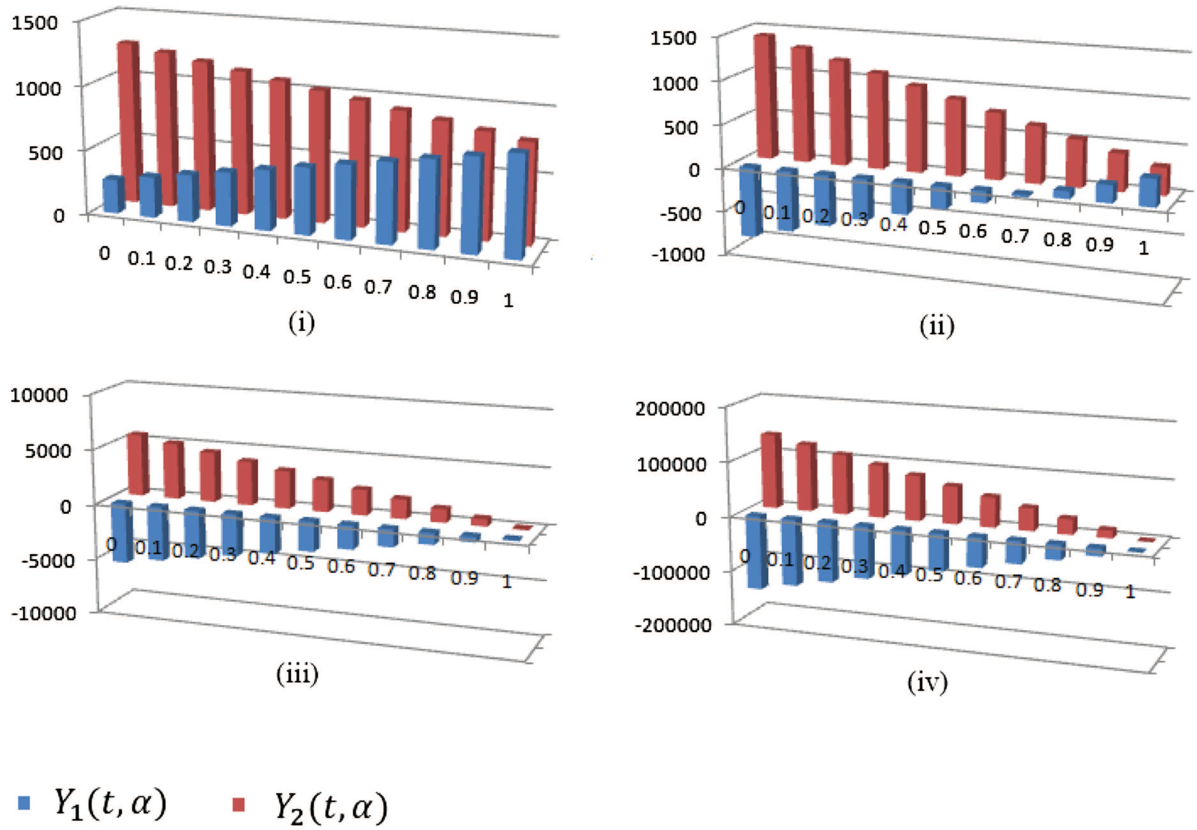


FIGURE 6. Figure of $Y_1(t, \alpha)$ and $Y_2(t, \alpha)$ for different values of t in Example 6.2 for Case 2.

7. Conclusions

In this paper, fuzzy difference equation and fuzzy differential equations were taken into account to construct and estimate the optimization of the national income in fuzzy environment, which were considered to be an important area of research. The approaches generalized Hukuhara difference and generalized Hukuhara derivative conception were applied to explain the imprecise results of the given model. Widely, the whole reflection reached its conclusion with the following interpretations:

– Representing national income model with fuzzy numbers enabled to meet the uncertain or imprecise parameters as well, which was positively advantageous for the researchers to analyze the bugged expenditures in a more precise way.
– The gH-difference and gH-derivative approach, having important place in fuzzy calculus, powerfully made it probable to attend the fuzzy solution of the principal model in both the cases, structured in crisp form or in fuzzy sense, accordingly.
– The national income determination model was considered in both discrete and continuous system in fuzzy environment which helped to differentiate models for different systems.
– The stability analysis for the models in different system in fuzzy environment were discussed.

It is strongly expected that the proposed concepts are properly applicable to different types of differential equation models in fuzzy environments.

TABLE 6. Solution for different values of t in Example 6.2 for Case 2.

$t = 1$ (yr)			$t = 2$ (yr)		
α	$Y_1(t,\alpha)$ (\$/yr)	$Y_2(t,\alpha)$ (\$/yr)	α	$Y_1(t,\alpha)$ (\$/yr)	$Y_2(t,\alpha)$ (\$/yr)
0	265.81	1267.30	0	−807.14	1421.71
0.1	316.80	1218.15	0.1	−695.36	1310.61
0.2	367.80	1168.99	0.2	−583.58	1199.50
0.3	418.79	1119.84	0.3	−471.80	1088.40
0.4	469.79	1070.68	0.4	−360.01	977.29
0.5	520.78	1021.53	0.5	−248.23	866.19
0.6	571.78	972.37	0.6	−136.45	755.09
0.7	622.77	923.22	0.7	−24.67	643.98
0.8	673.77	874.06	0.8	87.10	532.88
0.9	724.76	824.91	0.9	198.88	421.77
1	775.75	775.75	1	310.67	310.67
$t = 4$ (yr)			$t = 8$ (yr)		
0	−5443.64	5595.99	0	−135374.47	135455.79
0.1	−4891.61	5044.05	0.1	−121832.95	121914.28
0.2	−4339.59	4492.12	0.2	−108291.44	108372.77
0.3	−3787.56	3940.18	0.3	−94749.92	94831.25
0.4	−3235.53	3388.24	0.4	−81208.41	81289.74
0.5	−2683.50	2836.31	0.5	−67666.90	67748.23
0.6	−2131.47	2284.37	0.6	−54125.38	54206.72
0.7	−1579.45	1732.43	0.7	−40583.87	40665.20
0.8	−1027.42	1180.50	0.8	−27042.35	27123.69
0.9	−475.39	628.56	0.9	−13500.84	13582.18
1	76.63	76.63	1	40.67	40.67

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