

## AVERAGE COVERING NUMBER FOR SOME GRAPHS

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**Abstract.** The interconnection networks are modeled by means of graphs to determine the reliability and vulnerability. There are lots of parameters that are used to determine vulnerability. The average covering number is one of them which is denoted by  $\bar{\beta}(G)$ , where  $G$  is simple, connected and undirected graph of order  $n \geq 2$ . In a graph  $G = (V(G), E(G))$  a subset  $S_v \subseteq V(G)$  of vertices is called a cover set of  $G$  with respect to  $v$  or a local covering set of vertex  $v$ , if each edge of the graph is incident to at least one vertex of  $S_v$ . The local covering number with respect to  $v$  is the minimum cardinality of among the  $S_v$  sets and denoted by  $\beta_v$ . The average covering number of a graph  $G$  is defined as

$$\bar{\beta}(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} \beta_v.$$

In this paper, the average covering numbers of  $k$ th power of a cycle  $C_n^k$  and  $P_n \square P_m$ ,  $P_n \square C_m$ , cartesian product of  $P_n$  and  $P_m$ , cartesian product of  $P_n$  and  $C_m$  are given, respectively.

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### 1. INTRODUCTION AND PRELIMINARIES

Graph theory is now a very important part of mathematics having countless applications in computing, technology and science. Everything is linked in our world; countries are linked by streets, railways and flight networks. Scientists want to analyze, understand and optimize these networks and this can be done by using graph theory. Graph theory is applied in flight networks, air traffic controls, design of computer chips, internet, and so on. If any network can be modeled as a simple undirected, connected and unweighted graph  $G$  then deterministic measures tend to provide a worst-case analysis of some expectations of the overall disconnection process [6]. Throughout this paper, unless otherwise specified, we use terminology of [2]. There are many measures of reliability and vulnerability in graph theory. Connectivity is one of them, which is the best known and also the most useful one. Let  $G = (V(G), E(G))$  be a graph, where  $V(G)$  and  $E(G)$  are vertex and edge sets of  $G$ , respectively. The connectivity of a graph  $G$  is the minimum number of vertices, whose removal from  $G$  results in a disconnected or a trivial graph and denoted by  $\kappa(G)$  [2]. Integrity, toughness, tenacity, domination, covering are other examples of measures. The integrity of a graph  $G$  is denoted by  $I(G)$  and

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defined as  $I(G) := \min\{|S| + m(G-S) : S \subseteq V(G)\}$  where  $m(G-S)$  is the maximum order of the components of  $G-S$  [4]. On the other hand,  $T(G)$ , the tenacity of a graph  $G$  is defined by  $T(G) = \min\{\frac{|S| + \tau(G-S)}{\omega(G-S)}\}$ , where the minimum is taken over all vertex cutsets  $S$  of  $V(G)$ ,  $\omega(G-S)$  is the number of components of  $G-S$  and  $\tau(G-S)$  is the number of vertices in the largest component of the graph induced by  $G-S$  [9]. A graph has toughness  $t(G)$  if  $t(G)$  is the largest number  $t$  such that for any subset  $S$  of  $V(G)$ ,  $|S| > tc(G \setminus S)$ , whenever  $c(G \setminus S) > 1$  [3, 10]. Menger's classical theorem tells us that in a  $k$ -connected graph, every pair of vertices are joined by  $k$  internally disjoint paths. Two vertices  $u$  and  $v$  in a graph  $G$  are said to be  $k$ -connected if there are  $k$  or more pairwise internally disjoint paths between them. The  $(u, v)$ -connectivity of  $G$ , denoted by  $\kappa_G(u, v)$ , is defined to be the maximum value of  $k$  for which  $u$  and  $v$  are  $k$ -connected. It is a well-known fact that the connectivity  $\kappa(G)$  equals  $\min\{\kappa_G(u, v) : u, v \in V(G)\}$ . If the order of  $G$  is  $n$ , then the average connectivity of  $G$  is denoted by  $\bar{\kappa}(G)$  is defined to be  $\bar{\kappa}(G) = \sum_{u,v} \kappa_G(u, v) / \binom{n}{2}$  [1]. Toughness, integrity and tenacity are NP-hard computable, whereas the average connectivity can be computed in polynomial time, which makes it more attractive for applications [8]. A vertex and an edge are said to cover each other in a graph  $G$  if they are incident in  $G$ . A vertex cover in  $G$  is a set of vertices that covers all the edges of  $G$ . The minimum cardinality of a vertex cover in a graph  $G$  is called the vertex covering number of  $G$  and is denoted by  $\beta(G)$  [2]. In 2013, Dogan and Dundar have defined the average covering number, which is also NP-hard computable. In a graph  $G = (V(G), E(G))$  a subset  $S_v \subseteq V(G)$  of vertices is called a cover set of  $G$  with respect to  $v$  or a local covering set of vertex  $v$ , if each edge of the graph is incident to at least one vertex of  $S_v$ . The local covering number with respect to  $v$  is the minimum order of a cover that contains  $v$  and it is the minimum cardinality of among the  $S_v$  sets which is denoted by  $\beta_v$ . Hence,  $\beta_v$ -set shows that the set which has the minimum order of a cover that contains  $v$ . The average covering number  $\bar{\beta}(G)$  of a graph  $G$  is

$$\bar{\beta}(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} \beta_v$$

where  $n \geq 2$  is order of  $G$  and the sum is overall  $n$  vertices. So, the average covering number is defined by the mean of local covering numbers [5]. In 2016, Dogan and Dundar have studied on some graph operations by using this parameter [6].

The primary goals of this paper are to find the average covering numbers of  $k$ -th power of a cycle of order  $n$ , a grid graph  $P_n \square P_m$  and  $P_n \square C_m$ . The definitions and theorems that will be used in this paper are given below.

**Definition 1.1.** For an integer  $n \geq 3$ , the cycle  $C_n$  is a graph of order  $n$  and size  $m$  whose vertices can be labeled by  $v_1, v_2, \dots, v_n$  and whose edges are  $v_1v_n$  and  $v_i v_{i+1}$  for  $i = 1, 2, \dots, n-1$ . The cycle  $C_n$  is also referred to as an  $n$ -cycle and moreover the 3-cycle is also called a triangle [2].

**Definition 1.2.** For each positive integer  $k$ , the  $k$ th power  $G^k$  of a graph  $G$  is that graph with  $V(G^k) = V(G)$  and  $uv \in E(G^k)$  if and only if  $1 \leq d_G(u, v) \leq k$ . Thus  $G^1 = G$  and  $G^k = K_n$  if  $k \geq d$ . So the graph  $G^2$  is also called the square of  $G$ , while  $G^3$  is called the cube of  $G$  [2].

**Definition 1.3.** The distance  $d_G(u, v)$  from vertex  $u$  to a vertex  $v$  in a connected graph  $G$  is the smallest length of a  $u-v$  path in  $G$  [2].

**Definition 1.4.** The cartesian product  $G$  of two graphs  $G_1$  and  $G_2$  is commonly denoted by  $G_1 \square G_2$ , has vertex set  $V(G) = V(G_1) \times V(G_2)$  where two distinct vertices  $(u, v)$  and  $(x, y)$  of  $G_1 \square G_2$  are adjacent if either (1)  $u = x$  and  $vy \in E(G_2)$  or (2)  $v = y$  and  $ux \in E(G_1)$ . As expected,  $G_1 \square G_2 \cong G_2 \square G_1$  for all graphs  $G_1$  and  $G_2$  [2].

Now there are definitions about vertex transitive graph which will be used in the proof of Theorem 2.2.

**Definition 1.5.** An automorphism of a graph  $G$  is a bijection  $\varphi : V(G) \rightarrow V(G)$  taking neighbours to neighbours and non-neighbours to non-neighbours.  $G$  is vertex-transitive (respectively almost transitive) if  $V(G)$  has only one (respectively finitely many) orbit(s) under the automorphism group  $Aut(G)$  [9].

**Definition 1.6.** A graph that contains a single orbit is called a vertex transitive if and only if there exists an automorphism  $\varphi$  of  $G$  for every two vertices  $u$  and  $v$  of  $G$ , such that  $\varphi(u) = v$ . Necessarily then, every vertex transitive graph is regular [2].

### 2. MAIN RESULTS

In this section we mention about average covering number of  $C_n^k$ ,  $P_n \square P_m$  and  $P_n \square C_m$ , where  $n$  and  $m$  are orders of the graphs, respectively.

**Theorem 2.1.** Let  $C_n^k$  be  $k$ th power of a cycle  $C_n$  with  $n \geq 3$ . Then

$$\bar{\beta}(C_n^k) = n - \left\lfloor \frac{n}{k+1} \right\rfloor.$$

*Proof.* Let  $C_n^k$  be the  $k$ th power of a cycle  $C_n$ .  $k = 1$  is obvious. Let  $k \geq 2$ ,  $u_i \in V(C_n^2)$  and  $n \equiv r \pmod{(k+1)}$  with  $a = \left\lfloor \frac{n}{k+1} \right\rfloor$ . Now we give local covering numbers for each situation. Reader can see the labeling of  $C_n^2$  in Figure 1.

$$\begin{aligned} n \equiv 0 \pmod{(k+1)} &\Rightarrow \beta_{u_1}(C_n^k) = |\{ u_1, u_2, \dots, u_k, u_{k+2}, \dots, u_{2k+1}, u_{2k+3}, \\ &\quad \dots, u_{a(k+1)-1} \}| = n - \left\lfloor \frac{n}{k+1} \right\rfloor \\ n \equiv 1 \pmod{(k+1)} &\Rightarrow \beta_{u_1}(C_n^k) = |\{ u_1, u_2, \dots, u_k, u_{k+2}, \dots, u_{2k+1}, u_{2k+3}, \\ &\quad \dots, u_{a(k+1)-1}, u_{a(k+1)+1} \}| = n - \left\lfloor \frac{n}{k+1} \right\rfloor \\ &\quad \vdots \\ n \equiv k \pmod{(k+1)} &\Rightarrow \beta_{u_1}(C_n^k) = |\{ u_1, u_2, \dots, u_k, u_{k+2}, \dots, u_{2k+1}, u_{2k+3}, \\ &\quad \dots, u_{a(k+1)-1}, u_{a(k+1)+1}, \dots, u_{a(k+1)+k} \}| \\ &= n - \left\lfloor \frac{n}{k+1} \right\rfloor. \end{aligned}$$

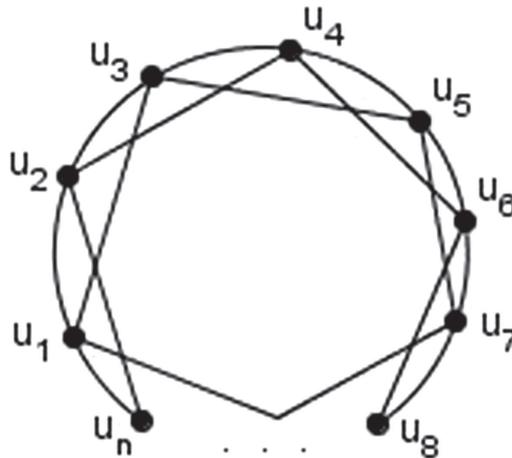


FIGURE 1. Graph  $C_n^2$ .

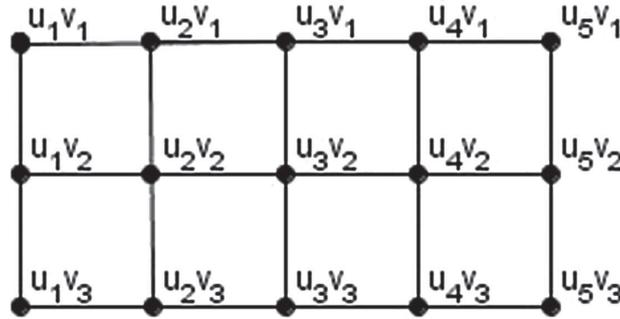


FIGURE 2. Graph  $P_5 \square P_3$ .

The graph  $C_n^k$  is regular and vertex transitive. Therefore each vertex of  $C_n^k$  has the same average covering number. Hence the average covering number of  $C_n^k$  is obtained as below

$$\bar{\beta}(C_n^k) = n - \left\lfloor \frac{n}{k+1} \right\rfloor.$$

□

**Theorem 2.2.** Let  $P_n, P_m$  be the paths of order  $n, m$  respectively where  $n, m$  are the positive integers. Then average covering number of  $P_n \square P_m$  is

$$\bar{\beta}(P_n \square P_m) = \begin{cases} \frac{mn}{2}, & \text{if } mn \text{ is even} \\ \frac{mn}{2} + \frac{1}{2mn}, & \text{otherwise} \end{cases}$$

*Proof.* Let  $P_n$  and  $P_m$  be the paths of order  $n, m$  respectively, with  $V(P_n) = \{u_1, u_2, \dots, u_n\}$  and  $V(P_m) = \{v_1, v_2, \dots, v_m\}$ . The cartesian product  $G$  of two graphs  $P_n$  and  $P_m$  has vertex set, such as  $V(G) = V(P_n) \times V(P_m)$  and  $|V(G)| = nm$ .

$$V(G) = \{u_1 v_1, u_2 v_1, u_3 v_1, \dots, u_n v_1, u_1 v_2, u_2 v_2, u_3 v_2, \dots, u_n v_2, \dots, u_1 v_m, u_2 v_m, u_3 v_m, \dots, u_n v_m\}.$$

We consider two cases which are obtained from parity of  $nm$ . Reader can see the labeling of  $P_5 \square P_3$  in Figure 2.

**Case 1:** Let  $nm$  be an even number. We take vertices  $u_i v_j$  ( $i = 1, \dots, n, j = 1, \dots, m$ ) to our local covering set where  $d(u_i v_j, u'_i v'_j) = 2$ . If  $n$  is odd then local covering set of  $u_i v_j \in V(G)$  is

$$\beta_{u_i v_j} = |\{u_1 v_1, u_3 v_1, u_5 v_1, \dots, u_n v_1, u_2 v_2, u_4 v_2, \dots, u_{n-1} v_2, u_1 v_3, u_3 v_3, \dots, u_n v_3, u_2 v_4, u_4 v_4, \dots, u_{n-1} v_4, \dots\}|$$

or

$$\beta_{u_i v_j} = |\{u_2 v_1, u_4 v_1, u_6 v_1, \dots, u_{n-1} v_1, u_1 v_2, u_3 v_2, \dots, u_n v_2, u_2 v_3, u_4 v_3, \dots, u_{n-1} v_3, u_1 v_4, u_3 v_4, \dots, u_n v_4, \dots\}|$$

and cardinality of these sets are equal to  $\frac{nm}{2}$  then  $\beta_{u_i v_j} \leq \frac{nm}{2}$ . If  $n$  is even then local covering set of  $u_i v_j \in V(G)$  is

$$\beta_{u_i v_j} = |\{u_2 v_1, u_4 v_1, u_6 v_1, \dots, u_{n-1} v_1, u_1 v_2, u_3 v_2, \dots, u_n v_2, u_2 v_3, u_4 v_3, \dots, u_{n-1} v_3, u_1 v_4, u_3 v_4, \dots, u_n v_4, \dots\}|$$

or

$$\beta_{u_i v_j} = |\{u_1 v_1, u_3 v_1, u_5 v_1, \dots, u_n v_1, u_2 v_2, u_4 v_2, \dots, u_{n-1} v_2, u_1 v_3, u_3 v_3, \dots, u_n v_3, u_2 v_4, u_4 v_4, \dots, u_{n-1} v_4, \dots\}|$$

and  $\beta_{u_i v_j} \leq \frac{nm}{2}$  for both sets. So, the average covering number is  $\bar{\beta}(G) \leq \frac{\frac{nm}{2}(nm)}{nm} = \frac{nm}{2}$ . Let  $S_{u_i v_j}$  be a minimal local covering set which contains  $u_i v_j$ . If a vertex  $u_i v_j$  is deleted from a graph  $G$  then every edge of  $G$  which are incident with  $u_i v_j$  are also deleted. If  $u_i v_j$  is equal to  $u'_i v'_j$  or  $u''_i v''_j$  then  $(u'_i v'_j, u''_i v''_j) \notin E(G - u_i v_j)$ . Otherwise,  $(u'_i v'_j, u''_i v''_j) \in E(G - u_i v_j)$  and the edge  $(u'_i v'_j, u''_i v''_j)$  has to be covered by a vertex in the set  $S_{u_i v_j}$ . Since  $S_{u_i v_j}$  is minimal then deletion a vertex from this set is not going to be a local covering set. Therefore,  $\bar{\beta}(G) \geq \frac{nm}{2}$ . Hence,

$$\bar{\beta}(G) = \frac{nm}{2}.$$

**Case 2:** Let  $nm$  be odd. We use the same process as the previous case. Then we obtain two different local covering sets. If  $i + j$  is even then local covering set is

$$\beta_{u_i v_j} = |\{u_1 v_1, u_3 v_1, u_5 v_1, \dots, u_n v_1, u_2 v_2, u_4 v_2, \dots, u_{n-1} v_2, u_1 v_3, u_3 v_3, \dots, u_n v_3, u_2 v_4, u_4 v_4, \dots, u_{n-1} v_4, \dots\}|$$

So, local covering number of  $u_i v_j$  is  $\frac{nm+1}{2}$  or if  $i + j$  is odd then local average covering set is

$$\beta_{u_i v_j} = |\{u_2 v_1, u_4 v_1, u_6 v_1, \dots, u_{n-1} v_1, u_1 v_2, u_3 v_2, \dots, u_n v_2, u_2 v_3, u_4 v_3, \dots, u_{n-1} v_3, u_1 v_4, u_3 v_4, \dots, u_n v_4, \dots\}|.$$

Therefore,  $\beta_{u_i v_j} \leq \frac{nm-1}{2}$ . Thus, the average covering number of  $G$  is

$$\begin{aligned} \bar{\beta}(G) &\leq \frac{\left(\frac{mn+1}{2}\right)\left(\frac{mn+1}{2}\right) + \left(\frac{mn-1}{2}\right)\left(\frac{mn-1}{2}\right)}{mn} \\ \bar{\beta}(G) &\leq \frac{2(mn)^2 + 4}{4mn} \\ \bar{\beta}(G) &\leq \frac{(mn)^2 + 1}{2mn}. \end{aligned}$$

Let  $S_{u_i v_j}$  be a minimal local covering set which contains  $u_i v_j$ . If a vertex  $u_i v_j$  is deleted from a graph  $G$ , then every edges of  $G$  which are incident with  $u_i v_j$  are also deleted. If  $u_i v_j$  is equal to  $u'_i v'_j$  or  $u''_i v''_j$  then  $(u'_i v'_j, u''_i v''_j) \notin E(G - u_i v_j)$ . Otherwise,  $(u'_i v'_j, u''_i v''_j) \in E(G - u_i v_j)$  and the edge  $(u'_i v'_j, u''_i v''_j)$  has to be covered by a vertex in the set  $S_{u_i v_j}$ . Since  $S_{u_i v_j}$  is minimal then deletion a vertex from this set not going to be a local covering set. Therefore,  $\bar{\beta}(G) \geq \frac{(mn)^2+1}{2mn}$ . Hence,

$$\bar{\beta}(G) = \frac{mn}{2} + \frac{1}{2mn}.$$

The average covering number of  $P_n \square P_m$  is obtained from the above two cases.

$$\bar{\beta}(P_n \square P_m) = \begin{cases} \frac{mn}{2}, & \text{if } mn \text{ is even} \\ \frac{mn}{2} + \frac{1}{2mn}, & \text{otherwise} \end{cases}$$

□

**Theorem 2.3.** Let  $n, m$  be the positive integers. Then,  $\bar{\beta}(P_n \square C_m) = n \lceil \frac{m}{2} \rceil$ .

*Proof.* Let  $P_n$  be the path of order  $n$ ,  $C_m$  be the cycle of order  $m$  and  $V(P_n) = \{u_1, u_2, \dots, u_n\}$ ,  $V(C_m) = \{v_1, v_2, \dots, v_m\}$ . Hence, we obtain vertex set of  $P_n \square C_m$  as

$$V(P_n \square C_m) = \{u_1v_1, u_2v_1, u_3v_1, \dots, u_nv_1, u_1v_2, u_2v_2, u_3v_2, \dots, u_nv_2, \dots, u_1v_m, u_2v_m, u_3v_m, \dots, u_nv_m\}.$$

We consider four cases here which are obtained from parity of  $n, m$ .

**Case 1:** Let  $n$  and  $m$  are both even. Then we take vertices  $u_iv_j$  ( $i = 1, \dots, n, j = 1, \dots, m$ ) to our local covering set such as

$$\beta_{u_iv_j} = |\{u_1v_1, u_3v_1, \dots, u_{n-1}v_1, u_2v_2, u_4v_2, \dots, u_nv_2, \dots, u_1v_{m-1}, u_3v_{m-1}, \dots, u_nv_{m-1}, u_2v_m, u_4v_m, \dots, u_nv_m\}|$$

or

$$\beta_{u_iv_j} = |\{u_2v_1, u_4v_1, \dots, u_nv_1, u_1v_2, u_3v_2, \dots, u_{n-1}v_2, u_2v_{m-1}, u_4v_{m-1}, \dots, u_{n-1}v_{m-1}, u_1v_m, u_3v_m, \dots, u_{n-1}v_m\}|.$$

This can be seen easily by covering number of  $C_m$ , and the cardinality of these sets are the same for each  $u_iv_j$  ( $i = 1, \dots, n, j = 1, \dots, m$ ),  $\beta_{u_iv_j} \leq \lceil \frac{m}{2} \rceil n$ . If we remove any vertex from our minimal  $S_{u_iv_j}$ , then the new set will not be a local covering set. So,  $\beta_{u_iv_j} \geq \lceil \frac{m}{2} \rceil n$  for  $n$  and  $m$  are both even. Therefore,

$$\bar{\beta}(P_n \square C_m) = n \lceil \frac{m}{2} \rceil.$$

**Case 2:** Let  $n$  and  $m$  are both odd. Then we take vertices  $u_iv_j$  ( $i = 1, \dots, n, j = 1, \dots, m$ ) to our local covering set such as

$$\beta_{u_iv_j} = |\{u_1v_1, u_3v_1, \dots, u_nv_1, u_2v_2, u_4v_2, \dots, u_{n-1}v_2, u_1v_{m-2}, u_3v_{m-2}, \dots, u_nv_{m-2}, u_2v_{m-1}, u_4v_{m-1}, \dots, u_{n-1}v_{m-1}, u_1v_m, u_2v_m, u_3v_m, \dots, u_nv_m\}|$$

or

$$\beta_{u_iv_j} = |\{u_1v_1, u_2v_1, u_3v_1, \dots, u_nv_1, u_2v_2, u_4v_2, \dots, u_{n-1}v_2, u_1v_{m-2}, u_3v_{m-2}, \dots, u_nv_{m-2}, u_2v_{m-1}, u_4v_{m-1}, \dots, u_{n-1}v_{m-1}, u_1v_m, u_3v_m, \dots, u_nv_m\}|.$$

The cardinality of these sets are the same for each  $u_iv_j$  ( $i = 1, \dots, n, j = 1, \dots, m$ ). We know that  $n$  vertices will cover any part of  $P_n \square C_m$  which is seen in Figure 3 as marked by dotted line and there are  $\lfloor \frac{m}{2} \rfloor$  number of parts. We should take  $n$  additional vertex to our local covering set to cover the rest of edges. Thus, for each vertex  $\beta_{u_iv_j} \leq n + \lfloor \frac{m}{2} \rfloor n = n(1 + \lfloor \frac{m}{2} \rfloor) = n \lceil \frac{m}{2} \rceil$ . If we remove any vertex from our minimal local covering set then the new set will not be a local covering set. Hence,  $\beta_{u_iv_j} \geq n \lceil \frac{m}{2} \rceil$  for  $n$  and  $m$  are both odd. Therefore,

$$\bar{\beta}(P_n \square C_m) = n \lceil \frac{m}{2} \rceil.$$

**Case 3:** Let  $n$  be even,  $m$  be odd. Then we take vertices  $u_iv_j$  ( $i = 1, \dots, n, j = 1, \dots, m$ ) to our local covering set such as

$$\beta_{u_iv_j} = |\{u_1v_1, u_3v_1, \dots, u_{n-1}v_1, u_2v_2, u_4v_2, \dots, u_nv_2, u_1v_{m-2}, u_3v_{m-2}, \dots, u_{n-1}v_{m-2}, u_2v_{m-1}, u_4v_{m-1}, \dots, u_nv_{m-1}, u_1v_m, u_2v_m, u_3v_m, \dots, u_nv_m\}|$$

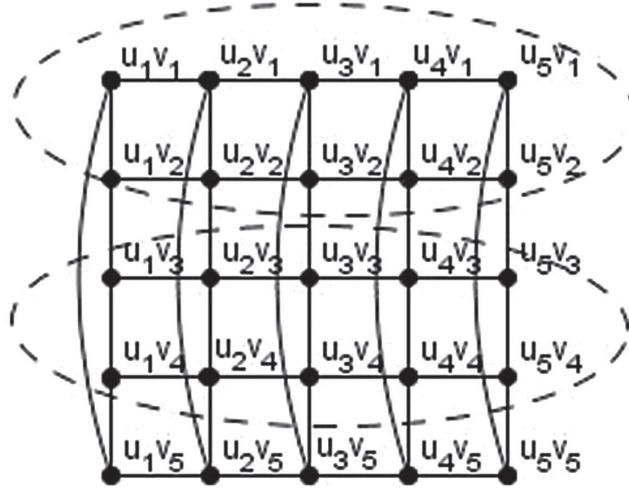


FIGURE 3. Graph  $P_5 \square C_5$ .

or

$$\beta_{u_i v_j} = |\{u_1 v_1, u_2 v_1, u_3 v_1, \dots, u_n v_1, u_2 v_2, u_4 v_2, \dots, u_n v_2, u_1 v_{m-2}, u_3 v_{m-2}, \dots, u_{n-1} v_{m-2}, u_2 v_{m-1}, u_4 v_{m-1}, \dots, u_n v_{m-1}, u_1 v_m, u_3 v_m, \dots, u_{n-1} v_m\}|.$$

As previous cases the cardinality of these sets are the same and it is equal to  $n + (m - 1) \frac{n}{2} = n (\frac{1+m}{2}) = n \lceil \frac{m}{2} \rceil$ . Thus, for each vertex  $\beta_{u_i v_j} \leq n \lceil \frac{m}{2} \rceil$ . If we remove any vertex from our minimal local covering set, then the new set will not be a local covering set. So,  $\beta_{u_i v_j} \geq n \lceil \frac{m}{2} \rceil$  for an even  $n$  and an odd  $m$ . Therefore,

$$\bar{\beta}(P_n \square C_m) = n \lceil \frac{m}{2} \rceil.$$

**Case 4:** Let  $n$  be odd,  $m$  be even. Then we take vertices  $u_i v_j$  ( $i = 1, \dots, n, j = 1, \dots, m$ ) to our local covering set such as

$$\beta_{u_i v_j} = |\{u_1 v_1, u_3 v_1, \dots, u_n v_1, u_2 v_2, u_4 v_2, \dots, u_{n-1} v_2, u_2 v_{m-2}, u_4 v_{m-2}, \dots, u_{n-1} v_{m-2}, u_1 v_{m-1}, u_3 v_{m-1}, \dots, u_n v_{m-1}, u_2 v_m, u_4 v_m, \dots, u_{n-1} v_m\}|$$

or

$$\beta_{u_i v_j} = |\{u_2 v_1, u_4 v_1, u_6 v_1, \dots, u_{n-1} v_1, u_1 v_2, u_3 v_2, \dots, u_n v_2, u_1 v_{m-2}, u_3 v_{m-2}, \dots, u_n v_{m-2}, u_2 v_{m-1}, u_4 v_{m-1}, \dots, u_{n-1} v_{m-1}, u_1 v_m, u_3 v_m, \dots, u_n v_m\}|.$$

As previous cases the cardinality of these sets are the same and it is equal to  $n \lceil \frac{m}{2} \rceil$ . Hence,  $\beta_{u_i v_j} \leq n \lceil \frac{m}{2} \rceil$  for each vertex. If we remove any vertex from our minimal local covering set, then the new set will not be a local covering set. Hence,  $\beta_{u_i v_j} \geq n \lceil \frac{m}{2} \rceil$  while  $n$  is odd and  $m$  is even. Therefore,

$$\bar{\beta}(P_n \square C_m) = n \lceil \frac{m}{2} \rceil.$$

Consequently, from all of the cases above we conclude that

$$\bar{\beta}(P_n \square C_m) = \frac{nmn \lceil \frac{m}{2} \rceil}{nm} = n \lceil \frac{m}{2} \rceil.$$

□

### 3. CONCLUSION

In this paper, the average covering numbers of  $C_n^k$  and  $P_n \square P_m, P_n \square C_m$  are given, respectively. The power of the graphs have same number of vertices as the graph, whereas the product of graphs have usually more vertices than the original graph. Cartesian product of two paths gives us grid graphs, which are clearly used to model city street layouts and more recently processor connections in multiprocessor systems. So, it has wide area of use in daily life. It is known that if  $\bar{\beta}(G_1) > \bar{\beta}(G_2)$ , then  $G_1$  is more stable than  $G_2$  from [5]. Hence, we can use the average covering number as a decision support tool which gives us a new aspect to vulnerability of a graph.

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