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A NOVEL APPROACH TO STOCHASTIC INPUT-OUTPUT MODELING

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Abstract. An approach to input-output modeling is proposed in which the inner consumption and the final demand are random. The main aspects of its novelty are: (a) the economy is allowed to be nonproductive with a certain probability $\varkappa \in [0, 1)$; (b) the economy can be open, which means that import of the corresponding commodities is included in the model. In this approach, the production-and-import plan is set to be feasible if the probability of not satisfying the final demand does not exceed a certain value $\alpha \in (0, 1)$. Then the problem of finding optimal plans consists in minimizing the production and import costs on the set of feasible plans. The solvability of this problem and properties of the solutions are studied and a concrete example of the stochastic input-output model is analyzed.

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1. The Approach

Input-output analysis (combined with other mathematical and statistical tools) offers a consistent framework for modeling inter-sectorial relationships in an economic system. It can also be employed to analyze relationships between the economy, the energy sector, ecology or society, see [16, 17, 20], and even to elaborating new technologies [7]. Traditional input-output models [14, 15, 22] describe an economy – a production system consisting of $N \geq 2$ sectors, each of which produces a single homogeneous commodity. All the commodities are measured in the same unit, and a part of them is used inside the economy for production purposes. That is, a part of the output for one sector can be used as an input of the other ones. The related problem consists in finding a *production plan* $x_1, \dots, x_N \geq 0$ such that the following conditions be satisfied

$$x_i - y_i \geq d_i, \qquad i = 1, \dots, N, \tag{1.1}$$

where y_i is the inner consumption of ith commodity, whereas $d_i \geq 0$ is the final (outer) demand of this commodity which has to be satisfied. In the simplest input-output model introduced in [14] by Wassily Leontief (awarded with Nobel Prize in Economics in 1973), each y_i is a linear function of $x_1, \dots, x_N$, that is,

$$y_i = \sum_{j=1}^{N} c_{ij} x_j, \qquad i = 1, \dots, N, \tag{1.2}$$

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where $c_{ij} \geq 0$ is the amount of ith commodity needed (consumed) for producing one unit of jth commodity. By considering column vectors X, Y and D with entries x_i, y_i and d_i, respectively, (1.1) and (1.2) can be combined into the following condition

$$X - Y = X - CX = (I - C)X \geq D, \tag{1.3}$$

where $C = (c_{ij})_{N\times N}$ is the inner consumption matrix and I is the $N \times N$ identity matrix. The inequality in (1.3) is supposed to hold entry-by-entry. If the matrix $I - C$ is invertible, the inequality in (1.3) can be solved by setting $X = (I - C)^{-1}D$. However, the solution makes sense only if all the entries of the matrix

$$B = (b_{ij})_{N\times N} = (I - C)^{-1} \tag{1.4}$$

are nonnegative. In this case, b_{ij} is the amount of commodity i which one ought to produce in order to satisfy the demand of one unit of commodity j, B is called *Leontiev's matrix* and the economy itself is considered as *productive*. In this work, however, we use a more formal definition consistent with this interpretation.

Definition 1.1. The economy is called productive if the spectral radius of C satisfies $r(C) < 1$.

In this case, $I - C$ is called a *nonsingular M-matrix*. For more on the theory of such matrices see Chapter 9 of [1]. Since $r(C) < 1$, the series in the right-hand side of the expansion

$$B = \sum_{n=0}^{\infty} C^n = I + C + C^2 + \cdots$$

converges, and thus the polynomials $B_n := I + C + \cdots + C^n$ can be used to approximate Leontiev's matrix.

In real world applications, however, the entries of the inner consumption matrix C are subject to various random effects. In order to take this fact into account, more realistic input-output models should consider C as a random matrix. This was understood quite a long time ago, see a review and historical remarks in [9, 21] and also in ([18], Sect. 2). It turns out that the most mathematically advanced works in this direction are concentrated on deriving information on the probability distribution of the entries of the Leontief matrix given the distribution of the entries of C is known, see [10, 23, 24]. A more modest task in this context is to estimate the coefficients b_{ij}, see [11–13, 18, 19] and ([22], Chap. 14). Of course, the very existence of the Leontief matrix is possible only if the economy is productive with probability one, which is assumed therein – more or less explicitly. For instance, in [10] this is done by employing Beta distributions on the interval $(0, 1)$. However, the almost sure productivity of the economy cannot be expected in general. In view of this, an actual and important task is to elaborate mathematical concepts and tools of the input-output modeling which covers the case of nonproductive economies. In this article, we pursue this task on both conceptual and technical levels. Namely, we propose an approach to stochastic input-output modeling, the novelty of which consists in the following:

(a) the economy is allowed to be nonproductive with a certain probability $\varkappa \in [0, 1)$;
(b) a (feasible) production plan is defined as such that the outer (possibly random) demand be satisfied with probability greater than or equal to $1 - \alpha$ for a prescribed $\alpha \in (0, 1)$;
(c) the economy is allowed to be open, which means that the corresponding commodities can also be imported;
(d) optimal plans are chosen among feasible production-and-import plans under the condition of minimizing objective functions given in the model.

In (a) and (b) we assume that the components of the inner consumption matrix C and the outer demands d_i are random variables defined on a common standard probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, see, *e.g.*, [6]. According to (a): the economy is productive with probability $1 - \varkappa > 0$, and hence for some nonnegative $x_1, \ldots, x_N$, the event

$$\mathcal{A}(x_1, \ldots, x_N): \qquad x_i - \sum_{j=1}^{N} c_{ij} x_j \geq d_i, \qquad i = 1, \ldots, N \tag{1.5}$$

can occur with a prescribed positive probability. In (b) we propose to define the collection of feasible production plans as the set of all nonnegative $x_1, \ldots, x_N$ satisfying the condition

$$\mathbb{P}\left(\mathcal{A}(x_1, \ldots, x_N)\right) \geq 1 - \alpha, \tag{1.6}$$

holding for some fixed $\alpha \in (0, 1)$. The relationship between $\varkappa$ and α are described in Theorem 2.5. In (c) we propose to include import of the commodities. This may be helpful in satisfying the demand as the amount z_i of the imported commodity of ith type should be added to the left-hand side of the balance condition in (1.5) in this case. Of course, now the feasibility relates to both production and import plans, see Definition 2.2. Then (d) means that optimal production-and-import plans are chosen to minimize the objective function $\varphi(x_1, \ldots, x_N) + \psi(z_1, \ldots, z_N)$ on the set of feasible plans. Here φ and ψ are production and import cost functions, respectively. An important advantage of the proposed approach is that here one deals with the matrix C only, and hence avoids a complex and tiresome procedure of getting information on the distribution of the Leontief matrix B. Moreover, our approach admits the following extensions:

(i) Along with random c_{ij} and d_i one can also allow the cost functions be random, which might lead to the corresponding modification of the optimization problem arising in (d).
(ii) One can include time (discrete or continuous) by considering production and import plans as Markov processes.

Noteworthy, considering probabilities as in (1.6) was suggested already in [8]. However, the discussion therein was restricted to the issue of constructing feasible production plans without any suggestion on how to select optimal ones. In Section 2, we propose and analyze an open stochastic input-output model which realizes the approach outlined above. In Section 3, we illustrate our approach by analyzing in detail a seemingly simple model of this kind with $N = 2$, $c_{11} = c_{22} = 0$ and random (independent) c_{12} and c_{21}. In spite of such simplifying assumptions – which allows us to make all the computations explicitly – this model turns out to be rich enough to reveal the main aspects of the theory.

2. The Open Stochastic Leontief Model

2.1. The model

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a standard probability space. We consider an economy with $N \geq 2$ sectors, as described above, in which the entries of both matrices C and D in (1.3) are random variables relative to the mentioned probability space. We assume that they have finite expectations and hence are almost surely bounded. We also assume that the commodities produced by the economy are available at an external market, and thus the economy is a part of an *open* economic system. Let $z_i \geq 0$, $i = 1, \ldots, N$, denote the amount of the imported commodity of type i. Then the balance condition, cf. (1.3), takes the form

$$X - C(\omega)X + Z \geq D(\omega), \qquad X \geq 0, \quad Z \geq 0, \tag{2.1}$$

where Z is the corresponding column vector, and the inequalities are supposed to hold entry-by-entry. Let $\mathcal{K} \subset \mathbb{R}^N \times \mathbb{R}^N$ denote the positive cone, *i.e.*, the set of all pairs (X, Z) with $X, Z \geq 0$. For a given $\omega \in \Omega$, the set of all those $(X, Z) \in \mathcal{K}$ for which (2.1) holds is denoted by $\mathcal{M}_\omega$. Clearly, $\mathcal{M}_\omega$ is closed and convex, however, may be empty. The convexity means that $(\theta X + (1-\theta)X', \theta Z + (1-\theta)Z') \in \mathcal{M}_\omega$ holding for each $(X, Z) \in \mathcal{M}_\omega$, $(X', Z') \in \mathcal{M}_\omega$ and $\theta \in (0, 1)$, that can be checked directly by (2.1). For $\mathcal{A} \in \mathfrak{F}$, we define

$$\mathcal{M}(\mathcal{A}) = \bigcap_{\omega \in \mathcal{A}} \mathcal{M}_\omega. \tag{2.2}$$

As the intersection of closed convex sets, $\mathcal{M}(\mathcal{A})$ is also closed and convex. For $\alpha \in (0, 1)$, we then put

$$\widetilde{\mathcal{M}}^\alpha = \bigcup_{\mathcal{A}: \mathbb{P}(\mathcal{A}) \geq 1 - \alpha} \mathcal{M}(\mathcal{A}). \tag{2.3}$$

The set $\widetilde{\mathcal{M}}^\alpha$ can be given the following interpretation. For $(X, Z) \in \mathcal{K}$, define

$$\mathcal{A}(X, Z) = \{\omega : (2.1) \text{ holds}\}. \tag{2.4}$$

It is clearly an element of $\mathfrak{F}$. Then $\widetilde{\mathcal{M}}^\alpha$ consists of all those pairs of production and import vectors (X, Z) for each of which

$$\mathbb{P}(\mathcal{A}(X, Z)) \geq 1 - \alpha. \tag{2.5}$$

In Theorem 2.4 below, we show that due to the openness of the model the sets $\widetilde{\mathcal{M}}^\alpha$ are nonempty for each $\alpha \in (0, 1)$. *A priori*, it may happen that the economy is such that $\mathcal{A}(X, 0) = \varnothing$ for all $X \geq 0$. This merely means that it is almost surely nonproductive, and the only possibility to satisfy the demand is to import the corresponding amounts of the commodities. In Theorem 2.5 and Section 3, we analyze such situations in detail. In particular, we find the relationship between α and the probability $\varkappa$ for the economy to be non-productive.

In contrast to $\mathcal{M}(\mathcal{A})$ defined in (2.2), the set $\widetilde{\mathcal{M}}^\alpha$ need not be convex. This fact can be an obstacle if one wants to employ methods of convex analysis in the study the corresponding optimization problem. If this is the case, as the set of feasible plans we propose to use a convex subset of $\widetilde{\mathcal{M}}^\alpha$ defined as follows. By (2.5) we have that $\widetilde{\mathcal{M}}^{\alpha'} \subset \widetilde{\mathcal{M}}^\alpha$ whenever $\alpha' < \alpha$. Indeed, if $(X, Z) \in \widetilde{\mathcal{M}}^{\alpha'}$, then $\mathbb{P}(\mathcal{A}(X, Z)) \geq 1 - \alpha' > 1 - \alpha$. Let $\mathrm{Conv}\mathcal{M}$ stand for the convex hull of a given $\mathcal{M} \subset \mathcal{K}$. That is, $\mathrm{Conv}\mathcal{M}$ is the intersection of all convex $\mathcal{M}' \subset \mathcal{K}$ such that $\mathcal{M} \subset \mathcal{M}'$.

Proposition 2.1. *For each $\alpha \in (0, 1)$, it follows that*

$$\mathcal{M}^\alpha := \mathrm{Conv}\widetilde{\mathcal{M}}^{\alpha/2} \subset \widetilde{\mathcal{M}}^\alpha.$$

Proof. By the very definition of the convex hull we have that each $(X, Z) \in \mathcal{M}^\alpha$ is a convex combination of some $(X', Z'), (X'', Z'') \in \widetilde{\mathcal{M}}^{\alpha/2}$, which means that the following holds

$$X = \theta X' + (1 - \theta)X'', \quad Z = \theta Z' + (1 - \theta)Z''$$

for some $\theta \in [0, 1]$. By (2.1) and (2.5) this means that

$$\mathcal{A}(X, Z) \supseteq \mathcal{A}(X', Z') \cap \mathcal{A}(X'', Z''). \tag{2.6}$$

Indeed, if ω belongs to the right-hand side of (2.6), then the inequality (2.1) holds for both (X', Z') and (X'', Z''), *cf.* (2.4). To employ this fact we use (2.1) twice: first multiply by θ both its sides written with (X', Z'), then multiply by $1 - \theta$ both sides written with (X'', Z''). Thereafter, add them side-by-side. This yields that (2.1) holds also for (X, Z) and hence $\omega \in \mathcal{A}(X, Z)$. By (2.6) and standard properties of the probability we then have

$$\begin{aligned} \mathbb{P}(\mathcal{A}(X, Z)) &\geq \mathbb{P}(\mathcal{A}(X', Z') \cap \mathcal{A}(X'', Z'')) \\ &\geq \mathbb{P}(\mathcal{A}(X', Z')) + \mathbb{P}(\mathcal{A}(X'', Z'')) - 1 \\ &\geq \left(1 - \frac{\alpha}{2}\right) + \left(1 - \frac{\alpha}{2}\right) - 1 = 1 - \alpha. \end{aligned}$$

Thus, $(X, Z) \in \widetilde{\mathcal{M}}^\alpha$, which completes the proof. □

Definition 2.2. For a given $\alpha \in (0, 1)$, $\widetilde{\mathcal{M}}^\alpha$ (resp. $\mathcal{M}^\alpha$) is called the set (resp. convex set) of *feasible* plans. The realization of a feasible plan allows one to satisfy the outer demand with probability not less than $1 - \alpha$.

For the production vector X (resp. the import vector Z), we denote the cost of its realization by $\varphi(X)$ (resp. $\psi(Z)$).

Definition 2.3. A feasible plan (X^*, Z^*) is called optimal if, for all other feasible plans (X, Z), the following condition holds

$$\varphi(X^*) + \psi(Z^*) \leq \varphi(X) + \psi(Z). \tag{2.7}$$

The ultimate goal of the analysis of the model which we thus propose is to find optimal plans. Its realization consists in constructing the set of feasible plans $\widetilde{\mathcal{M}}^\alpha$, and then solving the optimization problem (2.7) on this set. If both φ and ψ are convex functions, one can employ the convex set of feasible plans $\mathcal{M}^\alpha$ (instead of $\widetilde{\mathcal{M}}^\alpha$) and then apply powerful tools of convex optimization (see, *e.g.*, [5]). Note, however, that an optimal plan $(X^*, Z^*) \in \mathcal{M}^\alpha$ found thereby need not be optimal on the whole set $\widetilde{\mathcal{M}}^\alpha$ if $\mathcal{M}^\alpha$ is a proper subset of the latter, see Proposition 2.1. Clearly, this realization crucially depends on the concrete model and may be quite complex.

2.2. The properties

Here we analyze some aspects of the approach presented above and discuss possible ways to find optimal plans.

2.2.1. Estimating the probability of productivity

Let $r(C)$ denote the spectral radius of the inner consumption matrix C. It is clearly a random variable taking positive values with probability one. Since the entries of C are almost surely nonnegative, $r(C)$ is an eigenvalue of C such that any of its other eigenvalues, $\lambda(C)$, satisfies $|\lambda(C)| \leq r(C)$, see Theorem 1.1 of [1]. Thus, $r(C)$ is a root of the characteristic polynomial of C. The coefficients of this polynomial are random variables expressed in terms of the entries of C. In general, finding probability distributions of $r(C)$ as a root of a random polynomial is a hard problem, see [4]. Instead, we propose a more realistic method of estimating $r(C)$ based on the following arguments. For a fixed $\varepsilon \in (0, 1)$, let us consider the event

$$\mathcal{A}_\varepsilon = \{\omega : r(C(\omega)) \leq 1 - \varepsilon\}. \tag{2.8}$$

Let $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0, 1)$ be a decreasing sequence such that $\varepsilon_n \to 0$ as $n \to +\infty$. Then $\mathcal{A}_{\varepsilon_n} \subset \mathcal{A}_{\varepsilon_{n+1}}$ for all n, and the event

$$\mathcal{A} := \bigcup_{n\geq 1} \mathcal{A}_{\varepsilon_n} = \{\omega : r(C(\omega)) < 1\} \tag{2.9}$$

is merely the productivity of the economy, see Definition 1.1. Clearly, it does not depend on the particular choice of the sequence $\{\varepsilon_n\}$. The probability of $\mathcal{A}$ can be calculated if the probability distribution of $r(C)$ is known. The essence of the proposed method is that $\mathbb{P}(\mathcal{A})$ can be estimated from below as follows. Consider the random variables:

$$s_i = \sum_{j=1}^{N} c_{ij}, \quad s^* = \max_i s_i, \tag{2.10}$$

and the event

$$\mathcal{A}^*_\varepsilon = \{\omega : s^*(\omega) \leq 1 - \varepsilon\}. \tag{2.11}$$

Then by ([1], Thm. 2.35, p. 37) one has that

$$\mathbb{P}(\mathcal{A}^*_\varepsilon) \leq \mathbb{P}(\mathcal{A}_\varepsilon). \tag{2.12}$$

Since $\{\mathcal{A}_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is an ascending sequence, we have that $\mathbb{P}(\mathcal{A}_{\varepsilon_{n+1}}) \geq \mathbb{P}(\mathcal{A}_{\varepsilon_n})$, and

$$\mathbb{P}(\mathcal{A}) = \lim_{n\to+\infty} \mathbb{P}(\mathcal{A}_{\varepsilon_n}). \tag{2.13}$$

Also by (2.9) and (2.12) it follows that

$$1 - \varkappa := \mathbb{P}(\mathcal{A}) \geq \sup_{\varepsilon\in(0,1)} \mathbb{P}(\mathcal{A}^*_\varepsilon). \tag{2.14}$$

The random variables defined in (2.10) are order statistics; their probability distributions, and hence the probabilities of the events defined in (2.11), can be found in a standard way if the distribution of the entries c_{ij} is known (see *e.g.*, [2]). In Section 3, we return to discussing this issue.

2.2.2. Feasible plans and productivity

Our first result is a statement on the existence of feasible plans, see Definition 2.2.

Theorem 2.4. *For each $\alpha \in (0,1)$, the sets $\widetilde{\mathcal{M}}^{\alpha}$ and $\mathcal{M}^{\alpha}$ are nonempty.*

Proof. Recall that we have assumed the almost-sure boundedness of the entries of C and D. Let $Z \geq 0$ be such that the event $\mathcal{G}_Z := \{\omega : D(\omega) \leq Z\}$ satisfies $\mathbb{P}(\mathcal{G}_Z) \geq 1-\alpha$, which is possible for each $\alpha > 0$. Then $(0, Z) \in \widetilde{\mathcal{M}}^{\alpha}$. Next, take any $X \geq 0$ and then $Z \geq 0$ such that the event $\mathcal{H}_{XZ} := \{\omega : C(\omega)X + D(\omega) \leq X + Z\}$ satisfies $\mathbb{P}(\mathcal{H}_{XZ}) \geq 1-\alpha$. Then $(X, Z) \in \widetilde{\mathcal{M}}^{\alpha}$. This also yields that $\widetilde{\mathcal{M}}^{\alpha}$ is unbounded. Since $\widetilde{\mathcal{M}}^{\alpha/2}$ is nonempty, by $\widetilde{\mathcal{M}}^{\alpha/2} \subset \mathrm{Conv}\widetilde{\mathcal{M}}^{\alpha/2} = \mathcal{M}^{\alpha}$ we have that $\mathcal{M}^{\alpha} \neq \varnothing$, holding for all $\alpha \in (0,1)$. □

From this proof one can see the role of the import of commodities for securing the existence of feasible plans. Now let us analyze the possibility of having feasible plans with the zero import, *i.e.*, such that $Z = 0$. Recall that $\mathcal{A}$ is defined in (2.9).

Theorem 2.5. *Let the economy be productive with probability $\mathbb{P}(\mathcal{A}) = 1 - \varkappa$ for some $\varkappa \in (0,1)$, cf. (2.14). Then, for each $\delta > 0$, the set $\widetilde{\mathcal{M}}^{\varkappa+\delta}$ of feasible plans contains $(X, 0)$ with some $X \geq 0$.*

Proof. Take any decreasing sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon \to 0$. Then for a given $n \in \mathbb{N}$ and an arbitrary $\omega \in \mathcal{A}_{\varepsilon_n}$, there exists the Leontief matrix $B(\omega) = (I - C(\omega))^{-1}$. Its entries are nonnegative and, according to Corollary 2.9.4 of [3], there exists a norm on $\mathbb{R}^N$ such that the operator norm of $B(\omega)$ is equal to its spectral radius $r(B(\omega)) = [1 - r(C(\omega))]^{-1} \leq \varepsilon_n^{-1}$ (see (2.8)). According to our assumptions, the norm of $D(\omega)$ is almost-surely bounded. Hence, for an arbitrary $\delta > 0$ one can take $\eta > 0$ such that the event $\mathcal{C}_\eta = \{\omega : \|D(\omega)\| \leq \eta\}$ satisfies $\mathbb{P}(\mathcal{C}_\eta) = 1 - \delta/2$. By the additivity of $\mathbb{P}$ we then have

$$\begin{aligned}\mathbb{P}(\mathcal{A}_{\varepsilon_n} \cap \mathcal{C}_\eta) &= \mathbb{P}(\mathcal{A}_{\varepsilon_n}) + \mathbb{P}(\mathcal{C}_\eta) - \mathbb{P}(\mathcal{A}_{\varepsilon_n} \cup \mathcal{C}_\eta) \\ &\geq \mathbb{P}(\mathcal{A}_{\varepsilon_n}) + (1 - \delta/2) - 1 = \mathbb{P}(\mathcal{A}_{\varepsilon_n}) - \delta/2.\end{aligned}$$

For each $\omega \in \mathcal{A}_{\varepsilon_n} \cap \mathcal{C}_\eta$, we have that

$$\|B(\omega)D(\omega)\| \leq \|B(\omega)\| \cdot \|D(\omega)\| \leq \frac{\eta}{\varepsilon_n}. \tag{2.15}$$

Now we take $X \geq 0$ such that $X \geq W$ for all $W \geq 0$ satisfying $\|W\| \leq \eta/\varepsilon_n$. In view of (2.15), for each $\omega \in \mathcal{A}_{\varepsilon_n} \cap \mathcal{C}_\eta$ this vector X satisfies

$$X \geq B(\omega)D(\omega),$$

and hence $(X, 0) \in \widetilde{\mathcal{M}}^{\alpha_n}$ with $\alpha_n = 1 - \mathbb{P}(\mathcal{A}_{\varepsilon_n}) + \delta/2$. By (2.13) we have that $\mathbb{P}(\mathcal{A}_{\varepsilon_n}) \to 1 - \varkappa - 0$ as $n \to +\infty$. Thus, there exists n_δ such that $\alpha_n \leq 1 - \varkappa + \delta$ for all $n > n_\delta$. Hence, $\widetilde{\mathcal{M}}^{\alpha_n} \subset \widetilde{\mathcal{M}}^{\varkappa+\delta}$, that completes the proof. □

3. The Example

3.1. The setup

For the sake of simplicity, we take $N = 2$ and assume that only c_{12} and c_{21} are random, whereas d_1 and d_2 are deterministic. Moreover, we set $c_{11} = c_{22} = 0$. Next, we suppose that c_{12} and c_{21} are independent and uniformly distributed on the intervals $[0, a]$ and $[0, b]$, respectively. The latter fact yields

$$\mathbb{P}(c_{12} \leq \mu) = \frac{\mu}{a}, \qquad \mathbb{P}(c_{21} \leq \nu) = \frac{\nu}{b}, \tag{3.1}$$

holding for $\mu \in [0, a]$ and $\nu \in [0, b]$. Here a and b are positive parameters. Due to this choice of the model, most of the calculations can be performed explicitly. However, as follows from the analysis below, the theory of this seemingly simple model is not so simple, and the model itself is rich enough to be able to illustrate the main aspects of our approach. Considering more involved cases (more random c_{ij}, more complex laws of c_{ij}, random d_i, $N > 2$) would cause the necessity to use much more involved mathematical means, including numerical methods. This definitely does not correspond to our intension in this work and is far beyond of its scope.

3.1.1. Productivity

Due to the assumptions made above, the spectral radius of C can be calculated explicitly as the root of the characteristic polynomial

$$h_C(\lambda) = \lambda^2 - c_{12}c_{21},$$

and hence

$$r(C) = \sqrt{c_{12}c_{21}}. \tag{3.2}$$

We use this formula and the laws of c_{12} and c_{21} (see (3.1)) to find that

$$\mathbb{P}(r(C) \le x) = \begin{cases} 0, & x \le 0; \\ \frac{x^2}{ab}\left[1 + \ln a + \ln b - \ln x^2\right], & x \in (0, \sqrt{ab}]; \\ 1 & x > \sqrt{ab}. \end{cases} \tag{3.3}$$

If $ab < 1$, then $r(C) < 1$ and thus the economy is productive with probability one. Henceforth, we assume

$$ab > 1. \tag{3.4}$$

In view of (3.4), the economy can be nonproductive with positive probability $\varkappa$. To find this probability we use (2.9), (2.14), (3.3) and obtain

$$1 - \varkappa = \mathbb{P}(r(C) < x) = \mathbb{P}(r(C) \le x) = \frac{1}{ab}\left[1 + \ln(ab)\right],$$

which yields

$$\varkappa = \left[ab - 1 - \ln(ab)\right]/ab > 0. \tag{3.5}$$

Now we can use (3.5) to check the accuracy of the estimate in (2.14). By (2.10) we have that $s^* = \max\{c_{12}; c_{21}\}$. Then, for some $\varepsilon > 0$,

$$\mathcal{A}^*_\varepsilon = \{\omega : r(C(\omega)) \le 1 - \varepsilon\} = \{\omega : c_{12}(\omega) \le 1 - \varepsilon\} \cap \{\omega : c_{21}(\omega) \le 1 - \varepsilon\}.$$

Note that, in view of (3.2), $r(C) \le 1 - \varepsilon$ for each $\omega \in \mathcal{A}_\varepsilon$. By (2.14) we then have

$$1 - \varkappa \ge \mathbb{P}(\mathcal{A}^*_\varepsilon) = \mathbb{P}(c_{12} \le 1 - \varepsilon)\mathbb{P}(c_{21} \le 1 - \varepsilon) \tag{3.6}$$

$$= \begin{cases} \frac{(1-\varepsilon)^2}{ab}, \text{ if } & \min\{a; b\} > 1 - \varepsilon; \\ \frac{1-\varepsilon}{b}, \quad \text{if} & a \le 1 - \varepsilon, \quad b > 1 - \varepsilon; \\ \frac{1-\varepsilon}{a}, \quad \text{if} & b \le 1 - \varepsilon, \quad a > 1 - \varepsilon. \end{cases}$$

Here we have used that $ab > 1$ and also that c_{12} and c_{21} are independent and uniformly distributed on $[0, a]$ and $[0, b]$, respectively, see (3.1). Since ε in (3.6) is arbitrary, the best estimate for $\varkappa$ is attained for $\varepsilon = 0$, cf. (2.14). This yields

$$\varkappa \le 1 - \min\left\{\frac{1}{ab}; \frac{1}{a}; \frac{1}{b}\right\}. \tag{3.7}$$

For example, if $a > 1$ and $b > 1$, then by (3.7) we have that $\varkappa \le 1 - 1/ab$ whereas its exact value is $1 - 1/ab - \ln(ab)/ab$.

3.1.2. Feasible plans

For the considered example, the balance conditions (2.1) take the form:

$$\begin{cases} x_1 - c_{12}x_2 + z_1 \geq d_1 \\ x_2 - c_{21}x_1 + z_2 \geq d_2. \end{cases}$$

This imposes the corresponding bounds on the values of random c_{12} and c_{21}. Keeping them in mind we define

$$\mathcal{A}_1(x_1, x_2, z_1, z_2) = \{\omega : c_{12}(\omega) \in [0, (x_1 + z_1 - d_1)/x_2]\}, \tag{3.8}$$

$$\mathcal{A}_2(x_1, x_2, z_1, z_2) = \{\omega : c_{21}(\omega) \in [0, (x_2 + z_2 - d_2)/x_1]\}.$$

Since both c_{12} and c_{21} are nonnegative, we have that

$$\mathcal{A}_1(x_1, x_2, z_1, z_2) \neq \varnothing, \quad \mathcal{A}_2(x_1, x_2, z_1, z_2) \neq \varnothing,$$

holding if and only if the following conditions are satisfied:

$$x_1 + z_1 \geq d_1, \qquad x_2 + z_2 \geq d_2. \tag{3.9}$$

At the same time, the event defined in (2.4) now is

$$\mathcal{A}(X, Z) = \mathcal{A}_1(x_1, x_2, z_1, z_2) \cap \mathcal{A}_2(x_1, x_2, z_1, z_2).$$

Set

$$\pi_i(x_1, x_2, z_1, z_2) = \mathbb{P}[\mathcal{A}_i(x_1, x_2, z_1, z_2)], \quad i = 1, 2.$$

Since c_{12} and c_{21} are independent, we have that

$$\mathbb{P}(\mathcal{A}(X, Z)) = \pi_1(x_1, x_2, z_1, z_2)\pi_2(x_1, x_2, z_1, z_2).$$

Then (2.5) takes the form

$$\pi_1(x_1, x_2, z_1, z_2)\pi_2(x_1, x_2, z_1, z_2) \geq 1 - \alpha. \tag{3.10}$$

By (3.8) and (3.1) we get

$$\pi_1(x_1, x_2, z_1, z_2) = \begin{cases} (x_1 + z_1 - d_1)/ax_2, & \text{if } \; x_1 + z_1 - d_1 \leq ax_2; \\ 1, & \text{if } \; x_1 + z_1 - d_1 \geq ax_2. \end{cases} \tag{3.11}$$

$$\pi_2(x_1, x_2, z_1, z_2) = \begin{cases} (x_2 + z_2 - d_2)/bx_1, & \text{if } \; x_2 + z_2 - d_2 \leq bx_1; \\ 1, & \text{if } \; x_2 + z_2 - d_2 \geq bx_1. \end{cases}$$

Then the relevant pairs of conditions that appear in (3.11) are

$$\begin{aligned} &x_1 + z_1 - d_1 \leq ax_2 \quad \text{and} \quad x_2 + z_2 - d_2 \leq bx_1, \quad (a) \\ &x_1 + z_1 - d_1 \leq ax_2 \quad \text{and} \quad x_2 + z_2 - d_2 > bx_1, \quad (b) \\ &x_1 + z_1 - d_1 > ax_2 \quad \text{and} \quad x_2 + z_2 - d_2 \leq bx_1. \quad (c) \end{aligned} \tag{3.12}$$

We have excluded the combination of the second lines in (3.11) because the case where both inequalities turn into equalities, *i.e.*, where $z_i = d_i$ (and hence $x_i = 0$) for both $i = 1, 2$ (resp. for just one of i), is already included in case (a) (resp. (b) or (c)) of (3.12). For $d_i > z_i$ for both $i = 1, 2$, the corresponding system of inequalities is

$$\begin{cases} x_1 - ax_2 > d_1 - z_1 \\ -bx_1 + x_2 > d_2 - z_2. \end{cases}$$

In view of (3.4), however, it has no positive solution, and hence the mentioned combination of conditions should not be considered ass it cannot be satisfied for feasible (X, Z).

3.1.3. Optimal plans

Now we choose the cost functions, which we take in the form

$$\varphi(x_1, x_2) = \theta_1 x_1 + \theta_2 x_2, \qquad \psi(z_1, z_2) = \tau_1 z_1 + \tau_2 z_2, \tag{3.13}$$

with nonnegative parameters θ_i and τ_i, $i = 1, 2$. Here θ_i is the production cost of ith commodity and τ_i is the corresponding price at the outer market. Then the total cost of the production-and-import plan is

$$f = \theta_1 x_1 + \theta_2 x_2 + \tau_1 z_1 + \tau_2 z_2. \tag{3.14}$$

Clearly, the pure import plan $z_i = d_i$, $i = 1, 2$, is feasible, and its cost is $\tau_1 d_1 + \tau_2 f_2$. Then according to Definition 2.3 the cost f_* of an optimal plan should satisfy

$$f_* \leq \tau_1 d_1 + \tau_2 d_2. \tag{3.15}$$

In view of this, we will not consider those feasible (X, Z) for which the total cost (3.14) fails to satisfy (3.15). In particular, such that $z_i > d_i$ for at least one of $i = 1, 2$. It turns out that these are exactly cases (b) and (c) in (3.12). Indeed, by (b) we have that $x_2 > d_2 - z_2 + b(d_1 - z_1 + a x_2)$, which yields, see also (3.4),

$$(z_2 - d_2) + b(z_1 - d_1) > (ab - 1)x_2 \geq 0.$$

Hence, $z_i > d_i$ for at least one of $i = 1, 2$. Case (c) of (3.12) can be ruled out in the same way. Thus, from now on we take into account only case (a) of (3.12).

Instead of z_1 and z_2 it is convenient to introduce the following variables

$$u_1 = \begin{cases} 1, & \text{if } \ x_1 = x_2 = 0, \ z_1 = d_1 \\ \frac{x_1 + z_1 - d_1}{a x_2}, & \text{if } \ x_2 > 0; \end{cases} \tag{3.16}$$

$$u_2 = \begin{cases} 1, & \text{if } \ x_1 = x_2 = 0, \ z_2 = d_2 \\ \frac{x_2 + z_2 - d_2}{b x_1}, & \text{if } \ x_1 > 0. \end{cases} \tag{3.17}$$

In these new variables, the probabilities π_i defined in (3.11) take the form

$$\pi_i = \min\{u_i, 1\}, \qquad i = 1, 2. \tag{3.18}$$

By (3.16) and (3.17) we also have

$$z_1 = a u_1 x_2 - x_1 + d_1, \qquad z_2 = b u_2 x_1 - x_2 + d_2. \tag{3.19}$$

By these formulas we express our "old" variables z_i through the new ones. In view of $z_i \geq 0$ for both $i = 1, 2$, by (3.19) we get that x_i and u_i satisfy

$$\begin{cases} a u_1 x_2 - x_1 + d_1 \geq 0 \\ b u_2 x_1 - x_2 + d_2 \geq 0. \end{cases} \tag{3.20}$$

Conditions (3.10) and case (a) of (3.12) imply, see also (3.18), that the pairs (u_1, u_2) ought to belong to the following set of feasible values

$$\mathcal{U}_\alpha = \{(u_1, u_2) : 0 \leq u_1 \leq 1, \ 0 \leq u_2 \leq 1, \ u_1 u_2 \geq 1 - \alpha\}. \tag{3.21}$$

In the new variables, the total cost function (3.14) takes the form

$$\begin{aligned} f(x_1, x_2, u_1, u_2) &= \varphi(x_1, x_2) + \psi(a u_1 x_2 - x_1 + d_1, b u_2 - x_2 + d_2) \\ &= g(x_1, x_2, u_1, u_2) + \tau_1 d_1 + \tau_2 d_2, \end{aligned} \tag{3.22}$$

with

$$g(x_1, x_2, u_1, u_2) = \varrho_1 x_1 + \varrho_2 x_2, \tag{3.23}$$
$$\varrho_1 = \theta_1 - \tau_1 + b\tau_2 u_2,$$
$$\varrho_2 = \theta_2 - \tau_2 + a\tau_1 u_1.$$

Then by (3.15) an optimal collection x_1^*, x_2^*, u_1^* and u_2^* ought to satisfy

$$g(x_1^*, x_2^*, u_1^*, u_2^*) \leq 0. \tag{3.24}$$

3.2. Finding optimal solutions

In view of (3.23), both ϱ_i are increasing functions of u_1 and u_2. By (3.22) and (3.24) this means that optimal values of $(u_1, u_2) \in \mathcal{U}_\alpha$ lie on the arc

$$\Xi_\alpha = \{(u_1, u_2) : u_1 = \xi,\ u_2 = (1-\alpha)/\xi,\ \xi \in [1-\alpha, 1]\}, \tag{3.25}$$

of the hyperbola $u_1 u_2 = 1 - \alpha$, which is one of the boundaries of (3.21). On this arc, ϱ_i can be considered as functions of $\xi \in [1-\alpha, 1]$, which we denote by $\varrho_i(\xi)$. Hence,

$$\varrho_1(\xi) = \theta_1 - \tau_1 + b\tau_2(1-\alpha)/\xi, \tag{3.26}$$
$$\varrho_2(\xi) = \theta_2 - \tau_2 + a\tau_1 \xi.$$

In view of (3.24), the crucial role in minimizing g belongs to the signs of ϱ_i. With this regard let us consider the following cases:

$$(a) \qquad \inf_{\xi \in [1-\alpha, 1]} \varrho_i(\xi) \geq 0, \qquad \text{for both } i = 1, 2; \tag{3.27}$$
$$(b) \qquad \sup_{\xi \in [1-\alpha, 1]} \varrho_i(\xi) < 0, \qquad \text{for both } i = 1, 2;$$
$$(c) \qquad \varrho_i(\xi_i) = 0, \quad \xi_i \in (1-\alpha, 1) \quad \text{for at least one of } i = 1, 2.$$

Respectively, we have the following options in choosing optimal plans.

3.2.1. Import is preferable

Import is preferable if the solution $x_1 = x_2 = 0$ is optimal. By (3.9) we then get that $z_i = d_i$, $i = 1, 2$, in this case. By (3.22) and (3.24) the optimality of this solution is equivalent to case (a) in (3.27), which can be rewritten in the form

$$\tau_1 - \theta_1 \leq (1-\alpha) b\tau_2, \tag{3.28}$$
$$\tau_2 - \theta_2 \leq (1-\alpha) a\tau_1.$$

Conclusion 3.1. The solution $x_1 = x_2 = 0$ and $z_1 = d_1$, $z_2 = d_2$ is optimal if and only if the model parameters satisfy both conditions in (3.28). In this case the total cost is $f = \tau_1 d_1 + \tau_2 d_2$.

The economical meaning of the conditions in (3.28) can be seen from the following. By (3.1) the value which c_{21} does not exceed with probability $1-\alpha$ is $(1-\alpha)b$. Let us compare the costs of the following two options: (a) to import one unite of the first commodity, and hence to spend τ_1; (b) to produce the mentioned unit. Option

(b) is related to buying c_{21} units of the second commodity needed for the production. With probability $1-\alpha$ the related cost is $(1-\alpha)b\tau_2$, and thus the total cost of option (b) is $\theta_1+(1-\alpha)b\tau_2$. According to the first inequality in (3.28), option (a) is preferable. In a similar way, one can interpret the second inequality.

Clearly, both conditions in (3.28) are satisfied if the import prices are lower then the corresponding production costs. However, import is preferable even if the import prices are somewhat higher. The reason is that the production is related to inner production costs, that appear in the right-hand sides of the mentioned conditions. Since the case of $\tau_i \le \theta_i$ for both $i=1,2$ is completely described in Conclusion 3.1, from now on we assume that $\tau_i > \theta_i$ for at least one $i=1,2$.

3.2.2. Production in preferable

By (3.26) condition (b) in (3.27) takes the form

$$\tau_1-\theta_1 > b\tau_2, \qquad \tau_2-\theta_2 > a\tau_1. \tag{3.29}$$

Similarly as in the case of pure import, these conditions can be interpreted as follows. By (3.1) c_{21} does not exceed b almost surely. Then the cost of producing one unit of the first commodity – related to buying c_{21} units of the second one – is $\theta_1+b\tau_2$ in the worst case where c_{21} takes its maximum value. According to the first condition in (3.29), this is smaller than the corresponding import cost. If (3.29) is satisfied, then

$$g=\varrho_1(\xi)x_1+\varrho_2(\xi)x_2<0, \tag{3.30}$$

for all $x_i>0$. The optimal values of x_i and ξ are those which minimize g on the set, *cf.* (3.20),

$$\begin{cases} x_1-a\xi x_2 \le d_1,\\ -\frac{b(1-\alpha)}{\xi}x_1+x_2\le d_2. \end{cases} \tag{3.31}$$

with positive x_1, x_2 and $\xi\in[1-\alpha,1]$. This nonlinear optimization problem will be solved in the following steps. First, we fix ξ and consider the problem of minimizing g given in (3.30) on the set defined by the constraints in (3.31), that is a standard task of linear programming. Then the minimum value of g is attained at the corner point

$$\begin{aligned} x_1 &= x_1^*(\xi) := \frac{1}{\gamma}[d_1+a\xi d_2], \\ x_2 &= x_2^*(\xi) := \frac{1}{\gamma}[d_2+b(1-\alpha)d_1/\xi], \end{aligned} \tag{3.32}$$

that corresponds to the equalities in (3.31) and hence to the choice $z_1=z_2=0$. By this the first step of solving the mentioned nonlinear problem has been completed. Here

$$\gamma := 1-(1-\alpha)ab>0, \tag{3.33}$$

and its positivity is necessary for $x_1^*(\xi)\ge 0$ and $x_2^*(\xi)\ge 0$ to hold. Next, to find the optimal value of ξ we plug (3.32) in (3.30) and consider the function

$$\begin{aligned} w(\xi) &:= \varrho_1(\xi)x_1^*(\xi)+\varrho_2(\xi)x_2^*(\xi) \\ &= \frac{1}{\gamma}(\theta_1 d_1+\theta_2 d_2)+\frac{a\theta_1 d_2}{\gamma}\xi+\frac{(1-\alpha)bd_1\theta_2}{\gamma\xi} \\ &- \tau_1 d_1-\tau_2 d_2, \end{aligned} \tag{3.34}$$

with ξ taking values in $[1-\alpha, 1]$. Thus, the problem now is reduced to minimizing w on the latter interval. To this end we calculate

$$w'(\xi) = \frac{bd_1\theta_2}{\gamma}\left(\frac{\theta_1}{\theta_2}\cdot\frac{ad_2}{bd_1} - \frac{1-\alpha}{\xi^2}\right).$$

Depending on the value of the parameters in $(\cdots)$ in the latter formula, for this derivative as a function defined on $[1-\alpha, 1]$ one has the following options: (a) it has a simple root in $(1-\alpha, 1)$; (b) $w'(\xi) \le 0$ for all $\xi \in [1-\alpha, 1]$; (c) $w'(\xi) > 0$ for all $\xi \in [1-\alpha, 1]$. Let us analyze these cases. If the following holds

$$\frac{(1-\alpha)bd_1}{ad_2} < \frac{\theta_1}{\theta_2} < \frac{bd_1}{(1-\alpha)ad_2}, \tag{3.35}$$

then $w'(1-\alpha) < 0$ and $w'(1) > 0$. This means that option (a) takes place and hence w attains its minimum at

$$\xi_* = \sqrt{\frac{(1-\alpha)\theta_2 bd_1}{\theta_1 ad_2}} \in (1-\alpha, 1), \tag{3.36}$$

which readily follows by (3.35). For

$$\frac{\theta_1}{\theta_2} \le \frac{(1-\alpha)bd_1}{ad_2}, \tag{3.37}$$

$w'(\xi) \le 0$ for all $\xi \in [1-\alpha, 1]$; hence, the minimum of w is attained at $\xi_* = 1$. Finally, for

$$\frac{\theta_1}{\theta_2} \ge \frac{bd_1}{(1-\alpha)ad_2}, \tag{3.38}$$

$w'(\xi) \ge 0$ for all $\xi \in [1-\alpha, 1]$, and the minimum of w is attained at $\xi_* = 1-\alpha$. In all these cases, the optimal solution is obtained from (3.32) in the form

$$x_1^* = \frac{d_1}{\gamma} + \frac{ad_2\xi_*}{\gamma}, \qquad x_2^* = \frac{d_2}{\gamma} + \frac{(1-\alpha)bd_1}{\gamma\xi_*}, \tag{3.39}$$

with the corresponding value of ξ_*. Note that the first summands in these expressions correspond to the amounts of the produced commodities used directly to satisfying the external demand, whereas the second summands correspond to the amounts used for the inner consumption. These formulas may have the following economical interpretation. The condition in (3.37) can also be written in the form

$$\frac{ad_2\theta_1}{\gamma} \le \frac{(1-\alpha)bd_1\theta_2}{\gamma}.$$

If it holds, then the cost of the production of the first commodity for the inner consumption is smaller than that of the second commodity. Then the optimality of the solution is in favor of producing the first commodity. In a similar way, one may interpret the condition in (3.38) that favorites the production of the second commodity. Obviously, the conditions in (3.35) correspond to the intermediate case.

Conclusion 3.2. Let the conditions in (3.29) be satisfied. Then the solution $x_1 = x_1^*$, $x_2 = x_2^*$, $z_1 = z_2 = 0$ is optimal. Here x_i^*, $i = 1, 2$, are given in (3.39) with $\xi_* =$ RHS (3.36), $\xi_* = 1$, $\xi_* = 1-\alpha$ for the model parameters satisfying (3.35), (3.37) and (3.38), respectively.

3.2.3. The intermediate cases

Here we consider case (c) in (3.27). Assume that $\varrho_1(\xi_1) = 0$ for some $\xi_1 \in (1-\alpha, 1)$. Then

$$\xi_1 = \frac{(1-\alpha)b\tau_2}{\tau_1 - \theta_1}, \tag{3.40}$$

and the condition $\xi_1 \in (1-\alpha, 1)$ implies

$$(1-\alpha)b\tau_2 < \tau_1 - \theta_1 < b\tau_2. \tag{3.41}$$

That is, the import price τ_1 is intermediate as compared to the cases described in (3.29) and (3.28). With regard to ϱ_2 we have the following possibilities:

$$\begin{aligned} &(i) \qquad \varrho_2(\xi) \geq 0, \quad \text{for all } \xi \in [1-\alpha, 1]; \\ &(ii) \qquad \varrho_2(\xi) < 0, \quad \text{for all } \xi \in [1-\alpha, 1]; \\ &(iii) \qquad \varrho_2(\xi_2) = 0, \quad \text{for some } \xi_2 \in (1-\alpha, \alpha). \end{aligned} \tag{3.42}$$

In case (i), we have that $\varrho_1(\xi)x_1 + \varrho_2(\xi)x_2 \geq \varrho_1(\xi)x_1$, holding for all feasible x_i and ξ. Thus, we set $x_2^* = 0$, that by (3.20) yields $x_1 \leq d_1$. By the latter, for $\xi \geq \xi_1$ and hence for $\varrho_1(\xi) \leq 0$, we have $\varrho_1(\xi)x_1 \geq \varrho_1(1)d_1$, which means that the optimal solution is $x_1^* = d_1$ and $\xi_* = 1$. By (3.19) this yields $z_1 = 0$ and $z_2 = b(1-\alpha)d_1 + d_2$. Here we have taken into account that $u_2 = (1-\alpha)/\xi$, see (3.25).

In case (ii) in (3.42), we have to minimize $\varrho_1(\xi)x_1 + \varrho_2(\xi)x_2$ on the set of (x_1, x_2) satisfying (3.20) and $\xi \in (\xi_1, 1]$. The only difference between this problem and that in case (b) of (3.27) solved above (see Conclusion 3.2) is that in the latter one we dealt with $\xi \in [1-\alpha, 1]$. Therefore, we first obtain $x_1^*(\xi)$ and $x_2^*(\xi)$, see (3.32), which we plug in (3.30) and thus arrive at (3.34) with the only difference that w given therein has to be minimized on $(\xi_1, 1]$. In this task, we have the following cases: ξ_1 defined in (3.40) and ξ_* as in Conclusion 3.2 satisfy (a) $\xi_* < \xi_1$; (b) $\xi_* \geq \xi_1$. Case (a) (resp. (b)) naturally includes $\xi_* = 1-\alpha$ (resp. $\xi_* = 1$). Then by (3.36), (3.38) and (3.40) we conclude that case (a) takes place if the model parameters satisfy the following condition:

$$\tau_1 - \theta_1 < \tau_2 \sqrt{\frac{(1-\alpha)ab\theta_1 d_2}{\theta_2 b_1}}. \tag{3.43}$$

Clearly, case (b) corresponds to the opposite inequality in (3.43). In case (a) $w'(\xi) > 0$ for all (feasible) $\xi \in [\xi_1, 1]$. Hence the optimal solution is $\xi^* = \xi_1$. In case (b) the interval $[\xi_1, 1]$ contains ξ_* mentioned in Conclusion 3.2, which means that this ξ_* is the solution of the problem in question.

In case (iii) of (3.42), by (3.26) we find

$$\xi_2 = \frac{\tau_2 - \theta_2}{a\tau_1}, \tag{3.44}$$

which also implies, *cf.* (3.41)

$$(1-\alpha)a\tau_1 < \tau_2 - \theta_2 < a\tau_1. \tag{3.45}$$

For this ξ_2 and ξ_1 given in (3.40) we have the following possibilities: (a) $\xi_2 < \xi_1$; (b) $\xi_2 \geq \xi_1$. By (3.40) and (3.44) we get that case (a) corresponds to the following condition

$$\left(1 - \frac{\theta_1}{\tau_1}\right)\left(1 - \frac{\theta_2}{\tau_2}\right) < (1-\alpha)ab = 1 - \gamma, \tag{3.46}$$

see (3.33). Case (b) corresponds to the opposite sign in (3.46), which immediately yields that both $\tau_i > \theta_i$ (the case of both $\tau_i \leq \theta_i$ is described in Conclusion 3.1). In case (a) we have the following situation with the signs of $\varrho_i(\xi)$:

$$\begin{aligned} &(i) \quad \varrho_2(\xi) \leq 0, \quad \varrho_1(\xi) > 0, \quad \text{for} \quad \xi \in [1-\alpha, \xi_2], \\ &(ii) \quad \varrho_2(\xi) > 0, \quad \varrho_1(\xi) > 0, \quad \text{for} \quad \xi \in (\xi_2, \xi_1) \\ &(iii) \quad \varrho_2(\xi) > 0, \quad \varrho_1(\xi) \leq 0, \quad \text{for} \quad \xi \in [\xi_1, 1]. \end{aligned} \tag{3.47}$$

In view of the fact that $g = \varrho_1 x_1 + \varrho_2 x_2$, $g > 0$ for $\xi \in (\xi_2, \xi_1)$. For $\xi \in [1-\alpha, \xi_2]$, we take $x_1^* = 0$ and obtain, *cf.* (3.45),

$$g = \varrho_2(\xi)x_2 \geq \varrho_2(1-\alpha)d_2 = -(\tau_2 - \theta_2 - a\tau_1(1-\alpha))d_2. \tag{3.48}$$

For $\xi \in [\xi_1, 1]$, we take $x_2^* = 0$ and obtain, *cf.* (3.41),

$$g = \varrho_1(\xi)x_1 \geq \varrho_1(1)d_1 = -(\tau_1 - \theta_1 - b\tau_2(1-\alpha))d_1. \tag{3.49}$$

Then the optimal solution is found by comparing (3.49) and (3.48). For instance, if

$$(\tau_1 - \theta_1 - b\tau_2(1-\alpha))d_1 > (\tau_2 - \theta_2 - a\tau_1(1-\alpha))d_2, \tag{3.50}$$

then the optimal solution is $x_1^* = d_1$, $x_2^* = 0$, $z_1^* = 0$, $z_2^* = d_2$. In case of the opposite strict inequality in (3.50), one takes $x_1^* = 0$, $x_2^* = d_2$, $z_1^* = d_1$, $z_2^* = 0$. In case of equality in (3.50), both these solutions get optimal.

Let us now study the case where $\xi_2 \geq \xi_1$. If $\xi_2 = \xi_1$, which corresponds to the equality in (3.46), the optimal solution is $\xi = \xi_1 = \xi_2$, and hence $\varrho_1 = \varrho_2 = 0$. As in the case described in Conclusion 3.1, this corresponds to the pure import solution $x_1^* = x_2^* = 0$, $z_1^* = d_1$, $z_2^* = d_2$. For $\xi_2 > \xi_1$, similarly as in (3.47) we obtain that both $\varrho_i(\xi)$ are non-positive only if $\xi \in (\xi_1, \xi_2)$.

3.3. Concluding remarks

The intervals characterizing the probability distributions of c_{12} and c_{21} were chosen for simplicity of calculations to be $[0, a]$ and $[0, b]$, respectively. A more realistic version would be $[a_-, a_+]$ and $[b_-, b_+]$ for some $0 < a_- < a_+$ and $0 < b_- < b_+$. In that case, one allows the coefficients to randomly oscillate in the mentioned intervals. In principle, this version could also be treated by the method applied in this article with possible complications caused by the appearance of additional two parameters. Other choices of the probability laws of the entries of C were discussed in papers [8, 10], but only in the context of finding the laws of the entries of the Leontief matrix B given in (1.4), assuming that this matrix exists and thus the economy is productive with probability one. Nowhere the possibility for the economy be nonproductive was considered and, therefore, nowhere finding production plans for such an economy was formulated as an optimization problem. The choice of the objective functions in (3.13) was also dictated by our wish to make the calculations simple and transparent. Note that we have dealt with the whole set $\widetilde{\mathcal{M}}^\alpha$ defined in (2.3) as we managed to apply direct means of finding optimal solutions rather than relied on general methods of convex optimization.

The parameter γ introduced in (3.33) reflects the very essence of our approach. Namely, typically ab exceeds 1, *cf.* (3.4), but not too much. Then the "numerical effect" of passing to the condition in (1.6) is just multiplying ab by $1-\alpha$, and making thereby the economy productive with probability $1-\alpha$ since $\gamma > 0$. If γ is small, the economy is "nearly nonproductive", and hence the optimal plans given in (3.39) are much bigger than those corresponding to $\gamma = 1$. For $\gamma \leq 0$, the economy is almost surely nonproductive, and the solution $x_1 = x_2 = 0$ is the only feasible one in this case.

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