RAIRO-Oper. Res. 53 (2019) 1187–1195
https://doi.org/10.1051/ro/2018054

**RAIRO Operations Research**
www.rairo-ro.org

# SOME RESULTS ON THE $b$-CHROMATIC NUMBER IN COMPLEMENTARY PRISM GRAPHS

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**Abstract.** A $b$-coloring of a graph $G$ is a proper coloring of $G$ with $k$ colors such that each color class has a vertex that is adjacent to at least one vertex of every other color classes. The $b$-chromatic number is the largest integer $k$ for which $G$ has a $b$-coloring with $k$ colors. In this paper, we present some results on $b$-coloring in complementary prism graphs.

**Mathematics Subject Classification.** 05C15.

Received January 21, 2018. Accepted June 27, 2018.

## 1. Introduction

All graphs considered in this paper are undirected, finite and simple. For terminology and notation not defined here we refer to [3]. Let $G$ be a graph with vertex-set $V(G)$ and edge-set $E(G)$. The complement $\overline{G}$ of $G$ is the graph having the same vertex-set as $G$ such that two vertices are adjacent if and only if the same two vertices are non-adjacent in $G$. For a non-empty set $A \subseteq V(G)$, we denote by $G[A]$ the subgraph of $G$ induced by $A$, and by $G - A$ be the subgraph induced by $V(G) - A$. If $A = \{v\}$ we may write $G - v$ instead of $G - \{v\}$. For a vertex $v$ of $G$, the *open neighborhood* of $v$ is $N_G(v) = \{u \in V(G) : uv \in E(G)\}$, the closed neighborhood of $v$ is $N_G[v] = N_G(v) \cup \{v\}$ and the degree of $v$, denoted $d_G(v)$ is $|N_G(v)|$. By $\Delta(G)$ we denote the *maximum degree* of the graph $G$. Let $C_n$, $P_n$, $K_n$ denote, respectively, the *cycle*, the *path* and the *complete graph* of order $n$. The *complete bipartite* graph on $p + q$ vertices is denoted by $K_{p,q}$.

The *complementary prism graph* $G\overline{G}$ of $G$ is the graph formed from the disjoint union of $G$ and $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices of $G$ and $\overline{G}$. The complementary prism graphs are a sub-family of complementary products of graphs, which were first introduced by Haynes *et al.* [10] in 2007, as a generalization of cartesian products of graphs. In [10–12], T.W. Haynes *et al.* studied some parameters of complementary prism graphs such as the independence number, the domination number and the total domination number. Several different parameters have been studied in complementary prism graphs, see [1,17,26]. It is important to note that complementary prism graphs are a generalization of several known graphs. For example, the complementary prism of $C_5$ is the Petersen graph, and the complementary prism graph of $K_n$ is the corona $K_n \circ K_1$.

*Keywords.* $b$-coloring, $b$-chromatic number, complementary prism graph.

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A *proper coloring* of $G$ is an assignment of colors to the vertices of $G$ such that any two adjacent vertices have different colors. The *chromatic number* of a graph $G$, denoted by $\chi(G)$, is the minimum number for which there exist a proper coloring for $G$ with $\chi(G)$ colors.

A *b-coloring* of a graph $G$ is a proper coloring of $G$ such that each color class contains at least one vertex that has a neighbor in each of the other color classes, such a vertex is called *b-vertex*. The *b-chromatic number* of $G$, denoted $b(G)$, is defined as the largest integer $k$ for which $G$ admits a $b$-coloring with $k$ colors. Every proper coloring of a graph $G$ with $\chi(G)$ colors is a $b$-coloring. Otherwise, every vertex of a color class with no $b$-vertex can be recolored by some other color. This yields a proper coloring with less than $\chi(G)$ colors, which is a contradiction. Consequently, $\chi(G) \leq b(G)$.

Computing $b(G)$ is NP-complete in general [13,25], even when restricted to bipartite graphs [20]. In [13,25], a linear algorithm is given for the class of trees. This parameter, defined by Irving and Manlove [13,25], is studied extensively for various class of graphs, such as trees [13, 25], regular graphs [4], power graphs [6–8, 19], tight graphs [21,22], Kneser graphs [2,9,18], cactus graphs [5], outerplanar graphs [24], cartesian product graphs [23], the strong product graphs, the lexicographic product graphs and the direct product graphs [15]. For a recent survey on the $b$-chromatic number, see [16].

The *m-degree* of a graph $G$, denoted $m(G)$, is the largest integer $k$ such that $G$ has $k$ vertices of degree at least $k-1$. Let $\omega(G)$ be the size of a maximum clique of $G$. It is known that every graph $G$ satisfies

$$\omega(G) \leq b(G) \leq m(G). \tag{1.1}$$

A *dense vertex* is a vertex with degree at least $m(G)-1$.

In this paper, we first determine the $b$-chromatic number of some complementary prism graphs. Next, we show that for every nontrivial graph $G$ of order $n$, $b(G\overline{G}) \leq n$, and we give a characterization of triangle-free graphs that achieve equality in this bound.

In all figures, dark dots represent $b$-vertices, and the number in parentheses indicates the color of the corresponding vertex.

## 2. The $b$-chromatic number of some complementary prism graphs

In this section we give exact values for the $b$-chromatic number of complementary prism graph for some classical graphs, namely cycles, paths, complete bipartite graphs and complete graphs.

It was shown in [4] that the Petersen graph $C_5\overline{C_5}$ has $b$-chromatic number 3. Here, we determine the $b$-chromatic number of $C_n\overline{C_n}$, when $n \geq 3$ ($n \neq 5$).

**Theorem 2.1.** *Let $C_n$ be a cycle of order $n \geq 3$. Then,*

$$b(C_n\overline{C_n}) = \begin{cases} n & : 3 \leq n \leq 4 \\ n - \left\lceil \frac{n}{4} \right\rceil & : \quad 5 \leq n \end{cases}$$

*Proof.* Let $C_n$ be a cycle of order $n \geq 3$ with vertex-set $V(C_n) = \{u_1, u_2, \ldots, u_n\}$ and edge-set $E(C_n) = \cup_{i=1}^{n-1}\{u_iu_{i+1}\} \cup \{u_1u_n\}$, and let $\overline{C_n}$ be the complement of $C_n$ with vertex-set $V(\overline{C_n}) = \{v_1, v_2, \ldots, v_n\}$ and edge-set $E(\overline{C_n}) = \{v_iv_j : 1 \leq i < j \leq n \text{ and } u_iu_j \notin E(C_n)\}$. Let $G_n = C_n\overline{C_n}$ be the complementary prism graph of $C_n$ with vertex-set $V(G_n) = V(C_n) \cup V(\overline{C_n})$ and edge-set $E(G_n) = E(C_n) \cup E(\overline{C_n}) \cup (\cup_{i=1}^{n}\{u_iv_i\})$.

We start by proving that the statement is true for $n \in \{3,4,5\}$. So, when $n \in \{3,4\}$, it is not difficult to show that the $m$-degree of $G_n$ is equal to $n$, giving that $b(G_n) \leq n$. Hence, to prove equality, we exhibit a $b$-coloring of $G_n$ using $n$ colors, as follows. For $n=3$, we assign color $i$ to $u_i$ for $i \in \{1,2,3\}$ and we assign colors $2,1,1$ to $v_1, v_2, v_3$, respectively. For $n=4$, we assign color $i$ to $u_i$ for $i \in \{1,2,3,4\}$ and we assign colors $3,4,1,2$ to $v_1, v_2, v_3, v_4$ respectively. In each case, we obtain a $b$-coloring of $G_n$ with $n$ colors. Hence $b(G_n) = n$ for $n \in \{3,4\}$. Also, the statement is true for $n=5$, as proved in [4]. For $n=6$, to show that $b(G_6) \geq 4$, it suffices to give a $b$-coloring of $G_6$ using 4 colors. We do this as follows. Assign color 1 to $v_1, v_2, u_6$, color 2 to $u_2, v_3, u_4$, color 3 to $u_3, v_4, u_5$, color 4 to $v_5, v_6, u_1$. In this case the vertices $v_2, v_3, v_4$ and $v_6$ are the $b$-vertices of colors $1, 2, 3$ and $4$ respectively. Claim that $b(G_6) \leq 4$. Suppose the contrary that $b(G_6) > 5$. It is clear that $m(G_6) = 5$, so $b(G_6) = 5$. Hence, every $b$-vertex occurs in $\overline{C_6}$ since the degree of every vertex in $C_6$ is 3 and $b(G_6) = 5$. Let $c$ be a $b$-coloring of $G_6$ with 5 colors. Without loss of generality, suppose that $v_1$ is $b$-vertex of color 1, as all colors of $c$ appear in $N(v_1)$, so let us suppose that $c(u_1) = 2$, $c(v_3) = 3$, $c(v_4) = 4$ and $c(v_5) = 5$. The $b$-vertex of color 2 is either $v_2$ or $v_6$, say $v_2$, but in this case one of color 1 or 3 does not appear in $N(v_2)$ which contradict the fact that $v_2$ is a $b$-vertex. So, $b(G_6) \leq 4$.

Assume now that $n \geq 7$ and let $k = b(G_n)$. In order to show that the statement is true for $n \geq 7$, we first prove that $k \geq n - \lceil \frac{n}{4} \rceil$. To do this, let $B$ be a subset of $V\left(\overline{C_n}\right)$ defined as follows

$$B = \{v_j \in V\left(\overline{C_n}\right) : j = 1 + 4m, m \in \mathbb{N}_0\} \text{ with } \mathbb{N}_0 = \{0, 1, 2, \ldots\}. \tag{2.1}$$

It is clear that $|B| = \lceil \frac{n}{4} \rceil$.

Let $A = V\left(\overline{C_n}\right) - B$ and $t = |A|$. Then $t = n - \lceil \frac{n}{4} \rceil$. Hence, to show that $k \geq n - \lceil \frac{n}{4} \rceil$ for $n \geq 7$, it suffices to exhibit a $b$-coloring $c$ of $G_n$ with $t$ colors. We begin by coloring the vertices of $V\left(\overline{C_n}\right)$, as follows. We color each vertex of $A$ with a different color, and we color each vertex $v_i \in B - \{v_n\}$ with $c(v_{i+1})$. If $v_n \in B$, then we color it with $c\left(v_{n-1}\right)$. Hence, we get a partial coloring of $G_n$ with $t$ colors. Now, we color the vertices of $V\left(C_n\right)$ in a way that all vertices of $A$ become $b$-vertices of $c$ of distinct colors. Then for each vertex $v_j \in A - \{v_n\}$, we color its neighbor $u_j$ in $V(C_n)$ as follows. If $j = 2 + 4m$ or $j = 3 + 4m$, $(m \in \mathbb{N}_0)$, then we color $u_j$ with color $c(v_{j+1})$ and if $j = 4 + 4m$, $(m \in \mathbb{N}_0)$, we color it with color $c(v_{j-1})$. If $v_n \in A$, then we proceed as follows. If $n = 2 + 4m$, or $n = 3 + 4m$ then clearly $v_n$ is a $b$-vertex since it has all colors in its neighborhood at the end of the previous step of coloring. If $n = 4 + 4m$, $(m \in \mathbb{N}_0)$, then we color $u_n$ with color $c\left(v_{n-1}\right)$. It is not difficult to check that this coloring yields a partial $b$-coloring of $G_n$ with $t$ colors that can be easily extended to a $b$-coloring of $G_n$, because the number of used colors is at least 4 and the degree of the remaining uncolored vertices is at most 3. For example, Figure 1 shows a $b$-coloring of $C_7\overline{C}_7$ and $C_8\overline{C}_8$ using 5 and 6 colors, respectively. Hence

$$k \geq n - \left\lceil \frac{n}{4} \right\rceil \text{ for } n \geq 7. \tag{2.2}$$

Now, consider a $b$-coloring $c$ of $G_n$ with $k$ colors. Since the degree of every vertex in $C_n$ is 3 and since $k \geq 4$, we get that every $b$-vertex occurs in $\overline{C_n}$. Notice that, by the structure of $G_n$,

$$\text{each vertex of } V\left(\overline{C_n}\right) \text{ has exactly two non-neighbors in } V\left(\overline{C_n}\right). \tag{2.3}$$

In addition, we have

$$\begin{aligned} V\left(\overline{C_n}\right) &= \left(N\left[v_j\right] \cup N\left[v_{j+1}\right]\right) - \{u_j, u_{j+1}\} \text{ for } j \in \{1, \ldots, n-1\} \\ &= \left(N\left[v_1\right] \cup N\left[v_n\right]\right) - \{u_1, u_n\}. \end{aligned} \tag{2.4}$$

In view of (2.3) and (2.4), each color of $c$ can be repeated at most twice in $V\left(\overline{C_n}\right)$, because otherwise $c$ will not be a proper coloring. Let $T$ (respectively, $S$) denote the set of all vertices of $V\left(\overline{C_n}\right)$ whose colors appear twice (respectively, once) in $V\left(\overline{C_n}\right)$. Thus,

$$|T| \text{ is even and } |T| + |S| = n. \tag{2.5}$$

Furthermore, it is not difficult to see that

$$k = |S| + \frac{|T|}{2}. \tag{2.6}$$

Combining the second part of (2.5) with (2.6), we obtain

$$k = n - \frac{|T|}{2}. \tag{2.7}$$

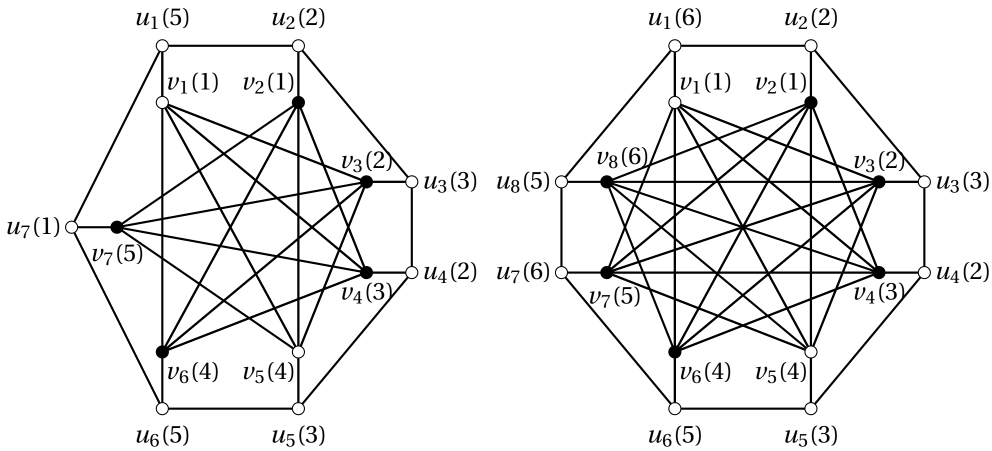


FIGURE 1. $b$-coloring of $C_7\overline{C}_7$ and $C_8\overline{C}_8$ with 5 and 6 colors, respectively.

Using (2.2) and (2.7), we get $|T| \leq 2\left\lceil \frac{n}{4} \right\rceil$.

We claim that $|T| = 2\left\lceil \frac{n}{4} \right\rceil$. Suppose not. Then since $|T|$ is even, it follows that $|T| \leq 2\left\lceil \frac{n}{4} \right\rceil - 2$. This together with the second part of (2.5) imply

$$|S| \geq n - 2\left\lceil \frac{n}{4} \right\rceil + 2. \tag{2.8}$$

We assert that there is a vertex in $S$ that is adjacent to all vertices of $T$. Suppose not, so every vertex of $S$ has a non-neighbor in $T$. Since every vertex $t$ of $T$ has a non-neighbor in $T$, $t$ has at most one non-neighbor in $S$. This means that $|T| \geq |S|$. Combining this with (2.5) and (2.8) one can obtain the following

$$n \geq 2|S| \geq 2n - 4\left\lceil \frac{n}{4} \right\rceil + 4. \tag{2.9}$$

As $\left\lceil \frac{n}{4} \right\rceil - 1 < \frac{n}{4}$, it follows from (2.9) that $n > n$, which is impossible. This contradiction finishes the proof of the assertion.

Hence, there is a vertex $s_i \in S$ that is adjacent to all vertices of $T$. Recall that such vertex has exactly one neighbor in $V(C_n)$, and it needs to see all colors of $c$ on its neighbors (except its own color). Hence, according to (2.3), $s_i$ has two non-neighbors in $S$ with distinct colors, say $j, l$. Therefore, since $j$ and $l$ appear exactly once in $V(\overline{C_n})$, one of them is missing in $N(s_i)$, which is impossible. This contradiction finishes the proof of the claim.

Consequently $|T| = 2\left\lceil \frac{n}{4} \right\rceil$, and thus (2.7) yields $k = n - \left\lceil \frac{n}{4} \right\rceil$. This finishes the proof of Theorem 2.1. □

Next, we determine the exact value of $b\left(P_n\overline{P_n}\right)$ by using the same steps as in Theorem 2.1.

**Theorem 2.2.** *Let $P_n$ be a path of order $n \geq 2$. Then,*

$$b(P_n\overline{P_n}) = \begin{cases} n & : 2 \leq n \leq 3 \\ n - \left\lfloor \frac{n+1}{4} \right\rfloor & : \quad 4 \leq n \end{cases}$$

*Proof.* Let $P_n$ be a path of order $n \geq 2$ with vertex-set $V(P_n) = \{u_1, u_2, \ldots, u_n\}$ and edge-set $E(P_n) = \cup_{i=1}^{n-1}\{u_iu_{i+1}\}$, and let $\overline{P_n}$ be the complement of $P_n$ with vertex-set $V(\overline{P_n}) = \{v_1, v_2, \ldots, v_n\}$ and edge-set $E(\overline{P_n}) = \{v_iv_j : 1 \leq i < j \leq n \text{ and } u_iu_j \notin E(P_n)\}$. Let $G_n = P_n\overline{P_n}$ the complementary prism graph of $P_n$ with vertex-set $V(G_n) = V(P_n) \cup V(\overline{P_n})$ and edge-set $E(G_n) = E(P_n) \cup E(\overline{P_n}) \cup (\cup_{i=1}^{n}\{u_iv_i\})$.

We first show that the statement is true for $n \in \{2, 3, 4, 5\}$. For $n \in \{2, 3\}$, we can easily see that $G_n$ is a $P_4$ or $C_5 + e$ where $e$ is an extra edge incident to exactly one vertex of $C_5$. In such cases, the statement is clearly satisfied. For $n = 4$, we have $m(G_4) = 4$ with dense vertex-set $D = \{u_2, u_3, v_1, v_4\}$, which means that $b(G_4) \leq 4$. We claim that $b(G_4) \leq 3$. Suppose to the contrary that $b(G_4) = 4$ and let $c$ be a $b$-coloring of $G_4$ with 4 colors. We should emphasize here that each vertex in $D$ is a $b$-vertex of $c$. Therefore, we can assume, without loss of generality, that $v_1, v_4, u_2, u_3$ are colored $1, 2, 3$ and 4, respectively. Since $v_1$ needs to see all colors on its neighbors (except its own color), $u_1$ must be colored with the color 4. But in this case $u_2$ will have a missing color in its neighborhood, which contradicts the fact that $u_2$ is a $b$-vertex of $c$. Hence $b(G_4) \leq 3$ and, because $G_4$ has an odd cycle, we get equality. For $n = 5$, we have $m(G_5) = 4$, which means that $b(G_5) \leq 4$. To show that $b(G_5) = 4$, it suffices to give a $b$-coloring of $G_5$ using 4 colors. We do this as follows. Assign color 1 to $v_5, u_4$, color 2 to $v_1, u_2, u_5$, color 3 to $u_1$, $u_3, v_2$ and color 4 to $v_3, v_4$, in this case the vertices $v_5, v_1, v_2$ and $v_3$ are $b$-vertices of colors $1, 2, 3$ and 4 respectively. Hence $b(G_5) = 4$. Assume now that $n \geq 6$ and let $k = b(G_n)$. Since, for $n \geq 6$, $k \geq 5$ and the degree of vertices of $P_n$ is 3, then all $b$-vertices of $c$ occurs in $\overline{P_n}$. As in Theorem 2.1, we start by proving that $k \geq n - \left\lfloor \frac{n+1}{4} \right\rfloor$. For this, we distinguish between two cases according to the value of $n$.

*Case 1*: $n \neq 4m + 3$ $(m \geq 1)$.

In this case $\left\lfloor \frac{n+1}{4} \right\rfloor = \left\lfloor \frac{n}{4} \right\rfloor$. Let $B$ be a subset of $V\left(\overline{P_n}\right)$ defined as follows

$$B = \{v_j \in V\left(\overline{P_n}\right) : j = 4 + 4m, m \in \mathbb{N}_0\} \text{ with } \mathbb{N}_0 = \{0, 1, 2, \ldots\}.$$

It is clear that $|B| = \left\lfloor \frac{n}{4} \right\rfloor$. Let $A = V(\overline{P_n}) - B$ and $t = |A|$, so $t = n - \left\lfloor \frac{n+1}{4} \right\rfloor$. To show that $k \geq t$, we exhibit a $b$-coloring $c$ of $G_n$ with $t$ colors. To do this, we start, as in Theorem 2.1, by coloring each vertex of $A$ with a different color, and assigning color $c\left(v_{i-1}\right)$ to $v_i \in B$. Hence, we obtain a partial coloring of $G_n$ with $t$ colors. Now, we color the remaining vertices of $G_n$ in a way that all vertices of $A$ become $b$-vertices of $c$ of distinct colors. Then for each vertex $v_j \in A - \{v_n\}$, we color its neighbor $u_j$ in $V(P_n)$ as follows. If $j = 2 + 4m$ or $j = 3 + 4m$, $(m \in \mathbb{N}_0)$, then we color $u_j$ with color $c\left(v_{j-1}\right)$ and if $j = 4m + 1$, $(m \in \mathbb{N}_0)$, then we color it with color $c\left(v_{j+1}\right)$. If $v_n \in A$, then we proceed as follows. If $n = 4m + 1$, then clearly $v_n$ is a $b$-vertex since it has all colors in its neighborhood at the end of the previous step of coloring. If $n = 2 + 4m$ or $n = 3 + 4m, (m \in \mathbb{N}_0)$, then we color $u_n$ with color $c\left(v_{n-1}\right)$. It is not difficult to check that this coloring yields a partial $b$-coloring of $G_n$ with $t$ colors that can be easily extended to a $b$-coloring of $G_n$, because the number of used colors is at least 4 and the degree of the remaining uncolored vertices is at most 3. So, $k \geq n - \left\lfloor \frac{n+1}{4} \right\rfloor$.

*Case 2*: $n = 4m + 3$ $(m \geq 1)$.

In this case $\left\lfloor \frac{n+1}{4} \right\rfloor = \left\lceil \frac{n}{4} \right\rceil$. Let $C_n\overline{C_n}$ be the complementary prism graph of $C_n$ with $V(C_n) = V(P_n)$ and $V(\overline{C_n}) = V(\overline{P_n})$. Observe that since, in $C_n\overline{C_n}$, $u_1u_n \in E(C_n\overline{C_n})$ and $v_1v_n \notin E(C_n\overline{C_n})$, it follows that $G_n$ can be obtained from $C_n\overline{C_n}$ by adding edge $v_1v_n$ and deleting edge $u_1u_n$. In the coloring constructed in the proof of Theorem 2.1, $v_1$ is in $B$, $v_n$ is in $A$ and $c(v_1) = c(v_2) \neq c(v_n)$. This implies that the constructed coloring is also a proper coloring of $G_n$ and, since all the $b$-vertices are in $\overline{C_n}$, they are still $b$-vertices in $G_n$. Hence any $b$-coloring of $C_n\overline{C_n}$ is a $b$-coloring of $G_n$, implying that $k \geq b(C_n\overline{C_n}) = n - \left\lceil \frac{n}{4} \right\rceil = n - \left\lfloor \frac{n+1}{4} \right\rfloor$. For example, Figure 2 shows a $b$-coloring of $P_8\overline{P}_8$ and $P_9\overline{P}_9$ using 6 and 7 colors, respectively.

Hence in each case, we have

$$k \geq n - \left\lfloor \frac{n+1}{4} \right\rfloor. \tag{2.10}$$

Now, consider a $b$-coloring $c$ of $G_n$ with $k$ colors and let $S$ and $T$ be two disjoint subsets of $V(\overline{P_n})$ defined as in Theorem 2.1 for $\overline{C_n}$. Proceeding in a way similar to Theorem 2.1 one can show that clauses (2.4)–(2.7) remain true here. Hence, by (2.7) and (2.10), we get $|T| \leq 2\left\lfloor \frac{n+1}{4} \right\rfloor$.

We claim that $|T| = 2\left\lfloor \frac{n+1}{4} \right\rfloor$. Suppose not. As $|T|$ is even, $|T| \leq 2\left\lfloor \frac{n+1}{4} \right\rfloor - 2$. This with the second part of (2.5) yield

$$|S| \geq n - 2\left\lfloor \frac{n+1}{4} \right\rfloor + 2. \tag{2.11}$$

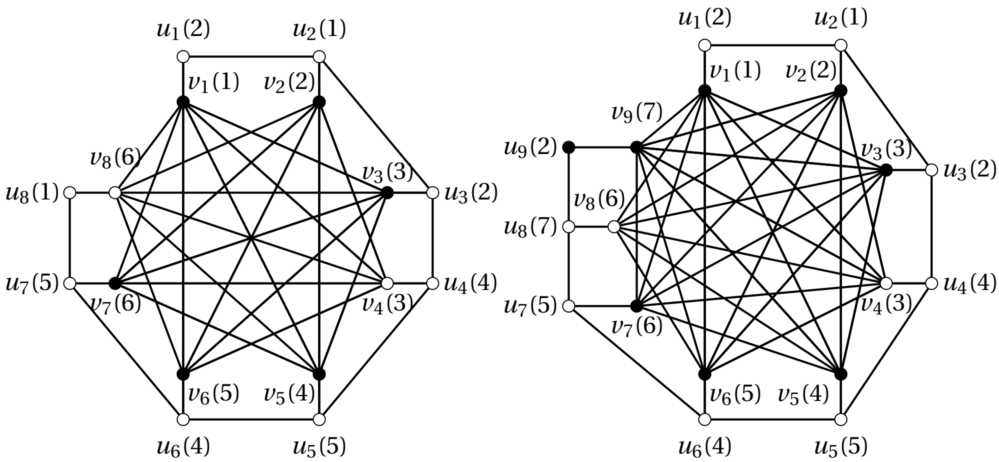


FIGURE 2. $b$-coloring of $P_8\overline{P}_8$ and $P_9\overline{P}_9$ with 6 and 7 colors, respectively.

As in Theorem 2.1, we assert that there is a vertex in $S$ having two non-neighbors in $S$. Suppose not. Using the same proof as for Theorem 2.1, one can show that $|T| \geq |S|$. This together with (2.5) and (2.11) imply that

$$n \geq 2\,|S| \geq 2n - 4\left\lfloor \frac{n+1}{4} \right\rfloor + 4.$$

Since $\left\lfloor \frac{n+1}{4} \right\rfloor \leq \frac{n+1}{4}$, $n \geq n+3$, which is impossible. This contradiction finishes the proof of the assertion.

Following the same arguments as in Theorem 2.1 (we omit the details), one can show that $S$ has a vertex with a missing color in its neighborhood, which leads to a contradiction. This contradiction finishes the proof of the claim.

Consequently $|T| = 2\left\lfloor \frac{n+1}{4} \right\rfloor$, and so by the second part of (2.5), we have $k = n - \left\lfloor \frac{n+1}{4} \right\rfloor$. This finishes the proof of Theorem 2.2. $\square$

We now give the $b$-chromatic number of the complementary prism graph of a complete bipartite graph $K_{p,q}$ $(p \geq q \geq 1)$.

**Theorem 2.3.** *Let $K_{p,q}$ be a complete bipartite graph with $p \geq q \geq 1$. Then,*

$$b(K_{p,q}\overline{K_{p,q}}) = \begin{cases} 4 & : 3 \leq p = q = 2 \\ p & : \quad q+2 \leq p \\ p+1 & : \quad p = q+1 \end{cases}$$

*Proof.* It is obvious to see that the statement is true for $p = 1$. Assume that $p \geq 2$, let $G_{p,q} = K_{p,q}\overline{K_{p,q}}$ be the complementary prism graph of $K_{p,q}$ with vertex-set $V(G_{p,q})$ and edge-set $E(G_{p,q})$. Note that $G_{p,q}$ has the following structure. Then $V(G_{p,q})$ can be partitioned into four disjoint subsets $X = \{x_1, x_2, \ldots, x_p\}$, $Y = \{y_1, y_2, \ldots, y_q\}$, $Z = \{z_1, z_2, \ldots, z_p\}$ and $T = \{t_1, t_2, \ldots, t_q\}$ such that $X$ and $Y$ are stable sets, $Z$ and $T$ are cliques, $X \cup Y$ induces a complete bipartite graph with bipartition $(X, Y)$ and $E(G_{p,q}) = E(G[X \cup Y]) \cup E(G[Z]) \cup E(G[T]) \cup (\cup_{i=1}^{p}\{x_i z_i\}) \cup (\cup_{i=1}^{q}\{y_i t_i\})$.

Let $k = b(G_{p,q})$ and $D$ be the set of dense vertices in $G_{p,q}$. We distinguish between three cases.

*Case 1*: $p = q$.

If $p = 2$, then $G_{2,2} = C_4\overline{C_4}$ and therefore $b(G_{2,2}) = 4$ by Theorem 2.1. Assume $p \geq 3$, so clearly $D = X \cup Y$ and $m(G_{p,p}) = p+2$. We shall show that $b(G_{p,p}) \leq p+1$. Suppose not. Then by (1.1), we have $b(G_{p,p}) = p+2$.

Let $c$ be a $b$-coloring of $G_{p,p}$ using $p+2$ colors. Up to symmetry and without loss of generality, we can suppose that $x_1$ is a $b$-vertex of $c$ of color 1. Hence clearly all vertices of $Y \cup \{z_1\}$ are colored differently. Assume that $c(z_1) = p+2$. Observe that, in every coloring of $G[X \cup Y]$, no color can appear in both $X$ and $Y$. Therefore, each vertex in $X - \{x_1\}$ is colored with the color 1 or $p+2$. This leads to a contradiction because there is no $b$-vertex of $c$ of color $i \in \{2, 3, \ldots, p+1\}$. Now, we shall show that $k = p+1$. For this, it suffices to exhibit a $b$-coloring of $G_{p,p}$ with $p+1$ colors. To do this, we first color all vertices of $X$ with color 1 and all vertices of $Y$ with color $p+1$. Next, for $i \in \{1, \ldots, p\}$, color $z_i$ with the color $i+1$, and color $t_i$ with the color $i$. It is easy to check that this yields a $b$-coloring of $G_{p,p}$ with $p+1$ colors, where $Z \cup \{t_1\}$ are $b$-vertices of this coloring. Hence $b(G_{p,p}) = p+1$. This finishes Case 1.

Observe that if $p > q$, then $\omega(G_{p,q}) = p$ and $m(G_{p,q}) = p+1$. Hence, according to (1.1), we have

$$p \leq k \leq p+1. \tag{2.12}$$

*Case 2*: $p = q+1$.

We shall show that $k = p+1$. For this, it suffices, in view of (2.12), to exhibit a $b$-coloring of $G_{p,q}$ with $k$ colors. We do this as follows. We first color all vertices of $X$ with the color $p+1$, and for $i \in \{1, \ldots, p\}$, color $z_i$ with the color $i$. Next, for $i \in \{1, \ldots, q\}$, color $y_i$ with the color $i+1$, and color $t_i$ with the color $i$. It is easy to check that this yields a $b$-coloring of $G_{p,q}$ with $k$ colors, where $Z \cup \{x_1\}$ are $b$-vertices of this coloring. Hence $k = p+1$.

*Case 3*: $p \geq q+2$.

We claim that $k = p$. Suppose not. Then by (2.12), we have $k = p+1$. Let $c$ be a $b$-coloring of $G_{p,q}$ with $k$ colors. It is easy to show that $D = Y \cup Z$. Since $Z$ is a clique, all its vertices are colored differently. So, assume without loss of generality that for $i \in \{1, \ldots, p\}$, $z_i$ is colored with $i$. Since $|Y| = q \leq p-2 < k$, $Z$ must contain at least one $b$-vertex of $c$, say without loss of generality that $z_1$ is such vertex. As $z_1$ needs to see all colors on its neighbors, $x_1$ must be colored $p+1$ and therefore no vertex in $Y \cup Z$ is colored with this color, a contradiction. Hence $k = p$. This finishes the proof of Theorem 2.3. □

In the special case when $G$ or $\overline{G}$ is a complete graph, the $b$-chromatic number of $G\overline{G}$ is equal to $n$. This follows by (1.1) since $\omega(G\overline{G}) = m(G\overline{G}) = n$.

**Proposition 2.4.** *If $G$ or $\overline{G}$ is a complete graph, then $b\left(G\overline{G}\right) = n$.*

## 3. Triangle-free graphs with $b(G\overline{G}) = |V(G)|$

In this section, we show that for every graph $G$ of order $n \geq 2$, $b(G\overline{G}) \leq n$, and we give a characterization of triangle-free graphs that achieve equality in this bound.

Let $H = G\overline{G}$ be the complementary prism graph of $G$ and let $V(G) = \{x_1, \ldots, x_n\}$ and $v(\overline{G}) = \{y_1, \ldots, y_n\}$. As $d_H(x_i) + d_H(y_i) = n+1$, it follows that $H$ has at least $n$ vertices of degree at least $n-1$ and does not have $n+1$ vertices that can be of degree at least $n$. This means that

$$m(G\overline{G}) \leq n, \tag{3.1}$$

so the following result is immediate.

**Observation 3.1.** *For every graph $G$ of order $n \geq 2$, $b(G\overline{G}) \leq n$.*

When $G$ is a trivial graph, clearly $b(G\overline{G}) = 2$. Recall that every graph $G$ satisfies $b(G) \leq \Delta(G)+1$ as observed in [13, 25]. Therefore, as $\Delta(G\overline{G}) = \max\{\Delta(G), \Delta(\overline{G})\} + 1$, it follows that $b(G\overline{G}) \leq \max\{\Delta(G), \Delta(\overline{G})\} + 2$.

**Theorem 3.2.** *Let $G$ be a triangle-free graph of order $n \geq 2$ and $G\overline{G}$ be the complementary prism graph of $G$. Then $b(G\overline{G}) = n$ if and only if $G$ or $\overline{G}$ is isomorphic to $P_2$, $P_3$ or $C_4$.*

*Proof.* The sufficiency follows from Theorems 2.1 and 2.2, so let us prove the necessity. Then $n = b(G\overline{G}) \leq \max\{\Delta(G), \Delta(\overline{G})\} + 2 \leq n+1$ since $\max\{\Delta(G), \Delta(\overline{G})\} \leq n-1$. We distinguish between two cases.

*Case 1*: $\max\{\Delta(G), \Delta(\overline{G})\} = n-1$. Assume without loss of generality that $\Delta(\overline{G}) \leq \Delta(G) = n-1$. Let $x$ be a vertex of $G$ such that $d_G(x) = \Delta(G)$. As $G$ is triangle-free, $G - x$ is without edge, meaning that $G = K_{1,n-1}$. According to Theorem 2.3, $b(G\overline{G}) = n$ if and only if $n = 2$ or 3. Hence $G$ or $\overline{G}$ is isomorphic to $P_2$ or $P_3$.

*Case 2*: $\max\{\Delta(G), \Delta(\overline{G})\} = n-2$. Assume without loss of generality that $\Delta(\overline{G}) \leq \Delta(G) = n-2$. Therefore, clearly, neither $G$ nor $\overline{G}$ contains an isolated vertex. Let $V(G) = \{x_1, \ldots, x_n\}$ and $V(\overline{G}) = \{y_1, \ldots, y_n\}$. Then $V(G\overline{G}) = V(G) \cup V(\overline{G})$ and $E(G\overline{G}) = E(G) \cup E(\overline{G}) \cup (\cup_{i=1}^{n}\{x_i y_i\})$. Assume that $d_G(x_n) = n-2$ and $x_{n-1}$ is the unique non-neighbor of $x_n$ in $G$. Let $X_1 = \{x_n, x_{n-1}\}$, $X_2 = V(G) - X_1$, $Y_1 = \{y_n, y_{n-1}\}$ and $Y_2 = V(\overline{G}) - Y_1$. If $n = 3$, then either $x_2$ is isolated or $x_1$ has degree $n-1$ in $G$, a contradiction since $\Delta(G) \leq n-2$, so this case cannot occur. If $n = 4$, then $X_1 = \{x_3, x_4\}$ and $X_2 = \{x_1, x_2\}$, and so $G$ is either a cycle $C_4$ $x_1$-$x_3$-$x_2$-$x_4$-$x_1$ or a path $P_4$ $x_i$-$x_3$-$x_j$-$x_4$, $\{i, j\} = \{1, 2\}$. Recall that by Theorems 2.1 and 2.2, we have $b(C_4\overline{C_4}) = 4$ and $b(P_4\overline{P_4}) = 3 \neq n$. Hence, $G$ or $\overline{G}$ is isomorphic to a cycle $C_4$. Assume now that $n \geq 5$, and let $p$ be the number of neighbors of $x_{n-1}$ in $X$. Then, clearly, $1 \leq p \leq n-2$. If $p = n-2$, then $G = K_{2,n-2}$. According to Theorem 2.3, $b(G\overline{G}) = n$ if and only if $n = 4$, which contradicts $n \geq 5$. Suppose now that $p \leq n-3$ and assume without loss of generality that $x_1, \ldots, x_p$ are the neighbors of $x_{n-1}$ in $X$. Since $b(G\overline{G}) = n$, (1.1) and (3.1) imply $m(G\overline{G}) = n$. On the other hand, it is a routine matter to check that if $p = 1$, then $d_{G\overline{G}}(z) = n-1$ for each $z \in \{x_n, y_2, y_3, \ldots, y_{n-1}\}$ and all the remaining vertices in $G\overline{G}$ have degree less than $n-1$. And if $p > 1$, then $d_{G\overline{G}}(z) = n-1$ for each $z \in \{x_n, y_{p+1}, \ldots, y_{n-2}\}$ and all the remaining vertices in $G\overline{G}$ have degree less than $n-1$. Therefore, in each case, $G\overline{G}$ has less than $n-1$ vertices of degree $n-1$, which is a contradiction with the fact that $m(G\overline{G}) = n$. Hence this case cannot occur. □

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