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# ANALYSIS OF A DISCRETE-TIME REPAIRABLE QUEUE WITH DISASTERS AND WORKING BREAKDOWNS

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**Abstract.** In this paper, we analyse a discrete-time queue with a primary server of high service capacity and a substitute server of low service capacity. Disasters that only arrive during the busy periods of the primary server remove all customers from the system and make the primary server breakdown. When the primary server fails and is being repaired, the substitute server handles arriving customers. Applying the embedded Markov chain technique and the supplementary variable method, we determine the distribution of the system length at departure epochs and the joint distribution of the queue length and server's state at an arbitrary instant. Then we derive the sojourn time distribution. We also provide the probability generating function of the time between failures. Some numerical examples are delivered to give an insight into the impact of system parameters on performance measures and a cost function.

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## 1. Introduction

Since discrete-time queues can be applied to evaluate the performance of various information and communication technology, the study of discrete-time queues has been extensively carried out (see Atencia [1], Gao and Wang [8], Lim *et al.* [17], Ndreca and Scoppola [18], Nobel [19], Takagi [21], Tian *et al.* [22] and references therein). Many queueing models incorporate unexpected events that cause the service interruption and the server breakdown (see Choudhury *et al.* [4], Ke and Lin [14], and Ke *et al.* [13]). Also, it is assumed that the occurrence of such an event removes either one customer or all customers from the system (see Gelenbe [9], Towsley and Tripathi [23]).

The term "disaster", when all the customers are removed, in the queueing theory was first introduced by Gelenbe [9]. Since 1989, many studies have been carried out on queues with disasters. Towsley and Tripathi [23] analysed an M/M/1 queue with disasters. Yang *et al.* [24] analysed the M/G/1 queue system with disasters. Later, the inclusion of the multi-phase random operative environment to the repaired M/G/1 queue system with disasters was performed by Jiang *et al.* [10]. Yang *et al.* [24] and Jiang *et al.* [10] considered a model where

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the server stops serving customers during the repair period. Kalidass and Kasturi [12] studied an M/M/1 queue with working breakdowns, in which the server may fail at any time in operation period. However, when the system fails, the service continues at a slower rate by a standby server. Kim and Lee [15] considered an M/G/1 queue with disasters and working breakdowns, in which the system is equipped with a substitute server during the repair period. Other works on queues with disasters can be found in Economou and Fakinos [5], Economou and Manou [6], Gani and Swift [7], Chakravarthy [3], Sudhesh [20].

The topic on disasters was recently extended to discrete-time queues. Atencia and Moreno [2] studied a discrete-time Geo/Geo/1 queue with negative customers and various killing strategies, in which an arriving customer may be a positive customer or a negative customer (that models a disaster). Jolai *et al.* [11] applied a Geo/Geo/1/1/N queue with disasters for the performance analysis of an email contact center. Yi *et al.* [25] discussed a discrete-time Geo/G/1 queue with disasters. However, in Yi *et al.* [25] did not specify how the server is repaired. Lee *et al.* [16] investigated a Geo/G/1 queue with disasters, where the server fails due to a disaster arrival and the repair time is generally distributed. However, all discrete-time single server queue models with disasters in the literature are analysed under an assumption that no service is possible during a repair period. In this paper, as an extension of Yi *et al.* [25] and Lee *et al.* [16], we consider a new discrete-time queue of two servers, where the primary server of high service capacity fails due to disasters, the repair times are geometrically distributed, and the substitute server of low service capacity is activated during the repair periods of the primary server, which is called working breakdown service. On the one hand, working breakdown service can handle emergencies occurring during the repair period and optimize utilization of the system. On the other hand, it can decrease the congestion and the cost of the customers who wait for the primary server to be repaired.

The queue under consideration can be used to model a manufacturing system. Consider a manufacturing system equipped with a substitute machine in preparation for possible main machine failures and repairs. The main machine may suddenly breakdown when it is in operation and all work in process is resulted in destruction and lost. The substitute machine operates at a lower service rate while the main machine is repaired. After repair completion, the main machine returns to the system and becomes available and the substitute machine stops working. Another example of our model is the computer system which is subjected to the attacks of viruses. The presence of a virus may sometimes clear all the work and slow down the performance of the computer system and cause it to be repaired. However, the computer system may still be able to perform various chores at a considerably slower rate. Such phenomenon can be modeled as disasters and working breakdowns in our paper.

The rest of this paper is organized as follows. The model description is given in Section 2. In Section 3, we present an embedded Markov chain. In Section 4, we determine the steady-state system size distribution at arbitrary epoch. In Section 5, we derive the sojourn time distribution of any test customer. In Section 6, We provide the probability generating function of time between failures. In Section 7, we give some numerical examples and the numerical analysis of a cost function. Section 8 presents a brief conclusion.

## 2. Model formulation

In this section, we consider a discrete-time queue with LAS-DA (Late Arrival System with Delayed Access), see the reference of Takagi [21] for details. Let the time slots be numbered by $n = 0, 1, 2, \ldots$. A potential customer arrives at instant $(n^-, n)$ just before the beginning of slot $n$ and a potential service completion occurs at instant $(n, n^+)$ just after the beginning of slot $n$. We assume that a potential disaster occurs after the arrival of a potential customer if two events simultaneously happen in time slot $n$. The system has two service units: the primary server and the substitute server. If a disaster arrives during the busy periods of the primary server, it removes all customers and causes the failure of the primary server. Note that the repairing process of the primary server is immediately started. The arrival of a disaster has no any effect on the system during either the idle or the repair periods of the primary sever. The repairing process lasts at least one slot and ends on the boundary of a slot. Therefore, the arrival of a disaster at instant $(n^-, n)$ just before the beginning of slot

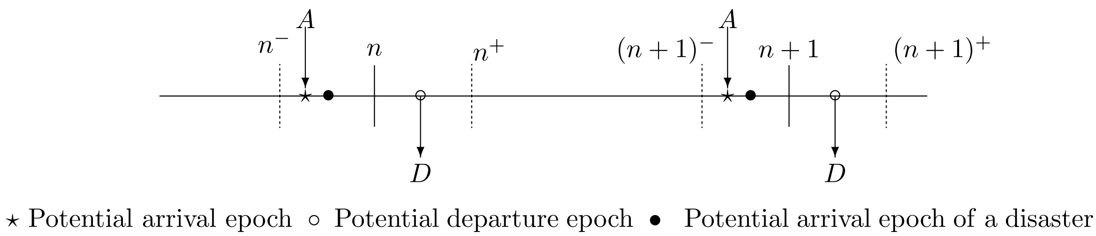


FIGURE 1. Various time epochs in a LAS-DA system.

$n$ has no effect on the server if the repairing process ends at the beginning of slot $n$. The services rendered by the primary sever and the substitute server are termed as normal service and working breakdowns service, respectively. A trajectory for time events is illustrated in Figure 1.

Thereinafter, we denote $\bar{x} = 1 - x$ for $\forall x \in (0,1)$. The detailed description of the model is given as follows:

(1) Independent and identically distributed (i.i.d.) inter-arrival times, $A$, follow the geometric distribution $P(A = k) = \lambda\bar{\lambda}^{k-1}$, $k \geq 1, 0 < \lambda < 1, \bar{\lambda} = 1 - \lambda$.
(2) Inter-arrival times, $D$, of disasters are independent and identically distributed $P(D = k) = \delta\bar{\delta}^{k-1}$, $k \geq 1, 0 < \delta < 1$.
(3) In the normal busy periods, the service times, denoted by $S_1$, are i.i.d. random variables with probability mass function (p.m.f.) $P(S_1 = k) = s_{1,k}, k \geq 1, E[S_1] = \mu^{-1}$ and probability generating function (p.g.f.) $G_{S_1}(z) = \sum_{k=1}^{\infty} z^k s_{1,k}$. The server can handle only one customer at a time according to the first-come, first-served (FCFS) discipline.
(4) The repair time of the primary server, denoted by $R$, follows a geometric distribution with p.m.f $P(R = k) = \theta\bar{\theta}^{k-1}$, $k \geq 1, 0 < \theta < 1$. The server is as good as a new one after a repair.
(5) The customers arrive during a repair period are handled by the substitute server. The working breakdown service time of a customer, denoted as $S_0$, has the following distribution $P(S_0 = k) = s_{0,k}, k \geq 1, E[S_0] = \nu^{-1}, 0 < \nu < \mu$ and $G_{S_0}(z) = \sum_{k=1}^{\infty} z^k s_{0,k}$.
(6) At a repair completion instant, if there are customers in the system, the primary server takes over the service from the substitute server, and the service level is restored. Otherwise, the primary server stays idle and waits for the next arriving customer.

**Remark 2.1.** (1) Since an arriving disaster can only remove the customers during the server's busy period, if both a customer and a disaster arrive at the same idle slot boundary, say in the same time interval $(n^-, n)$, the customer is not expelled from the system under our assumption LAS-DA, the arriving customer begins his service in the time interval $((n+1)^-, n+1)$.
(2) According to our assumptions, if the primary sever is repaired at a slot boundary $n$ and a substitute service ends in or after $(n, n^+)$, then the primary server immediately takes over the substitute service at instant $n$ and begins the normal service to the customer being served by the substitute server.

## 3. AN EMBEDDED MARKOV CHAIN

Let $t_k$ $(t_0 = 0)$ be the $k$th departure epoch when either a customer finishes the service or a disaster removes all customers, $N_k^+$ be the number of customers in the system at the time $t_k^+$, $J_k^+$ be the primary sever's state at time $t_k^+$. Note that either $J_k^+ = 0$ if the primary sever is under repair at time $t_k^+$, or $J_k^+ = 1$ if the primary sever is available at time $t_k^+$.

Process $\{(N_k^+, J_k^+), k \geq 0\}$ is a two-dimensional Markov chain with state space $\{(k,j) : k \geq 0, j = 0, 1\}$. The following one-step transition probabilities are introduced:

$$P_{(k,i)(m,j)} = P\{N_{n+1}^+ = m, J_{n+1}^+ = j | N_n^+ = k, J_n^+ = i\}, k, m \geq 0, i, j = 0, 1.$$

Let $(N^+, J^+)$ be the stationary limit of the Markov chain $\{(N_k^+, J_k^+), k \geq 0\}$, and

$$P_{k,j} = P(N^+ = k, J^+ = j) = \lim_{n\to\infty} P(N_n^+ = k, J_n^+ = j), k \geq 0, j = 0, 1.$$

**Remark 3.1.** The condition $0 < \delta < 1$ ensures that the queueing system is stable. All customers in the system are cleared out whenever a disaster arrives, the number of customers at arbitrary epoch does not go to $\infty$. (See the reference of Kim and Lee [15] for details.)

To obtain the stationary distribution at departure epoches, we introduce some notations:

$$\begin{aligned} q &= P(A < R) = \frac{\lambda\overline{\theta}}{1-\overline{\lambda}\overline{\theta}}, \\ \alpha &= P(R \leq S_0) = 1 - G_{S_0}(\overline{\theta}), \\ \beta &= P(D \leq S_1) = 1 - G_{S_1}(\overline{\delta}). \end{aligned}$$

$$\begin{aligned} a_k &= \sum_{n=\max(1,k)}^{\infty} s_{1,n}\overline{\delta}^n \binom{n}{k} \lambda^k \overline{\lambda}^{n-k}, \ k \geq 0, \\ b_k &= \sum_{n=\max(1,k)}^{\infty} s_{0,n}\overline{\theta}^n \binom{n}{k} \lambda^k \overline{\lambda}^{n-k}, \ k \geq 0, \\ v_k &= \sum_{n=\max(1,k)}^{\infty} \theta\overline{\theta}^{n-1} \binom{n}{k} \lambda^k \overline{\lambda}^{n-k} \sum_{m=n}^{\infty} s_{0,m}, \ k \geq 0. \end{aligned}$$

Then $a_k$ is the probability that there are $k$ customer arriving during $S_1$ without the occurrence of disasters, $b_k$ is the probability that $S_0 < R$ and $k$ customers arrive during $S_0$, and $v_k$ is the probability that $R \leq S_v$ and $k$ customers arrive during $R$.

Denote $\Lambda(z) = \overline{\lambda} + \lambda z$, the $z$-transforms of $\{a_k, k \geq 0\}$, $\{b_k, k \geq 0\}$ and $\{v_k, k \geq 0\}$ are, respectively, given as follows:

$$\begin{aligned} A(z) &= \sum_{k=0}^{\infty} a_k z^k = G_{S_1}(\overline{\delta}\Lambda(z)), \\ B(z) &= \sum_{k=0}^{\infty} b_k z^k = G_{S_0}(\overline{\theta}\Lambda(z)), \\ V(z) &= \sum_{k=0}^{\infty} v_k z^k = \frac{\theta\Lambda(z)}{1-\overline{\theta}\Lambda(z)} \left(1 - B(z)\right). \end{aligned}$$

Evidently,

$$A(1) = G_{S_1}(\overline{\delta}), B(1) = G_{S_0}(\overline{\theta}), \ V(1) = 1 - G_{S_0}(\overline{\theta}).$$

According to Rouche's theorem, for $|z| < 1$, each of the two equations $z - A(z) = 0$ and $z - B(z) = 0$ has a unique root, denoted by $z_1$ and $z_0$, respectively, *i.e.* $z_1 = A(z_1) = G_{S_1}(\overline{\delta}(\overline{\lambda} + \lambda z_1))$ and $z_0 = B(z_0) = G_{S_0}(\overline{\theta}(\overline{\lambda} + \lambda z_0))$.

Let $c_k = \sum_{j=0}^{k} v_j a_{k-j}, k \geq 0$, be the probability that $R \leq S_0, S_1 < D$ and $k$ customers arrive during $R + S_1$. Then, we obtain

$$C(z) = \sum_{k=0}^{\infty} c_k z^k = V(z)A(z).$$

With the above notations, we give the expressions of the one-step transition probabilities $P_{(k,i)(m,j)}$ as follows.

(1) The transition from $(0,0)$ to $(0,0)$ is caused by three cases: (i) no customers arrive during repair period and the next departure happens due to the arrival of a disaster; (ii) a customer arrives during repair period and his service time $S_0$ is less than the remaining repair time, and no customers arrive during $S_0$; (iii) there is a customer arrival during repair period and his service time $S_0$ is no less than the remaining repair time, and the customer is removed by the arrival a disaster during his normal service period. Then

$$P_{(0,0)(0,0)} = q(b_0 + \alpha\beta)) + \overline{q}\beta.$$

(2) The transition from $(0,0)$ to $(k,0)$, $k \geq 1$, occurs only if a customer arrives during repair period and his service time $S_0$ is less than the remaining repair time, and $k$ customers arrive during $S_0$. Then

$$P_{(0,0)(k,0)} = qb_k.$$

(3) The transition from $(0,0)$ to $(k,1)$, $k \geq 0$, is caused by either (i) with probability $qc_k$, there is a customer arriving during repair period and he leaves the system with his normal service completion, and there are $k$ customers at the departure epoch; or (ii) with probability $\overline{q}a_k$, no customer arrives during a repair period, the next customer arrives during the primary sever idle period and leaves the system with his normal service completion, $k$ customers arrive during the normal service time. Then

$$P_{(0,0)(k,1)} = qc_k + \overline{q}a_k.$$

(4) For the transition from $(1,0)$ to $(0,0)$, two cases exist: either (i) with probability $b_0$, the event $(S_0 < R)$ occurs and no customer arrives during $S_0$; or (ii) with probability $\alpha\beta$, the event $(R \leq S_0, D \leq S_1)$ occurs. Then

$$P_{(1,0)(0,0)} = b_0 + \alpha\beta.$$

(5) The transition from $(k,0)(k \geq 2)$ to $(0,0)$ occurs if and only if $(R \leq S_0, D \leq S_1)$ holds.

$$P_{(k,0)(0,0)} = \alpha\beta, k \geq 2.$$

(6) The transition from $(k,0)$ $(k \geq 1)$ to $(m,1)$ $(m \geq k-1)$ occurs if and only if $R \leq S_0$ and $S_1 < D$ and there are $k$ customers arriving during $R + S_1$. Then

$$P_{(k,0)(m,1)} = c_{m-k+1}.$$

(7) The transition from $(k,1)$ $(k \geq 0)$ to $(0,0)$ occurs if and only if a disaster arrives during the next normal service time. Then

$$P_{(k,1)(0,0)} = \beta.$$

(8) The transition from $(0,1)$ to $(k,1)(k \geq 0)$ occurs only $k$ customers arrive during the next service time without disaster arriving. Then

$$P_{(0,1)(k,1)} = a_k.$$

(9) Similarly, for the transitions from $(k,1)$ to $(m,1)$, $m \geq k-1 \geq 0$, and from $(k,0)$ to $(m,0)$, $k \geq 1, m \geq \max\{k-1, 1\}$ we have

$$P_{(k,1)(m,1)} = a_{m-k+1}\,, P_{(k,0)(m,0)} = b_{m-k+1}.$$

Based on the above probabilities, the transition probability matrix $\mathbf{P}$ of the embedded Markov chain $\{(N_n, J_n), n \geq 1\}$ can be written as

$$\mathbf{P} = \begin{bmatrix} \mathbf{B}_0 & \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \cdots \\ \mathbf{C}_0 & \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 & \cdots \\ \mathbf{C} & \mathbf{A}_0 & \mathbf{A}_1 & \mathbf{A}_2 & \cdots \\ \mathbf{C} & & \mathbf{A}_0 & \mathbf{A}_1 & \cdots \\ \vdots & & & \vdots & \vdots \end{bmatrix},$$

where

$$\mathbf{B}_0 = \begin{bmatrix} q(b_0+\alpha\beta)+\overline{q}\beta & qc_0+\overline{q}a_0 \\ \beta & a_0 \end{bmatrix},\ \mathbf{B}_k = \begin{bmatrix} qb_k & qc_k+\overline{q}a_k \\ 0 & a_k \end{bmatrix}, k \geq 1,$$

$$\mathbf{C}_0 = \begin{bmatrix} b_0+\alpha\beta & c_0 \\ \beta & a_0 \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} \alpha\beta & 0 \\ \beta & 0 \end{bmatrix},\ \mathbf{A}_k = \begin{bmatrix} b_k & c_k \\ 0 & a_k \end{bmatrix}, k \geq 0.$$

Our objective is to find the stationary distribution $\{P_{k,j}, k \geq 0, j = 0,1\}$ of the embedded Markov chain $\{(N_n, J_n), n \geq 1\}$. The Kolmogorov equations are:

$$P_{0,0} = (q(b_0+\alpha\beta)+\overline{q}\beta)P_{0,0} + (b_0+\alpha\beta)P_{1,0} + \alpha\beta\sum_{k=2}^{\infty} P_{k,0} + \beta\sum_{k=0}^{\infty} P_{k,1}, \tag{3.1}$$

$$P_{n,0} = qb_k P_{0,0} + \sum_{k=0}^{n} P_{n-k+1,0}b_k, n \geq 1, \tag{3.2}$$

$$P_{0,1} = (qc_0+\overline{q}a_0)P_{0,0} + a_0P_{0,1} + c_0P_{1,0} + a_0P_{1,1}, \tag{3.3}$$

$$P_{n,1} = (qc_n+\overline{q}a_n)P_{0,0} + a_nP_{0,1} + \sum_{k=0}^{n}(P_{n-k+1,0}c_k + P_{n-k+1,1}a_k), n \geq 1. \tag{3.4}$$

Define $P_j(z) = \sum_{n=0}^{\infty} P_{n,j}z^n, j = 0,1$. From (3.2) we obtain that

$$(z - B(z))(P_0(z) - P_{0,0}) = z(qB(z)P_{0,0} - b_0(P_{1,0} + qP_{0,0})). \tag{3.5}$$

Taking $z = z_0 = B(z_0)$ in (3.5) yields

$$b_0(P_{1,0} + qP_{0,0}) = qz_0P_{0,0}. \tag{3.6}$$

Substituting (3.6) into (3.5) leads to the following result

$$P_0(z) = P_{0,0}\left(1 + \frac{qz(B(z)-z_0)}{(z-B(z))}\right). \tag{3.7}$$

From (3.3) and (3.4), after carefully calculating one can get

$$(z - A(z))P_1(z) = z(qC(z) + \overline{q}A(z))P_{0,0} + (P_0(z) - P_{0,0})C(z) + (z-1)A(z)P_{0,1}. \tag{3.8}$$

By (3.7) and $C(z) = V(z)A(z)$, equation (3.8) can be written as follows:

$$(z - A(z))P_1(z) = A(z)\left(z\Big(\frac{qV(z)(z-z_0)}{z-B(z)} + \overline{q}\Big)P_{0,0} - (1-z)P_{0,1}\right). \tag{3.9}$$

Taking $z = z_1 = A(z_1)$ in (3.9) leads to

$$P_{0,1} = \frac{z_1}{1-z_1}\Big(\frac{qV(z_1)(z_1-z_0)}{z_1-B(z_1)} + \overline{q}\Big)P_{0,0} \overset{\triangle}{=} \sigma P_{0,0}, \tag{3.10}$$

where $\sigma = \frac{z_1}{1-z_1}\Big(\frac{qV(z_1)(z_1-z_0)}{z_1-B(z_1)} + \overline{q}\Big)$. Substituting (3.10) into (3.9), we obtain

$$P_1(z) = \frac{A(z)}{z-A(z)}\left(z\Big(\frac{qV(z)(z-z_0)}{z-B(z)} + \overline{q}\Big) - \sigma(1-z)\right)P_{0,0}. \tag{3.11}$$

Applying the normalizing condition $P_0(1) + P_1(1) = 1$ yields the expression

$$P_{0,0} = \left((1-qz_0)\frac{A(1)}{1-A(1)} + \frac{q(1-z_0)}{1-B(1)} + \overline{q}\right)^{-1}. \tag{3.12}$$

Summing above results, we have the following theorem.

**Theorem 3.2.** (1) *The generating functions of the stationary joint distribution of the Markov chain* $\{(N_n^+, J_n^+), n \geq 0\}$ *are given by:*

$$P_0(z) = P_{0,0}\left(1 + \frac{qz(B(z)-z_0)}{(z-B(z))}\right),$$
$$P_1(z) = \frac{A(z)}{z-A(z)}\left(z\Big(\frac{qV(z)(z-z_0)}{z-B(z)} + \overline{q}\Big) - \sigma(1-z)\right)P_{0,0},$$
$$P_{0,0} = \left((1-qz_0)\frac{A(1)}{1-A(1)} + \frac{q(1-z_0)}{1-B(1)} + \overline{q}\right)^{-1}.$$

(2) *The primary sever's state probabilities at the departure epoch are as follows:*

$$P(J^+ = 0) = P_0(1) = P_{0,0}\left(\frac{q(1-z_0)}{1-B(1)} + \overline{q}\right),$$
$$P(J^+ = 1) = P_1(1) = P_{0,0}(1-qz_0)\frac{A(1)}{1-A(1)}.$$

## 4. Steady-state system size distribution at arbitrary epoch

At arbitrary epoch $n^+$, let $N(n^+)$ be the system size, and $J(n^+)$ be the primary sever's state, where $J(n^+) = 0, 1$ according to whether the primary sever is under repair or not, respectively. Let $\zeta_0(n^+)$ be the remaining working breakdown service time when $J(n^+) = 0$ and $N(n^+) \geq 1$, $\zeta_1(n^+)$ be the remaining normal service time when $J(n^+) = 1$ and $N(n^+) \geq 1$. Then our system state at arbitrary epoch can be described by the Markov process $\{(N(n^+), J(n^+), \zeta_0(n^+), \zeta_1(n^+)), n \geq 0\}$ with state space $\mathcal{S} = \{(0,0),(0,1)\} \cup \{(n,j,k) : j = 0,1, n \geq 1, k \geq 1\}$.

To obtain the steady-state system size distribution, we define

$$Q_{0,j} = \lim_{t\to\infty} P(N(t^+) = 0, J(t^+) = j), j = 0,1,$$
$$Q_{n,j}(k) = \lim_{t\to\infty} P(N(t^+) = n, J(t^+) = j, \zeta_j(t^+) = k), j = 0,1, n,k \geq 1,$$
$$P_B = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} Q_{n,1}(k).$$

Considering the states of the system at two consecutive time epochs, we have the following set of equilibrium equations:

$$Q_{0,0} = \bar{\theta}\bar{\lambda}Q_{0,0} + \bar{\theta}\bar{\lambda}Q_{1,0}(1)\delta P_B, \tag{4.1}$$
$$Q_{n,0}(k) = \bar{\theta}\left(\delta_{n,1}\lambda Q_{0,0}s_{0,k} + \bar{\lambda}Q_{n,0}(k+1) + \lambda Q_{n-1,0}(k+1) + \bar{\lambda}Q_{n+1,0}(1)s_{0,k} + \lambda Q_{n,0}(1)s_{0,k}\right), n,k \geq 1, \tag{4.2}$$
$$Q_{0,1} = \bar{\lambda}Q_{0,1} + \theta\bar{\lambda}Q_{0,0} + \bar{\delta}\bar{\lambda}Q_{1,1}(1), \tag{4.3}$$
$$Q_{n,1}(k) = \delta_{n,1}\theta\lambda Q_{0,0}s_{1,k} + \delta_{n,1}\lambda Q_{0,1}s_{1,k} + \theta\left(\bar{\lambda}\sum_{i=1}^{\infty}Q_{n,0}(i) + \lambda\sum_{i=1}^{\infty}Q_{n-1,0}(i)\right)s_{1,k} + \bar{\delta}\left(\bar{\lambda}Q_{n,1}(k+1) + \lambda Q_{n-1,1}(k+1) + \bar{\lambda}Q_{n+1,1}(1)s_{1,k} + \lambda Q_{n,0}(1)s_{1,k}\right), n,k \geq 1, \tag{4.4}$$

where $\delta_{n,1}$ is the Kronecker delta function, $Q_{-1,1}(k) = Q_{-1,0}(k) = 0, k \geq 1$, and the normalization condition is

$$Q_{0,0} + Q_{0,1} + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(Q_{n,0}(k) + Q_{n,1}(k)) = 1.$$

To solve (4.1)–(4.4), we define the following generating functions:

$$\phi_j(k;z) = \sum_{n=1}^{\infty} z^n Q_{n,j}(k), k \geq 1, j = 0,1,$$
$$\Phi_j(x,z) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} x^k z^n Q_{n,j}(k) = \sum_{k=1}^{\infty} x^k \phi_j(k;z), j = 0,1.$$

Multiplying (4.2) and (4.4) by $z^n$ and summing over $n = 1, 2, \ldots$, respectively, we have

$$\phi_0(k;z) = \bar{\theta}\lambda z Q_{0,0} s_{0,k} + \bar{\theta}\Lambda(z)\phi_0(k+1;z) + \frac{\bar{\theta}\Lambda(z)}{z}\phi_0(1;z)s_{0,k} - \bar{\theta}\bar{\lambda}Q_{1,0}(1)s_{0,k}, \tag{4.5}$$

$$\begin{aligned}\phi_1(k;z) = {}& \lambda z(\theta Q_{0,0} + Q_{0,1})s_{1,k} + \bar{\delta}\Lambda(z)\phi_1(k+1,z) + \frac{\bar{\delta}\Lambda(z)}{z}\phi_1(1;z)s_{1,k} \\ & + \theta\Lambda(z)\Phi_1(1,z)s_{1,k} - \bar{\delta}\bar{\lambda}Q_{1,1}(1)s_{1,k}.\end{aligned} \tag{4.6}$$

Respectively multiplying (4.5) and (4.6) by $x^k$ and summing over $k = 1, 2, \cdots$, we get

$$\frac{x - \bar{\theta}\Lambda(z)}{x}\Phi_0(x,z) = \bar{\theta}G_{S_0}(x)(\lambda z Q_{0,0} - \bar{\lambda}Q_{1,0}(1)) - \frac{\bar{\theta}\Lambda(z)}{z}(z - G_{S_0}(x))\phi_0(1,z), \tag{4.7}$$

$$\begin{aligned}\frac{x - \bar{\delta}\Lambda(z)}{x}\Phi_1(x,z) = {}& G_{S_1}(x)\Big(\lambda z(\theta Q_{0,0} + Q_{0,1}) + \theta\Lambda(z)\Phi_0(1,z) - \bar{\delta}\bar{\lambda}Q_{1,1}(1)\Big) \\ & - \frac{\bar{\delta}\Lambda(z)}{z}(z - G_{S_1}(x))\phi_1(1;z).\end{aligned} \tag{4.8}$$

Taking $x = \bar{\theta}\Lambda(z)$ and $x = \bar{\delta}\Lambda(z)$, respectively, in (4.7) and (4.8), the left hand sides of the two equations vanish and we have

$$\frac{\bar{\theta}\Lambda(z)}{z}(z - B(z))\phi_0(1,z) = \bar{\theta}B(z)(\lambda z Q_{0,0} - \bar{\lambda}Q_{1,0}(1)), \tag{4.9}$$

$$\frac{\bar{\delta}\Lambda(z)}{z}(z - A(z))\phi_1(1;z) = A(z)\Big(\lambda z(\theta Q_{0,0} + Q_{0,1}) + \theta\Lambda(z)\Phi_0(1,z) - \bar{\delta}\bar{\lambda}Q_{1,1}(1)\Big). \tag{4.10}$$

Inserting $z = z_0$ into (4.9) yields

$$\bar{\lambda}Q_{1,0}(1) = \lambda z_0 Q_{0,0}. \tag{4.11}$$

Then equation (4.9) can be changed to

$$\phi_0(1,z) = \frac{\lambda z B(z)(z - z_0)}{\Lambda(z)(z - B(z))}Q_{0,0}. \tag{4.12}$$

Substituting (4.11) and (4.12) into (4.7) leads to

$$\frac{x - \bar{\theta}\Lambda(z)}{x}\Phi_0(x,z) = \frac{\lambda\bar{\theta}z(z - z_0)}{z - B(z)}\Big(G_{S_0}(x) - B(z)\Big)Q_{0,0}. \tag{4.13}$$

Then the marginal generating function of the system size when the primary sever is busy is given by

$$\Phi_0(1,z) = \frac{\lambda\bar{\theta}z(z - z_0)}{z - B(z)}\frac{1 - B(z)}{1 - \bar{\theta}\Lambda(z)}Q_{0,0}. \tag{4.14}$$

Similarly, inserting $z = z_1$ into (4.10) yields

$$\bar{\delta}\bar{\lambda}Q_{1,1}(1) = \lambda z_1(\theta Q_{0,0} + Q_{0,1}) + \theta\Lambda(z_1)\Phi_0(1, z_1). \tag{4.15}$$

Let $\psi(z) = \frac{\lambda\bar{\theta}z(z-z_0)}{z-B(z)}\frac{1-B(z)}{1-\bar{\theta}\Lambda(z)}$, combining (4.3) and (4.15) leads to

$$Q_{0,1} = \frac{\theta\Lambda(z_1)}{\lambda(1-z_1)}(1+\psi(z_1))Q_{0,0},$$

and

$$\lambda(Q_{0,1} + \theta Q_{0,0}) = \frac{\theta}{1-z_1}(1+\Lambda(z_1)\psi(z_1))Q_{0,0}. \tag{4.16}$$

From (4.8), (4.15) and (4.16), we have that

$$\frac{x-\bar{\delta}\Lambda(z)}{x}\Phi_1(x,z) = \frac{\theta z(G_{S_1}(x) - A(z))}{z - A(z)}\left(1 + \Lambda(z)\psi(z) - \frac{1-z}{1-z_1}(1+\Lambda(z_1)\psi(z_1))\right)Q_{0,0}. \tag{4.17}$$

Thus the marginal generating function of the system size when the substitute server is busy is given by

$$\Phi_1(1,z) = \frac{\theta z(1-A(z))}{(1-\bar{\delta}\Lambda(z))(z-A(z))}\left(1 + \Lambda(z)\psi(z) - \frac{1-z}{1-z_1}(1+\Lambda(z_1)\psi(z_1))\right)Q_{0,0}. \tag{4.18}$$

Using the normalization condition $Q_{0,0} + Q_{0,1} + \Phi_0(1,1) + \Phi_1(1,1) = 1$, we get

$$Q_{0,0} = \left(1 + \frac{1-\bar{\theta}\Lambda(z_0)}{\delta} + \frac{\lambda\bar{\theta}(1-z_0)}{\theta} + \frac{\theta\Lambda(z_1)}{\lambda(1-z_1)}(1+\psi(z_1))\right)^{-1}. \tag{4.19}$$

Based on the above analysis, we present some performance measures in the following theorem.

**Theorem 4.1.** (1) *The probability that the primary sever is under repair and the system is empty is given by*

$$\begin{aligned} Q_{0,0} &= \left(1 + \frac{1-\bar{\theta}\Lambda(z_0)}{\delta} + \frac{\lambda\bar{\theta}(1-z_0)}{\theta} + \frac{\theta\Lambda(z_1)}{\lambda(1-z_1)}(1+\psi(z_1))\right)^{-1} \\ &= \frac{\lambda\delta\theta(1-z_1)}{\lambda(\theta+\delta)(1-z_1)(\theta+\lambda\bar{\theta}(1-z_0)) + \delta\theta^2\Lambda(z_1)\big(1+\psi(z_1)\big)}. \end{aligned}$$

(2) *The probability that the substitute server is busy is as follows*

$$\Phi_0(1,1) = \frac{\lambda\bar{\theta}(1-z_0)}{\theta}Q_{0,0}.$$

(3) *The probability that the primary sever is idle is given by*

$$Q_{0,1} = \frac{\theta\Lambda(z_1)}{\lambda(1-z_1)}(1+\psi(z_1))Q_{0,0}.$$

(4) *The probability that the primary sever is busy is as follows*

$$\Phi_1(1,1) = \frac{1-\bar{\theta}\Lambda(z_0)}{\delta}Q_{0,0}.$$

(5) *The mean system size, denoted by* $L_\mathrm{s}$*, is given by*

$$\begin{aligned}L_\mathrm{s} &= \frac{\mathrm{d}}{\mathrm{d}z}\varPhi_0(1,z)\Big|_{z=1} + \frac{\mathrm{d}}{\mathrm{d}z}\varPhi_1(1,z)\Big|_{z=1} \\ &= \frac{Q_{0,0}}{\delta}\Bigg[\lambda^2\bar{\theta}(1-z_0) + \frac{(\delta+\theta)\lambda\bar{\theta}}{\theta}\Big(\frac{(1-z_0)(\theta+\lambda\bar{(\theta)})}{\theta} + \frac{z_0-B(1)}{1-B(1)}\Big) + \frac{\theta(1+\varLambda(z_1)\psi(z_1))}{1-z_1} \\ &\quad +(\theta+\lambda\bar{\theta}(1-z_0))\Big(\frac{\lambda\bar{\delta}}{\delta}+1-\frac{1}{1-A(1)}\Big)\Bigg].\end{aligned}$$

Let $\varPi_0$ denote the probability that the primary server is being repaired. Then from Theorem 4.1, we have

$$\varPi_0 = Q_{0,0} + \varPhi_0(1,1) = \frac{\theta+\lambda\bar{\theta}(1-z_0)}{\theta}Q_{0,0}. \tag{4.20}$$

## 5. Sojourn time distribution

In this section, we consider the FCFS sojourn time distribution of a test customer (TC), no matter whether it leaves the system by a successful normal service completion or a successful working breakdown service completion or a disaster arrival. Let $W$ be the TC's sojourn time, which is the total time that a TC is expected to spend in the system. Throughout this section, let $X^{*n}$ be the sum of $n$ i.i.d. nonnegative random variables $X_1, X_2, \ldots, X_n$. By convention, we assume $X^{*0} = 0$.

First, we give a Lemma without a proof.

**Lemma 5.1.** *Given two discrete-time random variables* $X, Y$*, where* $P(X=k) = \delta\bar{\delta}^{k-1}, k = 1,2,\ldots,$ *and* $P(Y=k) = y_k, k=1,2,\ldots.$ *The p.g.f of* $Y$ *is* $Y(z) = \sum_{k=1}^{\infty} z^k y_k$.

(1) *Under the condition* $Y < X$*, the p.g.f. of* $Y$ *is given by*

$$E[z^Y|Y<X] = \frac{Y(\bar{\delta}z)}{Y(\bar{\delta})}.$$

(2) *Under the condition* $X \le Y$*, the p.g.f. of* $X$ *is given by*

$$E[z^X|X\le Y] = \frac{\delta z}{1-\bar{\delta}z}\frac{1-Y(\bar{\delta}z)}{1-Y(\bar{\delta})}.$$

(3) *The p.g.f. of* $\min\{X,Y\}$ *is given by*

$$E[z^{\min\{X,Y\}}] = \frac{\delta z + (1-z)Y(\bar{\delta}z)}{1-\bar{\delta}z}.$$

(4) *The probability of the event* $(Y^{*(m-1)} < X \le Y^{*m})$ *is given by*

$$P(Y^{*(m-1)} < X \le Y^{*m}) = (Y(\bar{\delta}))^{m-1}(1-Y(\bar{\delta})),$$

*and the conditional generating function of* $X$ *given that the event* $(Y^{*(m-1)} < X \le Y^{*m})$ *occurs is*

$$E[z^X|Y^{*(m-1)} < X \le Y^{*m}] = \frac{\delta z}{1-\bar{\delta}z}\frac{Y(\bar{\delta}z))^{m-1}(1-Y(\bar{\delta}z))}{P(Y^{*(m-1)} < X \le Y^{*m})}.$$

Now, we derive the p.g.f of $W$, denoted by $G_W(z) = E[z^W]$. Six cases may exist for a TC's arrival which occurs in $(t^-, t)$.

- Case 1. The TC arrives when the primary sever is idle.

- Case 2. The TC arrives when the primary sever is busy and a disaster arrives.
- Case 3. The TC arrives when the primary sever is busy and a disaster does not arrive.
- Case 4. The TC arrives when the primary sever is under repair and the substitute server is idle and the repair does not end at time $t$.
- Case 5. The TC arrives while the substitute server is busy and the repair does not end at time $t$.
- Case 6. The TC arrives at a slot when which the primary server is under a repair and the repair ends at time $t$.

Let $W_i$ be the sojourn time of the TC in case $i$ and $G_{W_i}(z) = P(\text{case } i)E[z^{W_i}|\text{case } i]$, $i = 1,2,3,4,5,6$. Then $G_W(z) = E[z^W] = \sum_{i=1}^{6} G_{W_i}(z)$. Using Lemma 5.1, we find the expressions of $G_{W_i}(z)$ as follows.

In case 1, the TC's sojourn time is $\min\{S_1, D\}$. Then

$$\begin{aligned} G_{W_1}(z) &= \lim_{t\to\infty} P(N(t^-)=0, J(t^-)=1)E[z^{\min\{S_1,D\}}] \\ &= \lim_{t\to\infty} P(N((t-1)^+)=0, J((t-1)^+)=1)E[z^{\min\{S_1,D\}}] \\ &= Q_{0,1}\frac{\delta z + (1-z)G_{S_1}(\bar{\delta}z)}{1-\bar{\delta}z}. \end{aligned} \tag{5.1}$$

In Case 2, because the TC arrives finding the primary sever is busy and a disaster arrives. Then his sojourn time is zero and

$$\begin{aligned} G_{W_2}(z) &= \delta \lim_{t\to\infty} \sum_{n=1}^{\infty} P(N(t^-)=n, J(t^-)=1) \\ &= \delta \lim_{t\to\infty} \sum_{n=1}^{\infty} P(N((t-1)^+)=n, J((t-1)^+)=1) \\ &= \delta \Phi_1(1,1). \end{aligned} \tag{5.2}$$

In Case 3, by conditioning on the system size and the remaining service at the TC arrival epoch, we have

$$\begin{aligned} G_{W_3}(z) &= \bar{\delta} \lim_{t\to\infty} \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} P(N(t^-)=n, J(t^-)=1, \zeta_1(t^-)=k)E[z^{\min\{k+S_1^{*n},D\}}] \\ &= \bar{\delta} \lim_{t\to\infty} \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} P(N((t-1)^+)=n, J((t-1)^+)=1, \zeta_1((t-1)^+)=k+1)E[z^{\min\{k+S_1^{*n},D\}}] \\ &= \bar{\delta} \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} Q_{n,1}(k+1)\frac{\delta z + (1-z)E[(\bar{\delta}z)^{k+S_1^{*n}}]}{1-\bar{\delta}z} \\ &= \bar{\delta}\frac{\delta z\Phi_1(1,1) + (1-z)(\bar{\delta}z)^{-1}\Phi_1(\bar{\delta}z, G_{S_1}(\bar{\delta}z))}{1-\bar{\delta}z}. \end{aligned} \tag{5.3}$$

In Case 4, the TC can immediately receive his service rendered by the substitute server. There are two cases we should consider: $(S_0 < R)$ and $(S_0 \geq R)$.

$$\begin{aligned} &G_{W_4}(z) \\ &= \bar{\theta} \lim_{t\to\infty} P(N(t^-)=0, J(t^-)=0)\left(P(S_0<R)E[z^{S_0}|S_0<R] + P(S_0\geq R)E[z^{R+\min\{S_1,D\}}|S_0\geq R]\right) \\ &= \bar{\theta}Q_{0,0}\left(G_{S_0}(\bar{\theta}z) + \frac{\theta z}{1-\bar{\theta}z}(1-G_{S_0}(\bar{\theta}z))\frac{\delta z + (1-z)G_{S_1}(\bar{\delta}z)}{1-\bar{\delta}z}\right). \end{aligned} \tag{5.4}$$

In Case 5, the TC's sojourn time is decided by the following subcases:

- Case 5a. The unfinished work after the TC arrival is less than the remaining repair time. In this case, denote the sojourn time as $W_{5a}$ with p.g.f. $G_{W_{5a}}(z)$. Conditioning on the system size and the remaining working breakdown service time at the TC arrival epoch, we obtain

$$\begin{aligned}
G_{W_{5a}}(z) &= \bar{\theta} \lim_{t\to\infty} \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} P(N(t^-)=n, J(t^-)=1, \zeta_1(t^-)=k)P(k+S_0^{*n}<R)E[z^{k+S_0^{*n}}|k+S_0^{*n}<R]\\
&= \bar{\theta} \lim_{t\to\infty} \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} P(N((t-1)^+)=n, J((t-1)^+)=0, \zeta_1((t-1)^+)=k+1)(\bar{\theta}z)^k(G_{S_0}(\bar{\theta}z))^n\\
&= \bar{\theta}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} Q_{n,0}(k+1)(\bar{\theta}z)^k(G_{S_0}(\bar{\theta}z))^n\\
&= \bar{\theta}\frac{\Phi_0(\bar{\theta}z, G_{S_0}(\bar{\theta}z))}{\bar{\theta}z}.
\end{aligned}$$

- Case 5b. The unfinished work $U$ after the TC arrival is equal to or longer than the remaining repair time. In this case, denote the sojourn time as $W_{5b}$ with p.g.f. $G_{W_{5b}}(z)$. Suppose there are $n$ customers in the system and the remaining working breakdown service time is $k$ at time $t^-$, $n\geq 1, k\geq 0$. Then $U=k+S_0^{*n}$. We should consider the following cases:
(1) If $R\leq k$, then $W_{5b}=R+\min\{S_1^{*(n+1)}, D\}$;
(2) If $k+S_0^{*m-1}<R\leq k+S_0^{*m}, m=1,2,\cdots,n$, then $W_{5b}=R+\min\{S_1^{*(n+1-m)}, D\}$. Then by the memoryless property of geometric distribution and Lemma 5.1, we obtain

$$\begin{aligned}
&G_{W_{5b}}(z)\\
&= \bar{\theta}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} Q_{n,0}(k+1))P(R\leq k)E[z^{R+\min\{S_1^{*(n+1)},D\}}|R\leq k]\\
&\quad + \bar{\theta}\sum_{n=1}^{\infty}\sum_{m=1}^{n}\sum_{k=0}^{\infty} Q_{n,0}(k+1))P(k+S_0^{*m-1}<R\leq k+S_0^{*m})E[z^{R+\min\{S_1^{*(n+1-m)},D\}}|k+S_0^{*m-1}<R\leq k+S_0^{*m}]\\
&= \bar{\theta}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} Q_{n,0}(k+1))P(R\leq k)E[z^R|R\leq k]E[\min\{S_1^{*(n+1)}, D\}]\\
&\quad + \bar{\theta}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{m=1}^{n} Q_{n,0}(k+1))(\bar{\theta}z)^kP(S_0^{*m-1}<R\leq S_0^{*m})E[z^R|S_0^{*m-1}<R\leq S_0^{*m}]E[z^{\min\{S_1^{*(n+1-m)},D\}}]\\
&= \bar{\theta}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} Q_{n,0}(k+1))\frac{\theta z(1-(\bar{\theta}z)^k)}{1-\bar{\theta}z}\frac{\delta z+(1-z)(G_{S_1}(\bar{\delta}z))^{n+1}}{1-\bar{\delta}z}\\
&\quad + \bar{\theta}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{m=1}^{n} Q_{n,0}(k+1))(\bar{\theta}z)^k\frac{\theta z}{1-\bar{\theta}z}(G_{S_0}(\bar{\theta}z))^{m-1}(1-G_{S_0}(\bar{\theta}z))\frac{\delta z+(1-z)(G_{S_1}(\bar{\delta}z))^{n+1-m}}{1-\bar{\delta}z}\\
&= \bar{\theta}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} Q_{n,0}(k+1))\frac{\theta z(1-(\bar{\theta}z)^k)}{1-\bar{\theta}z}\frac{\delta z+(1-z)(G_{S_1}(\bar{\delta}z))^{n+1}}{1-\bar{\delta}z} + \frac{\bar{\theta}\theta z}{(1-\bar{\theta}z)(1-\bar{\delta}z)}\\
&\quad \times\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} Q_{n,0}(k+1))(\bar{\theta}z)^k\left(\delta z(1-(G_{S_0}(\bar{\theta}z))^n)+(1-z)G_{S_1}(\bar{\delta}z)(1-G_{S_0}(\bar{\theta}z))\frac{(G_{S_1}(\bar{\delta}z))^n-(G_{S_0}(\bar{\theta}z))^n}{G_{S_1}(\bar{\delta}z)-G_{S_0}(\bar{\theta}z)}\right)\\
&= \frac{\bar{\theta}\theta z}{(1-\bar{\theta}z)(1-\bar{\delta}z)}\left(\delta z\Phi_0(1,1)-\frac{\delta}{\bar{\theta}}\Phi_0(\bar{\theta}z,1)+(1-z)G_{S_1}(\bar{\delta}z)\left(\Phi_0(1,G_{S_1}(\bar{\delta}z))-\frac{\Phi_0(\bar{\theta}z,G_{S_1}(\bar{\delta}z))}{\bar{\theta}z}\right)\right.\\
&\quad \left.+\frac{\delta}{\bar{\theta}}\left(\Phi_0(\bar{\theta}z,1)-\Phi_0(\bar{\theta}z,G_{S_0}(\bar{\theta}z))\right)+\frac{(1-z)G_{S_1}(\bar{\delta}z)(1-G_{S_0}(\bar{\theta}z))}{G_{S_1}(\bar{\delta}z)-G_{S_0}(\bar{\theta}z)}\frac{1}{\bar{\theta}z}\left(\Phi_0(\bar{\theta}z,G_{S_1}(\bar{\delta}z))-\Phi_0(\bar{\theta}z,G_{S_0}(\bar{\theta}z))\right)\right).
\end{aligned}$$

Combining Case 5a and Case 5b, we get

$$\begin{aligned} G_{W_5}(z) &= G_{W_{5a}}(z) + G_{W_{5b}}(z) \\ &= \bar{\theta}\left(\frac{\Phi_0(\bar{\theta}z, G_{S_0}(\bar{\theta}z))}{\bar{\theta}z} + \frac{\theta z}{(1-\bar{\theta}z)(1-\bar{\delta}z)}\left(\delta z\Phi_0(1,1) - \frac{\delta}{\bar{\theta}}\Phi_0(\bar{\theta}z,1) + (1-z)G_{S_1}(\bar{\delta}z)\left(\Phi_0(1, G_{S_1}(\bar{\delta}z)) - \frac{\Phi_0(\bar{\theta}z, G_{S_1}(\bar{\delta}z))}{\bar{\theta}z}\right)\right.\right. \\ &\left.\left. + \frac{\delta}{\bar{\theta}}\left(\Phi_0(\bar{\theta}z,1) - \Phi_0(\bar{\theta}z, G_{S_0}(\bar{\theta}z))\right) \qquad + \frac{(1-z)G_{S_1}(\bar{\delta}z)(1-G_{S_0}(\bar{\theta}z))}{G_{S_1}(\bar{\delta}z) - G_{S_0}(\bar{\theta}z)}\frac{1}{\bar{\theta}z}\left(\Phi_0(\bar{\theta}z, G_{S_1}(\bar{\delta}z)) - \Phi_0(\bar{\theta}z, G_{S_0}(\bar{\theta}z))\right)\right)\right). \end{aligned} \tag{5.5}$$

In Case 6, at time $t^+$ the primary sever restarts to serve all the customers on FCFS. Conditioning on the system size at $t^-$, we obtain

$$\begin{aligned} G_{W_6}(z) &= \theta \lim_{t\to\infty}\sum_{n=0}^{\infty} P(N(t^-) = n, J(t^-) = 1)E[z^{\min\{S_1^{*(n+1)}, D\}}] \\ &= \theta\left(Q_{0,0}\frac{\delta z + (1-z)G_{S_1}(\bar{\delta}z)}{1-\bar{\delta}z} + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} Q_{n,0}(k)\frac{\delta z + (1-z)(G_{S_1}(\bar{\delta}z))^{(n+1)}}{1-\bar{\delta}z}\right) \\ &= \theta\left(Q_{0,0}\frac{\delta z + (1-z)G_{S_1}(\bar{\delta}z)}{1-\bar{\delta}z} + \frac{\delta z\Phi_0(1,1) + (1-z)G_{S_1}(\bar{\delta}z)\Phi_0(1, G_{S_1}(\bar{\delta}z))}{1-\bar{\delta}z}\right). \end{aligned} \tag{5.6}$$

Based on equations (5.1)–(5.6), we can get the expression of $G_W(z) = E[z^W] = \sum_{i=1}^{6} G_{W_i}(z)$.

## 6. The time between failures

Let $\Theta$ denote the time between two consecutive failures, which is called a regeneration cycle. Note that $\Theta$ consists of many normal busy periods and idle periods and a repair period. In this paper, the duration in which the primary (or substitute) server works at the normal (or working breakdown) service rate continuously is called a normal (or working breakdown) busy period. Let $B_N$ and $B_W$ with p.g.f. $G_{B_N}(z)$ and $G_{B_W}(z)$, respectively, represent the lengths of normal and working breakdown busy periods beginning with only one customer, by Tian *et al.* [22], we know that $G_{B_N}(z) = G_{S_1}(z(\bar{\lambda} + \lambda G_{B_N}(z)))$ and $G_{B_W}(z) = G_{S_0}(z(\bar{\lambda} + \lambda G_{B_W}(z)))$. It can be proved that $z_1 = G_{B_N}(\bar{\delta})$ and $z_0 = G_{B_W}(\bar{\theta})$.

To conduct the cycle analysis, let $K$ be the system size at the beginning epoch of a new normal busy period, which begins immediately at the end of a repair period with a customer's working breakdown service interrupted. Then we having the following Lemma 6.1.

**Lemma 6.1.** *The p.g.f.* $G_K(z) = \sum_{n=1}^{\infty} z^n P(K = n)$ *is given by*

$$G_K(z) = \frac{1}{1-z_0}\frac{z(z-z_0)}{z-B(z)}V(z). \tag{6.1}$$

*Proof.* Define $B_R$ be the event that a normal busy period begins immediately after the repair period ends with a customer's working breakdown service interrupted, then

$$P(B_R) = \theta\Phi_0(1,1) = \lambda\bar{\theta}(1-z_0)Q_{0,0}.$$

Then we can get the distribution of $K$ as follows:

$$P(K = n) = \frac{1}{P(B_R)}\theta\left(\bar{\lambda}\sum_{k=1}^{\infty} Q_{n,0}(k) + \lambda\sum_{k=1}^{\infty} Q_{n-1,0}(k)\right), n \geq 1.$$

From (4.14), we obtain the p.g.f. of $K$ as follows

$$\begin{aligned} G_K(z) &= \sum_{n=1}^{\infty} z^n P(K=n) = \frac{\theta}{P(B_R)} \sum_{n=1}^{\infty} z^n \left( \bar{\lambda} \sum_{k=1}^{\infty} Q_{n,0}(k) + \lambda \sum_{k=1}^{\infty} Q_{n-1,0}(k) \right) \\ &= \frac{\theta \Lambda(z)}{P(B_R)} \Phi_0(1,z) = \frac{1}{1-z_0} \frac{z(z-z_0)}{z-B(z)} \frac{\theta \Lambda(z)}{1-\bar{\theta}\Lambda(z)} (1-B(z)) \\ &= \frac{1}{1-z_0} \frac{z(z-z_0)}{z-B(z)} V(z). \end{aligned}$$

This ends the proof of Lemma 6.1. □

Now, we derive the expression of the p.g.f. $G_\Theta(z) = E[z^\Theta]$.

Depending on the first customer arrival time after the primary sever enters into the repair period, the time $\Theta$ may be divided into three types:

- Type-1, denoted by $\Theta_1$ ($= \Theta | A > R$): the first customer arrives after the end of repair period with probability $p_1 = P(A > R) = \frac{\bar{\lambda}\theta}{1-\bar{\lambda}\bar{\theta}}$.
- Type-2, denoted by $\Theta_2$ ($= \Theta | A = R$): the first customer arrives at the end of repair period with probability $p_2 = P(A = R) = \frac{\lambda\theta}{1-\bar{\lambda}\bar{\theta}}$.
- Type-3, denoted by $\Theta_3$ ($= \Theta | A < R$): the first customer arrives during repair period with probability $p_3 = P(A < R) = \frac{\lambda\bar{\theta}}{1-\bar{\lambda}\bar{\theta}}$.

Define $G_{\Theta_i}(z) = E[z^{\Theta_i}], i = 1,2,3$, then $G_\Theta(z) = \sum_{i=1}^{3} p_i G_{\Theta_i}(z)$. So we dedicate to derive the expressions of $\Theta_i(z) = E[z^{\Theta_i}], i = 1,2,3$.

(1) First, we derive the p.g.f. $G_{\Theta_1}(z)$. For $\Theta_1$, the primary sever immediately enters into idle period after the end of repair period. Let $L$ be the length of the duration from the epoch at which the the primary sever idle period begins to the epoch at which a new repair period begins. Then $\Theta_1 = R|(R < A) + L$ and $G_{\Theta_1}(z) = E[z^R | R < A] E[z^L]$.

When the first customer arrives in the system, a normal busy period begins. If $(D \le B_N)$, then $L = A_1 + D$. Otherwise, if $(B_N < D)$ occurs, after the end of normal busy period, the primary sever enters into idle period again, then $L = A_1 + B_N + \widetilde{L}$, where $\widetilde{L}$ is a new random random and distributed with $L$. Thus the p.g.f. $G_L(z)$ of $L$ is given by:

$$\begin{aligned} G_L(z) &= P(D \le B_N) E[z^{A_1 + D} | D \le B_N] + P(B_N < D) E[z^{A_1 + B_N + \widetilde{L}} | B_N < D] \\ &= E[z^{A_1}] \Big( P(D \le B_N) E[z^D | D \le B_N] + P(B_N < D) E[z^{B_N} | B_N < D] G_L(z) \Big) \\ &= \frac{\lambda z}{1-\bar{\lambda} z} \left( \frac{\delta z}{1-\bar{\delta} z} (1 - G_{B_N}(\bar{\delta} z)) + G_{B_N}(\bar{\delta} z) G_L(z) \right), \end{aligned}$$

which leads to

$$G_L(z) = \left( \frac{\lambda z}{1-\bar{\lambda} z} \frac{\delta z}{1-\bar{\delta} z} (1 - G_{B_N}(\bar{\delta} z)) \right) \left( 1 - \frac{\lambda z}{1-\bar{\lambda} z} G_{B_N}(\bar{\delta} z) \right)^{-1}, \tag{6.2}$$

and

$$E[L] = \frac{1}{\delta} + \frac{1}{\lambda(1 - G_{B_N}(\bar{\delta}))}. \tag{6.3}$$

**Remark 6.2.** Let $L_1$, with p.g.f $G_{L_1}(z)$, be the time duration from the epoch at which the normal busy period begins with only one customer to the epoch at which the next repair period begins, then $L = A + L_1$ and $G_L(z) = \frac{\lambda z}{1-\bar{\lambda} z} G_{L_1}(z)$. It follows from equation (6.2) that

$$G_{L_1}(z) = \left(\frac{\delta z}{1-\bar{\delta} z}(1 - G_{B_N}(\bar{\delta} z))\right)\left(1 - \frac{\lambda z}{1-\bar{\lambda} z} G_{B_N}(\bar{\delta} z)\right)^{-1}. \tag{6.4}$$

From (6.2), the p.g.f. $G_{\Theta_1}(z)$ is given by

$$\begin{aligned} G_{\Theta_1}(z) &= E[z^R | R < A] E[z^L] \\ &= \frac{1}{p_1} \frac{\theta \bar{\lambda} z}{1 - \bar{\theta}\bar{\lambda} z}\left(\frac{\lambda z}{1-\bar{\lambda} z}\frac{\delta z}{1-\bar{\delta} z}(1 - G_{B_N}(\bar{\delta} z))\right)\left(1 - \frac{\lambda z}{1-\bar{\lambda} z} G_{B_N}(\bar{\delta} z)\right)^{-1}, \end{aligned} \tag{6.5}$$

and the mean value of $\Theta_1$ is

$$\begin{aligned} E[\Theta_1] &= E[R | R < A] + E[L] \\ &= \frac{1}{1 - \bar{\lambda}\bar{\theta}} + \frac{1}{\delta} + \frac{1}{\lambda(1 - G_{B_N}(\bar{\delta}))}. \end{aligned} \tag{6.6}$$

(2) For $\Theta_2$, a new normal busy period begins with only one customer at the end of the repair period. Then $\Theta_2 = R|(R = A) + L_1$. Using (6.4) yields

$$\begin{aligned} G_{\Theta_2}(z) &= E[z^R | R = A] L_1(z) \\ &= \frac{1}{p_2} \frac{\lambda \theta z}{1 - \bar{\lambda}\bar{\theta} z}\left(\frac{\delta z}{1-\bar{\delta} z}(1 - G_{B_N}(\bar{\delta} z))\right)\left(1 - \frac{\lambda z}{1-\bar{\lambda} z} G_{B_N}(\bar{\delta} z)\right)^{-1}, \end{aligned} \tag{6.7}$$

and

$$E[\Theta_2] = \frac{1}{1 - \bar{\lambda}\bar{\theta}} + \frac{1}{\delta} + \frac{G_{B_N}(\bar{\delta})}{\lambda(1 - G_{B_N}(\bar{\delta}))}. \tag{6.8}$$

(3) For $\Theta_3$, the working breakdown busy period immediately begins at the arrival epoch of the first customer. Let $F$ denotes the time interval from the epoch at which the working breakdown busy period begins to the epoch at which a new repair period begins. Then $\Theta_3 = A|(A < R) + F$. We have the following two cases.

- Case 3a: If $(B_W < R)$ occurs, at the end of the working breakdown busy period during the repair period, a new unconditional cycle restarts because of the memoryless property of the geometric distribution. In this case, $F = B_W|(B_W < R) + \Theta$.
- Case 3b: If $(R \le B_W)$ occurs, at the end of the repair period, a normal busy period with $K$ customers immediately begins, the p.g.f. $G_K(z)$ is given by (6.1).

Let $B_K$ be length of the normal busy period begin with $K$ customers, then the p.g.f. $G_{B_K}(z) = G_K(G_{B_N}(z))$, and the equality $G_{B_K}(\bar{\delta}) = G_K(G_{B_N}(\bar{\delta})) = G_K(z_1)$ holds.

If $(D \le B_K)$ happens, then $F = R|(R \le B_W) + D|(D \le B_K)$. If $(B_K < D)$ occurs, then $F = R|(R \le B_W) + B_K|(D \le B_K) + L$. Based on the above two cases, the p.g.f. $G_F(z)$ is derived by

$$\begin{aligned} G_F(z) &= P(B_W < R) E[z^{B_W} | (B_W < R)] G_\Theta(z) + P(R \le B_W) E[z^R | R \le B_W]\Big(P(D \le B_K) E[z^D | D \le B_K] \\ &\quad + P(B_K < D) E[z^{B_K} | B_K < D] G_L(z)\Big) \\ &= G_{B_W}(\bar{\theta} z) G_\Theta(z) + \frac{\theta z}{1 - \bar{\theta} z}(1 - G_{B_W}(\bar{\theta} z))\left(\frac{\delta z}{1 - \bar{\delta} z}(1 - G_{B_K}(\bar{\delta} z)) + G_{B_K}(\bar{\delta} z) G_L(z)\right), \end{aligned} \tag{6.9}$$

and

$$E[F] = G_{B_W}(\bar{\theta})E[\Theta] + (1 - G_{B_W}(\bar{\theta}))\left(\frac{1}{\theta} + \frac{1}{\delta}(1 - G_{B_K}(\bar{\delta})) + G_{B_K}(\bar{\delta})E[L]\right). \tag{6.10}$$

By equations (6.9) and (6.10), we obtain

$$\begin{aligned} G_{\Theta_3}(z) &= E[z^A | A < R]G_F(z) \\ &= \frac{1}{p_3}\frac{\lambda\bar{\theta}z}{1 - \bar{\lambda}\bar{\theta}z}\left(G_{B_W}(\bar{\theta}z)G_\Theta(z) + \frac{\theta z}{1 - \bar{\theta}z}(1 - G_{B_W}(\bar{\theta}z))\left(\frac{\delta z}{1 - \bar{\delta}z}(1 - G_{B_K}(\bar{\delta}z)) + G_{B_K}(\bar{\delta}z)G_L(z)\right)\right), \end{aligned} \tag{6.11}$$

and

$$E[\Theta_3] = \frac{1}{1 - \bar{\lambda}\bar{\theta}} + G_{B_W}(\bar{\theta})E[\Theta] + (1 - G_{B_W}(\bar{\theta}))\left(\frac{1}{\theta} + \frac{1}{\delta} + \frac{G_{B_K}(\bar{\delta})}{\lambda(1 - G_{B_N}(\bar{\delta}))}\right). \tag{6.12}$$

Combining (6.5), (6.7) and (6.11), we get the p.g.f. $G_\Theta(z)$ of $\Theta(z)$: $G_\Theta(z) = \sum_{i=1}^{3} p_i G_{\Theta_i}(z)$, and from (6.6), (6.8) and (6.12), we obtain that

$$\begin{aligned} E[\Theta] &= \sum_{i=1}^{3} p_i E[\Theta_i] \\ &= \frac{1}{1 - \bar{\lambda}\bar{\theta}} + \frac{\bar{\lambda}\theta}{1 - \bar{\lambda}\bar{\theta}}\left(\frac{1}{\delta} + \frac{1}{\lambda(1 - G_{B_N}(\bar{\delta}))}\right) + \frac{\lambda\theta}{1 - \bar{\lambda}\bar{\theta}}\left(\frac{1}{\delta} + \frac{G_{B_N}(\bar{\delta})}{\lambda(1 - G_{B_N}(\bar{\delta}))}\right) \\ &\quad + \frac{\lambda\bar{\theta}}{1 - \bar{\lambda}\bar{\theta}}\left(G_{B_W}(\bar{\theta})E[\Theta] + (1 - G_{B_W}(\bar{\theta}))\left(\frac{1}{\theta} + \frac{1}{\delta} + \frac{G_{B_K}(\bar{\delta})}{\lambda(1 - G_{B_N}(\bar{\delta}))}\right)\right), \end{aligned}$$

which leads to

$$\begin{aligned} E[\Theta] &= \frac{1}{\theta} + \frac{1}{\delta} + \frac{\theta\big(\bar{\lambda} + \lambda G_{B_N}(\bar{\delta})\big) + \lambda\bar{\theta}G_{B_K}(\bar{\delta})\big(1 - G_{B_W}(\bar{\theta})\big)}{\lambda\big(1 - G_{B_N}(\bar{\delta})\big)\big(\theta + \lambda\bar{\theta}(1 - G_{B_W}(\bar{\theta}))\big)} \\ &= \frac{1}{\theta} + \frac{1}{\delta} + \frac{\theta\Lambda(z_1) + \lambda\bar{\theta}G_K(z_1)\big(1 - z_0\big)}{\lambda\big(1 - z_1\big)\big(\theta + \lambda\bar{\theta}(1 - z_0)\big)}. \end{aligned} \tag{6.13}$$

By the alternating renewal theorem, we have the following results

$$\begin{aligned} \Pi_0 = \frac{E[R]}{E[\Theta]} &= \frac{\delta\lambda\big(1 - z_1\big)\big(\theta + \lambda\bar{\theta}(1 - z_0)\big)}{\lambda(\delta + \theta)(1 - z_1)\big(\theta + \lambda\bar{\theta}(1 - z_0)\big) + \delta\theta^2\Lambda(z_1)(1 + \psi(z_1))} \\ &= \frac{\theta + \lambda\bar{\theta}(1 - z_0)}{\theta}Q_{0,0}, \end{aligned}$$

which agrees with equation (4.20).

## 7. Numerical examples

In this section we investigate the impact of system parameter $\theta, \delta$ on some performance measures $P_0, L_s, E[\Theta]$.

- The performance measures of the Geo/(NB, Geo)/1 queue are plotted in Figure 2 for a case where the customer arrival rate is $\lambda = 0.35$, the normal service time $S_1$ follows negative binomial distribution $\mathrm{NB}(2, 0.25)$, working breakdown service time $S_0$ follows geometric distribution $\mathrm{Geo}(0.05)$.

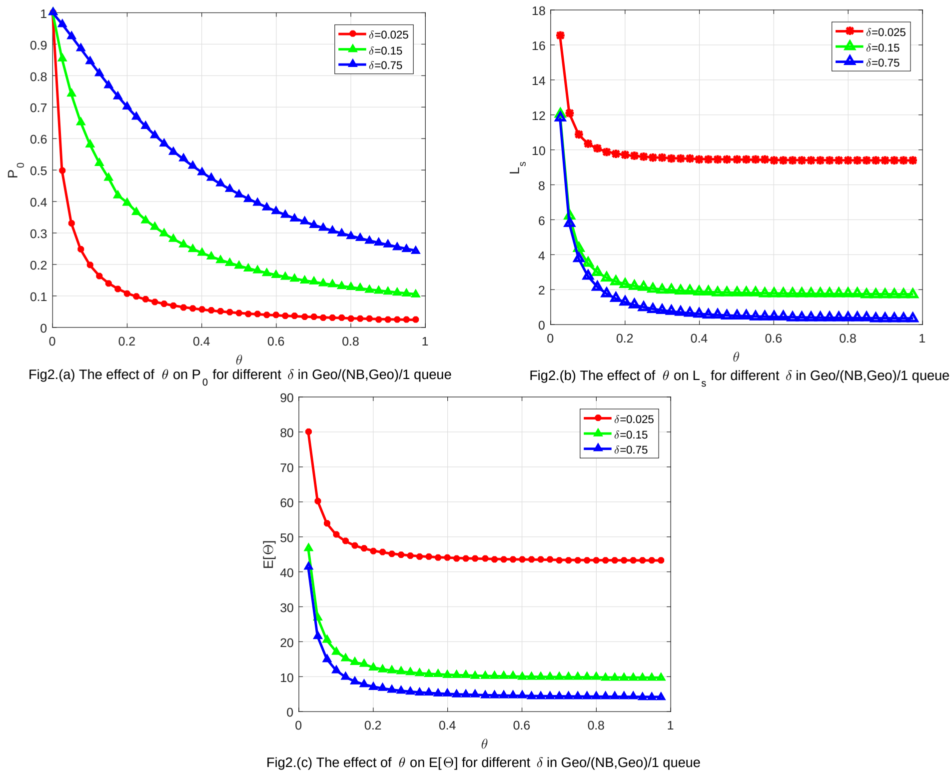


FIGURE 2. $P_0$, $L_\mathrm{s}$ and $E[\Theta]$ *vs.* $\theta$ for different $\delta$ in Geo/(NB, Geo)/1 queue.

- The performance measures of the Geo/(PH, NB)/1 queue are depicted in Figure 3 for a case where $\lambda = 0.35$, the normal service time $S_1$ has a discrete Phase-type distribution $\mathrm{PH}_d(\alpha, T)$, $\alpha = (\frac{2}{7}, \frac{5}{7}), T = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$, working breakdown service time $S_0$ has a negative binomial distribution $\mathrm{NB}(2, 0.25)$.

Figures 2(a) and 3(a) show that given other parameter fixed, the probability that the primary sever is under repair, $P_0$, decreases monotonously as the repair rate increases, and when $\theta \to 0$, the value of $P_0$ tends to 1, which agrees with our expectations. On the other hand, from Figures 2(a) and 3(a), we can see that as the arrival rate of a disaster increases, the chance that the primary sever enters into a repair period increases, because the arrival of a disaster not only removes all the customers, but also causes the primary sever breakdown. As expected, Figures 2(b), (c) and 3(b), (c) demonstrate that $L_\mathrm{s}$ and $E[\Theta]$ are monotonically decreasing as functions of $\theta$ and $\delta$, respectively. The reason is that when the value of $\theta$ or $\delta$ increases, a repair period is shortened, the probability that the customers are served by the primary sever or removed by a disaster is increased, which leads to the decrement of the mean system length and the mean length of the time between two failures.

In practice, operating cost control has strong practical significance and application value for economic activities. In the following, our objective is to study the cost minimization problem based on a given cost structure.

Cost elements are defined as follows:

$C_\mathrm{h} \equiv$ holding cost for each customer in the system per unit time;

$C_\mathrm{s} \equiv$ setup cost per cycle;

$C_0 \equiv$ cost for keeping the substitute server available per unit time;

$C_1 \equiv$ cost for keeping the primary sever available per unit time.

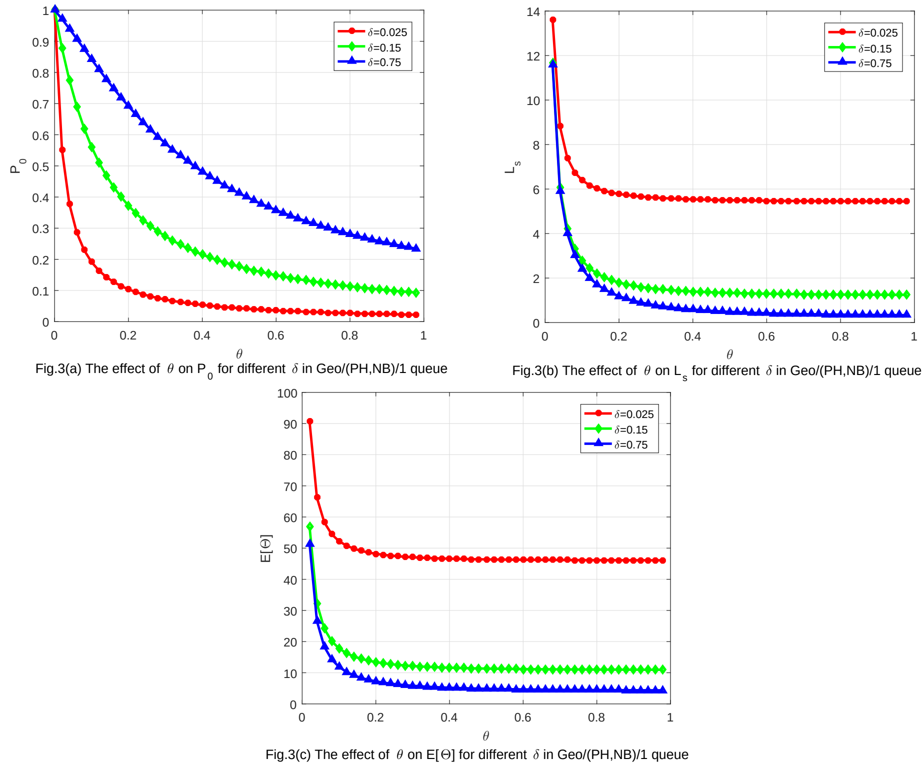


FIGURE 3. $P_0$, $L_{\mathrm{s}}$ and $E[\Theta]$ *vs.* $\theta$ for different $\delta$ in Geo/(PH, NB)/1 queue.

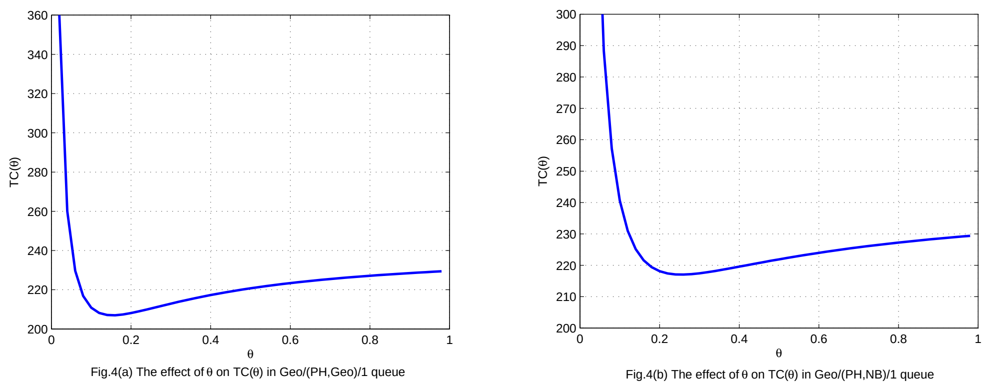


FIGURE 4. TC($\theta$) *vs.* $\theta$ in Geo/(PH, Geo)/1 and Geo/(PH, NB)/1 queue.

Taking the repair rate $\theta$ as a decision variable, the per unit time total expected cost function is given by

$$\mathrm{TC}(\theta) = C_{\mathrm{h}} L_{\mathrm{s}} + C_0 \Pi_0 + C_1(1 - \Pi_0) + C_{\mathrm{s}} \frac{1}{E(C)}.$$

The cost minimization problem can be mathematically illustrated by $\min_\theta \mathrm{TC}(\theta)$ subject to $0 < \theta < 1$. Due to highly non-linear and complex of the cost function TC($\theta$), it's not easy to get the derivatives of it. With various numerical examples, see Figure 4(a) and (b), we find the optimum value of $\theta$, say $\theta^*$ by Matlab.

In Figure 4(a) and (b), we take $C_\mathrm{s} = 75, C_\mathrm{h} = 40, C_0 = 100, C_1 = 200$ and $\lambda = 0.35, d = 0.25$, $S_1 \sim \mathrm{PH}_d(\alpha, T)$ as in Figure 3. $S_0 \sim$ Geo(0.25) in Figure 4(a) and $S_0 \sim$ NB(2, 0.25) in Figure 4(b). Figure 4(a) and (b), respectively, depict the effect of $\theta$ on the cost function TC($\theta$) in the Geo/(PH, Geo)/1 queue and in the Geo/(PH, NB)/1 queue.

Figure 4 shows that there is an optimal repair rate $\theta$ which minimizing the operating cost. Implementing the parabolic method in the computer software MATLAB and taking the error $\varepsilon = 10^{-6}$ , we find the solution $\theta^* = 0.153093$ with TC($\theta^*$) = 206.9618 in Figure 4(a), $\theta^* = 0.253981$ with TC($\theta^*$) = 217.0406 in Figure 4(b).

## 8. Conclusions and future research

In this paper, we have analyzed a repairable discrete-time Geo/G/1 queueing system with disasters and a substitute server, in which disaster only arrives in the primary sever busy period, a disaster arrival not only removes all customers, but also causes the primary sever breakdown. During the repair period of the primary sever, customers are served by the substitute server at a lower service rate. The working breakdown service during repair period contributes to a more realistic service schedule to the study of the discrete-time queue with disasters and breakdowns. We have derived the principal characteristics of our model, for example, the embedded Markov, the steady-state system size, the sojourn time of a test customer in the system and the time cycle.

This topic in this paper can be extended to the discrete-time retrial queues with negative customers or disaster, which will be our future research direction.

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