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MEAN-VARIANCE PORTFOLIO SELECTION WITH AN UNCERTAIN EXIT-TIME IN A REGIME-SWITCHING MARKET

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Abstract. In this paper, we deal with multi-period mean-variance portfolio selection problems with an exogenous uncertain exit-time in a regime-switching market. The market is modelled by a non-homogeneous Markov chain in which the random returns of assets depend on the states of the market and investment time periods. Applying the Lagrange duality method, we derive explicit closed-form expressions for the optimal investment strategies and the efficient frontier. Also, we show that some known results in the literature can be obtained as special cases of our results. A numerical example is provided to illustrate the results.

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Introduction

The single-period mean-variance (M-V) portfolio selection model proposed by Markowitz [18] is the foundation of modern portfolio theory. In this model, an investor seeks to minimize the variance (as the measure of risk) of his/her portfolio return for a given desired expected portfolio return, or, to maximize the expectation of his/her portfolio return for a given level of risk. The analytic solution of the M-V portfolio selection problem when the short selling is allowed is given by Merton [21]. The original single-period M-V model was extended to the multi-period case, and for the first time, Li and Ng [12] derived the optimal investment strategy. The continuous time M-V problem is also analytically solved by Zhou and Li [38].

The main assumption in Li and Ng [12] is that the asset returns during consecutive time periods are independent. This assumption is not true in a realistic setting. Therefore, some sort of dependence between the asset returns across successive time periods should be considered. Recently, regime-switching models have been considered by authors to deal with this deficiency. In such models, there is a finite set of market states (regimes) which reflects the stochastic market environment. For example, market states can be divided into bullish and bearish. The asset returns and key market parameters, such as the bank interest rate, or stocks appreciation and volatility rates depend on the market state. Zhou and Yin [39] and Yin and Zhou [34] considered dynamic continuous-time and discrete-time versions of Markowitz's model, respectively, when the market state

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process follows a Markov chain. Also, Cakmak and Ozekici [4] investigated multi-period M-V portfolio selection problems in Markovian regime-switching markets.

In the all of the above papers, and in many other related works, the market exit-time has been considered deterministic. In the real world, however, the exit-time is uncertain. Actually, the market exit-time is affected by many exogenous and endogenous factors. The early work on this subject was done by Yaari [26], who considered the problem of optimal consumption for a consumer with uncertain lifetime when the market contains a riskless asset. Hakansson [8, 9] extended this model to a discrete-time multi-period model with risky assets and an uncertain lifetime. Merton [20] investigated a continuous-time investment-consumption problem for an investor when the time of retirement is the first jump time of an independent Poisson process. Karatzas and Wang [10] studied a utility maximization problem when the exit-time is a stopping time of asset price filtration. Liu and Loewenstein [15] considered a utility maximization problem with uncertain time-horizon and transaction costs. Blanchet-Scalliet *et al.* [3] investigated an optimal investment problem when the uncertain exit-time is correlated with the returns of risky assets. Lv *et al.* [17] considered a continuous-time M-V portfolio selection model in an incomplete market when market parameters and time horizon are random. The static M-V portfolio selection model with an uncertain exit-time is studied by Martellini and Urosevic [19]. Keykhaei [11] extended the model of Martellini and Urosevic [19] to the case where each asset has individual uncertain exit-time. The multi-period M-V portfolio selection model when the exit-time is uncertain is studied by Guo and Hu [6]. Zhang and Li [36] extended this model to the case with one risky asset and one riskless asset where the returns of risky asset are serially correlated. Yi *et al.* [33] investigated a multi-period asset-liability management model with an uncertain exit-time. An infinite-period M-V portfolio selection model with an uncertain exit-time is studied by Guo and Cai [7]. Wu and Li [23] studied a multi-period M-V portfolio selection model in a Markovian regime-switching market with an exogenous uncertain exit-time. Wu *et al.* [25] extended this model to the case when the exit-time depends on the current market state. Yao *et al.* [28] considered a multi-period M-V portfolio selection problem with endogenous liabilities and an uncertain exit-time in a regime-switching market.

A multi-period M-V portfolio optimization problem, with a certain exit-time T, has one of the following two standard formulations,

$$\tilde{P}(\mu):\begin{cases}\min\limits_{\pi}\ \mathbb{V}\mathrm{ar}\big[W_T\big]\\ \text{s.t. } \mathbb{E}\big[W_T\big]=\mu,\\ \qquad W_{n+1}=W_n+R_n'\pi_n,\end{cases}\qquad \tilde{P}(\sigma):\begin{cases}\max\limits_{\pi}\ \mathbb{E}\big[W_T\big]\\ \text{s.t. } \mathbb{V}\mathrm{ar}\big[W_T\big]=\sigma^2,\\ \qquad W_{n+1}=W_n+R_n'\pi_n,\end{cases}$$

where $\pi=\{\pi_0,\pi_1,\ldots,\pi_{T-1}\}$ is the portfolio strategy and R_n is the vector of asset rate of returns in period n $(n=0,1,\ldots,T-1)$ and W_n denotes the wealth amount available for investment at time n $(n=0,1,\ldots,T)$. In this paper, the superscript $'$ denotes the transpose of a matrix or a vector. Formulation $\tilde{P}(\mu)$ is used when an investor seeks to minimize his/her investment risk, measured by the variance of the terminal wealth, and his desired expected wealth is μ. Also, formulation $\tilde{P}(\sigma)$ is used when an investor seeks to maximize his/her expected terminal wealth for a specified risk level σ^2. An equivalent formulation to these standard problems is the following trade-off formulation

$$\tilde{P}(\omega):\begin{cases}\max\limits_{\pi} & \mathbb{E}\big[W_T\big]-\omega\mathbb{V}\mathrm{ar}\big[W_T\big]\\ \text{s.t.} & W_{n+1}=W_n+R_n'\pi_n,\end{cases}$$

defined parametrically for trade-off parameter $\omega>0$. Formulation $\tilde{P}(\omega)$ is used when an investor specifies his/her desirable trade-off between the expectation and the variance of the terminal wealth.

All the above problems are non-separable in the sense of dynamic programming. This is due to the fact that while the expectation term satisfies the smoothing property, that is, if $s<t$, then $\mathbb{E}\big[\mathbb{E}\big[.|\mathcal{F}_t\big]|\mathcal{F}_s\big]=\mathbb{E}\big[.|\mathcal{F}_s\big]$, but the variance term does not, *i.e.*, $\mathbb{V}\mathrm{ar}\big[\mathbb{V}\mathrm{ar}\big[.|\mathcal{F}_t\big]|\mathcal{F}_s\big]\neq\mathbb{V}\mathrm{ar}\big[.|\mathcal{F}_s\big]$, where $\mathcal{F}_s$ and $\mathcal{F}_t$ are information sets available at times s and t, respectively. Therefore, the principle of dynamic programming no longer applies. Li and Ng [12] overcame this difficulty. In order to solve problems $\tilde{P}(\mu)$ and $\tilde{P}(\sigma)$ they used equivalent trade-off problem

$\tilde{P}(\omega)$, and embedded problem $\tilde{P}(\omega)$ into the following family of tractable auxiliary problems, parametrized in terms of λ and ω,

$$\tilde{P}(\lambda,\omega):\begin{cases}\max\limits_{\pi} & \mathbb{E}\big[-\omega W_T^2+\lambda W_T\big]\\ & W_{n+1}=W_n+R_n'\pi_n.\end{cases}$$

Note that problem $\tilde{P}(\lambda,\omega)$ is separable in the sense of dynamic programming, and, therefore, can be solved by using dynamic programming approach. Since then, this embedding technique has been widely used to solve multi-period M-V portfolio selection problems. Though the embedding technique helps to tackle the non-separable multi-period M-V problems, it is complicated. The optimal solution of problem $\tilde{P}(\omega)$ can be derived *via* identifying parameter λ^* under which the solution of $\tilde{P}(\lambda^*,\omega)$ also solves $\tilde{P}(\omega)$. After solving the auxiliary problem, the expectation and the variance of the terminal wealth must be computed in terms of $\gamma=\frac{\lambda}{\omega}$. To solve problem $\tilde{P}(\omega)$ it is necessary to find the appropriate γ^* in terms of ω and then compute the expectation and the variance of the terminal wealth in terms of ω. The efficient frontier can be obtained by eliminating ω in the expectation and the variance of the terminal wealth. To solve problems $\tilde{P}(\mu)$ and $\tilde{P}(\sigma)$ the associated ω must be calculated in terms of μ and σ. This procedure will be more complicated when the model involves uncertain exit-time and regime-switching.

Recently, some researchers have applied the Lagrange duality method to solve multi-period M-V portfolio selection problems with formulation $\tilde{P}(\mu)$. Compared to the embedding technique, the Lagrange duality method is more visible with simpler procedure and less calculation factors. Specially, there is no need to compute the mean and the variance of the terminal wealth in the auxiliary problem which is essential in the embedding technique. For example, see [5, 24, 27–29, 31, 32] for discrete time models and [1, 2, 13, 14, 22, 30, 35, 37, 39] for continuous time models.

In this paper, we assume that the asset returns over each time period depend on the market state (regime). As mentioned before, the market exit-time is affected by many exogenous and endogenous factors, and therefore is uncertain. Here, we assume that the investor may be forced to exit the market for some exogenous reasons, such as death and serious illness, before his/her planned exit-time. Then, the exit-time is uncertain (for example, the investor's time of death, such as in Yaari [26] and Hakansson [8, 9]). So, we consider multi-period M-V portfolio selection problems with an exogenous uncertain exit-time in a Markov regime-switching market. Applying Lagrange duality method, the explicit closed form expressions for the optimal investment strategies and the efficient frontier are derived. Unlike the embedding technique, we first solve the standard problems and after that we solve the trade-off problem. Also, some especial cases are investigated. It is shown that some known models in the literature can be considered as special cases of our model.

The rest of this paper is organized as follows. In Section 1, we present the basic notations, model assumptions and problem formulations. In Section 2, we describe the Lagrange duality approach. An auxiliary problem is introduced and finally the optimal investment strategy and the efficient frontier of the original problems are derived. Some especial cases are investigated in Section 3 and finally a numerical example is given in the last section.

1. Problem formulations

We consider a financial market consisting of $N+1$ risky assets. An investor with investment time-horizon T joins the market at time 0 with an initial wealth $W_0=w_0$. However, the investor may be forced to exit the market at time τ before T for some uncontrollable reasons. We assume that the uncertain exit-time τ is an exogenous discrete random variable with the probability mass function $q_n=\mathrm{P}(\tau=n), n=1,2,\ldots$. Therefore, the real exit-time is $T\wedge\tau:=\min\{T,\tau\}$. Define

$$p_n:=\mathrm{P}(T\wedge\tau=n)=\begin{cases}q_n, & n=1,2,\ldots,T-1,\\ 1-\sum\limits_{k=1}^{T-1}q_k, & n=T.\end{cases}$$

Without loss of generality, we assume that $p_T > 0$, otherwise it is not necessary to consider T as the time-horizon.

We assume that the market state process follows a Markov chain, denoted by $\{X_n : 0, 1, 2, \ldots\}$, with a discrete state space $S = \{1, 2, \ldots, L\}$ and transition probability matrices Q_n with one-step transition probabilities

$$Q_n(i,j) = \mathrm{P}(X_n = j | X_{n-1} = i),$$

for $i, j \in S$ and $n = 1, 2, \ldots$. Denote the return of asset k over period n when the market state is $X_n = i$ by $r_{n,k}(i)$, and the vector of returns by $R_n(i) = \big(r_{n,0}(i), r_{n,1}(i), \ldots, r_{n,N}(i)\big)'$. The vector of excess returns is denoted by $\tilde{R}_n(i) = \big(r_{n,1}(i) - r_{n,0}(i), \ldots, r_{n,N}(i) - r_{n,0}(i)\big)'$. We assume that return vectors $R_0(i), \ldots, R_{T-1}(i)$ have different distribution functions. Also, $R_n(i)$ is independent of $R_m(j)$, when $n \neq m$, for all $i, j \in S$. Moreover, we assume that $R_n(X_n)$ and X_{n+1} are independent. Also, $\mathbb{E}\big[\tilde{R}_n(i)\big] \neq \mathbf{0}$ and $\mathbb{E}\big[R_n(i)R_n(i)'\big]$ is positive definite for all $i \in S$ and $n = 0, 1, \ldots, T-1$. We further assume that: short selling is allowed, the investment strategies are self-financing, *i.e.*, no capital addition or withdrawals are allowed, and finally, there is no transaction cost.

Denote the portfolio during period n by $\pi_n = (\pi_{n,1}, \pi_{n,2}, \ldots, \pi_{n,N})' \in \mathbb{R}^N$, where $\pi_{n,i}$ denotes the amount invested in assets i. A multi-period portfolio strategy is an investment sequence $\pi = \{\pi_0, \pi_1, \ldots, \pi_{T-1}\}$. Denote the wealth at time n by W_n. Then, the self-financing property implies that

$$W_{n+1} = r_{n,0}(X_n)W_n + \tilde{R}_n(X_n)'\pi_n,$$

for $n = 0, 1, \ldots, T-1$.

In the following $\mathbb{E}_{i_0}\big[.\big] = \mathbb{E}\big[.|X_0 = i_0\big]$ and $\mathbb{V}\mathrm{ar}_{i_0}\big[.\big] = \mathbb{E}_{i_0}\big[.\big]^2 - \mathbb{E}^2_{i_0}\big[.\big]$ denote the conditional expectation and variance given that initial market state is i_0. In this paper, we consider the following portfolio selection problems,

$$P(\mu) : \begin{cases} \min\limits_{\pi} \mathbb{V}\mathrm{ar}_{i_0}\big[W_{T\wedge\tau}\big] \\ \text{s.t. } \mathbb{E}_{i_0}\big[W_{T\wedge\tau}\big] = \mu, \\ \qquad W_{n+1} = r_{n,0}(X_n)W_n + \tilde{R}_n(X_n)'\pi_n, \end{cases}$$

$$P(\sigma) : \begin{cases} \max\limits_{\pi} \mathbb{E}_{i_0}\big[W_{T\wedge\tau}\big] \\ \text{s.t. } \mathbb{V}\mathrm{ar}_{i_0}\big[W_{T\wedge\tau}\big] = \sigma^2, \\ \qquad W_{n+1} = r_{n,0}(X_n)W_n + \tilde{R}_n(X_n)'\pi_n, \end{cases}$$

$$P(\omega) : \begin{cases} \max\limits_{\pi} \mathbb{E}_{i_0}\big[W_{T\wedge\tau}\big] - \omega\mathbb{V}\mathrm{ar}_{i_0}\big[W_{T\wedge\tau}\big] \ (\omega > 0) \\ \text{s.t. } W_{n+1} = r_{n,0}(X_n)W_n + \tilde{R}_n(X_n)'\pi_n. \end{cases}$$

A strategy π^* is said to be *efficient* if there exists no other strategy π such that $\mathbb{E}_{i_0}\big[W_{T\wedge\tau}\big]\big|_{\pi} \geq \mathbb{E}_{i_0}\big[W_{T\wedge\tau}\big]\big|_{\pi^*}$ and $\mathbb{V}\mathrm{ar}_{i_0}\big[W_{T\wedge\tau}\big]\big|_{\pi} \leq \mathbb{V}\mathrm{ar}_{i_0}\big[W_{T\wedge\tau}\big]\big|_{\pi^*}$ and at least one of the inequalities is strict. Each efficient strategy π^* introduces an *efficient point* $\big(\mathbb{E}_{i_0}\big[W_{T\wedge\tau}\big]\big|_{\pi^*}, \mathbb{V}\mathrm{ar}_{i_0}\big[W_{T\wedge\tau}\big]\big|_{\pi^*}\big)$ in the Mean-Variance plane. The *efficient frontier* is the set of all efficient points (that are generated by various values of μ in $P(\mu)$ or σ in $P(\sigma)$ or ω in $P(\omega)$).

Applying embedding technique, Wu and Li [23] solved the trade-off problem $P(\omega)$, when the market contains a riskless asset and the return vectors $R_0(i), R_1(i), \ldots, R_{T-1}(i)$ are independent and identically distributed. As mentioned before, different form Wu and Li [23], we assume that all assets are risky and the return vectors are not identically distributed. Using the Lagrange duality approach, we solve problem $P(\mu)$, and finally, derive the solution of problems $P(\sigma)$ and $P(\omega)$.

2. Lagrange duality approach

2.1. Auxiliary problem

Note that $\mathbb{E}_{i_0}\big[W_{T\wedge\tau}\big] = \mu$ is equivalent to $\mathbb{E}_{i_0}\big[W_{T\wedge\tau} - \mu\big] = 0$. Also, $\mathbb{E}_{i_0}\big[W_{T\wedge\tau}\big] = \mu$, implies that $\mathbb{V}\mathrm{ar}_{i_0}$ $\big[W_{T\wedge\tau}\big] = \mathbb{E}_{i_0}\big[(W_{T\wedge\tau} - \mu)^2\big]$. For convenience, we define $p_0 = 0$. Since the exit-time is independent of portfolio behavior, we have

$$\mathbb{E}_{i_0}\big[W_{T\wedge\tau}-\mu\big]=\mathbb{E}_{i_0}\left[\sum_{n=0}^{T}p_n(W_n-\mu)\right],$$
$$\mathbb{E}_{i_0}\big[(W_{T\wedge\tau}-\mu)^2\big]=\mathbb{E}_{i_0}\left[\sum_{n=0}^{T}p_n(W_n-\mu)^2\right].$$

Therefore, problem $P(\mu)$ can be rewritten as

$$P(\mu):\begin{cases}\min\limits_{\pi}\ \mathbb{E}_{i_0}\left[\sum_{n=0}^{T}p_n(W_n-\mu)^2\right]\\ \text{s.t. } \mathbb{E}_{i_0}\left[\sum_{n=0}^{T}p_n(W_n-\mu)\right]=0,\\ \qquad W_{n+1}=r_{n,0}(X_n)W_n+\tilde{R}_n(X_n)'\pi_n.\end{cases}$$

To solve problem $P(\mu)$ we apply the Lagrange multiplier method. To this end, we dualize the equality constraint $\mathbb{E}_{i_0}\left[\sum_{n=0}^{T}p_n(W_n-\mu)\right]=0$ by using a Lagrange multiplier 2λ and solve the following auxiliary problem,

$$P(\mu,\lambda):\begin{cases}\min\limits_{\pi}\ \mathbb{E}_{i_0}\left[\sum_{n=0}^{T}p_n(W_n-\mu)^2\right]+2\lambda\mathbb{E}_{i_0}\left[\sum_{n=0}^{T}p_n(W_n-\mu)\right]\\ \text{s.t. } W_{n+1}=r_{n,0}(X_n)W_n+\tilde{R}_n(X_n)'\pi_n.\end{cases}$$

Introducing the notations $a=\lambda-\mu$ and $b=\mu^2-2\lambda\mu$, problem $P(\mu,\lambda)$ can be rewritten in the following formulation,

$$P(\mu,\lambda):\begin{cases}\min\limits_{\pi}\ \mathbb{E}_{i_0}\left[\sum_{n=0}^{T}p_nW_n^2+2ap_nW_n+b\right]\\ \text{s.t. } W_{n+1}=r_{n,0}(X_n)W_n+\tilde{R}_n(X_n)'\pi_n.\end{cases}$$

2.2. Solution to the auxiliary problem

In the following, for any matrix $A_{n\times n}$ and vector $a\in\mathbb{R}^n$, $A(i,j)$ and $a(j)$ denote the component (i,j) and component j in A and a, respectively. Define the matrix A_a such that $A_a(i,j)=A(i,j)a(j)$. Also, let $\bar{A}$ be a column vector whose ith component is $\bar{A}(i)=\sum_{j=1}^{n}A(i,j)$. Obviously, $\bar{A}=A\mathbf{1}$, where $\mathbf{1}=(1,1,\ldots,1)'\in\mathbb{R}^n$. For any vector $a_n\in\mathbb{R}^L$, define $Q_{a_n}=(Q_n)_{a_n}$. Furthermore, denote $\sum_{\emptyset}A_k=\mathbf{0}$ and $\prod_{\emptyset}A_k=I$, where I is an identity matrix.

Lemma 2.1. *Suppose that A and B are $L\times L$ matrices. Let, $a,a_n\in\mathbb{R}^L$ where $a_n=c\mathbf{1}$ for some constant c. Then*

(I) $A\overline{B}=\overline{AB}$
(II) $\overline{AQ_{a_n}}=c\overline{A}$
(III) $\overline{Q_{a_n}A}=c\overline{Q_nA}$
(IV) $\overline{A_a}=Aa$

Proof. The proof is obvious. ◻

Lemma 2.2. *Let $b_n, c_n \in \mathbb{R}^L$, for $n = 0, 1, \ldots, T-1$. Suppose that the recursive sequence $\{a_n(i)\}_{n=0}^T$ is defined backwardly by*

$$a_n(i) = b_n(i) + c_n(i)\overline{Q_{a_{n+1}}}(i),$$

for all $i \in \{1, \ldots, L\}$. If we define $b_T(i) = a_T(i)$, then

$$a_n(i) = b_n(i) + c_n(i) \sum_{k=n+1}^{T} \overline{\prod_{m=n+1}^{k-1} Q_{c_m} Q_{b_k}}(i),$$

for $n = 0, 1, \ldots, T-1$.

Proof. The proof proceeds by backward induction on n. Let $n = T-1$. Then

$$\begin{aligned} a_{T-1}(i) &= b_{T-1}(i) + c_{T-1}(i) \sum_{j \in S} Q_T(i,j) a_T(j) \\ &= b_{T-1}(i) + c_{T-1}(i) \sum_{j \in S} Q_T(i,j) b_T(j) \\ &= b_{T-1}(i) + c_{T-1}(i) \overline{Q_{b_T}}(i). \end{aligned}$$

Suppose now the induction hypothesis holds for $n = k+1$. Applying the first part of Lemma 2.1, for $n = k$ we have

$$\begin{aligned} a_k(i) &= b_k(i) \\ &\quad + c_k(i) \sum_{j \in S} Q_{k+1}(i,j) \left\{ b_{k+1}(j) + c_{k+1}(j) [\overline{Q_{b_{k+2}}}(j) + \sum_{l=k+3}^{T} \overline{\prod_{m=k+2}^{l-1} Q_{c_m} Q_{b_l}}(j)] \right\} \\ &= b_k(i) + c_k(i) \left[\overline{Q_{b_{k+1}}}(i) + (Q_{c_{k+1}} \overline{Q_{b_{k+2}}})(i) + \sum_{l=k+3}^{T} (Q_{c_{k+1}} \overline{\prod_{m=k+2}^{l-1} Q_{c_m} Q_{b_l}})(i) \right] \\ &= b_k(i) + c_k(i) \left[\overline{Q_{b_{k+1}}}(i) + \overline{Q_{c_{k+1}} Q_{b_{k+2}}}(i) + \sum_{l=k+3}^{T} \overline{\prod_{m=k+1}^{l-1} Q_{c_m} Q_{b_l}}(i) \right]. \end{aligned}$$

This completes the proof. □

For $n = 0, 1, \ldots, T-1$, define $\epsilon_n, \nu_n, \kappa_n \in \mathbb{R}^L$ such that,

$$\epsilon_n(i) = \mathbb{E}\big[\tilde{R}_n(i)'\big]\mathbb{E}^{-1}\big[\tilde{R}_n(i)\tilde{R}_n(i)'\big]\mathbb{E}\big[\tilde{R}_n(i)\big], \tag{2.1}$$

$$\nu_n(i) = \mathbb{E}\big[r_{n,0}(i)\big] - \mathbb{E}\big[\tilde{R}_n(i)'\big]\mathbb{E}^{-1}\big[\tilde{R}_n(i)\tilde{R}_n(i)'\big]\mathbb{E}\big[r_{n,0}(i)\tilde{R}_n(i)\big], \tag{2.2}$$

$$\kappa_n(i) = \mathbb{E}\big[r_{n,0}^2(i)\big] - \mathbb{E}\big[r_{n,0}(i)\tilde{R}_n(i)'\big]\mathbb{E}^{-1}\big[\tilde{R}_n(i)\tilde{R}_n(i)'\big]\mathbb{E}\big[r_{n,0}(i)\tilde{R}_n(i)\big]. \tag{2.3}$$

Furthermore, for $n = 0, 1, \ldots, T+1$, define $\theta_n \in \mathbb{R}^L$ such that

$$\theta_n(i) = \frac{\left(\sum_{k=n}^{T} p_k \overline{\prod_{m=n}^{k-1} Q_{\nu_m}}(i)\right)^2}{\sum_{k=n}^{T} p_k \overline{\prod_{m=n}^{k-1} Q_{\kappa_m}}(i)} \epsilon_{n-1}(i), \qquad n = 1, \ldots, T, \tag{2.4}$$

$$\theta_{T+1}(i) = -b/a^2, \tag{2.5}$$

$$\theta_0(i) = \sum_{k=1}^{T} \big((\varPi_{m=1}^{k-1} Q_m)\theta_k\big)(i). \tag{2.6}$$

The following lemme guaranties that $\theta_n(i)$ defined in (2.4) is well defined.

Lemma 2.3. $\mathbb{E}\big[\tilde{R}_n(i)\tilde{R}_n(i)'\big]$ *is positive definite and* $\epsilon_n(i), \kappa_n(i) > 0$ *for all* $i \in S$ *and* $n = 0, 1, \dots, T-1$.

Proof. The proof of positive definiteness of $\mathbb{E}\big[\tilde{R}_n(i)\tilde{R}_n(i)'\big]$ and $\kappa_n(i) > 0$ are similar to those in [12]. To prove $\epsilon_n(i) > 0$ note that $\mathbb{E}^{-1}\big[\tilde{R}_n(i)\tilde{R}_n(i)'\big]$ is positive definite and $\mathbb{E}\big[\tilde{R}_n(i)\big] \neq \mathbf{0}$. □

Define the value function of problem $P(\mu, \lambda)$ at time n by

$$v_n(i, w_n) = \min_{\pi_n, \dots, \pi_{T-1}} \mathbb{E}_{i,w_n}\left[\sum_{k=n}^{T} p_k W_k^2 + 2ap_k W_k + b\right],$$

for $n = 0, 1 \dots, T$, where $\mathbb{E}_{i,w_n}\big[.\big] = \mathbb{E}\big[.|X_n = i, W_n = w_n\big]$. According to the principle of dynamic programming we have

$$\begin{aligned} v_n(i, w_n) &= \min_{\pi_n} \mathbb{E}_{i,w_n}\left[p_n w_n^2 + 2ap_n w_n + v_{n+1}(X_{n+1}, W_{n+1})\right] \\ &= p_n w_n^2 + 2ap_n w_n + \min_{\pi_n} \sum_{j \in S} Q_{n+1}(i,j)\mathbb{E}\left[v_{n+1}\big(j, r_{n,0}(i)w_n + \tilde{R}_n(i)'\pi_n\big)\right] \end{aligned}$$

for $n = 0, 1 \dots, T-1$, with the boundary condition

$$v_T(i, w_T) = p_T w_T^2 + 2ap_T w_T + b.$$

Theorem 2.4. *The value function* $v_n(i, w_n)$ *is given by*

$$v_n(i, w_n) = \alpha_n(i) w_n^2 + 2a\beta_n(i) w_n + \gamma_n(i) \tag{2.7}$$

and the corresponding optimal investment strategy is given by

$$\begin{aligned} \pi_n^*(i, w_n) = &-\mathbb{E}^{-1}\big[\tilde{R}_n(i)\tilde{R}_n(i)'\big]\Bigg(w_n \mathbb{E}\big[r_{n,0}(i)\tilde{R}_n(i)\big] \\ &+ a\frac{\sum_{k=n+1}^{T} p_k \overline{\prod_{m=n+1}^{k-1} Q_{\nu_m}}(i)}{\sum_{k=n+1}^{T} p_k \overline{\prod_{m=n+1}^{k-1} Q_{\kappa_m}}(i)} \mathbb{E}\big[\tilde{R}_n(i)\big]\Bigg), \end{aligned} \tag{2.8}$$

where

$$\alpha_n(i) = p_n + \kappa_n(i) \sum_{k=n+1}^{T} p_k \overline{\prod_{m=n+1}^{k-1} Q_{\kappa_m}}(i), \tag{2.9}$$

$$\beta_n(i) = p_n + \nu_n(i) \sum_{k=n+1}^{T} p_k \overline{\prod_{m=n+1}^{k-1} Q_{\nu_m}}(i), \tag{2.10}$$

$$\gamma_n(i) = -a^2\theta_{n+1}(i) - a^2 \sum_{k=n+1}^{T} \big((\Pi_{m=n+1}^{k} Q_m)\theta_{k+1}\big)(i). \tag{2.11}$$

Proof. The proof proceeds by induction on n. Obviously (2.7) holds for $n=T$. For $n=T-1$,

$$\begin{aligned}
v_{T-1}(i,w_{T-1}) &= \min_{\pi_{T-1}} \mathbb{E}\Big[p_{T-1}w_{T-1}^2 + 2ap_{T-1}w_{T-1} \\
&\quad + \sum_{j\in S} Q_T(i,j)v_T\big(j, r_{T-1,0}(i)w_{T-1} + \tilde{R}_{T-1}(i)'\pi_{T-1}\big)\Big] \\
&= p_{T-1}w_{T-1}^2 + 2ap_{T-1}w_{T-1} \\
&\quad + \min_{\pi_{T-1}} \mathbb{E}\Big\{\overline{Q_{\alpha_T}}(i)\big[r_{T-1,0}(i)w_{T-1} + \tilde{R}_{T-1}(i)'\pi_{T-1}\big]^2 \\
&\quad + 2a\overline{Q_{\beta_T}}(i)\big[r_{T-1,0}(i)w_{T-1} + \tilde{R}_{T-1}(i)'\pi_{T-1}\big] + \overline{Q_{\gamma_T}}(i)\Big\} \\
&= p_{T-1}w_{T-1}^2 + 2ap_{T-1}w_{T-1} \\
&\quad + \overline{Q_{\alpha_T}}(i)\mathbb{E}\big[r_{T-1,0}^2(i)\big]w_{T-1}^2 + 2a\overline{Q_{\beta_T}}(i)\mathbb{E}\big[r_{T-1,0}(i)\big]w_{T-1} \\
&\quad + \overline{Q_{\gamma_T}}(i) \\
&\quad + \min_{\pi_{T-1}} \Big\{\Big(2\overline{Q_{\alpha_T}}(i)w_{T-1}\mathbb{E}\big[r_{T-1,0}(i)\tilde{R}_{T-1}(i)'\big] \\
&\quad + 2a\overline{Q_{\beta_T}}(i)\mathbb{E}\big[\tilde{R}_{T-1}(i)'\big]\Big)\pi_{T-1} \\
&\quad + \overline{Q_{\alpha_T}}(i)\pi_{T-1}'\mathbb{E}\big[\tilde{R}_{T-1}(i)\tilde{R}_{T-1}(i)'\big]\pi_{T-1}\Big\}.
\end{aligned} \tag{2.12}$$

Here $\overline{Q_{\alpha_T}}(i)=\overline{Q_{\beta_T}}(i)=p_T>0$ and $\overline{Q_{\gamma_T}}(i)=b$. Moreover, $\mathbb{E}\big[\tilde{R}_{T-1}(i)\tilde{R}_{T-1}(i)'\big]$ is positive definite by Lemma 2.3. Therefore, the necessary and sufficient optimality condition is as follows:

$$\begin{aligned}
\overline{Q_{\alpha_T}}(i)w_{T-1}\mathbb{E}\big[r_{T-1,0}(i)\tilde{R}_{T-1}(i)\big] + a\overline{Q_{\beta_T}}(i)\mathbb{E}\big[\tilde{R}_{T-1}(i)\big] \\
+ \overline{Q_{\alpha_T}}(i)\mathbb{E}\big[\tilde{R}_{T-1}(i)\tilde{R}_{T-1}(i)'\big]\pi_{T-1} = \mathbf{0}.
\end{aligned}$$

Then, one can obtain

$$\pi_{T-1}^*(i,w_{T-1}) = -\mathbb{E}^{-1}\big[\tilde{R}_{T-1}(i)\tilde{R}_{T-1}(i)'\big]\left(w_{T-1}\mathbb{E}\big[r_{T-1,0}(i)\tilde{R}_{T-1}(i)\big] + a\frac{\overline{Q_{\beta_T}}(i)}{\overline{Q_{\alpha_T}}(i)}\mathbb{E}\big[\tilde{R}_{T-1}(i)\big]\right)$$

as the optimal portfolio. Substituting $\pi_{T-1}^*(i,w_{T-1})$ in (2.12) yields

$$\begin{aligned}
v_{T-1}(i,w_{T-1}) &= \big[p_{T-1} + \kappa_{T-1}(i)\overline{Q_{\alpha_T}}(i)\big]w_{T-1}^2 + 2a\big[p_{T-1} + \nu_{T-1}(i)\overline{Q_{\beta_T}}(i)\big]w_{T-1} \\
&\quad - a^2\frac{\big(\overline{Q_{\beta_T}}(i)\big)^2}{\overline{Q_{\alpha_T}}(i)}\epsilon_{T-1}(i) + \overline{Q_{\gamma_T}}(i) \\
&= \alpha_{T-1}(i)w_{T-1}^2 + 2a\beta_{T-1}(i)w_{T-1} + \gamma_{T-1}(i),
\end{aligned}$$

where

$$\begin{aligned}
\alpha_{T-1}(i) &= p_{T-1} + \kappa_{T-1}(i)\overline{Q_{\alpha_T}}(i) = p_{T-1} + \kappa_{T-1}(i)p_T > 0, \\
\beta_{T-1}(i) &= p_{T-1} + \nu_{T-1}(i)\overline{Q_{\beta_T}}(i) = p_{T-1} + \nu_{T-1}(i)p_T, \\
\gamma_{T-1}(i) &= -a^2\theta_T(i) + \overline{Q_{\gamma_T}}(i) = -a^2\theta_T(i) + b.
\end{aligned}$$

Assume that (2.7) and (2.8) hold for $n = k+1$, and $\alpha_{k+1}(j) > 0$ for all $j \in S$. Then, for $n = k$,

$$\begin{aligned} v_k(i,w_k) &= \min_{\pi_k} \mathbb{E}\Big[p_k w_k^2 + 2ap_k w_k \\ &\quad + \sum_{j\in S} Q_{k+1}(i,j) v_{k+1}\big(j, r_{k,0}(i)w_k + \tilde{R}_k(i)'\pi_k\big)\Big] \\ &= p_k w_k^2 + 2ap_k w_k \\ &\quad + \min_{\pi_k} \mathbb{E}\Big\{\overline{Q_{\alpha_{k+1}}}(i)\big[r_{k,0}(i)w_k + \tilde{R}_k(i)'\pi_k\big]^2 \\ &\quad + 2a\overline{Q_{\beta_{k+1}}}(i)\big[r_{k,0}(i)w_k + \tilde{R}_k(i)'\pi_k\big] + \overline{Q_{\gamma_{k+1}}}(i)\Big\} \\ &= p_k w_k^2 + 2ap_k w_k \\ &\quad + \overline{Q_{\alpha_{k+1}}}(i)\mathbb{E}\big[r_{k,0}^2(i)\big]w_k^2 + 2a\overline{Q_{\beta_{k+1}}}(i)\mathbb{E}\big[r_{k,0}(i)\big]w_k + \overline{Q_{\gamma_{k+1}}}(i) \\ &\quad + \min_{\pi_k}\big\{\big(2\overline{Q_{\alpha_{k+1}}}(i)w_k\mathbb{E}\big[r_{k,0}(i)\tilde{R}_k(i)'\big] + 2a\overline{Q_{\beta_{k+1}}}(i)\mathbb{E}\big[\tilde{R}_k(i)'\big]\big)\pi_k \\ &\quad + \overline{Q_{\alpha_{k+1}}}(i)\pi_k'\mathbb{E}\big[\tilde{R}_k(i)\tilde{R}_k(i)'\big]\pi_k\big\}. \end{aligned} \tag{2.13}$$

Since $\alpha_{k+1}(j) > 0$ for all $j \in S$, then $\overline{Q_{\alpha_{k+1}}}(i) > 0$. Moreover, $\mathbb{E}\big[\tilde{R}_k(i)\tilde{R}_k(i)'\big]$ is positive definite. As it can be seen, the minimization problem (2.13) has the same structure as the one in (2.12). Therefore, the same argument can be repeated to obtain

$$\pi_k^*(i,w_k) = -\mathbb{E}^{-1}\big[\tilde{R}_k(i)\tilde{R}_k(i)'\big]\Big(w_k\mathbb{E}\big[r_{k,0}(i)\tilde{R}_k(i)\big] + a\frac{\overline{Q_{\beta_{k+1}}}(i)}{\overline{Q_{\alpha_{k+1}}}(i)}\mathbb{E}\big[\tilde{R}_k(i)\big]\Big) \tag{2.14}$$

as the optimal portfolio. Substituting $\pi_k^*(i,w_k)$ in (2.13) yields

$$\begin{aligned} v_k(i,w_k) &= \big[p_k + \kappa_k(i)\overline{Q_{\alpha_{k+1}}}(i)\big]w_k^2 + 2a\big[p_k + \nu_k(i)\overline{Q_{\beta_{k+1}}}(i)\big]w_k \\ &\quad - a^2\frac{\big(\overline{Q_{\beta_{k+1}}}(i)\big)^2}{\overline{Q_{\alpha_{k+1}}}(i)}\epsilon_k(i) + \overline{Q_{\gamma_{k+1}}}(i) \\ &= \alpha_k(i)w_k^2 + 2a\beta_k(i)w_k + \gamma_k(i), \end{aligned}$$

where

$$\begin{aligned} \alpha_k(i) &= p_k + \kappa_k(i)\overline{Q_{\alpha_{k+1}}}(i) > 0, \\ \beta_k(i) &= p_k + \nu_k(i)\overline{Q_{\beta_{k+1}}}(i), \\ \gamma_k(i) &= -a^2\theta_{k+1}(i) + \overline{Q_{\gamma_{k+1}}}(i). \end{aligned}$$

Finally, Lemmas 2.1 and 2.2 complete the proof. For instance, to obtain (2.11), let $b_n(i) = -a^2\theta_{n+1}(i)$ and $c_n(i) = 1$. □

2.3. Solution to problems $P(\mu)$, $P(\sigma)$ and $P(\omega)$

Considering the Equation (2.7), the optimal value of problem $P(\mu,\lambda)$ corresponding to the initial state i_0 is

$$\begin{aligned} v_0(i_0,w_0) &= \alpha_0(i_0)w_0^2 + 2a\beta_0(i_0)w_0 + \gamma_0(i_0) \\ &= \alpha_0(i_0)w_0^2 + 2a\beta_0(i_0)w_0 - a^2\Big[\theta_1(i_0) + \sum_{k=1}^{T-1}\big((\Pi_{m=1}^k Q_m)\theta_{k+1}\big)(i_0)\Big] + b \\ &= \alpha_0(i_0)w_0^2 + 2(\lambda-\mu)\beta_0(i_0)w_0 - (\lambda-\mu)^2\theta_0(i_0) + \mu^2 - 2\lambda\mu. \end{aligned} \tag{2.15}$$

Define

$$L(\lambda, w_0, i_0) := v_0(i_0, w_0).$$

Denote the optimal value of problem $P(\mu)$ by $\mathbb{V}\text{ar}^*_{i_0}\left[W_{T\wedge\tau}\right]$. The Lagrange Duality Theorem (see [16]) yields that

$$\mathbb{V}\text{ar}^*_{i_0}\left[W_{T\wedge\tau}\right] = \max_{\lambda} L(\lambda, w_0, i_0). \tag{2.16}$$

Lemma 2.3 yields that $\theta_T(j) = p_T\epsilon_{T-1}(j) > 0$, for all $j \in S$. Then

$$\theta_0(i_0) = \sum_{k=1}^{T} \left((\Pi_{m=1}^{k-1} Q_m)\theta_k\right)(i_0) \geq \left((\Pi_{m=1}^{T-1} Q_m)\theta_T\right)(i_0) > 0.$$

Therefore, the optimal solution in formulation (2.16) exists. Differentiating from (2.15) with respect to λ and setting to zero yields the optimal solution as

$$\lambda^* = \frac{\beta_0(i_0)w_0 - \mu}{\theta_0(i_0)} + \mu.$$

The optimal strategy of problem $P(\mu)$ can be computed by setting

$$a = \lambda^* - \mu = \frac{\beta_0(i_0)w_0 - \mu}{\theta_0(i_0)} \tag{2.17}$$

in (2.8). Substituting λ^* in (2.15), one can get the minimum variance in problem $P(\mu)$, denoted by $\mathbb{V}\text{ar}^*_{i_0}\left[W_{T\wedge\tau}\right]$, as follows

$$\mathbb{V}\text{ar}^*_{i_0}\left[W_{T\wedge\tau}\right] = \frac{1-\theta_0(i_0)}{\theta_0(i_0)}\left(\mathbb{E}_{i_0}\left[W_{T\wedge\tau}\right] - \frac{\beta_0(i_0)w_0}{1-\theta_0(i_0)}\right)^2 + \left(\alpha_0(i_0) - \frac{(\beta_0(i_0))^2}{1-\theta_0(i_0)}\right)w_0^2. \tag{2.18}$$

As it can be seen in Equation (2.18), $\mathbb{V}\text{ar}^*_{i_0}\left[W_{T\wedge\tau}\right]$ is a parabola with respect to $\mathbb{E}_{i_0}\left[W_{T\wedge\tau}\right]$. Therefore, for any optimal portfolio strategy with $\mathbb{E}_{i_0}\left[W_{T\wedge\tau}\right] < \frac{\beta_0(i_0)w_0}{1-\theta_0(i_0)}$, one can find an optimal portfolio strategy with the same $\mathbb{V}\text{ar}^*_{i_0}\left[W_{T\wedge\tau}\right]$ but a higher $\mathbb{E}_{i_0}\left[W_{T\wedge\tau}\right]$. This fact yields the following theorem.

Theorem 2.5. *The optimal investment strategy of problem $P(\mu)$ is given by (2.8) where*

$$a = \frac{\beta_0(i_0)w_0 - \mu}{\theta_0(i_0)}\cdot$$

Also, the efficient frontier is represented by (2.18) where $\mathbb{E}_{i_0}\left[W_{T\wedge\tau}\right] \geq \frac{\beta_0(i_0)w_0}{1-\theta_0(i_0)}$.

Corollary 2.6. *The global minimum variance strategy, that is the strategy that achieves the lowest obtainable variance among all feasible strategies, is given by (2.8) where*

$$a = \frac{\beta_0(i_0)w_0}{\theta_0(i_0) - 1} \tag{2.19}$$

for which the mean and the variance of the final wealth are as follows,

$$\widehat{\mathbb{E}}_{i_0}\left[W_{T\wedge\tau}\right] = \frac{\beta_0(i_0)w_0}{1-\theta_0(i_0)}, \qquad \widehat{\mathbb{V}ar}_{i_0}\left[W_{T\wedge\tau}\right] = \left(\alpha_0(i_0) - \frac{(\beta_0(i_0))^2}{1-\theta_0(i_0)}\right)w_0^2.$$

Corollary 2.6 indicates that the global minimum variance is not zero. However, in the case of no regime-switching, when the market contains a riskless asset and the exit-time is certain, as it is assumed in [12], the global minimum variance is zero, which can be obtained by the total investment in the riskless asset (see Eq. (76) in [12]). Then, the optimal standard deviation is a linear function in terms of the expected terminal wealth. So, the efficient frontier is a straight line, which is called the *capital market line*, in the Standard Deviation-Mean plane. Under the assumptions of our model, we still cannot obtain zero global minimum variance even in the presence of a riskless asset. The reason is, although for a given market state, the return of the riskless asset is deterministic (which depends on the market state), we don't know the future market states from now. So, investment in the riskless asset has a random return and, consequently, a positive variance (risk). This is true even we assume that the return of the riskless asset is constant for all market states over all time periods. In this case, investment in the riskless asset has a random return again, since the length of the investment time is uncertain. This issue is investigated through a numerical example in the last section.

In the following, it is shown that how the solution of problems $P(\sigma)$ and $P(\omega)$ can be found by using the obtained results for problem $P(\mu)$.

Corollary 2.7. *The optimal investment strategy of problem $P(\sigma)$ is given by (2.8) where*

$$a = \frac{\beta_0(i_0)w_0}{\theta_0(i_0)-1} - \sqrt{\frac{1}{\theta_0(i_0)\big(1-\theta_0(i_0)\big)}\left[\sigma^2 + \left(\frac{(\beta_0(i_0))^2}{1-\theta_0(i_0)} - \alpha_0(i_0)\right)w_0^2\right]}.$$

Proof. Note that if π^* solves $P(\sigma)$, then π^* solves $P(\mu)$ where $\mu = \mathbb{E}_{i_0}\big[W_{T\wedge\tau}\big]\big|_{\pi^*}$. On the other hand, if $\widetilde{\pi}$ solves $P(\mu)$, then $\widetilde{\pi}$ solves $P(\sigma)$ where $\sigma^2 = \mathbb{V}\mathrm{ar}_{i_0}\big[W_{T\wedge\tau}\big]\big|_{\widetilde{\pi}}$. The value of μ can be computed in terms of σ^2, by using Equation (2.18), as follows,

$$\mu = \sqrt{\frac{\theta_0(i_0)}{1-\theta_0(i_0)}\left[\sigma^2 + \left(\frac{(\beta_0(i_0))^2}{1-\theta_0(i_0)} - \alpha_0(i_0)\right)w_0^2\right]} + \frac{\beta_0(i_0)w_0}{1-\theta_0(i_0)}\cdot$$

Now the result follows from substituting this μ into (2.17). □

Corollary 2.8. *The optimal investment strategy of problem $P(\omega)$ is given by (2.8) where*

$$a = \frac{1+2\omega\beta_0(i_0)w_0}{2\omega\big(\theta_0(i_0)-1\big)}\cdot \tag{2.20}$$

Proof. Note that if $\widetilde{\pi}$ solves $P(\omega)$, then $\widetilde{\pi}$ solves $P(\mu)$ where $\mu = \mathbb{E}_{i_0}\big[W_{T\wedge\tau}\big]\big|_{\widetilde{\pi}}$ and $\widetilde{\pi}$ solves $P(\sigma)$ where $\sigma^2 = \mathbb{V}\mathrm{ar}_{i_0}\big[W_{T\wedge\tau}\big]\big|_{\widetilde{\pi}}$. Therefore, the pair $\big(\mathbb{E}_{i_0}\big[W_{T\wedge\tau}\big]\big|_{\widetilde{\pi}}, \mathbb{V}\mathrm{ar}_{i_0}\big[W_{T\wedge\tau}\big]\big|_{\widetilde{\pi}}\big)$ is on the efficient frontier of problem $P(\mu)$ and satisfies Equation (2.18). Denote the objective function of problem $P(\omega)$ by

$$U\big(\mathbb{E}_{i_0}\big[W_{T\wedge\tau}\big], \mathbb{V}\mathrm{ar}_{i_0}\big[W_{T\wedge\tau}\big]\big) = \mathbb{E}_{i_0}\big[W_{T\wedge\tau}\big] - \omega\mathbb{V}\mathrm{ar}_{i_0}\big[W_{T\wedge\tau}\big].$$

So, U takes its optimal value on the efficient frontier (2.18). Then, at the optimal solution $\widetilde{\pi}$, we have

$$\frac{1}{\omega} = \frac{\partial\mathbb{V}\mathrm{ar}_{i_0}\big[W_{T\wedge\tau}\big]}{\partial\mathbb{E}_{i_0}\big[W_{T\wedge\tau}\big]} = \frac{2\big(1-\theta_0(i_0)\big)}{\theta_0(i_0)}\left(\mathbb{E}_{i_0}\big[W_{T\wedge\tau}\big] - \frac{\beta_0(i_0)w_0}{1-\theta_0(i_0)}\right).$$

Now the result follows after computing $\mathbb{E}_{i_0}\big[W_{T\wedge\tau}\big]$ in terms of ω and substituting $\mu = \mathbb{E}_{i_0}\big[W_{T\wedge\tau}\big]$ in (2.17). □

The parameter ω in the formulation of problem $P(\omega)$ denotes the amount of the risk aversion of the investor. In fact, the more risk averse the investor, the bigger his/her risk aversion parameter $\omega > 0$. It can be seen from Equation (2.20) that the value of a in (2.20) converges to the value of a in (2.19) as ω goes to infinity. In other words, when the risk aversion parameter is extremely high, the investor chooses the portfolio with the lowest obtainable risk, *i.e.*, the global minimum variance portfolio. This fact is more observable when the objective function in problem $P(\omega)$ is replaced equivalently by

$$\min_{\pi} \ \mathbb{V}\mathrm{ar}_{i_0}\big[W_{T\wedge\tau}\big] - \frac{1}{\omega}\mathbb{E}_{i_0}\big[W_{T\wedge\tau}\big].$$

As it is seen, by applying the Lagrange duality approach, the optimal solution of problem $P(\mu)$ and, consequently, the optimal solutions of problems $P(\sigma)$ and $P(\omega)$, can be obtained with simpler procedure, compared to the embedding technique.

3. Special cases

In this section we investigate some special cases and show that some known models can be obtained as special cases of our model.

3.1. Portfolio selection in the presence of one riskless asset

Let the market contains one riskless asset. Also, let $r_{n,0}(i)$ be the return of the riskless asset in the time period n and the market state i. Then, Equations (2.1)–(2.3) can be rewritten as follows,

$$\begin{aligned}
&\epsilon_n(i) = \mathbb{E}\big[\tilde{R}_n(i)'\big]\mathbb{E}^{-1}\big[\tilde{R}_n(i)\tilde{R}_n(i)'\big]\mathbb{E}\big[\tilde{R}_n(i)\big],\\
&\nu_n(i) = r_{n,0}(i)\big(1-\epsilon_n(i)\big),\\
&\kappa_n(i) = \big(r_{n,0}(i)\big)^2\big(1-\epsilon_n(i)\big),
\end{aligned}$$

for $n = 0,1,\ldots,T-1$. Assume that the Markov chain is time-homogeneous and $Q_n = Q$ for all $n = 1,2,\ldots$. If for any $i \in S$, the random returns $R_0(i), R_1(i),\ldots,R_{T-1}(i)$ are identically distributed and the riskless asset has the same return $r_0(i)$ in each period (as it is assumed in [23]), then $\epsilon_n(i), \nu_n(i)$ and $\kappa_n(i)$ will be constant for all periods. For any $i \in S$, denote

$$\begin{aligned}
&\epsilon(i) = \epsilon_n(i),\\
&\nu(i) = r_0(i)\big(1-\epsilon(i)\big),\\
&\kappa(i) = \big(r_0(i)\big)^2\big(1-\epsilon(i)\big).
\end{aligned}$$

Then, $\theta_0(i_0)$ can be computed by substituting

$$\theta_n(i) = \frac{\big(\sum_{k=n}^{T} p_k \overline{Q_{\nu}^{k-n}}(i)\big)^2}{\sum_{k=n}^{T} p_k \overline{Q_{\kappa}^{k-n}}(i)}\epsilon(i),$$

obtained from (2.4), in (2.6). Also, from Equations (2.9) and (2.10), we have

$$\begin{aligned}
\alpha_0(i_0) &= \kappa(i_0)\sum_{k=1}^{T} p_k \overline{Q_{\kappa}^{k-1}}(i_0),\\
\beta_0(i_0) &= \nu(i_0)\sum_{k=1}^{T} p_k \overline{Q_{\nu}^{k-1}}(i_0).
\end{aligned}$$

Applying the above values for $\theta_0(i_0), \alpha_0(i_0)$ and $\beta_0(i_0)$ in Theorem 2.5 and Corollaries 2.7 and 2.8, the efficient frontier and the optimal strategies of problems $P(\mu)$, $P(\sigma)$ and $P(\omega)$ can be computed.

Besides the above conditions, let the exit-time is certain (as it is assumed in [4]), *i.e.*, $p_1 = p_2 =, \dots, = p_{T-1} = 0$ and $p_T = 1$. Then, one can get the values of $\alpha_0(i_0)$ and $\beta_0(i_0)$ as follows,

$$\alpha_0(i_0) = \kappa(i_0)\overline{Q_\kappa^{T-1}}(i_0),$$
$$\beta_0(i_0) = \nu(i_0)\overline{Q_\nu^{T-1}}(i_0).$$

Also, $\theta_0(i_0)$ can be computed by substituting

$$\theta_n(i) = \frac{\left(\overline{Q_\nu^{T-n}}(i)\right)^2}{\overline{Q_\kappa^{T-n}}(i)}\epsilon(i)$$

in (2.6). Using the obtained values for $\theta_0(i_0), \alpha_0(i_0)$ and $\beta_0(i_0)$, the efficient frontier and optimal strategies of problems $P(\mu)$, $P(\sigma)$ and $P(\omega)$ can be computed.

3.2. No regime-switching

Assume that there is no Markov regime-switching (as it is assumed in [6, 32]). Then, the parameters in (2.1)–(2.4) reduce to the following forms,

$$\epsilon_n = \mathbb{E}\big[\tilde{R}_n'\big]\mathbb{E}^{-1}\big[\tilde{R}_n\tilde{R}_n'\big]\mathbb{E}\big[\tilde{R}_n\big], \tag{3.1}$$

$$\nu_n = \mathbb{E}\big[r_{n,0}\big] - \mathbb{E}\big[\tilde{R}_n'\big]\mathbb{E}^{-1}\big[\tilde{R}_n\tilde{R}_n'\big]\mathbb{E}\big[r_{n,0}\tilde{R}_n\big], \tag{3.2}$$

$$\kappa_n = \mathbb{E}\big[r_{n,0}^2\big] - \mathbb{E}\big[r_{n,0}\tilde{R}_n'\big]\mathbb{E}^{-1}\big[\tilde{R}_n\tilde{R}_n'\big]\mathbb{E}\big[r_{n,0}\tilde{R}_n\big], \tag{3.3}$$

$$\theta_n = \frac{\left(\sum_{k=n}^T p_k \prod_{m=n}^{k-1}\nu_m\right)^2}{\sum_{k=n}^T p_k \prod_{m=n}^{k-1}\kappa_m}\epsilon_{n-1}. \tag{3.4}$$

Also, we have

$$\theta_0 = \sum_{k=1}^T \theta_k,$$
$$\alpha_0 = \sum_{k=1}^T p_k \prod_{m=0}^{k-1}\kappa_m,$$
$$\beta_0 = \sum_{k=1}^T p_k \prod_{m=0}^{k-1}\nu_m.$$

Moreover, the optimal strategy in (2.8) reduces to

$$\pi_n^*(w_n) = -\mathbb{E}^{-1}\big[\tilde{R}_n\tilde{R}_n'\big]\left(w_n\mathbb{E}\big[r_{n,0}\tilde{R}_n\big] + a\frac{\sum_{k=n+1}^T p_k\prod_{m=n+1}^{k-1}\nu_m}{\sum_{k=n+1}^T p_k\prod_{m=n+1}^{k-1}\kappa_m}\mathbb{E}\big[\tilde{R}_n\big]\right),$$

for $n = 0, 1, \dots, T-1$.

3.3. No regime-switching with certain exit-time

Problems $\tilde{P}(\mu)$, $\tilde{P}(\sigma)$ and $\tilde{P}(\omega)$ can be considered, respectively, as special cases of problems $P(\mu)$, $P(\sigma)$ and $P(\omega)$ when there is no Markov regime-switching and the exit-time is certain (see [12]). In this case, we have Equations (3.1)–(3.3) and

$$\theta_n = \frac{\left(\prod_{m=n}^{T-1}\nu_m\right)^2}{\prod_{m=n}^{T-1}\kappa_m}\epsilon_{n-1}.$$

Also,

$$\theta_0 = \sum_{k=1}^{T} \theta_k, \;\; \alpha_0 = \prod_{m=0}^{T-1} \kappa_m, \;\; \beta_0 = \prod_{m=0}^{T-1} \nu_m.$$

Moreover, the optimal strategy in (2.8) reduces to

$$\pi_n^*(w_n) = -\mathbb{E}^{-1}\big[\tilde{R}_n \tilde{R}_n'\big] \left(w_n \mathbb{E}\big[r_{n,0}\tilde{R}_n\big] + a \frac{\prod_{m=n+1}^{T-1} \nu_m}{\prod_{m=n+1}^{T-1} \kappa_m} \mathbb{E}\big[\tilde{R}_n\big] \right),$$

for $n = 0, 1, \ldots, T-1$.

4. Numerical example

Consider a market consisting of one risky asset and one riskless asset. It is assumed that the Markov chain is time-homogeneous. Let $T = 3$ and there are two market states, state 1 and state 2, with transition probability matrix $Q = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$. For the given time period n $(n = 0, 1, 2)$ and the market state i $(i = 1, 2)$, the return of the riskless asset is $r_{n,0}(i) = 1 + 0.01(n+1)i$, and the return of the risky asset is log-normal distributed such that $\log r_{n,1}(i) \sim N(\frac{2.5+i}{10}, \frac{2+n}{10})$. Using these assumptions, we have

$$A = \begin{pmatrix} 0.5583 & 0.7133 \\ 0.6287 & 0.7821 \\ 0.7033 & 0.8555 \end{pmatrix}, \quad B = \begin{pmatrix} 0.8563 & 1.1739 \\ 1.3463 & 1.7733 \\ 1.9721 & 2.5366 \end{pmatrix},$$

where $A(n+1, i) = \mathbb{E}[\tilde{R}_n(i)]$ and $B(n+1, i) = \mathbb{E}[(\tilde{R}_n(i))^2]$, for $n = 0, 1, 2$ and $i = 1, 2$.

Assume that the probability mass function of the real exit-time $T \wedge \tau$ is $(p_1, p_2, p_3) = (0.2, 0.3, 0.5)$. We further have

$$C = \begin{pmatrix} 0.6837 & 0.6751 \\ 0.6832 & 0.6806 \\ 0.5000 & 0.5000 \end{pmatrix}, \; D = \begin{pmatrix} 0.7077 & 0.7023 \\ 0.6981 & 0.6988 \\ 0.5000 & 0.5000 \end{pmatrix},$$

where

$$C(n, i) = \sum_{k=n}^{3} p_k \overline{\prod_{m=n}^{k-1} Q_{\nu_m}}(i), \;\; D(n, i) = \sum_{k=n}^{3} p_k \overline{\prod_{m=n}^{k-1} Q_{\kappa_m}}(i).$$

Also, $\alpha_0 = (0.4591, 0.4140)'$, $\beta_0 = (0.4391, 0.3902)'$ and $\theta_0 = (0.5792, 0.6314)'$.

Assume that an investor has an initial wealth $w_0 = 1$. Optimal strategies of problems $P(\mu)$, $P(\sigma)$ and $P(\omega)$ can be computed after substituting appropriate values for a in the following,

$$\pi_n^*(i, w_n) = -\left[w_n r_{n,0}(i) + a \frac{C(n+1, i)}{D(n+1, i)} \right] B(n+1, i)^{-1} A(n+1, i), \;\; n = 0, 1, 2.$$

Also, the efficient frontier for the given initial state i_0 becomes

$$\sigma^2(\mu) = \frac{1 - \theta_0(i_0)}{\theta_0(i_0)} \left(\mu - \frac{\beta_0(i_0)}{1 - \theta_0(i_0)} \right)^2 + \alpha_0(i_0) - \frac{\big(\beta_0(i_0)\big)^2}{1 - \theta_0(i_0)},$$

where $\sigma^2(\mu)$ is the optimal terminal variance corresponding to the expected terminal wealth μ. The efficient frontiers for initial states $i_0 = 1, 2$, corresponding to the certain and uncertain exit-time are demonstrated in

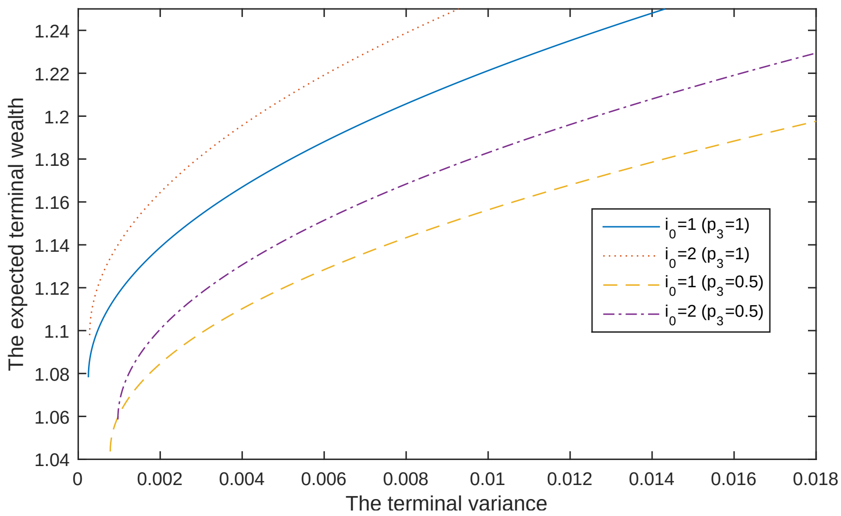

FIGURE 1. Efficient frontiers for $i_0 = 1, 2$, corresponding to the certain and uncertain exit-time.

Figure 1. Figure 1 implies that the uncertain exit-time increases the risk of the investment. It is obvious from Figure 1 that the investor with the certain exit-time will suffer less risk than the one with the uncertain exit-time to achieve the same expected terminal wealth. Moreover, it can be seen that the global minimum variance portfolios have non-zero risk over $T=3$ periods, in spite of the existence of the riskless asset. This is due to the fact that investing in the riskless asset over three periods has a random return since the market states are stochastic.

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