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A VENDOR MANAGED INVENTORY ON CONSIGNMENT WITH PENALTY AS A SUPPLY CHAIN COORDINATION ARRANGEMENT

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Abstract. This paper presents a mathematical two echelon vendor-managed inventory (VMI) model on consignment scheme in supply chain, in which the vendor pays a penalty for every extra unit which exceeds a specific upper limit to the buyer. In this arrangement the vendor can decide about the batch size he wants to transship to the buyer. Two cases are discussed in this paper; single vendor-single buyer and single vendor-two buyers. In this study after a review of coordination in supply chain and VMI on consignment, we discuss vendor-managed inventory scheme in comparison with traditional inventory control system. The optimal batch size is calculated and the paper investigates how a VMI on consignment arrangement with penalty can coordinate the supply chain. Numerical examples and sensitivity analysis is presented to illustrate the performance of model and the results.

Mathematics Subject Classification. 90B05.

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1. Introduction and literature review

Traditional supply chain is a system includes suppliers, manufacturers, distributors, retailers and customers that the members are linked to each other by forward flow of materials and backward flow of information. In the traditional supply chain, each member is accountable for controlling the inventory policy. The most important issue that all these members must deal with is the order quantity in order to satisfy the customers' demand [13]. In a brief statement, a traditional supply chain has some disadvantages such as long delivery times, multiple decision points, opaque information and low coordination [14].

To survive in the global market, businesses are struggling to remain competitive by reducing costs and improving service to their customers. Matching supply with demand, prevent inventory shortages and improve the performance of the transportation system, are common goals in most factories. More ever the success of supply chains depends on the efficient flow of products to the final customer. Although in some cases the lack of coordination between members may disturb the performance of the supply chain [17]. Most of the literature reviews in the field of supply chain seek to enhance the coordination and integration among members of the supply chain [3, 12, 25].

Keywords. Vendor managed inventory, supply chain management, coordination, consignment stock.

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1.1. Coordination through vendor managed inventory in supply chain

Coordination in the supply chain can be categorized into two modes, "strong" and "weak". If all the actions can take benefits to the whole supply chain, the supply chain coordinated as "strong". If implementing a set of actions make none of the members' situation worse off, and at least cause benefits for one member, the supply chain coordinated as "weak" [6]. Achieving strong coordination in the supply chain due to the asymmetry of information and members conflicting policies is difficult. That is why most of the research papers focus on the weak coordination [1].

In vendor-managed inventory (VMI), the vendor is responsible for satisfying the buyers' demand. Also the vendor controls the inventory level and the replenishment cycle, instead of the customer. The VMI advantages include integrated lower costs for the buyer and vendor [15], reducing the uncertainty in demand [29], the responsiveness to market demand [27] and the "bullwhip effect", by removing a layer of decision making, the delay and lag in the flow of information [13]. Opportunistic acts and behaviors by the vendor scare the buyers, who reluctant to sign a VMI contract [29]. In particular, the buyers may be scared of the case which vendor stores more goods in their warehouse to reduce the risk of inventory shortage [27].

VMI is divided in two modes based on the owner of the inventory in buyer's warehouse. The first is the traditional method which the buyer is the owner of the inventory after delivering the goods. The latter one which is recognized as "vendor managed on consignment inventory" is an approach in which the vendor is owner of the inventory until the goods received from the final customer. In general, the optimal decision about ownership of the goods depends on the ability of the vendor and the customer in controlling the risk. This paper considers supply chain coordination through VMI on consignment strategy. The basic concept of VMI was presented by Magee in 1958. That case was raised about who should be responsible for inventory control. But interest in the concept of VMI's increased in the 90s [26]. The researches considering VMI, which started in the second half of the nineties, can be classified in two categories:

1. Pure and applied management issues of VMI, based on the simulation approach.
2. Mathematical models on VMI and its economic analysis.

Dong and Xu [15] presented a mathematical model with deterministic parameters for comparison between traditional supply chain and supply chain with implementing VMI, in short and long term period. They conclude that VMI is an effective strategy for the supply chain, which could actualize many of the partners' benefits.

Goyal [21] was the first person who investigated the coordination benefits in replenishment decisions with considering a two-echelon supply chain. Fry *et al.* [16] investigated the cost savings by implementing VMI agreement and Darwish and Odah [10] studied a mathematical model assuming a single vendor and multiple buyers under VMI and used the penalty cost for inventories exceeding an upper limit on stock level. Also they presented an effective algorithm to find the global optimal solution. Darwish and Goyal [11] studied a VMI setting between a single vendor, with a finite production rate, and single buyer with limited storage space. Hariga *et al.* [24] extended a single vendor and multiple buyers' model which operates under a VMI arrangement and determines inventory limits on buyers' stock levels. Chakraborty *et al.* [7] considered a VMI model for single vendor, single buyer with considering a penalty for shipments exceeding an upper limit. They showed that how the VMI arrangement with penalty can be exerted as a supply chain coordination arrangement. They also assumed that ownership of inventory transferred to the buyer immediately after delivery. But in some situations the vendor wants to have the ownership of the inventory until the final customer purchases them.

1.2. Vendor managed inventory on consignment

In the field of "Vendor Managed on consignment" category, Gumus *et al.* [23] developed a single vendor, single buyer VMI model with considering the consignment inventory. Ben-Daya *et al.* [5] developed a supply chain model for a single vendor and multiple buyers under VMI on consignment setting and deterministic demand. They concluded that under VMI on consignment inventory policy, vendor's best policy is to send more frequent shipments with smaller batch sizes. Wang *et al.* [31] examined a single periodic VMI on consignment with revenue sharing contract with considering uncertainty and price-elastic demand. Almehdawe and Mantin [2] extended a VMI on consignment model with revenue sharing contract with considering one manufacturer and multi-retailers in a deterministic setting. Chen [9] investigated a VMI on consignment model with revenue sharing with considering deterioration and price dependent demand in the online market. Wong *et al.* [32] studied a VMI on consignment contract with sales rebate with considering one supplier and multi-retailers. In that proposed contract the retailers get a rebate for each unit sold beyond a specific limit. Zanoni and Jaber [33] presented a VMI on consignment model in which the demand at the buyer side is stock dependent. Chen *et al.* [8] assumed a supply chain with one wholesaler and a retailer with considering multi-stores and various business scenarios which one of them was VMI on consignment and revenue sharing. Guan and Zhao [22] assumed a VMI on consignment with revenue sharing mechanism with considering suggested bargaining procedures for selecting contract parameters. Gerchak *et al.* [20] and Gerchak and Khmelinitsky [18] developed a VMI on consignment contract in which the retailer in each period, shares the sales information like sales volume with the supplier, who in turn decides about deliver quantity for the next period. After that, the revenue is shared between two members based on sales information. Gerchak and Wang [19] assumed a VMI on consignment model in a stochastic single period mode in which the assembler shares the revenue with his suppliers. Khan *et al.* [28] considered a supply chain where the vendor transports every production batch in multiple lots to the buyer's warehouse. The buyer asses the products quality while fulfilling the market demands. They investigate the effect of different fractions of defective items, holding costs as well as disposal schemes. A comprehensive study in the field of VMI and VMI on consignment can be found in Lee *et al.* [30].

Table 1 represented the summary of literature review in the field of VMI on consignment policy.

TABLE 1. The summary of literature review in the field of VMI on consignment policy.

Reference	Coordination mechanism	Inventory model	Determinism	Structure
Gumus *et al.* [23]	Consignment contract	EOQ	Deterministic	Single vendor _ Single buyer
Ben-Daya *et al.* [5]	Consignment contract	EOQ	Deterministic	Single vendor _ Multiple buyers
Wang *et al.* [31]	Revenue sharing	Single-period	Stochastic	Single vendor _ Single buyer
Almehdawe and Mantin [2]	Revenue sharing	EOQ	Deterministic	Single vendor _ Multiple buyers
Chen [9]	Revenue sharing	EOQ	Deterministic	Single vendor _ Single buyer
Wong *et al.* [32]	Sales-Rebate	Single-period	Stochastic	Single vendor _ Multiple buyers
Zanoni and Jaber [33]	Consignment contract	Continuous review	Stochastic	Single vendor _ Single buyer
Chen *et al.* [8]	Revenue sharing	EOQ	Deterministic	Single vendor _ Single buyer
Guan and Zhao [22]	Revenue sharing	Continuous review	Stochastic	Single vendor _ Single buyer
Gerchak *et al.* [20]	Revenue sharing	Single-period	Stochastic	Single vendor _ Single buyer
Gerchak and Khmelinitsky [18]	Revenue sharing	Periodic review	Stochastic	Single vendor _ Single buyer
Gerchak and Wang [19]	Revenue sharing	Single period	Stochastic	Single vendor _ Single buyer
Khan *et al.* [28]	Consignment contract	EOQ	Deterministic	Single vendor _ Single buyer
This paper	Penalty scheme	EOQ	Deterministic	Single vendor _ Single buyer and two buyers

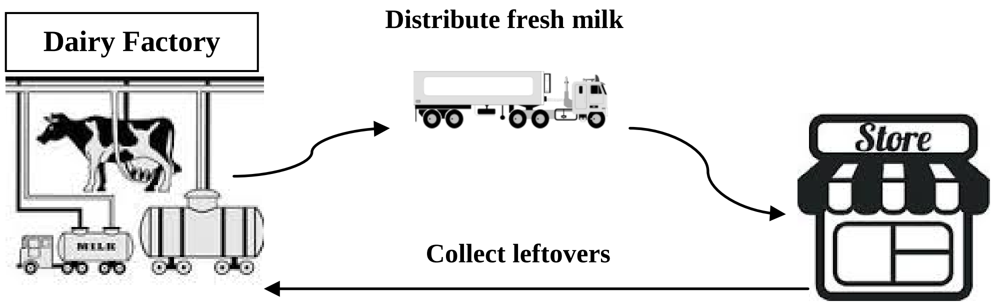

FIGURE 1. The conceptual model of the example.

As shown in Table 1, there are many articles which consider revenue sharing and consignment contracts for their coordination mechanism. However, to the best of our knowledge, there is no paper that considers penalty scheme as a coordination mechanism on VMI on consignment arrangement. So in this paper, a vendor managed inventory on consignment with penalty as a supply chain coordination arrangement is proposed.

1.3. Purpose

In practice, implementation of VMI on consignment strategy needs an effective approach for resolving the problems among supply chain partners. The benefits of this strategy are obvious for the buyer, because the responsibility of inventory holding and transportation costs transfer to the vendor. Vice versa, the benefits of this strategy are not clear for the vendor. But in some cases depending on market conditions, the vendor has no other alternative. The ownership of inventory may be a strategic decision for the vendor. For instance the vendor can be sure that the retailer keeps enough goods to response to customers anytime. Also ownership of the inventory can be a solution for vendors of fashion products. For the buyers purchase of such goods has been associated with a significant risk.

A good example of this situation is the local dairy food companies. They distribute their fresh milk, with different grades of fat, to local supermarkets on a VMI on consignment agreement. After two days, they collect and return the leftovers to the factories for further processing into other products such as cheese and butter milk. In Figure 1 the conceptual model of this example is illustrated.

In this paper, a mathematical two echelon vendor-managed inventory scheme with considering consignment stocks is presented. In this proposed model the vendor is responsible for ordering and holding costs instead of the buyer. The two cases include: (1) single vendor-single buyer and (2) single vendor-two buyers are considered. In this paper, we answer this question that "How to calculate the batch size with considering and analyzing the penalty and stock limit?" We want to find the optimum amount of batch size which the vendor must produce and prepare for the buyer in order to minimize his costs in which both sides have better situation than the traditional system after signing the vendor managed inventory on consignment contract. So we can categorize our contribution to the literature as following:

1. Considering penalty scheme in the realm of VMI on consignment. In such a scheme, the penalty cost charges the vendor, when the retailer's inventory level exceeds an upper level, such as the cost of renting an extra storage space.
2. Formulating the problem for single vendor-single buyer and single vendor-two buyers.
3. Obtaining a closed form solution for the batch size.

The rest of this paper is categorized as following. In Section 2 the notations, assumptions and mathematical model of defined problem presented. Numerical examples and sensitivity analysis are performed respectively in

Section 3 and managerial insights mentioned in Section 4. Finally conclusion and future researches recommendation are presented in Section 5.

2. Problem statement

In this paper, we consider a two echelon supply chain where the vendor produces a specific item and send it to buyers. We develop the model in two cases: (1) single buyer, (2) two buyers. With applying the VMI on consignment contract, the vendor is responsible for replenishment decision and the batch size. Also he is responsible for holding and ordering costs. In another words, he is the owner of the inventory until it being received by the final customers. As we mentioned, the buyer fears opportunistic behavior from vendor, wherein the vendor places excessive inventory at buyer's premise. Although the vendor incurs all the holding and ordering costs but the buyer may lose the chance of making contracts with other vendors or sell other kinds of products due to lack of space. The buyer usually wants to have an appropriate assortment plan which cover lots of brands and different products and satisfy the needs of his customers. Therefore, he imposes a penalty cost for excessive inventory to the vendor in this contract. Using of preset inventory levels is one of the policies against this buyers fear and reluctance. In each case we compare the traditional system and the new system with implementing VMI on consignment agreement and find the optimal batch size and order quantity. The following assumptions are used throughout the paper:

1. Demand is constant over time.
2. Shortages are not permitted.
3. The production rate is infinite.
4. The vendor incurs holding costs and ordering costs (including transportation costs).
5. The buyer inflicts a penalty per unit, for every unit that exceeds an upper limit.
6. The transportation time is zero (or equivalently the transportation speed is infinite).
7. The vendor makes decision about batch size and time of replenishments.
8. For every batch production, the vendor incurs a setup cost which consists of labor costs, machine installation costs and cost of materials required.

Also the following notations are considered throughout the paper:

Q: Order quantity.
D: Demand rate.
C_0: The ordering cost for vendor in single vendor-single buyer mode.
C_1: The ordering cost for vendor in single vendor-two buyers mode.
C_s: The ordering cost for buyer.
h: Holding cost per unit per unit of time.
w: Wholesale price for purchased items which are charged by the vendor.
z_i: Specific upper limit for the buyer number i ($i \in \{1, 2\}$).
x_i: Unit penalty charged by the buyer number i for every unit extra than the upper limit ($i \in \{1, 2\}$).
k: The coefficient for increasing the batch size (for example if $k = 2$ it means that the vendor will send $2Q$ items instead Q).
α_i: The fraction of k which is dedicated for buyer number $i (i \in \{1, 2\})$.

2.1. Case 1: Single vendor-single buyer

In this section, we assume a two echelon supply chain in two modes. First, we investigate the traditional system where both parties, the buyer and vendor, decide based on their own optimal strategies and maximize their own profit. In Section 2.1.1 the traditional system is presented. Then we introduce a new system where each member has better situations than traditional system with implementing VMI on consignment scheme. In Section 2.1.2 we proposed the new system with implementing VMI on consignment.

2.1.1. Traditional system

In traditional system, also known as decentralized supply chain, both parties choose their optimal strategies without considering the situation of other members. For example, in an inventory control model, the buyers will choose EOQ in order to minimize their cost. In our proposed model, the buyer will order based on his economic order quantity. His initial order is $Q=\sqrt{\frac{2DC_0}{h}}$, and his initial cost is as follows:

$$B_{\text{noVMI}}=\frac{D}{Q}C_0+Q\frac{h}{2}=\sqrt{2DC_0h} \tag{2.1}$$

Also the initial profit of vendor is as follows (his strategy is lot-for-lot):

$$S_{\text{noVMI}}=wD-\frac{D}{Q}C_s \tag{2.2}$$

2.1.2. The system with implementing VMI

Chakaboraty *et al.* [7], presented a system without considering consignment contract and find the optimal vendor and buyer costs as below:

$$B_{\text{V}}=\frac{kQ}{2}h+\frac{D}{kQ}C_0-\frac{x_1(kQ-z_1)^2}{2kQ}, \tag{2.3}$$

$$S_{\text{V}}=wD-\frac{D}{kQ}C_s-\frac{x_1(kQ-z_1)^2}{2kQ}. \tag{2.4}$$

2.1.3. New system with implementing VMI on consignment

In VMI system, the vendor wants to maximize his profit under implementing the VMI. In turn, if the vendor carries extra goods, the buyer imposes a per unit penalty, x_1, for the extra units which are more than a predefined upper limit z. According to the treatment in previous studies, we also assume that these values are determined based on the negotiations between the parties (*e.g.* Hariga *et al.* [24]). The vendor decides to change his production and transportation policy to produce and carry kQ units instead of Q units. If $kQ \le z_1$ the vendor doesn't pay any penalty cost. On the other hand, if $kQ > z_1$, this larger batch size may lead to penalty cost which is showed in Figure 2. Thus the amount of penalty in charged according to Figure 2 is as follows:

$$p=x_1.\frac{(kQ-z_1)}{2}.\frac{(kQ-z_1)}{kQ}=\frac{x_1(kQ-z_1)^2}{2kQ} \tag{2.5}$$

With implementing VMI on consignment the buyer doesn't pay any holding or ordering cost. In order to show the changes in situation of buyer's cost we represent the new buyer's cost as following:

$$B_{\text{VMI}}=-\frac{x_1(kQ-z_1)^2}{2kQ}<0 \tag{2.6}$$

Thus, the change in buyer's costs between traditional and VMI system is:

$$\varDelta B(k)=B_{\text{VMI}}-B_{\text{noVMI}} \tag{2.7}$$

$$\varDelta B(k)=-\frac{x_1(kQ-z_1)^2}{2kQ}-\frac{D}{Q}C_0-Q\frac{h}{2} \tag{2.8}$$

The decision problem for the vendor with known inventory upper bound and penalty is to determine the k for maximizing his profit. Vendor's profit with implementing VMI on consignment is:

$$S_{\text{VMI}}=wD-\frac{D}{kQ}C_s-\frac{kQ}{2}h-\frac{D}{kQ}C_0-\frac{x_1(kQ-z_1)^2}{2kQ} \tag{2.9}$$

The changes in vendor's profit between traditional and VMI system is:

$$\varDelta S = S_{\mathrm{VMI}} - S_{\mathrm{noVMI}} \tag{2.10}$$

$$\varDelta S(k) = -\frac{D}{kQ}C_s - \frac{kQ}{2}h - \frac{D}{kQ}C_0 - \frac{x_1(kQ-z_1)^2}{2kQ} + \frac{D}{Q}C_s \tag{2.11}$$

We want to show how implementing VMI with consignment stock and penalty can be an incentive mechanism for coordination in supply chain. According to Cachon [6], a contract can implement coordination in supply chain, if the set of supply chain actions is Nash equilibrium which means no positive unidirectional deviance from the set of the supply chain action exists from both the vendor and the buyer. According to (2.6) it is obvious that the buyer's cost in VMI system is less than the traditional system which means $\varDelta B(k) \leq 0$. Thus, for coordinating the supply chain the main problem is how we can maximize the vendor's profit. We can express the problem as follows:

$$\begin{array}{l} \text{Max } \varDelta S(k) \\ \text{subject to :} \\ \begin{cases} kQ \geq z_1 \\ k \geq 1 \end{cases} \end{array} \tag{2.12}$$

With constructing the Lagrangian function we have:

$$L\left(k,\lambda,\mu\right) = -\varDelta S(k) + \lambda(z_1 - kQ) + \mu(1-k) \tag{2.13}$$

$$L\left(k,\lambda,\mu\right) = \frac{D}{kQ}C_s + \frac{kQ}{2}h + \frac{D}{kQ}C_0 + \frac{x_1(kQ-z_1)^2}{2kQ} - \frac{D}{Q}C_s + \lambda(z_1 - kQ) + \mu(1-k) \tag{2.14}$$

The Karush Kuhn Tucker (KKT) conditions, provided some regularity first-order necessary conditions which should be satisfied in a problem in order that a solution in nonlinear mathematical formulation becomes

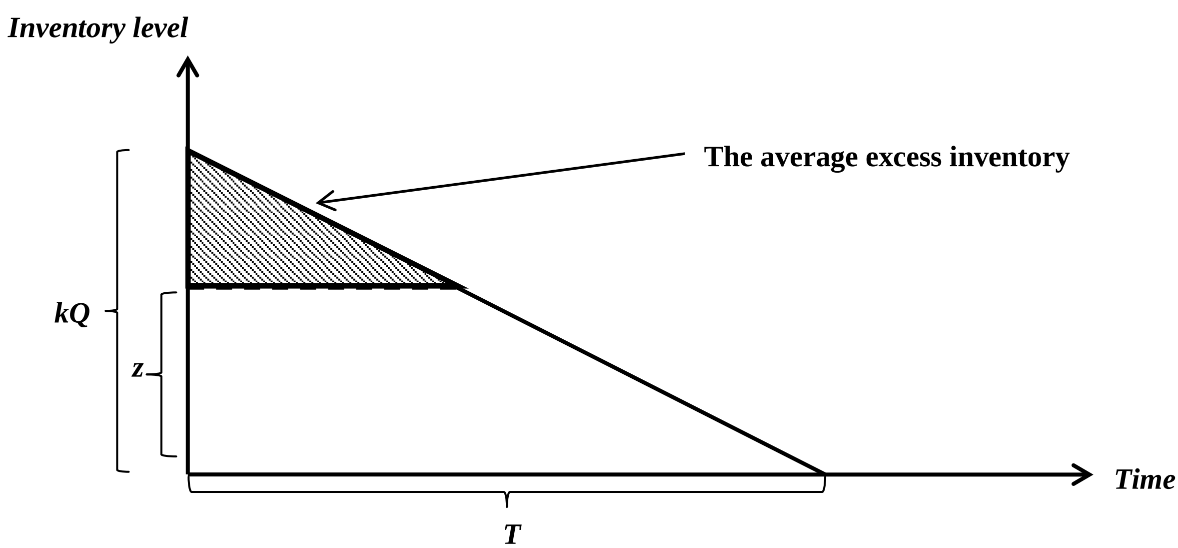

FIGURE 2. level of inventory at the buyer if $kQ > z$.

optimal [4]. The KKT conditions for equation (2.13) are as follows:

$$\frac{\partial L}{\partial k}=0\Rightarrow-\frac{D}{k^2Q}C_s+\frac{2x_1(kQ-z_1)kQ-Qx_1(kQ-z_1)^2}{2k^2Q^2}+h\frac{Q}{2}-\frac{D}{k^2Q}C_0-\lambda Q-\mu=0 \tag{2.15}$$

$$\lambda(z_1-kQ)=0 \tag{2.16}$$

$$\mu(1-k)=0 \tag{2.17}$$

$$\lambda\geq 0,\quad \mu\geq 0$$

The cases which we have $\lambda\neq 0$ or $\mu\neq 0$ are obvious solutions and their results are $kQ=z$ or $k=1$. In these situations the VMI arrangement is not very important. Thus, we will only investigate the system where $\lambda=0$ and $\mu=0$. According to (2.15) the optimal k is calculated and is as follows:

$$\bar{k}=\frac{1}{Q}\sqrt{\frac{z_1^2x_1+2D(C_s+C_0)}{x_1+h}} \tag{2.18}$$

As the function $-\Delta S(k)=\frac{D}{kQ}C_s+\frac{kQ}{2}h+\frac{D}{kQ}C_0+\frac{x_1(kQ-z)^2}{2kQ}-\frac{D}{Q}C_s$ is convex at $\bar{k}$, we can conclude that the KKT point is the global optimum point.

2.2. Case 2: Single vendor-two buyers

In this section, we assume a two echelon supply chain with single vendor and two suppliers. In this case, the vendor uses the joint replenishment cycle for carrying the items to the first and second buyer, which means $\frac{Q_1}{D_1}=\frac{Q_2}{D_2}$. In each cycle, α_1 percent of k is dedicated to the first buyer, and α_2 percent of the k is dedicated to the second buyer. Also for vendor's ordering cost C_1, we have $C_0<C_1<2\ C_0$. This condition represents that in normal situation, the vendor's ordering cost for serving both buyers in a same time is more than C_0 in serving just one buyer and less than 2 C_0 because of saving in transportation costs and packaging costs. In this case the first buyer determines the penalty equals to x_1 and the upper limit of inventory set as z_1. Also the second buyer determines penalty equals to z_2 and the upper limit of inventory set as x_2. As in the above case, we can show that for each buyer we have $\Delta B(k)=B_{\text{VMI}}-B_{\text{noVMI}}<0$. Now for coordinating the supply chain, the problem is how we can maximize the vendor's profit. We can model the problem as follows:

$$\begin{aligned}&\text{Max } \Delta S(k)=-\tfrac{D}{kQ}C_s-\tfrac{kQ}{2}h-\tfrac{D}{kQ}C_1-\tfrac{x_1(\alpha_1kQ_1-z_1)^2}{2\alpha_1kQ_1}-\tfrac{x_2(\alpha_2kQ_2-z_2)^2}{2\alpha_2kQ_2}+\tfrac{D}{Q}C_s\\&\text{subject to:}\\&\begin{cases}\alpha_1kQ_1\geq z_1\\ \alpha_2kQ_2\geq z_2\\ k\geq 1\\ \alpha_1+\alpha_2=1\\ Q_1+Q_2=Q\end{cases}\end{aligned} \tag{2.19}$$

Now, we construct the Lagrangian function:

$$L\left(k,\lambda,\varepsilon,\mu\right)=-\Delta S(k)+\lambda(z_1-\alpha_1kQ_1)+\varepsilon(z_2-\alpha_2kQ_2)+\mu(1-k) \tag{2.20}$$

After some manipulation we have:

$$\begin{aligned}L\left(k,\lambda,\varepsilon,\mu\right)=&\frac{D}{kQ}C_s+\frac{kQ}{2}h+\frac{D}{kQ}C_1+\frac{x_1(\alpha_1kQ_1-z_1)^2}{2\alpha_1kQ_1}+\frac{x_2(\alpha_2kQ_2-z_2)^2}{2\alpha_2kQ_2}\\&-\frac{D}{Q}C_s+\lambda(z_1-\alpha_1kQ_1)+\varepsilon(z_2-\alpha_2kQ_2)+\mu(1-k)\end{aligned} \tag{2.21}$$

The KKT conditions for (2.21) are described in (2.22)–(2.25) as follows:

$$\frac{\partial L}{\partial k} = 0 \Rightarrow -\frac{D}{k^2 Q}C_s + \frac{2\alpha_1 k x_1 Q_1(\alpha_1 k Q_1 - z_1) - x_1(\alpha_1 k Q_1 - z_1)^2}{2\alpha_1 k^2 Q_1} + \frac{2\alpha_2 k x_2 Q_2(\alpha_1 k Q - z_1) - x_2(\alpha_2 k Q_2 - z_2)^2}{2\alpha_2 k^2 Q_2}$$
$$+h\frac{Q}{2} - \frac{D}{k^2 Q}C_1 - \lambda Q - \mu = 0 \quad (2.22)$$
$$\lambda(z_1 - \alpha k Q_1) = 0 \quad (2.23)$$
$$\varepsilon(u_1 - \beta k Q_2) = 0 \quad (2.24)$$
$$\mu(1-k) = 0 \quad (2.25)$$
$$\lambda \geq 0, \quad \varepsilon \geq 0, \quad \mu \geq 0$$

The cases which we have $\lambda \neq 0$, $\mu \neq 0$ or $\varepsilon \neq 0$ are trivial ones and their results are $\alpha_1 k Q_1 = z$, $\alpha_2 k Q_2 = u$ and $k = 1$. In these situations the VMI arrangement is not very important. Thus, we will only investigate the system where $\lambda = 0$, $\mu = 0$ and $\varepsilon = 0$. According to (2.22) the optimal k is calculated and is as follows:

$$\bar{k} = \sqrt{\frac{2\alpha_1\alpha_2 Q_1 Q_2 D(C_s + C_1) + \alpha_1 x_1 Q z_1^2 + \alpha_2 x_2 Q z_2^2}{Q(\alpha_1^2 Q_1^2 \alpha_2 x_1 + \alpha_2^2 Q_2^2 \alpha_1 x_2 + \alpha_1\alpha_2 Q_1 Q_2 h Q)}} \quad (2.26)$$

As the function $-\Delta S(k)$ is convex at $\bar{k}$, we can conclude that the KKT point is the global optimum point.

3. Numerical examples and sensitivity analysis

To demonstrate the performance of the model given above and study the effects of model parameters on the optimal k or $\bar{k}$ we will exert a numerical example and then investigate the impact of shifting in parameters on the optimal solutions.

3.1. Single vendor-single buyer

For the first case let $D = 1000$, $C_0 = 10$, $C_S = 300$, $h = 2$, $x_1 = 3$ and $z_1 = 150$. Using 2.18 we have $\bar{k} = 3.708$ and from equations (2.7) and (2.10) we get respectively $\Delta B = -397.291$ and $\Delta S = 1595.950$. In the following paragraphs, we perform a sensitivity analysis with respect to vendor setup cost, stock upper limit and penalty.

3.1.1. Changes in C_s

Table 2 shows the effect of changing in vendors' set up cost C_s on the batch size and the condition of each party. From this table, the following conclusions can be drawn:

- An increase to the vendor's set up cost, C_s, results the vendor to send more items to the buyer, since it becomes more beneficial for the vendor to increase the batch size (k increases).
- An increase to the vendor's set up cost, C_s results the vendor to pay a larger amount of penalty to the buyer (p increases).
- It's obvious that both parties have better conditions than traditional system. The buyers' cost in this scheme is less than the traditional system ($\Delta B < 0$) and also the vendors' profit increases ($\Delta S \geq 0$).

3.1.2. Changes in z_1 and x_1

The impact of changes in upper limit of the inventory, z_1, is shown in Table 3, while Table 4 shows the effect of changes in penalty charged per unit, x_1, on the batch size. We can get the following conclusions from these two tables.

- As upper limit of the inventory is increased by the buyer, the vendor can distribute batches of larger size. Under this situation, the vendor increases the batch size without incurring any excessive cost. Additional, the penalty behooved on the vendor also decreases. Also p is moderately sensitive to the changes in stock

TABLE 2. The effect of changing C_s.

C_s	S_{noVMI}	B_{noVMI}	k	p	ΔS	ΔB	$\%\Delta S$	$\%\Delta B$
300	3000	200	3.708	197.232	1595.95	−397.232	53.198	−198.616
400	4000	200	4.213	262.069	2343.462	−462.069	58.586	−231.034
500	5000	200	4.663	321.921	3118.155	−521.921	62.363	−260.960
600	6000	200	5.074	377.676	3912.777	−577.676	65.213	−288.838
700	7000	200	5.454	430.03	4722.822	−630.030	67.469	−315.015
800	8000	200	5.809	479.516	5545.262	−679.516	69.315	−339.758
900	9000	200	6.144	526.546	6377.948	−726.546	70.866	−363.273

TABLE 3. The effect of changing z_1.

z_1	S_{noVMI}	B_{noVMI}	k	p	ΔS	ΔB	$\%\Delta S$	$\%\Delta B$
100	3000	200	3.605	282.435	1497.224	−482.435	49.907	−241.217
110	3000	200	3.623	263.544	1518.509	−463.544	50.617	−231.772
120	3000	200	3.642	245.605	1539.011	−445.605	51.3	−222.802
130	3000	200	3.662	228.591	1558.743	−428.591	51.958	−214.295
140	3000	200	3.684	212.476	1577.718	−412.476	52.59	−206.238
150	3000	200	3.708	197.232	1595.95	−397.232	53.198	−198.616
160	3000	200	3.733	182.828	1613.452	−382.828	53.781	−191.414

TABLE 4. The effect of changing x_1.

x_1	S_{noVMI}	B_{noVMI}	k	p	ΔS	ΔB	$\%\Delta S$	$\%\Delta B$
3	3000	200	3.708	197.232	1595.95	−397.232	53.198	−198.616
3.5	3000	200	3.564	209.229	1564.611	−409.229	52.153	−204.614
4	3000	200	3.44	218.807	1536.023	−418.807	51.2	−209.403
4.5	3000	200	3.331	226.472	1509.792	−426.472	50.326	−213.236
5	3000	200	3.235	232.601	1485.601	−432.601	49.52	−216.300
5.5	3000	200	3.149	237.487	1436.194	−437.481	48.773	−218.740
6	3000	200	3.072	241.338	1442.359	−441.338	48.078	−220.669

upper limit. With increasing in z_1, the amount of penalty, which the vendor should pay to the buyer decreases.

- On the hand, as the penalty charged per unit increases by the buyer, the vendor prefers to decrease the size of batches (k is reduced) in order to counter the increased penalty. Also p is moderately sensitive to the changes in x_1. With increasing in x_1, the amount of penalty which the vendor should pay to the buyer increases.

3.2. Single vendor-two buyers

In this case let the first buyers' demand $D_1 = 800$, the second buyers' demand $D_2 = 1200$, $\alpha_1 = 0.4$, $\alpha_2 = 0.6$, $C_1 = 16$, $C_S = 300$, $h = 2$, $x_2 = 4$, $z_1 = 150$ and $z_2 = 170$. Using (2.26) we get $\bar{k} = 4.404$ and from equation (2.19) we get $\Delta S = 1643.69$. In the following paragraph, we perform a sensitivity analysis with respect to vendor setup cost.

TABLE 5. The effect of changing C_s.

C_s	S_{noVMI}	k	p_1	p_2	p	ΔS	$\%\Delta S$
300	6000	4.404	6.34	93.324	99.664	1643.69	27.395
400	8000	5.049	0.216	151.053	151.269	2468.88	30.861
500	10000	5.62	1.308	207.16	208.468	3315.32	33.153
600	12000	6.139	6.1	260.961	267.061	4179.67	34.831
700	14000	6.617	12.963	312.419	325.382	5059.05	36.136
800	16000	7.063	21.044	361.683	382.727	5951.14	37.195
900	18000	7.482	29.859	408.942	438.801	6854.14	38.079

3.2.1. Changes in C_s

Table 5 shows the effect of changing in vendors' set up cost C_s on the batch size and the condition of each party. From this table, the following conclusions can be drawn:

- The k is moderately sensitive to the shifts in C_s. When C_s increases, the vendor prefers to send a larger batch size to the buyer.
- The p is also moderately sensitive to the changes in C_s. With increasing in C_s, the total amount of penalty, which the vendor should pay to the buyers, increases.

4. Managerial insights

Based on the performed sensitivity analysis, when the setup cost of the vendor increases, the vendor must make the batch size larger. However, vendor must notice that with an increasing in batch size, the amount of penalty that he should pay to the buyer increases. For example, the dairy food company must analyze its setup cost as well as the amount of penalty is incurred, and according to that to decide about the optimal batch size.

In this mechanism, the buyers should try decrease the upper limit of inventory level in order to gain more amount of money. From vendors' perspective, they should consider to increase the upper limit of inventory in order to pay less money as penalty. In dairy food company example, the local stores try to select the lowest level of upper limit of inventory. Vice versa, the dairy company encourages the supermarkets to choose higher level of upper limit of inventory.

Also, with increasing in penalty charged per unit, the amount of penalty increases but the batch size decreases. Meaning that the higher penalty charged per unit is more profitable for the buyers. Vice versa, the vendors prefer lower penalty charged per unit in order to increase the batch size. In the case of dairy food company example, the firm should encourage the local stores to select the lowest amount of penalty.

For example in the case of local dairy foods companies which mentioned in introduction, these companies should choose supermarkets and local markets with minimum upper limit of inventory and minimum penalty charged per unit in order to maximize their profits.

Also there are several other industries which this strategy is very useful. In apparel industry, many startup fashion designers in order to increase their brand awareness, build relationships with customers and gain larger profit margins, want to make contracts with famous boutiques or department stores. Many of vendors want to put their merchandise or products in these stores, just to make people know who they are. Furthermore, being in these stores ensure their profit due to the trust of people to these stores. On the other hand, these stores cover lots of brands and different products which confront them with lack of space in warehouse or in the floor. However, they are strong enough to impose penalty to their vendors in order to control their opportunistic behaviors.

In each country, the shoes market is very competitive. Some new shoes factories make VMI on consignment contracts with penalty with famous department stores. They believe that this strategy ensure their sale, increase their brand awareness and reduce the risk of being old fashioned in the end of each season. Although the amount of penalty cost is very high but both sides admit that VMI on consignment with penalty is a good motivation to control the batch size of each shipment and ensure their profits. On the other hand the fear of buyer about excessive inventory is reduced.

We suggest this agreement for new startup businesses in competitive markets. Also it can help powerful retailers such as department stores to control the suppliers' shipments. Especially where there are some space constraints in warehouse or store. Managers can investigate and analyze their own contract parameters such as unit set-up cost, unit penalty charged by the buyer and the specific upper limit and their impacts on decision variables such as the coefficient for increasing the batch size and changes in profit with the suggested structure and approach in this paper.

5. Conclusions

Many researchers compared the traditional inventory model with vendor managed inventory system and the comparison showed that in most cases the VMI is better than the traditional system. VMI on consignment is an effective strategy in many supply chains which is implemented and deployed in practice. Implementing of VMI on consignment strategy requires an effective approach for resolving the problems among supply chain partners.

In this paper, we investigated a vendor managed inventory with consignment model in two cases and highlighted the role of penalty in coordinating the supply chain. We first considered one vendor and one buyer and find the optimal batch size. We showed that, after implementing this coordination mechanism, both parties have better operation conditions than their traditional systems. After that, we considered one vendor and two buyers. Same as case 1, we showed that the parties are satisfied with this arrangement and have better operation conditions than before. Also in this paper, we investigated how an upper inventory limit can act as a key factor for implementing the VMI successfully. We show the efficiency of the model with some numerical examples and a sensitivity analysis is done to illustrate the parameters' changes impact on decision variables. Then we mentioned some managerial insights based on the sensitivity analysis and the dairy food company example.

There is still some work to be done in the future. For example, researchers can focus on pre-requisites, inhibitors and drivers in an integrated vendor managed inventory system and also incorporating incentives like advance payment and quantity discount mechanisms with VMI. Also investigating the vendor managed inventory scheme in a multi-echelon supply chain can be another suggestion for future directions.

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