RAIRO-Oper. Res. 53 (2019) 29–38
https://doi.org/10.1051/ro/2018061

RAIRO Operations Research
www.rairo-ro.org

A PENALTY METHOD FOR NONLINEAR PROGRAMMING

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Abstract. This paper presents a variant of logarithmic penalty methods for nonlinear convex programming. If the descent direction is obtained through a classical Newton-type method, the line search is done on a majorant function. Numerical tests show the efficiency of this approach *versus* classical line searches.

Mathematics Subject Classification. 90C25, 90C30.

Received November 2, 2017. Accepted July 11, 2018.

1. Introduction

In this paper, we propose a logarithmic penalty method using a new line search technic to solve the problem

$$\alpha = \min\left[\, f(x) \,:\, x \in D \,\right] \text{ with } \; D = \{x \in \mathbb{R}^n \,:\, x \geq 0,\, Ax = b\,\}, \tag{P}$$

under the following assumptions

1. f is convex and twice continuously differentiable on D.
2. There exists $x_0 > 0$ such that $Ax_0 = b$.
3. $b \in \mathbb{R}^p$, A is a $(p \times n)$ full rank matrix.
4. The set of optimal solutions of (P) is nonempty and bounded.

We deduce from the optimality conditions that $\bar{x}$ is an optimal solution of (P) if only if there exists $\bar{u} \in \mathbb{R}^p$ and $\bar{v} \in \mathbb{R}^n$ such that

$$\nabla f(\bar{x}) + A^t\bar{u} = \bar{v} \geq 0, \quad A\bar{x} = b, \quad \langle \bar{v}, \bar{x} \rangle = 0. \tag{1.1}$$

2. The penalization

2.1. The perturbed problem

Let us define the function $\gamma : \mathbb{R}^2 \to (-\infty, +\infty]$ by

$$\gamma(r,t) = \begin{cases} r\ln(r) - r\ln(t) & \text{if} \quad t > 0 \text{ and } r > 0, \\ 0 & \text{if} \quad r = 0 \text{ and } t \geq 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Keywords. Logarithmic penalty method, method of majorant functions, convex programming.

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The function γ is convex, lower semicontinuous and proper. Let us introduce the function φ defined on $\mathbb{R} \times \mathbb{R}^n$ by

$$\varphi(r,x) = \begin{cases} f(x) + \sum_{i=1}^n \gamma(r, x_i) \text{ if } & x \geq 0,\ Ax = b, \\ \\ +\infty & \text{if not.} \end{cases}$$

This function is also convex, lower semicontinuous and proper. Finally, let us introduce the function h defined by

$$h(r) = \inf_x \left[\, \varphi_r(x) := \varphi(r,x)\ :\ x \in \mathbb{R}^n \right]. \tag{P_r}$$

This function h is convex since φ is convex. Observe that the problems (P_0) and (P) coincide and thereby

$$\alpha = h(0).$$

$r > 0$ being fixed, the function φ_r is convex, lower semicontinuous and proper. Recall that the asymptotic function of φ_r is defined by

$$[\varphi_r]_\infty(d) = \lim_{t \to +\infty} \frac{\varphi_r(x_0 + td) - \varphi_r(x_0)}{t}.$$

The asymptotic functions of φ and f are related by

$$[\varphi_r]_\infty(d) = \begin{cases} f_\infty(d) \text{ if } & d \geq 0,\ Ad = 0, \\ +\infty & \text{if not.} \end{cases}$$

On the other hand, assumption 4 is equivalent to

$$\{d \in \mathbb{R}^n\ :\ [f]_\infty(d) \leq 0,\ d \geq 0,\ Ad = 0\} = \{0\}.$$

Hence,

$$\{d \in \mathbb{R}^n\ :\ [\varphi_r]_\infty(d) \leq 0\} = \{0\}.$$

This condition implies that the set of optimal solutions of problem (P_r) is a nonempty closed convex set. Taking into account that the function φ_r is strictly convex, (P_r) has one unique solution denoted by x_r or $x(r)$. Since r is strictly positive one has

$$x_r \in \tilde{D} = \{x \in \mathbb{R}^n\ :\ x > 0,\ Ax = b\,\}.$$

2.2. Convergence

Let $r > 0$. It follows from the necessary and sufficient optimality conditions the existence of $u_r = u(r) \in \mathbb{R}^p$, such that

$$\nabla f(x_r) - rX_r^{-1}e + A^t u_r = 0, \tag{2.1}$$

$$Ax_r - b = 0. \tag{2.2}$$

Observe that u_r is uniquely defined in reason of assumption 3. In fact, the couple (x_r, u_r) is the solution of the equation $H(x,u) = 0$ where

$$H(x,u) = \begin{pmatrix} \nabla f(x) - rX^{-1}e + A^t u \\ Ax - b \end{pmatrix},$$

where X is the diagonal matrix with diagonal entries $X_{ii} = x_i$ for all i.

According to the implicit function theorem, the functions $r :\longrightarrow x(r) := x_r$ and $r :\longrightarrow u(r) := u_r$ are differentiable on $(0,\infty)$. One has

$$\begin{pmatrix} \nabla^2 f(x(r)) + rX^{-2}(r) & A^t \\ A & 0 \end{pmatrix} \begin{pmatrix} x'(r) \\ u'(r) \end{pmatrix} = \begin{pmatrix} X^{-1}(r)e \\ 0 \end{pmatrix}. \tag{2.3}$$

It follows that function h is differentiable on $(0, \infty)$. Recall that

$$h(r) = nr \ln(r) + f(x(r)) - r \sum_{i=1}^{n} \ln(x_i(r)),$$

and therefore

$$h'(r) = n + n \ln(r) - \sum_{i=1}^{n} \ln(x_i(r)) + \langle \nabla f(x(r)) - rX^{-1}(r)e, x'(r) \rangle.$$

In view of (2.1) and (2.3)

$$h'(r) = n + n \ln(r) - \sum_{i=1}^{n} \ln(x_i(r)) - \langle A^t u(r), x'(r) \rangle,$$

$$h'(r) = n + n \ln(r) - \sum_{i=1}^{n} \ln(x_i(r)) - \langle u(r), Ax'(r) \rangle,$$

$$h'(r) = n + n \ln(r) - \sum_{i=1}^{n} \ln(x_i(r)).$$

Since $x(r) \in D$ and h is convex we obtain

$$f(x(r)) \geq \alpha = h(0) \geq h(r) + (0 - r)h'(r) = f(x(r)) - nr.$$

Consequently, we have

$$\alpha \leq f(x(r)) \leq \alpha + nr.$$

Now, let us turn our interest on the trajectory of $\{x(r)\}$ when $r \to 0$.

i) **The case where f is strongly convex with coefficient** $\tau > 0$. Then (P) has one unique optimal solution $\bar{x}$ and

$$nr \geq f(x(r)) - f(\bar{x}) \geq \langle \nabla f(\bar{x}), x(r) - \bar{x} \rangle + \frac{\tau}{2} \|x(r) - \bar{x}\|^2.$$

According to (1.1),

$$nr \geq \langle \bar{v}, x(r) \rangle + \frac{\tau}{2} \|x(r) - \bar{x}\|^2 \geq \frac{\tau}{2} \|x(r) - \bar{x}\|^2,$$

$$\|x(r) - \bar{x}\| \leq \sqrt{\frac{2nr}{\tau}}.$$

The convergence of $x(r)$ to $\bar{x}$ is of order 0.5.

ii) **The case where f is only convex.** The situation is more complex. Firstly, note that for $r \leq 1$,

$$x(r) \in \Omega := \{x \, : \, x \geq 0, \;\; Ax = b, \; f(x) \leq n + \alpha\}.$$

Ω is closed, convex and non empty, it is also bounded because its asymptotic cone is

$$\{d \in \mathbb{R}^n \, : \, [f]_\infty(d) \leq 0, \, d \geq 0, \, Ad = 0\} = \{0\}.$$

It follows that each accumulation point of $x(r)$ when $r \to 0$ is an optimal solution of (P).

The prototype of the method is the classical one of penalty methods. Starting from $(x_0, r_0) \in \tilde{D} \times (0, \infty)$, the iteration scheme consists of

1. Find an approximate solution x_{k+1} of the perturbed problem (P_{r_k}). It is expected that $\varphi(r_k, x_{k+1}) < \varphi(r_k, x_k)$.
2. Choose $r_{k+1} \in (0, r_k)$.

The iterations continue until a satisfactory approximation of $h(0)$ is obtained.

3. Solving the perturbed problem

Recall that the perturbed problem is

$$h(r) = \min_x [\, \varphi_r(x) = f(x) + \sum_{i=1}^{n} \gamma(r, x_i) \,:\, x \geq 0,\ Ax = b\,]. \tag{P_r}$$

3.1. The descent direction

At $x \in \tilde{D}$, the Newton descent direction d is given by solving the following quadratic convex optimization problem

$$\min_d [\, \langle \nabla \varphi_r(x), d\rangle + \frac{1}{2}\langle \nabla^2 \varphi_r(x) d, d\rangle : Ad = 0\,].$$

It suffices to solve the linear system with $n + p$ equations

$$\begin{pmatrix} \nabla^2 f(x) + rX^{-2} & A^t \\ A & 0 \end{pmatrix} \begin{pmatrix} d \\ v \end{pmatrix} = \begin{pmatrix} rX^{-1}e - \nabla f(x) \\ 0 \end{pmatrix}. \tag{SL}$$

It easy to show that the system (SL) is non singular. We have

$$(d^t, 0) \begin{pmatrix} \nabla^2 f(x) + rX^{-2} & A^t \\ A & 0 \end{pmatrix} \begin{pmatrix} d \\ v \end{pmatrix} = (d^t, 0) \begin{pmatrix} rX^{-1}e - \nabla f(x) \\ 0 \end{pmatrix}.$$

From which we obtain

$$\langle \nabla^2 f(x) d, d\rangle + \langle \nabla f(x), d\rangle = r[\, \langle X^{-1}d, e\rangle - \|X^{-1}d\|^2\,]. \tag{3.1}$$

The system can be equivalently written as

$$\begin{pmatrix} X\nabla^2 f(x)X + rI & XA^t \\ AX & 0 \end{pmatrix} \begin{pmatrix} X^{-1}d \\ v \end{pmatrix} = \begin{pmatrix} re - X\nabla f(x) \\ 0 \end{pmatrix}. \tag{3.2}$$

The descent direction being obtained, the next task consists in a line search. The function of one real variable to be minimized is

$$\theta(t) = \frac{1}{r}[\, \varphi_r(x + td) - \varphi_r(x)\,] = \frac{1}{r}[f(x + td) - f(x)\,] - \sum_{i=1}^{n} \ln(1 + ty_i),$$

where $y = X^{-1}d$. This function θ is convex. We have

$$\theta'(t) = \frac{1}{r}\langle \nabla f(x + td), d\rangle - \sum_{i=1}^{n} \frac{y_i}{1 + ty_i},$$

$$\theta''(t) = \frac{1}{r}\langle \nabla^2 f(x + td)d, d\rangle + \sum_{i=1}^{n} \frac{y_i^2}{(1 + ty_i)^2}.$$

We deduce from (3.1) that $\theta'(0) + \theta''(0) = 0$, what is expected since d is a Newton descent direction.

Recall that our problem consists in finding some $\bar{t}$ giving a significant decrease of the function $\theta(t)$. In the case where f is a linear function, this would be equivalent to solving a polynomial equation of degree $n+1$. We could, of course, use a classical iterative method but the computational cost becomes high when n is large. Of course, the same problem appears in the non convex case. The method that we shall introduce below intends to reduce the computational cost of the line search.

3.2. A majorant function

The method that we propose is based on the use of a majorant function θ_1 of the function θ. It is based on the following result.

Theorem 3.1. *[3] Let $w_i > 0, for\ i = 1, 2, \cdots, n$ then,*

$$\sum_{i=1}^{n} \ln(w_i) \geq (n-1)\ln(\bar{w} + \frac{\sigma_w}{\sqrt{n-1}}) + \ln(\bar{w} - \sigma_w\sqrt{n-1})$$

where $\bar{w} = \frac{1}{n}\sum_{i=1}^{n} w_i$ and $\sigma_w = \sqrt{\frac{1}{n}\|w\|^2 - \bar{w}^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(w_i - \bar{w})^2}$.

Set

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i, \quad \sigma_y = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2}, \quad w_i = 1 + ty_i,$$

$$\zeta = \bar{y} + \frac{\sigma_y}{\sqrt{n-1}}, \quad \beta = \bar{y} - \sigma_y\sqrt{n-1}.$$

Then, $\bar{w} = 1 + t\bar{y}$ and $\sigma_w = t\sigma_y$.

For all $t > 0$ for which the two functions θ and θ_1 are defined one has

$$\theta(t) \leq \theta_1(t) := \frac{f(x+td) - f(x)}{r} - (n-1)\ln(1+t\zeta) - \ln(1+t\beta),$$

$$\theta_1'(t) = \frac{\langle \nabla f(x+td), d\rangle}{r} - (n-1)\frac{\alpha}{1+t\zeta} - \frac{\beta}{1+t\beta}.$$

It follows that the domain of θ contains the domain of θ_1. This domain is $(0, \ddot{t}\,)$ where

$$\ddot{t} = \max\,[\,t\,:\,1+\zeta t > 0,\, 1+\beta t > 0\,].$$

Let us remark that

$$\theta_1(0) = \theta(0) = 0 \quad 0 < \theta''(0) = \theta_1''(0) = -\theta_1'(0) = -\theta'(0).$$

Therefore, θ_1 is a good approximation of θ in a neighbourhood of 0. θ_1 is strictly convex and therefore reaches its minimum at one unique point $\bar{t}$. This point $\bar{t}$ is the unique root of the equation $\theta_1'(t) = 0$ which belongs to the domain of θ_1. Since θ is bounded from above by θ_1 one has

$$\theta(\,\bar{t}\,) \leq \theta_1(\,\bar{t}\,) < 0.$$

Thus, with $\bar{t}$, we obtain a significant decrease of the function θ.

3.3. Minimisation of an auxiliary function

We are now interested in minimizing the function

$$\xi(t) = n\delta t - (n-1)\ln(1+\zeta t) - \ln(1+\beta t).$$

We have,

$$\xi'(t) = n\delta - \frac{(n-1)\zeta}{1+\zeta t} - \frac{\beta}{1+\beta t},$$

$$\xi''(t) = \frac{(n-1)\zeta^2}{(1+\zeta t)^2} + \frac{\beta^2}{(1+\beta t)^2},$$

$$\xi(0) = 0, \qquad \xi'(0) = n(\delta - \bar{y}), \quad \xi''(0) = n(\bar{y}^2 + \sigma_y^2) = \|y\|^2.$$

We impose the conditions $\xi'(0) < 0$ and $\xi''(0) > 0$. The function ξ is strictly convex. Its minimum is reached at $\bar{t}$ such that $\xi'(\bar{t}) = 0$. This $\bar{t}$ is one root of the equation

$$\zeta\beta\delta t^2 + t(\delta\zeta + \delta\beta - \zeta\beta) + \delta - \bar{y} = 0. \tag{3.3}$$

- If $\zeta = 0$, we have $\bar{t} = \frac{\bar{y}-\delta}{\delta\beta}$.
- If $\beta = 0$, we have $\bar{t} = \frac{\bar{y}-\delta}{\delta\zeta}$.
- If $\delta = 0$, we have $\bar{t} = \frac{-\bar{y}}{\zeta\beta}$.
- If $\zeta\beta\delta \neq 0$, $\bar{t}$ is the only root of the equation of the second degree which belongs to the domain of definition of ξ. Both roots are

$$t_- = \frac{1}{2}\left(\frac{1}{\delta} - \frac{1}{\zeta} - \frac{1}{\beta} - \sqrt{\Delta}\right), \quad t_+ = \frac{1}{2}\left(\frac{1}{\delta} - \frac{1}{\zeta} - \frac{1}{\beta} + \sqrt{\Delta}\right),$$

  where

$$\Delta = \frac{1}{\zeta^2} + \frac{1}{\beta^2} + \frac{1}{\delta^2} - \frac{2}{\zeta\beta} + \left(\frac{2n-4}{n}\right)\left[\frac{1}{\beta\delta} - \frac{1}{\delta\zeta}\right].$$

3.4. When f is linear

It exists $c \in \mathbb{R}^n$ such that $f(x) = \langle c, x\rangle$ for all x. Take $\delta = n^{-1}\langle c, d\rangle$ in the auxiliary function ξ. The two functions ξ and θ_1 coincide. The minimum of θ_1 is reached at the unique root $\bar{t}$ of the equation $\xi'(t) = 0$. Then,

$$\theta(\bar{t}) \le \theta_1(\bar{t}) < \theta_1(0) = \theta(0) = 0.$$

With $\bar{t}$ we have obtained a significant decrease of the function φ_r along the descent direction d.

It is interesting to note that the condition $\theta_1'(0) + \theta_1''(0) = 0$ implies

$$0 < \theta_1''(0) = n(\bar{y}^2 + \sigma_y^2) = \|y\|^2 = -\theta_1'(0) = n(\bar{y} - \delta).$$

3.5. When f is only convex

$\nabla f(x + td)$ is no longer constant and the equation $\theta_1'(t) = 0$ is not reduced to one equation of second degree. We look at another function θ_2 greater than θ. Given $\hat{t} \in (0, \ddot{t})$, we have for all $t \in (0, \hat{t}\,]$

$$\frac{f(x+td) - f(x)}{r} \le \frac{f(x+\hat{t}d) - f(x)}{r\hat{t}}\, t, \tag{3.4}$$

$$\theta(t) \le \theta_2(t) := \frac{f(x+\hat{t}d) - f(x)}{r\hat{t}}\, t - (n-1)\ln(1+\zeta t) - \ln(1+\beta t).$$

Take for δ in the auxiliary function ξ the quantity

$$\delta = \frac{f(x+\hat{t}d) - f(x)}{nr\hat{t}},$$

and compute the root $\bar{t}$ of the equation $\xi(t) = 0$.

In case where $\bar{t} \le \hat{t}$, one has $\theta(\bar{t}) \le \theta_1(\bar{t}) \le \theta_2(\bar{t}) < 0$ and therefore a significant decrease of the function φ_r along the direction d. The quality of the approximation of θ by θ_2 being better for small values of $\hat{t}$, it may be interesting to repeat the operation with a new value of $\hat{t}$ smaller than the former $\hat{t}$ and a little greater than $\ddot{t}$, the cost of the additional computation is small since it is the cost of one evaluation of f and the determination of the root of one equation of second degree.

In case where $\bar{t} > \hat{t}$, choose another $\hat{t}$ inside the interval $(\bar{t}, \ddot{t}\,)$ and compute again $\bar{t}$ for the new auxiliary function. Repeat as long as necessary until $\bar{t} \le \hat{t}$.

As initial value for $\hat{t}$, it can be thought of 1 (or a quantity a little greater) when $1 < \ddot{t}$ since 1 is the point where the second order approximation of θ reaches its minimum.

4. Algorithm

Three threshold parameters $\varepsilon > 0$, $\hat{r} > 0$ and $\omega \in (0, 1)$ are given.

1. Start with some $x \in \tilde{D}$ and $r > \hat{r}$.
2. Compute d and $y = X^{-1}d$.
3. If $\|y\| > \varepsilon$ compute $\bar{y}, \sigma, \zeta$ and β. Then determine $\bar{t}$ following 3.4 or 3.5 according to the linear or nonlinear case. Do $x = x + \bar{t}d = X(e + \bar{t}y)$ and return to 2.
4. If $\|y\| \leq \varepsilon$ we have obtained a good approximation of $h(r)$.
   (a) If $r \geq \hat{r}$, and $r = \omega r$ and go to 2.
   (b) If $r < \hat{r}$, STOP: we have obtained a good approximation of the optimal solution.

5. Numerical experiments

In the following tables, the column iter Indicates the number of iterations that have been executed, the column times Indicates the time measured in seconds, the div means that the algorithm has not converged. Method 1 corresponds to the method of majorant function introduced in this paper, method 2 corresponds to a classical Armijo-Goldstein line search.

5.1. Nonlinear convex objectif

We consider two examples that are written in the following form:

$$\alpha = \min \left[f(x) \,:\, x \geq 0,\, Ax = b \right]$$

Example 5.1.

$$f(x) = x_1^2 + x_2^2 - 3x_1 - 5x_2,$$

$$A = \begin{bmatrix} 1\;2\;1\;0 \\ 3\;1\;0\;1 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

The optimal solution is $x^* = \left(1, \frac{3}{2}, 0, \frac{5}{2}\right)$, the optimal value is $\alpha = -7.25$.

Example 5.2.

$$f(x) = x_1^2 + x_2^2 - 2x_1 - 4x_2,$$

$$A = \begin{bmatrix} 1\;4\;1\;0 \\ 2\;3\;0\;1 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

The optimal solution is: $x^* = \left(\frac{13}{17}, \frac{18}{17}, 0, \frac{22}{17}\right)$, the optimal value is $\alpha = -\frac{1104}{272} = -4.059$.

Table 1.

	Method 1			Method 2		
Example	Minimum	Iter	Times(s)	Minimum	Iter	Times(s)
1	−7.17	4	0.6×10^{-3}	−7.18	4	0.01
2	−4.05	39	0.4×10^{-2}	−4.05	39	0.01

5.2. Example with variable size

5.2.1. The objective function is linear

We consider a linear optimization problem:

$$\alpha = \min\,[\,c^t x \,:\, x \geq 0,\; Ax = b\,]$$

where A is the $m \times 2m$ matrix defined by

$$A[i,j] = \begin{cases} 1 \text{ if } i = j \text{ or } j = i+m, \\ 0 \text{ if not.} \end{cases}$$

The vectors $c \in \mathbb{R}^{2m}$ and $b \in \mathbb{R}^{m}$ are defined by

$$c[i] = -1,\; c[i+m] = 0,\; b[i] = 2 \quad \forall\, i = 1, \cdots, m.$$

The optimal value is $\alpha = -2m$.

Table 2.

	Method 1			Method 2		
m	Minimum	Iter	Times(s)	Minimum	Iter	Times(s)
5	-9.96	9	0.01×10^{-1}	-9.79	9	0.18×10^{-1}
25	-49.80	9	0.03	-49.67	11	0.049
50	-99.60	9	0.17	-99.35	11	0.22
500	-999.902	14	488.698	-999.835	17	602.661

5.2.2. The objective function is nonlinear

Example 5.3. Erikson's problem [7]

We consider the convex problem, with $n = 2m$

$$\alpha = \min\,[\,f(x) \,:\, x \geq 0,\; Ax = b\,]$$

where $f(x) = \sum_{i=1}^{n} x_i \ln(\frac{x_i}{a_i})$,

$a_i \in \mathbb{R}$, $b_i \in \mathbb{R}$ are fixed and,

$$A[i,j] = \begin{cases} 1 \text{ if } i = j \text{ or } j = i+m, \\ 0 \text{ if not} \end{cases}$$

We test this example for different values of n, a_i and b_i. Table 3 corresponds to the case where ($a_i = 1, \forall\, i = 1, \cdots, n$, $b_i = 6$, $\forall\, i = 1, \cdots, m$).

Table 4 to the case where ($a_i = 2, \forall\, i = 1, \cdots, n$, $b_i = 4$, $\forall\, i = 1, \cdots, m$).

Example 5.4. Quadratic case [6].

We consider the following quadratic problem with $n = m + 2$

$$\alpha = \min\,[\,f(x) \,:\, x \geq 0,\; Ax = b\,]$$

where $f(x) = \frac{1}{2} < x, Qx >$.

TABLE 3.

	Method 1			Method 2		
n	Minimum	Iter	Times(s)	Minimum	Iter	Times(s)
10	32.95	2	0.0005	32.95	4	0.003
50	164.79	2	0.002	164.79	5	0.02
100	329.58	2	0.007	329.58	5	0.04
1000	3295.83	3	4.49	3295.83	6	9.47

TABLE 4.

	Method 1			Method 2		
n	Minimum	Iter	Times(s)	Minimum	Iter	Times(s)
10	0.74×10^{-4}	2	0.05×10^{-2}	0.89×10^{-3}	3	0.2×10^{-2}
50	0.17×10^{-8}	3	0.03×10^{-1}	0.67×10^{-3}	4	0.14×10^{-1}
100	0.70×10^{-8}	3	0.09×10^{-1}	0.1×10^{-2}	4	0.43×10^{-1}
1000	0.7×10^{-7}	3	4.52	0.15×10^{-2}	5	7.38

TABLE 5.

	Method 1			Method 2		
n	Minimum	Iter	Times(s)	Minimum	Iter	Times(s)
4	0.571	31	0.7×10^{-2}	div	div	div
50	10.744	24	0.28	div	div	div
100	21.855	24	1.93	div	div	div
500	110.744	24	251.881	div	div	div
1000	221.855	24	2032.863	div	div	div

With

$$Q[i,j] = \begin{cases} 2 \text{ if } i=j=1 \text{ or } i=j=m, \\ 4 \text{ if } i=j \quad \text{ and } i \neq \{1,m\}, \\ 2 \text{ if } i=j-1 \text{ or } i=j+1, \\ 0 \text{ otherwise.} \end{cases}$$

$$A[i,j] = \begin{cases} 1 \text{ if } & i=j, \\ 2 \text{ if } & i=j-1, \\ 3 \text{ if } & i=j-2, \\ 0 \text{ otherwise.} \end{cases}$$

$b_i = 1, \forall\, i = 1, \cdots, m.\quad \forall\, i = 1, \cdots, n.$
We test this example for different value of n.

6. CONCLUSION

The numerical tests above show that the technic of majorant functions that we have developed in this paper leads to a significant reduction in computational time and an improvement in the quality of the results in comparison with the classical line search.

References

[1] F. Alizadeh, Interior point methods in semi-definite programming with application to combinatorial optimization. *SIAM J. Optimiz.* **5** (1995) 13–55.

[2] J.-F. Bonnans, J.-C. Gilbert, C. Lemaréchal and C. Sagastizàbal, Numerical optimization: theoretical and practical aspects. Mathematics and Applications, Vol. 27. Springer-Verlag, Berlin (2003).

[3] J.-P. Crouzeix, B. Merikhi, A logarithm barrier method for semi-definite programming. *RAIRO: OR* **42** (2008) 123–139.

[4] J.-P. Crouzeix and A. Seeger, New bounds for the extreme values of a finite sample of real numbers. *J. Math. Anal. Appl.* **197** (1996) 411–426.

[5] B. Merikhi, *Extension de quelques méthodes de points intérieurs pour la programmation semi-définie*. Thèse de doctorat, Université de Sétif (2006).

[6] M. Ouriemchi, *Résolution de problèmes non linéaires par les méthodes de points intérieurs. Théorie et algorithmes*. Thèse de doctorat, Université du Havre, France (2006).

[7] E. Shannon, A mathematical theory of communication. *Bell Syst. Tech. J.* **27** (1948) 379–423 and 623–656.

[8] H. Wolkowicz and G.-P.-H. Styan, Bounds for eigenvalues using traces. *Linear Algebra Appl.* **29** (1980) 471–506.