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BOUNDS FOR SIGNED DOUBLE ROMAN k-DOMINATION IN TREES

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Abstract. Let $k \geq 1$ be an integer and G be a simple and finite graph with vertex set $V(G)$. A *signed double Roman k-dominating function* (SDRkDF) on a graph G is a function $f : V(G) \rightarrow \{-1, 1, 2, 3\}$ such that (i) every vertex v with $f(v) = -1$ is adjacent to at least two vertices assigned a 2 or to at least one vertex w with $f(w) = 3$, (ii) every vertex v with $f(v) = 1$ is adjacent to at least one vertex w with $f(w) \geq 2$ and (iii) $\sum_{u \in N[v]} f(u) \geq k$ holds for any vertex v. The *weight* of a SDRkDF f is $\sum_{u \in V(G)} f(u)$, and the minimum weight of a SDRkDF is the *signed double Roman k-domination number* $\gamma^k_{sdR}(G)$ of G. In this paper, we investigate the signed double Roman k-domination number of trees. In particular, we present lower and upper bounds on $\gamma^k_{sdR}(T)$ for $2 \leq k \leq 6$ and classify all extremal trees.

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1. Introduction

All graphs considered in this paper are finite, simple, and undirected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The integers $n(G) = |V(G)|$ and $m(G) = |E(G)|$ are the *order* and the *size* of the graph G, respectively. For every vertex $v \in V(G)$, the *open neighborhood* $N_G(v)$ is the set $\{u \in V(G) \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V(G)$ is $\deg_G(v) = \deg(v) = |N(v)|$. We write P_n for the *path* of order n. A *tree* is an acyclic connected graph. A *leaf* of a tree T is a vertex of degree 1, a *support vertex* is a vertex adjacent to a leaf and a *strong support vertex* is a vertex adjacent to at least two leaves. The set of all leaves adjacent to a support vertex v is denoted by L_v. For a vertex v in a rooted tree T, let $C(v)$ denote the set of children of v, $D(v)$ denote the set of descendants of v and $D[v] = D(v) \cup \{v\}$. Also, the *depth* of v, depth(v), is the largest distance from v to a vertex in $D(v)$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A double star with respectively p and q leaves attached at

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each support vertex is denoted by $\mathrm{DS}_{p,q}$. The *corona* $\mathrm{cor}(H)$ of a graph H, is the graph obtained from H by adding a pendant edge to each vertex of H. For a subset $S \subseteq V(G)$ and a function $f : V(G) \to \mathbb{R}$, we define $f(S) = \sum_{x\in S} f(x)$. For a vertex v, we denoted $f(N[v])$ by $f[v]$ for notional convenience.

A *double Roman dominating function* (DRDF) is a function $f : V(G) \to \{0,1,2,3\}$ having the property that if $f(v) = 0$, then vertex v must have at least two neighbors assigned 2 under f or one neighbor with $f(w) = 3$, and if $f(v) = 1$, then vertex v must have at least one neighbor with label at least 2. The *weight* of a double Roman dominating function f is $\omega(f) = \sum_{v\in V(G)} f(v)$. The *double Roman dominating number* of G is the minimum weight of a double Roman dominating function on G. The double Roman domination was introduced by Beeler *et al.* [7] and has been studied in [1–3, 6, 15, 16, 20].

A *signed Roman k-dominating function* (SRkDF) on a graph G is a function $f : V(G) \to \{-1,1,2\}$ satisfying the conditions that (i) $\sum_{x\in N[v]} f(x) \geq k$ for each vertex $v \in V(G)$, and (ii) every vertex u for which $f(u) = -1$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of an SRkDF is the sum of its function values over all vertices. The *signed Roman k-domination number* of G, denoted $\gamma^k_{sR}(G)$, is the minimum weight of an SRkDF in G. The signed Roman k-domination number was introduced by Henning and Volkman in [9] and has been studied in [10–12, 17–19]. The special case $k = 1$ was introduced and investigated in [5] and has been studied in [13, 14].

In this paper, we continue the study of double Roman dominating functions on graphs. Inspired by the previous research on the signed Roman k-domination number [9, 10], we define the signed double Roman k-domination as follows.

Let $k \geq 1$ be an integer. A function $f : V(G) \to \{-1,1,2,3\}$ is *a signed double Roman k-dominating function* (SDRkDF) of G if the following conditions are fulfilled:

(i) $\sum_{x\in N[v]} f(x) \geq k$ for every vertex $v \in V(G)$,
(ii) If $f(v) = -1$, then vertex v must have at least two neighbors with label 2 or one neighbor with label 3,
(iii) If $f(v) = 1$, then vertex v must have at least one neighbor with label 2 or label 3.

The *weight* of a SDRkDF is the sum of its function values over all vertices. The *signed double Roman k-domination number* of G, denoted $\gamma^k_{sdR}(G)$, is the minimum weight of a SDRkDF in G. The special case $k = 1$ has been studied by Ahangar *et al.* [4]. As the assumption $\delta(G) \geq k/3 - 1$ is necessary, we always assume that when we discuss $\gamma^k_{sdR}(G)$, all graphs involved satisfy $\delta(G) \geq k/3 - 1$. We observe that for any graph G with $\delta(G) = 1$, for instance all non-trivial trees, $k = 1,2,3,4,5,6$ are all the possible values for k.

A SDRkDF f can be represented by the ordered quadrant (V_{-1}, V_1, V_2, V_3) where $V_i = \{v \in V(G) \mid f(v) = i\}$ for $i \in \{-1,1,2,3\}$. In this representation, its weight is $\omega(f) = |V_1| + 2|V_2| + 3|V_3| - |V_{-1}|$.

In this paper, we investigate the signed double Roman k-domination number of trees. In particular, we present lower and upper bounds on $\gamma^k_{sdR}(T)$ for $2 \leq k \leq 6$ and classify all extremal trees.

The following facts are easy to prove.

Observation 1.1. Let T be a tree and let f be an SDRkDF on T where $k \geq 2$. Then the following holds.

(i) If v is a leaf and u is its support vertex, then $f(v) \geq 1$ or $f(u) \geq 1$.
(ii) If $k \geq 3$ and v is a leaf or a support vertex in T, then $f(v) \geq 1$.
(iii) If v is a leaf and u is its support vertex, then $f(v) + f(u) \geq k$.
(iv) If $k = 6$ and v is a leaf or a support vertex in T, then $f(v) = 3$.
(V) For $n \geq 1$, then $\gamma^3_{sdR}(K_{1,n}) = n+2$, $\gamma^4_{sdR}(K_{1,n}) = n+3$, $\gamma^5_{sdR}(K_{1,n}) = 2n+3$ and $\gamma^6_{sdR}(K_{1,n}) = 3(n+1)$.
(vi) If $n(T) = 1$, then $\gamma^3_{sdR}(T) = 3$.

2. Paths

Ahangar *et al.* [4] show that

$$\gamma_{sdR}(P_n) = \begin{cases} n/3 & \text{if } n \equiv 0 \pmod 3, \\ \lceil n/3 \rceil + 1 & \text{if } n \equiv 1,2 \pmod 3. \end{cases}$$

In this section, we determine the signed double Roman k-domination number of paths for $k=2,3,4,5,6$ that are all possible values for k.

Proposition 2.1. *For $n \geq 2$, $\gamma^2_{sdR}(P_n)=n$.*

Proof. Let $P_n := v_1v_2\ldots v_n$. If $n \leq 4$, then clearly $\gamma^2_{sdR}(P_n)=n$. Let $n \geq 5$. To show that $\gamma^2_{sdR}(P_n) \leq n$, define $f : V(P_n) \to \{-1,1,2,3\}$ by $f(v_{3i+1}) = -1$, $f(v_{3i+2})=3$ and $f(v_{3i+3})=1$ for $0 \leq i \leq \frac{n-3}{3}$ when $n \equiv 0 \pmod 3$, by $f(v_{3i+3})=f(v_n)= -1$, $f(v_{3i+2})=f(v_{3i+4})=2$ for $0 \leq i \leq \frac{n-7}{3}$, $f(v_{n-1})=3$ and $f(v_1)=f(v_{n-2})=1$ when $n \equiv 1 \pmod 3$, and by $f(v_{3i+3})= -1$, $f(v_{3i+2})=f(v_{3i+4})=2$ for $0 \leq i \leq \frac{n-5}{3}$ and $f(v_1)=f(v_n)=1$ if $n \equiv 2 \pmod 3$. Clearly, f is an SDR2DF on P_n of weight n and hence $\gamma^2_{sdR}(P_n) \leq n$.

Now we show that $\gamma^2_{sdR}(P_n) \geq n$. We proceed by induction on n. It is not hard to see that $\gamma^2_{sdR}(P_n)=n$ for $n \leq 11$. Let $n \geq 12$ and let the statement hold for all paths of order less than n. Let $f=(V_{-1},V_1,V_2,V_3)$ be a $\gamma^2_{sdR}(P_n)$-function such that $|V_3|$ is as small as possible. Since $\gamma^2_{sdR}(P_n) \leq n$, we have $V_{-1} \neq \emptyset$. We first show that each vertex in V_3 is either a leaf or a support vertex. Suppose, to the contrary, that $f(v_i)=3$ where $3 \leq i \leq n-2$. By the choice of f, we must have $f(v_{i-1})= -1$ or $f(v_{i+1})= -1$. Assume without loss of generality that $f(v_{i-1})= -1$. Then we must have $f(v_{i-2})=1$ and $f(v_{i+1}) \geq 1$. By reassigning a value 2 to v_i, v_{i-2}, we obtain a $\gamma^2_{sdR}(P_n)$-function contradicting the choice of f. Therefore $V_3 \subseteq \{v_1,v_2,v_{n-1},v_n\}$. We consider three cases.

Case 1. $f(v_i)= -1$ for some $3 \leq i \leq n-2$.
First let v_i have a neighbor in V_3. Assume without loss of generality that $f(v_{i-1})=3$. Then we must have $i=3$. Since f is a $\gamma^2_{sdR}(P_n)$-function, we must have $f(v_1)=f(v_4)=1$. By reassigning a value 2 to v_2,v_4, we obtain a $\gamma^2_{sdR}(P_n)$-function contradicting the choice of f. Assume now that $f(v_{i-1})=f(v_{i+1})=2$. Then clearly $f(v_{i-2}) \geq 1$ and $f(v_{i+2}) \geq 1$. If $f(v_{i+2}) \geq 2$ (the case $f(v_{i-2}) \geq 2$ is similar), then let P_{n-3} be the path obtained from P_n by removing the vertices v_{i-1},v_i,v_{i+1} and adding an edge $v_{i+2}v_{i-2}$. Obviously, the function f, restricted to P_{n-3} is an SDR2DF on P_{n-3} and by the induction hypothesis we have $\gamma^2_{sdR}(P_n)=\omega(f)=3+\omega(f|_{P_{n-3}}) \geq n$ as desired. Assume that $f(v_{i-2})=f(v_{i+2})=1$. If $f(v_{i+3}) \geq 1$ (the case $f(v_{i-3}) \geq 1$ is similar), then let P_{n-1} be the path obtained from P_n by removing the vertex v_{i+2} and adding an edge $v_{i+1}v_{i+3}$. Clearly, the function f, restricted to P_{n-1} is an SDR2DF on P_{n-1} and by the induction hypothesis we have $\gamma^2_{sdR}(P_n)=\omega(f)=1+\omega(f|_{P_{n-1}}) \geq n$ as desired. Let $f(v_{i-3})=f(v_{i+3})= -1$. Then we must have $f(v_{i-4})=f(v_{i+4})=3$. Thus v_{i+4} (resp. v_{i-4}) is either a support vertex or a leaf. This implies that $n \leq 11$ which is a contradiction.

Case 2. $f(v_1)= -1$ (the case $f(v_n)= -1$ is similar).
Then we must have $f(v_2)=3$ and $f(v_3) \geq 1$. If $f(v_4) \geq 1$, then let P_{n-1} be the path obtained from P_n by removing the vertex v_3 and adding an edge v_2v_4. Clearly, the function f, restricted to P_{n-1} is an SDR2DF on P_{n-1} and by the induction hypothesis we have $\gamma^2_{sdR}(P_n)=\omega(f) \geq 1+\omega(f|_{P_{n-1}}) \geq n$ as desired. Assume that $f(v_4)= -1$. If $f(v_3)=1$, then we must have $f(v_5)=3$ and so v_5 is either a leaf or a support vertex yielding $n \leq 6$, a contradiction. Hence $f(v_3)=2$. This implies that $f(v_5) \geq 2$. As above, we can see that $f(v_5)=2$. Since $f(v_4)+f(v_5)+f(v_6) \geq 2$, we must have $f(v_6) \geq 1$. Let P_{n-3} be the path obtained from P_n by removing the vertices v_3,v_4,v_5 and adding an edge v_2v_6. Clearly, the function f, restricted to P_{n-3} is an SDR2DF on P_{n-3} and by the induction hypothesis we have $\gamma^2_{sdR}(P_n)=\omega(f) \geq 3+\omega(f|_{P_{n-3}}) \geq n$ as desired.

Case 3. $f(v_2)= -1$ (the case $f(v_{n-1})= -1$ is similar).
Since $f(v_1)+f(v_2) \geq 2$, we must have $f(v_1)=3$. By exchanging the values of v_1 and v_2, we stay on Case 2 and the result follows. This completes the proof. □

Now we determine the signed double Roman 3-domination number of paths. Obviously, $\gamma^3_{sdR}(P_1)=3$, $\gamma^3_{sdR}(P_2)=3$, and $\gamma^3_{sdR}(P_3)=4$.

Proposition 2.2. *For $n \geq 4$, $\gamma^3_{sdR}(P_n)=n+2$.*

Proof. If $n=4$, then clearly $\gamma^3_{sdR}(P_n)=6=n+2$. Suppose $n \geq 5$ and define $f : V(P_n) \to \{-1,1,2,3\}$ by $f(v_n)=1$, $f(v_{3i})= -1$ for $1 \leq i \leq \frac{n-3}{3}$ and $f(x)=2$ otherwise, when $n \equiv 0 \pmod 3$, by $f(v_n)=f(v_1)=1$,

$f(v_{3i+1}) = -1$ for $1 \leq i \leq \frac{n-4}{3}$ and $f(x) = 2$ otherwise, when $n \equiv 1 \pmod 3$, and by $f(v_n) = 3, f(v_{n-1}) = 1$, $f(v_{3i}) = -1, f(v_{3i-1}) = 3$ and $f(v_{3i-2}) = 1$ for $1 \leq i \leq \frac{n-2}{3}$ when $n \equiv 2 \pmod 3$. It is easy to verify that f is an SDR3DF of P_n of weight $n+2$ and so $\gamma^3_{sdR}(P_n) \leq n+2$.

To prove the inverse inequality, we proceed by induction on n. Clearly, the results hold for $n \leq 7$. Let $n \geq 8$ and let the statement hold for all paths of order less than n. Assume $P_n := v_1v_2v_3 \dots v_n$ be a path of order n and $f = (V_{-1}, V_1, V_2, V_3)$ is a $\gamma^3_{sdR}(P_n)$-function such that $|V_3|$ is as small as possible. If $V_{-1} = \emptyset$, then there are at least $\lceil \frac{n}{3} \rceil$ vertices in $V_2 \cup V_3$ yielding $\gamma^3_{sdR}(P_n) \geq n + \lceil \frac{n}{3} \rceil$ and this leads to the desired bound. Hence, we assume $V_{-1} \neq \emptyset$. Let $v_i \in V_{-1}$. By Observation 1.1, v_i is neither a leaf nor a support vertex and by definition of SDR3DF, v_i must have two neighbors in V_2 or one neighbor in V_3.

First let v_i have a neighbor in V_3. Assume without loss of generality that $f(v_{i+1}) = 3$. Since $f[v_{i+1}] \geq 3$ and $f[v_i] \geq 3$, we have $f(v_{i+2}) \geq 1$ and $f(v_{i-1}) \geq 1$. If $n = i+2$ (resp. $i-2 = 1$), then the function f, restricted to $P_{n-3} = P_n - \{v_n, v_{n-1}, v_{n-2}\}$ (resp. $P_{n-3} = P_n - \{v_1, v_2, v_3\}$), is an SDR3DF of P_{n-3} and by the induction hypothesis we have

$$\gamma^3_{sdR}(P_n) \geq 3 + \omega(f|_{P_{n-3}}) \geq 3 + \gamma^3_{sdR}(P_{n-3}) \geq 3 + (n-3+2) = n+2.$$

Let $3 \leq i \leq n-3$ and P_{n-3} be the path obtained from P_n by removing the vertices v_i, v_{i+1}, v_{i+2} and adding the edge $v_{i-1}v_{i+3}$. Clearly, the function $g : V(P_{n-3}) \to \{-1, 1, 2, 3\}$ by $g(v_{i-1}) = \max\{f(v_{i-1}), f(v_{i+2})\}$ and $g(x) = f(x)$ otherwise, is an SDR3DF of P_{n-3} and by the induction hypothesis we obtain

$$\gamma^3_{sdR}(P_n) \geq 3 + \omega(g) \geq 3 + \gamma^3_{sdR}(P_{n-3}) \geq 3 + (n-3+2) = n+2.$$

Now let v_i have two neighbors in V_2. That is $f(v_{i-1}) = f(v_{i+1}) = 2$. Since $f[v_{i+1}] \geq 3$ and $f[v_{i-1}] \geq 3$, we must have $f(v_{i+2}) \geq 2$ and $f(v_{i-2}) \geq 2$. Let P_{n-3} be the path obtained from P_n by removing the vertices v_i, v_{i-1}, v_{i+1} and adding the edge $v_{i-2}v_{i+2}$. Clearly, the function f, restricted to P_{n-3}, is an SDR3DF of P_{n-3} and the result follows by the induction hypothesis as above. This completes the proof. □

Obviously, $\gamma^4_{sdR}(P_2) = 4$ and $\gamma^4_{sdR}(P_3) = 5$.

Proposition 2.3. *For $n \geq 4$, $\gamma^4_{sdR}(P_n) = \lceil \frac{4n}{3} \rceil + 2$.*

Proof. Let $P_n := v_1v_2 \dots v_n$. If $n = 4$, then clearly $\gamma^4_{sdR}(P_n) = 8 = \lceil \frac{4n}{3} \rceil + 2$. Let $n \geq 5$. To show that $\gamma^4_{sdR}(P_n) \leq \lceil \frac{4n}{3} \rceil + 2$, define $f : V(P_n) \to \{-1, 1, 2, 3\}$ by $f(v_{3i}) = f(v_{3i+1}) = 1$ for $1 \leq i \leq \lfloor \frac{n-3}{3} \rfloor$ and $f(x) = 2$ otherwise when $n \equiv 0, 1 \pmod 3$, and by $f(v_{3i}) = f(v_{3i+1}) = f(v_{n-2}) = 1$ for $1 \leq i \leq \lfloor \frac{n-3}{3} \rfloor$ and $f(x) = 2$ otherwise when $n \equiv 2 \pmod 3$. Clearly, f is an SDR4DF on P_n yielding $\gamma^4_{sdR}(P_n) \leq \lceil \frac{4n}{3} \rceil + 2$.

Now we show that $\gamma^4_{sdR}(P_n) \geq \lceil \frac{4n}{3} \rceil + 2$. We proceed by induction on n. It is not hard to see that $\gamma^4_{sdR}(P_n) = \lceil \frac{4n}{3} \rceil + 2$ for $n \leq 6$. Let $n \geq 7$ and let the statement hold for all paths of order less than n. Assume $f = (V_{-1}, V_1, V_2, V_3)$ is a $\gamma^4_{sdR}(P_n)$-function. Suppose first that $V_{-1} \neq \emptyset$ and let $v_i \in V_{-1}$. Clearly, v_i is neither a support vertex nor a leaf. Since $f[v_i] \geq 4$, $f[v_{i+1}] \geq 4$ and $f[v_{i-1}] \geq 4$, we have $f(v_{i-1}) + f(v_{i+1}) \geq 5$, $f(v_{i+1}) + f(v_{i+2}) \geq 5$ and $f(v_{i-1}) + f(v_{i-2}) \geq 5$. Let $P' = P_{n-3}$ be the path obtained from P_n by removing the vertices v_{i-1}, v_i, v_{i+1} and adding the edge $v_{i-2}v_{i+2}$, and define $g : V(P') \to \{-1, 1, 2, 3\}$ by $g(x) = f(x)$ for $x \in V(P')$ when $f(v_{i+1}) + f(v_{i-1}) = 5$, and by $g(v_{i-2}) = \min\{f(v_{i-2}) + 1, 3\}$ and $g(x) = f(x)$ otherwise, when $f(v_{i+1}) + f(v_{i-1}) = 6$. Clearly, g is an SDR4DF on P' and by the induction hypothesis we have

$$\gamma^4_{sdR}(P_n) \geq \omega(g) + 4 \geq \left\lceil \frac{4(n-3)}{3} \right\rceil + 6 = \left\lceil \frac{4n}{3} \right\rceil + 2,$$

as desired. Now, let $V_{-1} = \emptyset$. Then obviously $f(v_1) + f(v_2) \geq 4$. Let $P' = P_{n-3}$ be the path obtained from P_n by removing the vertices v_1, v_2, v_3 and define $g : V(P') \to \{-1, 1, 2, 3\}$ by $g(v_4) = \min\{f(v_4) + f(v_3), 3\}$ and $g(x) = f(x)$ otherwise. Clearly, g is an SDR4DF on P' of weight at most $\omega(f) - 4$. By the induction hypothesis, we have

$$\gamma^4_{sdR}(P_n) \geq \omega(g)+4 \geq \left\lceil \frac{4(n-3)}{3} \right\rceil + 6 = \left\lceil \frac{4n}{3} \right\rceil + 2,$$

and the proof is complete. ☐

Proposition 2.4. *For $n \geq 3$,*

$$\gamma^5_{sdR}(P_n) = \begin{cases} \lceil \frac{5n}{3} \rceil + 2 \text{ if } n \equiv 0,\ 2 \pmod 3 \\ \lceil \frac{5n}{3} \rceil + 3 \text{ if } \ n \equiv 1 \pmod 3). \end{cases}$$

Proof. Let $P_n := v_1v_2\dots v_n$. If $n \leq 4$, then the results hold. Let $n \geq 5$. Define the function $f : V(P_n) \rightarrow \{-1,1,2,3\}$ by $f(v_1)=f(v_n)=2$, $f(v_{3i+2})=f(v_{n-1})=3$ for $0 \leq i \leq \lfloor \frac{n-4}{3} \rfloor$ and $f(x)=1$ otherwise. Clearly, f is an SDR5DF on P_n yielding

$$\gamma^5_{sdR}(P_n) \leq \begin{cases} \lceil \frac{5n}{3} \rceil + 2 \text{ if } n \equiv 0,\ 2 \pmod 3 \\ \lceil \frac{5n}{3} \rceil + 3 \text{ if } \ n \equiv 1 \pmod 3). \end{cases}$$

To prove the inverse inequality, we proceed by induction on n. Clearly, the results hold for $n \leq 6$. Let $n \geq 7$ and let the statements hold for all paths of order less than n. Assume $f=(V_{-1},V_1,V_2,V_3)$ is a $\gamma^5_{sdR}(P_n)$-function. Suppose first that $V_{-1} \neq \emptyset$ and let $v_i \in V_{-1}$. By Observation 1.1, v_i is neither a support vertex nor a leaf. Now, $f[v_{i+1}] \geq 5$ and $f[v_{i-1}] \geq 5$ imply that $f(v_{i-2})=f(v_{i-1})=f(v_{i+1})=f(v_{i+2})=3$. Let $P'=P_{n-3}$ be the path obtained from P_n by removing the vertices v_{i-1}, v_i, v_{i+1} and adding the edge $v_{i-2}v_{i+2}$. Obviously, the function f, restricted to P' is an SDR5DF of P' and by the induction hypothesis we have

$$\gamma^5_{sdR}(P_n) \geq \omega(f|_{P'})+5 \geq \left\lceil \frac{5(n-3)}{3} \right\rceil + 5 + x = \left\lceil \frac{5n}{3} \right\rceil + x,$$

where $x=2$ when $n \equiv 0,2 \pmod 3$ and $x=3$ when $n \equiv 1 \pmod 3$.

Now, let $V_{-1}=\emptyset$. By Observation 1.1 we have $f(v_1)+f(v_2) \geq 5$. Let $P'=P_{n-3}$ be the path obtained from P_n by removing the vertices v_1, v_2, v_3 and define $g : V(P') \rightarrow \{-1,1,2,3\}$ by $g(v_4)=f(v_4)+\min\{3-f(v_4), f(v_3)\}$, $g(v_5)=f(v_5)+\min\{3-f(v_5), \min\{3-f(v_4), f(v_4)+\min\{3-f(v_4), f(v_3)\}\}\}$ and $g(x)=f(x)$ otherwise. Clearly, g is an SDR5DF on P' of weight at most $\omega(f)-5$. Now the result follows as above and the proof is complete. ☐

Finally, we determine the signed double Roman 6-domination number of paths. Clearly, $\gamma^6_{sdR}(P_2)=6$.

Proposition 2.5. *For $n \geq 3$,*

$$\gamma^6_{sdR}(P_n) = \begin{cases} 2n+3 \text{ if } n \equiv 0,\ 2 \pmod 3 \\ 2n+4 \text{ if } \ n \equiv 1 \pmod 3). \end{cases}$$

Proof. Let $P_n := v_1v_2\dots v_n$. If $n \leq 5$, then clearly the results hold. Let $n \geq 6$ and define the function $f : V(P_n) \rightarrow \{-1,1,2,3\}$ by $f(v_{3i+1})=2, f(v_{3i})=1$ for $1 \leq i \leq \lfloor \frac{n-4}{3} \rfloor$ and $f(x)=3$ otherwise when $n \equiv 1 \pmod 3$, by $f(v_{3i+1})=2, f(v_{3i})=f(x_{n-2})=1$ for $1 \leq i \leq \lfloor \frac{n-4}{3} \rfloor$ and $f(x)=3$ otherwise when $n \equiv 2 \pmod 3$, and by $f(v_{3i+1})=2, f(v_{3i})=1$ for $1 \leq i \leq \frac{n-3}{3}$ and $f(x)=3$ otherwise when $n \equiv 0 \pmod 3$. Clearly, f is an SDR6DF on P_n yielding

$$\gamma^6_{sdR}(P_n) \leq \begin{cases} 2n+3 \text{ if } n \equiv 0,\ 2 \pmod 3 \\ 2n+4 \text{ if } \ n \equiv 1 \pmod 3). \end{cases}$$

To prove the inverse inequality, we proceed by induction on n. Clearly, the results hold for $n \leq 6$. Let $n \geq 7$ and let the statements hold for all paths of order less than n. Assume $f=(V_{-1},V_1,V_2,V_3)$ is a $\gamma^6_{sdR}(P_n)$-function. Obviously, $|V_{-1}|=0$. By Observation 1.1, $f(v_1)+f(v_2)=6$. Let $P'=P_{n-3}$ be the path obtained from P_n by removing the vertices v_1, v_2, v_3 and define $g : V(P') \rightarrow \{-1,1,2,3\}$ by $g(v_4)=f(v_4)+\min\{3-f(v_4), f(v_3)\}$, $g(v_5)=f(v_5)+\min\{3-f(v_5), \min\{3-f(v_4), f(v_4)+\min\{3-f(v_4), f(v_3)\}\}\}$ and $g(x)=f(x)$ otherwise. Clearly, g is an SDR6DF on P' of weight at most $\omega(f)-6$. By the induction hypothesis we have

$$\gamma^6_{sdR}(P_n) \geq \omega(g)+6 \geq 2(n-3)+6+x = 2n+x,$$

where $x=3$ when $n \equiv 0,2 \pmod 3$ and $x=4$ when $n \equiv 1 \pmod 3$. This completes the proof. ☐

3. Trees

In [4], Ahangar *et al.* prove that for any tree T of order $n \geq 2$, $\frac{-5n+24}{9} \leq \gamma_{sdR}(T) \leq n$ and they characterize all trees achieving the lower and the upper bounds. Our aim in this section is to establish lower and upper bounds on the signed double Roman k-domination number of a tree in terms of its order for $k=2,3,4,5,6$.

3.1. $k=2$

First we determine the signed double Roman 2-domination number of stars.

Proposition 3.1. *For $n \geq 3$, $\gamma^2_{sdR}(K_{1,n})=2$.*

Proof. Let $\{v, v_1, \dots, v_n\}$ be the vertex set of $K_{1,n}$, where v is the central vertex of $K_{1,n}$. By definition, we have $\gamma^2_{sdR}(K_{1,n}) = \sum_{u\in N[v]} f(u) \geq 2$. Define $g : V(K_{1,n}) \to \{-1,1,2,3\}$ by $g(v)=3$ and $g(v_i)=(-1)^i$ for $1 \leq i \leq n$ when n is odd, and by $g(v)=3, g(v_1)=2, g(v_i)= -1$ for $i=2,3,4$ and $g(v_i)=(-1)^i$ for $5 \leq i \leq n$ when n is even. It is easy to see that g is an SDR2DF of $K_{1,n}$ of weight 2, implying that $\gamma^2_{sdR}(K_{1,n})=2$. □

Proposition 3.2. *For $r \geq s \geq 1$, we have*

$$\gamma^2_{sdR}(\mathrm{DS}_{r,s}) \geq \frac{-(r+s+2)}{2} + 3.$$

The equality holds if and only if $T=\mathrm{DS}_{4,4}$.

Proof. Let $T=\mathrm{DS}_{r,s}$ and f be a $\gamma^2_{sdR}(T)$-function. Suppose u, v are the non-leaf vertices of T, $u_1,\dots,u_s$ are the leaves adjacent to u and $v_1,\dots,v_r$ are the leaves adjacent to v. If $r=s=1$, then $\mathrm{DS}_{1,1}=P_4$ and from Proposition 2.1, $\gamma^2_{sdR}(\mathrm{DS}_{1,1})=4 > \frac{-(r+s+2)}{2}+3$. Let $r \geq 2$. If $f(u) \leq 2$ (the case $f(v) \leq 2$ is similar), then we must have $f(u_i) \geq 1$ for each $1 \leq i \leq s$ because f is a SDR2DF of T and this implies that

$$\gamma^2_{sdR}(T)=\omega(f)= \sum_{x\in N[v]} f(x) + \sum_{i=1}^{s} f(u_i) \geq 2+s \geq 3 > \frac{-(r+s+2)}{2} + 3$$

as desired. Assume that $f(u)=f(v)=3$. Since f is an SDR2DF of T, we must have

$$2 \leq \sum_{x\in N[u]} f(x) = f(u)+f(v)+\sum_{i=1}^{s} f(u_i) = 6+\sum_{i=1}^{s} f(u_i) \tag{3.1}$$

and

$$2 \leq \sum_{x\in N[v]} f(x) = f(u)+f(v)+\sum_{i=1}^{s} f(v_i) = 6+\sum_{i=1}^{s} f(v_i). \tag{3.2}$$

Using inequalities (3.1) and (3.2), and the fact that f is $\gamma^2_{sdR}(T)$-function, we obtain $\sum_{i=1}^{s} f(u_i)= -4$ and $\sum_{i=1}^{r} f(v_i)=-4$ when $r \geq s \geq 6$, $\sum_{i=1}^{s} f(u_i)=-3$ and $\sum_{i=1}^{r} f(v_i)=-4$ when $s=5$ and $r \geq 6$, $\sum_{i=1}^{s} f(u_i)=-s$ and $\sum_{i=1}^{r} f(v_i)= -4$ when $s \leq 4$ and $r \geq 6$, $\sum_{i=1}^{s} f(u_i)= \sum_{i=1}^{r} f(v_i)= -3$ when $r,s=5$, $\sum_{i=1}^{s} f(u_i)= -s$ and $\sum_{i=1}^{r} f(v_i)= -3$ when $r=5$ and $s \leq 4$, and $\sum_{i=1}^{s} f(u_i)= -s$ and $\sum_{i=1}^{r} f(v_i)= -r$ when $r,s \leq 4$. Since

$$\gamma^2_{sdR}(T)=\omega(f)=f(u)+f(v)+\sum_{i=1}^{s} f(u_i)+\sum_{i=1}^{r} f(v_i),$$

we have $\gamma^2_{sdR}(T)=6-s-r$ if $r,s \leq 4$, $\gamma^2_{sdR}(T)=0$ if $r=s=5$, $\gamma^2_{sdR}(T)=3-s$ if $r=5, s \leq 4$, $\gamma^2_{sdR}(T)=2-s$ if $s \leq 4$ and $r \geq 6$, $\gamma^2_{sdR}(T)= -1$ if $s=5$ and $r \geq 6$, and $\gamma^2_{sdR}(T)= -2$ when $r \geq s \geq 6$. This implies that $\gamma^2_{sdR}(\mathrm{DS}_{r,s}) \geq \frac{-(r+s+2)}{2}+3$ with equality if and only if $T=\mathrm{DS}_{4,4}$. □

Let G be a connected graph which is not complete, let S be a vertex cut of G, and let X be the vertex set of a component of $G-S$. The subgraph H of G induced by $S\cup X$ is called an *S-component* of G.

For any tree T, assume F_T^2 is the tree obtained from T by adding $3\deg_T(v)+1$ pendant edges at v for each $v\in V(T)$. Assume that $\mathcal{T}_2=\{F_T^2 \mid T \text{ is a tree}\}$.

Theorem 3.3. *Let T be a tree of order $n\geq 2$. Then*

$$\gamma_{sdR}^2(T)\geq \frac{-n+6}{2}$$

with equality if and only if $T\in\mathcal{T}_2$.

Proof. The proof is by induction on n. The cases $n=2$ and $n=3$ follows from Proposition 2.1. Let $n\geq 4$ and assume that the statement is true for all trees of order less than n. Let T be a tree of order n. If $\mathrm{diam}(T)=2$, then T is a star and by Proposition 3.1 we have $\gamma_{sdR}^2(T)=2>\frac{-n+6}{2}$. If $\mathrm{diam}(T)=3$, then T is a double star $\mathrm{DS}_{p,q}$ with $q\geq p\geq 1$ and by Proposition 3.2, we have $\gamma_{sdR}^2(T)\geq\frac{-n+6}{2}$ with equality if and only if $T=\mathrm{DS}_{4,4}$. Therefore, we assume that $\mathrm{diam}(T)\geq 4$. Suppose $f=(V_{-1},V_1,V_2,V_3)$ is a $\gamma_{sdR}^2(T)$-function.

If there is a non-pendant edge uv such that $u,v\in V_{-1}$, then let T_1 and T_2 be the components of $T-uv$. Clearly, the function $f_i=f|_{T_i}$ is an SDR2DF of T_i for $i=1,2$, and we conclude from the induction hypothesis and the fact $w(f)=w(f_1)+w(f_2)$, that

$$\gamma_{sdR}^2(T)=\omega(f_1)+\omega(f_2)\geq\frac{-|V(T_1)|+6}{2}+\frac{-|V(T_2)|+6}{2}>\frac{-n+6}{2}\cdot$$

Henceforth, we assume that there is no non-pendant edge uv for which $u,v\in V_{-1}$. On the other hand, for any pendant edge uv, we must have $f(u)\geq 1$ or $f(v)\geq 1$ by Observation 1.1. Hence, we may assume that V_{-1} is an independent set. We consider the following cases.

Case 1. There is a non-leaf vertex v with $f(v)=-1$.

By definition, f must assign a 3 to any leaf adjacent to v. Suppose $T_1,\dots,T_r$ are the components of $T-v$ of order at least two. Since $\mathrm{diam}(T)\geq 4$, we have $r\geq 1$. Also the function $f_i=f|_{T_i}$ is an SDR2DF of T_i for each $i\in\{1,\dots,r\}$. If v is not a support vertex, then $r\geq 2$, and by the induction hypothesis we have

$$\gamma_{sdR}^2(T)=\sum_{i=1}^{r}\omega(f_i)+f(v)\geq\sum_{i=1}^{r}\frac{-|V(T_i)|+6}{2}+f(v)=\frac{-n+6}{2}+\frac{6r-5}{2}-1>\frac{-n+6}{2}\cdot$$

If v is a support vertex, then let $L_v=\{u_1,\dots,u_s\}$. If $s=1$, then we have

$$\gamma_{sdR}^2(T)=f(v)+f(u_1)+\sum_{i=1}^{r}\omega(f_i)\geq\sum_{i=1}^{r}\frac{-|V(T_i)|+6}{2}+2=\frac{-(n-2)+6r}{2}+2>\frac{-n+6}{2}\cdot$$

If $s\geq 2$, then let T' be the subtree of T induced by $L_v\cup\{v\}$. Obviously, the function $f'=f|_{T'}$ is an SDR2DF of T' and by the induction hypothesis we have

$$\begin{aligned}\gamma_{sdR}^2(T)&=\omega(f')+\sum_{i=1}^{r}\omega(f_i)\\&=(3s-1)+\sum_{i=1}^{r}\omega(f_i)\\&\geq\frac{-(s+1)+6}{2}+\sum_{i=1}^{r}\frac{-|V(T_i)|+6}{2}\\&=\frac{-n+6(r+1)}{2}\\&>\frac{-n+6}{2}\cdot\end{aligned}$$

By Case 1, we may assume that every vertex in V_{-1} is a leaf.

Case 2. There is a non-leaf vertex v with $f(v)=1$.
By definition, f must assign at least a 2 to any leaf adjacent to v, and since $\mathrm{diam}(T)\geq 4$, $T-v$ must have at least a component of order at least two. Let $T_1,\ldots,T_r$ be the components of $T-v$ of order at least two and let $v_i\in V(T_i)$ be the vertex adjacent to v for each $i\in\{1,\ldots,r\}$. Since v_i is not a leaf, we deduce from the assumption that $v_i\in\cup_{i=1}^{3}V_i$ and so $f(v_i)\geq 1$ for each $i\in\{1,\ldots,r\}$. Suppose $f(v_1)=\min\{f(v_i)\mid i=1,\ldots,r\}$ and let F_1 and F_2 are the components of $T-vv_1$ containing v_1 and v, respectively. If $f(v_1)=1$ or 2, then the function $f_2=f|_{F_2}$ is an SDR2DF of F_2 and the function $g:V(F_1)\to\{-1,1,2,3\}$ defined by $g(v_1)=f(v_1)+1$ and $g(x)=f(x)$ for $x\in V(F_1)-\{v_1\}$, is an SDR2DF of F_1. By the induction hypothesis, we obtain

$$\gamma^2_{sdR}(T)=\omega(g)+\omega(f_2)-1\geq \frac{-|V(F_1)|+6}{2}+\frac{-|V(F_2)|+6}{2}-1=\frac{-n+6}{2}+2>\frac{-n+6}{2}\cdot$$

Assume that $f(v_1)=3$. Then $f(v_i)=3$ for each $i\in\{1,\ldots,r\}$. Let F_1' be the tree obtained from F_1 by adding a pendant edge v_1v' and define $g:V(F_1')\to\{-1,1,2,3\}$ by $g(v')=1$ and $g(x)=f(x)$ for $x\in V(F_1')-\{v'\}$. Obviously, g is an SDR2DF of F_1' and the function $f_2=f|_{F_2}$ is an SDR2DF of F_2. It follows from the induction hypothesis that

$$\gamma^2_{sdR}(T)=\omega(g)+\omega(f_2)-1\geq \frac{-|V(F_1')|+6}{2}+\frac{-|V(F_2)|+6}{2}-1=\frac{-n+9}{2}>\frac{-n+6}{2}\cdot$$

Considering Cases 1 and 2, we may assume that all non-leaf vertices of T are assigned 2 or 3 under f.

Case 3. There is a non-leaf vertex v with $f(v)=2$.
Then any leaf adjacent to v (if any) must be assigned at least 1 under f. If v is a support vertex and $v'\in L_v$, then the function $f'=f|_{T-v'}$ is an SDR2DF of $T-v'$, and we conclude from the induction hypothesis that

$$\gamma^2_{sdR}(T)\geq\omega(f')+1\geq\frac{-(n-1)+6}{2}+1>\frac{-n+6}{2}\cdot$$

Assume v is not a support vertex, $N(v)=\{v_1,v_2,\ldots,v_r\}$ and T_i is the $\{v\}$-components of T containing v_i for each $i\in\{1,\ldots,r\}$. Obviously, the function $f_i=f|_{T_i}$ is an SDR2DF of T_i for each $i\in\{1,2,\ldots,r\}$. Note that $n=\sum_{i=1}^{r}|V(T_i)|-(r-1)$ and $\omega(f)=(\sum_{i=1}^{r}\omega(f_i))-2(r-1)$. By the induction hypothesis, we have

$$\begin{aligned}\gamma^2_{sdR}(T)&=\left(\sum_{i=1}^{r}\omega(f_i)\right)-2(r-1)\\&\geq\left(\sum_{i=1}^{r}\frac{-|V(T_i)|+6}{2}\right)-2(r-1)\\&=\frac{-n+6}{2}+\frac{5(r-1)}{2}-2(r-1)\\&>\frac{-n+6}{2}\cdot\end{aligned}$$

Considering above Cases, we may assume that all non-leaf vertices of T are assigned a 3 under f.

Case 4. There is a non-leaf vertex v such that v is not a support vertex.
Let $N(v)=\{v_1,v_2,\ldots,v_k\}$. By assumption we have $N[v]\subseteq V_3$. Let T^* be the graph obtained from $T-v$ by adding the edges $v_1v_2,v_2v_3,\ldots,v_{k-1}v_k$. Clearly, T^* is a tree of order $n-1$ and the function $f^*=f|_{T^*}$ is an SDR2DF on T^* of weight $\omega(f)-3$. We conclude from the induction hypothesis that

$$\gamma^2_{sdR}(T)=\omega(f^*)+3\geq\frac{-|V(T^*)|+6}{2}+3>\frac{-n+6}{2}\cdot$$

Case 5. Each vertex of T is either a leaf or a support vertex.
Obviously, for any support vertex, f assigns a -1 to at least one leaf adjacent to it. Let v be a support vertex and let $L_v = \{u_1, u_2, \ldots, u_s\}$.

If $f(u_i) = 1$ and $f(u_j) = -1$ for some i, j, then let $T' = T - \{u_i, u_j\}$. Clearly, the function f, restricted to T' is an SDR2DF of T' and by the induction hypothesis we have

$$\gamma^2_{sdR}(T) = \omega(f|_{T'}) \geq \frac{-(n-2)+6}{2} > \frac{-n+6}{2}\cdot$$

If $f(u_i) = t \in \{2, 3\}$ and $f(u_j) = -1$ for some i, j, then let $T' = T - u_j$. Obviously, the function $g : V(T') \to \{-1, 1, 2, 3\}$ defined by $g(u_i) = t - 1$ and $g(x) = f(x)$ otherwise, is an SDR2DF of T'. It follows from the induction hypothesis that $\gamma^2_{sdR}(T) = \omega(g) \geq \frac{-(n-1)+6}{2} > \frac{-n+6}{2}\cdot$

Henceforth, we assume that all leaves of T belong to V_{-1}. Recall that all support vertices belong to V_3. For every support vertex v, let $l_v = |L_v|$. Suppose T' is the tree obtained from T by removing all leaves of T. Since for every support vertex v, $f[v] \geq 2$, we must have $l_v \leq 3\deg_{T'}(v) + 1$. Hence

$$\sum_{v\in V(T')} l_v \leq \sum_{v\in V(T')} (3\deg_{T'}(v) + 1) = 7n(T') - 6. \tag{3.3}$$

On the other hand, since $n = n(T') + \sum_{v\in V(T')} l_v$ and $\gamma^2_{sdR}(T) = 3n(T') - \sum_{v\in V(T')} l_v$, it follows from (3.3) that

$$\gamma^2_{sdR}(T) = 3n(T') - \sum_{v\in V(T')} l_v \geq \frac{-(n(T') + \sum_{v\in V(T')} l_v) + 6}{2} = \frac{-n+6}{2}\cdot \tag{3.4}$$

If moreover $\gamma^2_{sdR}(T) = \frac{-n+6}{2}$, then all inequalities occurring in (3.3) and (3.4) become equality. In particular, we must have $l_v = 3\deg_{T'}(v) + 1$ for each $v \in V(T')$ yielding $T = F^2_{T'} \in \mathcal{T}_2$.

Conversely, let $T \in \mathcal{T}_2$. Clearly, the function $g : V(T) \to \{-1, 1, 2, 3\}$ that assigns a -1 to each leaf and a 3 to each support vertex, is an SDR2DF of T yielding $\gamma^2_{sdR}(T) \leq \frac{-n+6}{2}$. This implies that $\gamma^2_{sdR}(T) = \frac{-n+6}{2}$ and the proof is complete. □

Theorem 3.4. For any tree T of order $n \geq 2$, $\gamma^2_{sdR}(T) \leq n$.

Proof. The proof is by induction on n. The result is immediate for $n \leq 3$ by Proposition 2.1. Assume $n \geq 4$ and let the statement hold for all trees of order less than n. Suppose T is a tree of order n. If $\mathrm{diam}(T) = 2$, then the result is trivial by Proposition 3.1. If $\mathrm{diam}(T) = 3$, then T is a double star $\mathrm{DS}_{p,q}$ with $q \geq p \geq 1$ and by the upper bounds presented in the proof of Proposition 3.2, we have $\gamma^2_{sdR}(T) \leq n$. Therefore, we assume that $\mathrm{diam}(T) \geq 4$.

Assume $v_1v_2\ldots v_k$ ($k \geq 5$) is a diametrical path in T such that $\deg_T(v_2)$ is as large as possible and root T at v_k. If $\deg_T(v_2) \geq 3$, then let $T' = T - T_{v_2}$ and let f be a $\gamma^2_{sdR}(T')$-function. Define $g : V(T) \to \{-1, 1, 2, 3\}$ by $g(v_1) = -1$, $g(v_2) = 3$, $g(x) = 1$ for $x \in L_{v_2} - \{v_1\}$ and $g(x) = f(x)$ for $x \in V(T')$. Clearly, g is an SDR2DF of T and by the induction hypothesis we have

$$\gamma^2_{sdR}(T) \leq \omega(g) = \gamma^2_{sdR}(T') + |V(T_{v_2})| \leq |V(T')| + |V(T_{v_2})| = n.$$

Hence, we assume that $\deg_T(v_2) = 2$. By the choice of diametrical path, all children of v_3 have degree at most two. Let $C_2(v_3) = \{z_1 = v_2, z_2, \ldots, z_t\}$ be the set of all children of v_3 with degree two and let z'_i be the leaf adjacent to z_i for each i. Clearly, $C(v_3) = C_2(v_3) \cup L_{v_3}$. Let $T' = T - T_{v_3}$ and f be a $\gamma^2_{sdR}(T')$-function.

If $L_{v_3} \neq \emptyset$, then let $w \in L_{v_3}$ and define $g : V(T) \to \{-1, 1, 2, 3\}$ by $g(w) = -1, g(v_3) = g(z_i) = 3$, $g(z'_i) = -1$ for $1 \leq i \leq t$, $g(x) = 1$ for $x \in L_{v_3} - \{w\}$, and $g(x) = f(x)$ for $x \in V(T')$. Clearly, g is an SDR2DF of T and by the induction hypothesis we have

$$\gamma^2_{sdR}(T) \leq \omega(g) = \gamma^2_{sdR}(T') + |V(T_{v_3})| \leq n.$$

If $L_{v_3} = \emptyset$, then define $g : V(T) \to \{-1, 1, 2, 3\}$ by $g(v_3) = 1, g(z_i) = 3$, $g(z'_i) = -1$ for $1 \le i \le t$, and $g(x) = f(x)$ for $x \in V(T')$. It is easy to see that g is an SDR2DF of T and we deduce from the induction hypothesis that $\gamma^2_{sdR}(T) \le \omega(g) = \gamma^2_{sdR}(T') + |V(T_{v_3})| \le n$. This completes the proof. □

3.2. $k = 3$

Here, we present lower and upper bounds on the signed double Roman 3-domination number of a tree T in terms of its order.

Theorem 3.5. *Let T be a tree of order $n \ge 1$. Then*

$$\gamma^3_{sdR}(T) \ge \frac{4n+7}{5}$$

with equality if and only if $T = P_2$.

Proof. We proceed by induction on n. If $n = 1$ then $\gamma^3_{sdR}(T) = 3 > \frac{4n+7}{5}$, if $n = 2$ then $\gamma^3_{sdR}(T) = 3 = \frac{4n+7}{5}$, and if $n = 3$ then $\gamma^3_{sdR}(T) = 4 > \frac{4n+7}{5}$ by Observation 1.1 (part (v)). Assume that $n \ge 4$ and the statement holds for all trees of order less than n. Suppose T is a tree of order n. If $\mathrm{diam}(T) = 2$, then $T = K_{1,n-1}$ and by Observation 1.1 (part (v)) we have $\gamma^3_{sdR}(T) = n + 1 > \frac{4n+7}{5}$. If $\mathrm{diam}(T) = 3$, then T is a double star $\mathrm{DS}_{p,q}$ with $q \ge p \ge 1$ and by Observation 1.1 (part (iii)), we have $\gamma^3_{sdR}(T) \ge p + q + 4 > \frac{4(p+q+2)+7}{5}$. Hence, we assume that $\mathrm{diam}(T) \ge 4$. Suppose $f = (V_{-1}, V_1, V_2, V_3)$ is a $\gamma^3_{sdR}(T)$-function. If $V_{-1} = \emptyset$, then clearly $\gamma^3_{sdR}(T) \ge n+1 > \frac{4n+7}{5}$. Henceforth, we assume $V_{-1} \ne \emptyset$. If there is a non-pendant edge uv such that $u, v \in V_{-1}$, then using an argument similar to that described in Theorem 3.3 we obtain $\gamma^3_{sdR}(T) > \frac{4n+7}{5}$. Assume that there is no non-pendant edge uv for which $u, v \in V_{-1}$. As in the proof of Theorem 3.3, we may assume that V_{-1} is independent. Also, by Observation 1.1 (part (ii)), each vertex in V_{-1} is neither a leaf nor a support vertex.

Let $v \in V_{-1}$ and let $T_1, \dots, T_r$ $(r \ge 2)$ be the components of $T - v$. Clearly, the function $f_i = f|_{T_i}$ is an SDR3DF of T_i for each i. If $r \ge 3$, then we deduce from the induction hypothesis that

$$\begin{aligned}\gamma^3_{sdR}(T) &= \sum_{i=1}^{r} \omega(f_i) + f(v)\\ &\ge \sum_{i=1}^{r} \frac{4|V(T_i)|+7}{5} + f(v)\\ &\ge \frac{4(n-1)+7r}{5} - 1\\ &= \frac{4n+7}{5} + \frac{-4+7(r-1)}{5} - 1\\ &> \frac{4n+7}{5}.\end{aligned}$$

Let $r = 2$. If $|V_{-1}| = 1$, then clearly $\omega(f|_{T_1}) \ge |V(T_1)| + 1$ and $\omega(f|_{T_2}) \ge |V(T_2)| + 1$. This implies that $\gamma^3_{sdR}(T) = \omega(f|_{T_1}) + \omega(f|_{T_2}) - 1 \ge n + 1 > \frac{4n+7}{5}$. Hence we assume $|V_{-1}| \ge 2$. Let $u \in V_{-1}$ be a vertex such that $d(u, v)$ is as large as possible and let $T'_1, \dots, T'_s$ $(s \ge 2)$ be the components of $T - u$ and let $f'_i = f|_{T'_i}$ for each i. Assume without loss of generality that $v \in V(T'_1)$. If $s \ge 3$, then we have $\gamma^3_{sdR}(T) > \frac{4n+7}{5}$ as above. Suppose $s = 2$. By the choice of u, we must have $V_{-1} \cap V(T'_2) = \emptyset$. Then we have $\omega(f'_2) \ge |V(T_2)| + 1$. Now, by the induction hypothesis we obtain

$$\gamma^3_{sdR}(T) = \omega(f'_1) + \omega(f'_2) - 1 \ge \frac{4|V(T'_1)|+7}{5} + |V(T'_2)| + 1 - 1 > \frac{4n+7}{5}$$

and the proof is complete. □

Let $\mathcal{F} = \{\text{cor}(T) \mid T \text{ is a tree}\}$.

Theorem 3.6. *Let T be a tree of order $n \geq 2$. Then*

$$\gamma^3_{sdR}(T) \leq \frac{3n}{2}$$

with equality if and only if $T \in \mathcal{F}$.

Proof. The proof is by induction on n. The cases $n=2$ and $n=3$ follows from Proposition 2.2. Let $n \geq 4$ and let the statement hold for all trees of order less than n. Assume T is a tree of order n. If $\text{diam}(T)=2$, then $T=K_{1,n-1}$ and by Observation 1.1 (part (v)), we have $\gamma^3_{sdR}(T)=n+1 < \frac{3n}{2}$. If $\text{diam}(T)=3$, then T is a double star $\text{DS}_{p,q}$ with $q \geq p \geq 1$ and $\gamma^3_{sdR}(T)=n+2 \leq \frac{3n}{2}$ with equality if and only if $T=P_4=\text{cor}(P_2)$. Therefore, we assume that $\text{diam}(T) \geq 4$. Let $v_1v_2\dots v_k$ ($k \geq 5$) be a diametrical path in T such that $\deg_T(v_2)$ is as large as possible and root T at v_k.

If $\deg_T(v_2) \geq 3$, then let $T'=T-v_1$ and let $f=(V_{-1},V_1,V_2,V_3)$ be a $\gamma^3_{sdR}(T')$-function. Since v_2 is a support vertex in T', by Proposition 1.1 (part (iii)) we may assume that $f(v_2) \geq 2$. Then $g=(V_{-1},V_1 \cup \{v_1\},V_2,V_3)$ is an SDR3DF of T and by the induction hypothesis we have $\gamma^3_{sdR}(T) \leq \omega(g)=\gamma^3_{sdR}(T')+1 \leq \frac{3(n-1)}{2}+1 < \frac{3n}{2}$.

Assume that $\deg_T(v_2)=2$. Then by the choice of diametrical path, all children of v_3 with depth 1, must have degree 2. We consider two cases.

Case 1. $\deg_T(v_3) \geq 3$.
First let v_3 be a support vertex. Let $T'=T-\{v_1,v_2\}$ and let $f=(V_{-1},V_1,V_2,V_3)$ be a $\gamma^3_{sdR}(T')$-function. Since v_3 is a support vertex, we may assume that $f(v_3) \geq 2$. Then $g=(V_{-1},V_1 \cup \{v_1\},V_2 \cup \{v_2\},V_3)$ is an SDR3DF of T and by the induction hypothesis we have

$$\gamma^3_{sdR}(T) \leq \omega(g)=\gamma^3_{sdR}(T')+3 \leq \frac{3(n-2)}{2}+3 \leq \frac{3n}{2}\cdot$$

Note that the equality holds if and only if $\gamma^3_{sdR}(T')=\frac{3(n-2)}{2}$, i.e., if and only if $T' \in \mathcal{F}$, and this if and only if $T \in \mathcal{F}$. Henceforth, we may assume that all children of v_3 have degree 2. Let $T'=T-T_{v_3}$ and let $f=(V_{-1},V_1,V_2,V_3)$ be a $\gamma^3_{sdR}(T')$-function. Then $g=(V_{-1},V_1 \cup (D[v_3]-C(v_3)),V_2 \cup C(v_3),V_3)$ is an SDR3DF of T and by the induction hypothesis we have

$$\gamma^3_{sdR}(T) \leq \omega(g)=\gamma^3_{sdR}(T')+3|C(v_3)|+1 \leq \frac{3(n-(2|C(v_3)|+1))}{2}+3|C(v_3)|+1 < \frac{3n}{2}\cdot$$

Case 2. $\deg_T(v_3)=2$.
If $\deg(v_4)=2$ or v_4 is a support vertex, then let $T'=T-\{v_1,v_2,v_3\}$ and let $f=(V_{-1},V_1,V_2,V_3)$ be a $\gamma^3_{sdR}(T')$-function. Since v_4 is a leaf or a support vertex in T', we have $f(v_4) \geq 1$ by Proposition 1.1 (part (ii)). Then $g=(V_{-1},V_1 \cup \{v_1,v_3\},V_2 \cup \{v_2\},V_3)$ is an SDR3DF of T and by the induction hypothesis we obtain $\gamma^3_{sdR}(T) \leq \omega(g)=\gamma^3_{sdR}(T')+4 \leq \frac{3(n-3)}{2}+4 < \frac{3n}{2}$.

Assume that $\deg(v_4) \geq 3$ and that v_4 is not a support vertex. Using above arguments, we may assume that all vertices in T_{v_4} except v_4 have degree at most two. If $\text{diam}(T)=4$, then clearly $\gamma^3_{sdR}(T) < \frac{3n}{2}$. Assume that $\text{diam}(T) \geq 5$, $T'=T-T_{v_4}$ and $f=(V_{-1},V_1,V_2,V_3)$ is a $\gamma^3_{sdR}(T')$-function. Let $v_4x_3^jx_2^jx_1^j$ ($1 \leq j \leq r$) be the paths of length 3 in T_{v_4} and $v_4y_2^jy_1^j$ ($1 \leq j \leq s$) be the paths of length 2 in T_{v_4}, if any. If $s \geq 1$, then define $g: V(T) \to \{-1,1,2,3\}$ by $g(v_4)=1, g(x_3^j)=g(x_1^j)=1$, $g(x_2^j)=2$ for $1 \leq j \leq r$, $g(y_2^j)=2, g(y_1^j)=1$ for $1 \leq j \leq s$, and $g(x)=f(x)$ for $x \in V(T')$. Clearly, g is an SDR3DF of T and we deduce from the induction hypothesis that

$$\gamma^3_{sdR}(T) \leq \gamma^3_{sdR}(T')+3s+4r+1 \leq \frac{3(n-(3r+2s+1))}{2}+3s+4r+1 < \frac{3n}{2}\cdot$$

If $s=0$, then define $g : V(T) \to \{-1,1,2,3\}$ by $g(v_4)=2, g(x_3^j)=g(x_1^j)=1,\ g(x_2^j)=2$ for $1 \le j \le r$, and $g(x)=f(x)$ for $x \in V(T')$. Clearly, g is an SDR3DF of T and by the induction hypothesis we have

$$\gamma_{sdR}^3(T) \le \gamma_{sdR}^3(T') + 4r + 2 \le \frac{3(n-(3r+1))}{2} + 4r + 2 < \frac{3n}{2}\cdot$$

This completes the proof. □

3.3. $k=4$

First we present an upper bound on $\gamma_{sdR}^4(T)$ and characterize all extreme trees. Let $\mathcal{F}=\{\mathrm{cor}(T) \mid T \text{ is a tree}\}$.

Theorem 3.7. *For any tree T of order $n \ge 2$,*

$$\gamma_{sdR}^4(T) \le 2n.$$

The equality holds if and only $T \in \mathcal{F}$.

Proof. Let T be a tree of order n. Clearly, the function $f : V(T) \to \{-1,1,2,3\}$ defined by $f(x)=2$ for $x \in V(T)$, is an SDR4DF of T of weight $2n$ and hence $\gamma_{sdR}^4(T) \le 2n$.

If $T \in \mathcal{F}$, then we deduce from Observation 1.1 (part (iii)) that $\gamma_{sdR}^4(T) \ge 2n$ and so $\gamma_{sdR}^4(T)=2n$.

Conversely, let $\gamma_{sdR}^4(T)=2n$. If T has a strong support vertex, say v, then the function $g : V(T) \to \{-1,1,2,3\}$ defined by $g(v)=3, g(x)=1$ for $x \in L_v$ and $g(x)=2$ otherwise, is an SDR4DF of T of weight at most $2n-1$ which leads to a contradiction. If T has a non-support vertex of degree at least two, say v, then the function $g : V(T) \to \{-1,1,2,3\}$ defined by $g(v)=1$ and $g(x)=2$ otherwise, is an SDR4DF of T of weight at most $2n-1$ that leads to a contradiction. Thus each vertex of T is either a leaf or a support vertex which is not strong. This implies that $T=\mathrm{cor}(T')$ for some tree T' and so $T \in \mathcal{F}$. This completes the proof. □

Next we present a lower bound on the signed double Roman 4-domination number of trees T and characterize all extreme trees.

Theorem 3.8. *For any tree T of order $n \ge 1$, $\gamma_{sdR}^4(T) \ge n+2$.*

Proof. The proof is by induction on n. The cases $n=2$ and $n=3$ follows from Proposition 2.3. Let $n \ge 4$ and let the statement hold for all trees of order less than n. Assume T is tree of order n. If $\mathrm{diam}(T)=2$, then $T=K_{1,n-1}$ and by Observation 1.1 (part (v)) we have $\gamma_{sdR}^4(T)=n+2$ and if $\mathrm{diam}(T)=3$, then T is a double star $\mathrm{DS}_{p,q}$ with $q \ge p \ge 1$ and by Observation 1.1 (part (iii)), we have $\gamma_{sdR}^4(T) \ge p+q+6 > n+2$. Therefore, we assume that $\mathrm{diam}(T) \ge 4$. Suppose $f=(V_{-1},V_1,V_2,V_3)$ is a $\gamma_{sdR}^4(T)$-function. If $V_{-1}=\emptyset$, then obviously $\gamma_{sdR}^4(T) > n+2$. Henceforth, we assume $V_{-1} \ne \emptyset$. If there is a non-pendant edge uv such that $u,v \in V_{-1}$, then applying an argument similar to that described in Theorem 3.3 we obtain $\gamma_{sdR}^4(T) > n+2$. Assume that there is no non-pendant edge uv for which $u,v \in V_{-1}$. As in the proof of Theorem 3.3, we may assume that V_{-1} is independent. Also, it follows from Observation 1.1 (part (iii)) that each vertex in V_{-1} is neither a leaf nor a support vertex.

Let $v \in V_{-1}$ and $T_1,\dots,T_r$ $(r \ge 2)$ be the components of $T-v$. Clearly, the function $f_i=f|_{T_i}$ is an SDR4DF of T_i for each i and by the induction hypothesis we obtain

$$\begin{aligned}\gamma_{sdR}^4(T) &= \textstyle\sum_{i=1}^r \omega(f_i) + f(v)\\ &\ge (\textstyle\sum_{i=1}^r |V(T_i)| + 2) + f(v)\\ &= n-1+2r-1\\ &\ge n-1+4-1\\ &= n+2,\end{aligned} \tag{3.5}$$

and the proof is complete. □

To characterize the extreme trees we introduce the following family of trees. For any tree T of order at least 2, let $S(T)$ be the tree obtained from T by subdividing all non-pendant edges of T once. Let $\mathcal{T}$ be the family of trees T such that $|L_v| \geq \frac{\deg(v)+1}{2}$ for any non-leaf vertex $v \in V(T)$, and let $\mathcal{ST} = \{S(T) \mid T \in \mathcal{T}\}$. Note that $\mathcal{ST}$ contains all non-trivial stars.

Lemma 3.9. *If $T \in \mathcal{ST}$, then $\gamma^4_{sdR}(T) = n(T)+2$. Moreover, if $n(T) \geq 3$, then T has a unique $\gamma^4_{sdR}(T)$-function.*

Proof. Let $T \in \mathcal{ST}$. It is easy to see that the function $g : V(T) \to \{-1, 1, 2, 3\}$ assigning a 1 to all leaves, a 3 to all support vertices and a -1 to the remaining vertices, is an SDR4DF of T of weight $n(T)+2$ and so $\gamma^4_{sdR}(T) \leq n(T)+2$. Thus $\gamma^4_{sdR}(T) = n(T)+2$ by Theorem 3.8.

Assume that $T \in \mathcal{ST}$ is a tree of order $n \geq 3$. To show that T has a unique $\gamma^4_{sdR}(T)$-function, we proceed by induction on n. If $n = 3$, then clearly T has a unique $\gamma^4_{sdR}(T)$-function. Assume that $n \geq 4$ and the statement is true for all trees in $\mathcal{ST}$ of order less than n. Suppose $T \in \mathcal{ST}$ is a tree of order n. If $\operatorname{diam}(T) = 2$, then T is a star and clearly T has a unique $\gamma^4_{sdR}(T)$-function. Let $\operatorname{diam}(T) \geq 3$. Since $T \in \mathcal{ST}$, there exists a tree $T' \in \mathcal{T}$ such that $T = S(T')$. It follows that $\operatorname{diam}(T) \geq 4$. Assume $f = (V_{-1}, V_1, V_2, V_3)$ is a $\gamma^4_{sdR}(T)$-function and let $v_1v_2 \dots v_k; (k \geq 5)$ be a diametrical path of T. Root T at v_k. Clearly, $v_2, v_4 \in V(T')$ and so $|L_{v_i}| \geq 2$ for $i = 2, 4$ and $\deg(v_3) = 2$. It follows from Proposition 1.1 (part (iii)) that $f(v_2) = f(v_4) = 3$ and $f(x) = 1$ for each $x \in L_{v_2} \cup L_{v_4}$. Since f is a $\gamma^4_{sdR}(T)$-function, we must have $f(v_3) = -1$. Let $T_1 = T - T_{v_3}$. Obviously $T_1 \in \mathcal{ST}$ and the function f, restricted to T_1 is an SDR4DF of T_1. By Theorem 3.8, we have

$$n + 2 = \gamma^4_{sdR}(T) = \omega(f|_{T_1}) + |V(T_{v_3})| \geq |V(T_1)| + 2 + |V(T_{v_3})| = n + 2.$$

This implies that $\gamma^4_{sdR}(T_1) = |V(T_1)| + 2$ and so $f|_{T_1}$ is a $\gamma^4_{sdR}(T_1)$-function. We deduce from the induction hypothesis that $f|_{T_1}$ is the unique γ^4_{sdR}-function of T_1 and hence f is the unique γ^4_{sdR}-function of T. This completes the proof. □

Theorem 3.10. *Let T be a tree of order n. Then $\gamma^4_{sdR}(T) = n + 2$ if and only $T \in \mathcal{ST}$.*

Proof. According to Lemma 3.9, we only need to prove necessity. Let $\gamma^4_{sdR}(T) = n+2$. The proof is by induction on n. If $n = 2$, then $T = P_2 \in \mathcal{ST}$. Let $n \geq 3$ and let the statement hold for all trees of order less than n. Assume T is tree of order n. As in the proof of Theorem 3.8, we can see that either T is a star or $\operatorname{diam}(T) \geq 4$ and all inequalities occurring in (3.5) are equalities. If T is a star, then clearly $T \in \mathcal{ST}$. Let $\operatorname{diam}(T) \geq 4$ and all inequalities occurring in (3.5) are equalities. This implies that $r = 2$, $\gamma^4_{sdR}(T_i) = |V(T_i)| + 2$ and f_i is a $\gamma^4_{sdR}(T_i)$-function for $i = 1, 2$. By the induction hypothesis we have $T_1, T_2 \in \mathcal{ST}$ and Lemma 3.9 implies that f_i is the unique γ^4_{sdR}-function of T_i for $i = 1, 2$. Suppose $T_1 = S(T'_1)$ and $T_2 = S(T'_2)$ where $T'_1, T'_2 \in \mathcal{T}$. Let $v_i \in T_i$ be the neighbor of v for $i = 1, 2$. Clearly, $v_i \in V(T'_i)$ for $i = 1, 2$. Assume T' is the tree obtained from trees T'_1, T'_2 by adding the edge v_1v_2. Since v_i is a non-leaf vertex of T_i, we have $|L_{v_i}| \geq \frac{\deg_{T'_i}(v_i)+1}{2}$ for $i = 1, 2$. If $|L_{v_i}| = \frac{\deg_{T'_i}(v_i)+1}{2}$ for some i, then we have $f(N[v_i]) \leq 3$ which is a contradiction. Thus $|L_{v_i}| \geq \frac{\deg_{T'_i}(v_i)+2}{2} = \frac{\deg_{T'}(v_i)+1}{2}$ for $i = 1, 2$, yielding $T' \in \mathcal{T}$. Hence $T = S(T') \in \mathcal{ST}$ and the proof is complete. □

3.4. k = 5

Here, we establish upper and lower bounds on the signed double Roman 5-domination in trees. Let $\mathcal{F} = \{\operatorname{cor}(T) \mid T \text{ is a tree}\}$.

Theorem 3.11. *For any T of order $n \geq 2$,*

$$\gamma^5_{sdR}(T) \leq \frac{5n}{2}.$$

The equality holds if and only if $T \in \mathcal{F}$.

Proof. The proof is by induction on n. If $n=2$, then clearly $\gamma^5_{sdR}(T)=5=\frac{5n}{2}$, and if $n=3$ then by Proposition 2.4 we have $\gamma^5_{sdR}(T)=7<\frac{5n}{2}$. Let $n\geq 4$ and let the statement hold for all trees of order less than n. Assume T is a tree of order n. If $\operatorname{diam}(T)=2$, then $T=K_{1,n-1}$ and by Observation 1.1 (part (v)) we have $\gamma^5_{sdR}(T)=2n+1<\frac{5n}{2}$. If $\operatorname{diam}(T)=3$, then T is a double star $\mathrm{DS}_{p,q}$ for some $q\geq p\geq 1$ and we deduce from Observation 1.1 (part (iii)) that $\gamma_{sdR}(T)=2n+2\leq\frac{5n}{2}$ with equality if and only if $T=P_4\in\mathcal{F}$. Therefore, we assume that $\operatorname{diam}(T)\geq 4$. Let $v_1v_2\dots v_k$ $(k\geq 5)$ be a diametrical path in T such that $\deg_T(v_2)$ is as large as possible and root T at v_k.

If $\deg_T(v_2)\geq 3$, then let $T'=T-v_1$ and let $f=(V_{-1},V_1,V_2,V_3)$ be a $\gamma^5_{sdR}(T')$-function such that $f(v_2)$ is as large as possible. Then $f(v_2)=3$ and the function $g=(V_{-1},V_1,V_2\cup\{v_1\},V_3)$ is an SDR5DF of T and by the induction hypothesis we have $\gamma^5_{sdR}(T)\leq\omega(g)=\gamma_{sdR}(T')+2\leq\frac{5(n-1)}{2}+2<\frac{5n}{2}$.

Assume that $\deg_T(v_2)=2$. By the choice of diametrical path, we may assume that all children of v_3 with depth 1, have degree 2. We consider two cases.

Case 1. $\deg_T(v_3)\geq 3$.
Suppose first that v_3 is a support vertex and $v'\in L_{v_3}$. Let $T'=T-\{v_1,v_2\}$ and let $f=(V_{-1},V_1,V_2,V_3)$ be a $\gamma^5_{sdR}(T')$-function. Since $f(v_3)+f(v')\geq 5$, we may assume that $f(v_3)=3$. Now the function $g=(V_{-1},V_1,V_2\cup\{v_1\},V_3\cup\{v_2\})$ is an SDR5DF of T and by the induction hypothesis we have $\gamma^5_{sdR}(T)\leq\omega(g)=\gamma^5_{sdR}(T')+5\leq\frac{5(n-2)}{2}+5\leq\frac{5n}{2}$. The equality holds if and only if $\gamma^5_{sdR}(T')=\frac{5(n-2)}{2}$, i.e, if and only if $T'\in\mathcal{F}$, and this if and only if $T\in\mathcal{F}$.

Henceforth, we assume that all children of v_3 are of depth 1 and degree 2. Let $T'=T-T_{v_3}$ and let $f=(V_{-1},V_1,V_2,V_3)$ be a $\gamma^5_{sdR}(T')$-function. Then $g=(V_{-1},V_1\cup\{v_3\},V_2\cup(D(v_3)-C(v_3)),V_3\cup C(v_3))$ is an SDR5DF of T and by the induction hypothesis we have

$$\gamma^5_{sdR}(T)\leq\omega(g)=\gamma^5_{sdR}(T')+5|C(v_3)|+1\leq\frac{5(n-(2|C(v_3)|+1))}{2}+5|C(v_3)|+1<\frac{5n}{2}.$$

Case 2. $\deg_T(v_3)=2$.
If v_4 is a support vertex or $\deg(v_4)=2$, then assume $T'=T-\{v_1,v_2,v_3\}$ and let $f=(V_{-1},V_1,V_2,V_3)$ be a $\gamma^5_{sdR}(T')$-function. Since v_4 is a leaf or a support vertex in T', we have $f(v_4)\geq 2$. Now the function $g=(V_{-1},V_1\cup\{v_3\},V_2\cup\{v_1\},V_3\cup\{v_2\})$ is an SDR5DF of T of weight $\gamma^5_{sdR}(T')+6$ and it follows from the induction hypothesis that $\gamma^5_{sdR}(T)\leq\gamma^5_{sdR}(T')+6\leq\frac{5(n-3)}{2}+6<\frac{5n}{2}$.

Now assume $\deg(v_4)\geq 3$ and that v_4 is not a support vertex. Using above arguments, we may assume that all vertices in T_{v_4} except v_4 have degree at most two. Assume that $T'=T-T_{v_4}$ and $f=(V_{-1},V_1,V_2,V_3)$ is a $\gamma^5_{sdR}(T')$-function. Let $v_4x^j_3x^j_2x^j_1$ $(1\leq j\leq r)$ be the paths of length 3 in T_{v_4} and $v_4y^j_2y^j_1$ $(1\leq j\leq s)$ be the paths of length 2 in T_{v_4}, if any. If $s\geq 1$, then define $g:V(T)\to\{-1,1,2,3\}$ by $g(v_4)=2$, $g(x^j_3)=g(x^j_1)=2$, $g(x^j_2)=3$ for $1\leq j\leq r$, $g(y^j_2)=3, g(y^j_1)=2$ for $1\leq j\leq s$ and $g(x)=f(x)$ for $x\in V(T')$. Clearly, g is an SDR5DF of T and we deduce from the induction hypothesis that

$$\gamma^5_{sdR}(T)\leq\gamma^5_{sdR}(T')+5s+7r+2\leq\frac{5(n-(3r+2s+1))}{2}+5s+7r+2<\frac{5n}{2}\cdot$$

If $s=0$, then define $g:V(T)\to\{-1,1,2,3\}$ by $g(v_4)=2, g(x^j_3)=g(x^j_1)=2$, $g(x^j_2)=3$ for $1\leq j\leq r$, and $g(x)=f(x)$ for $x\in V(T')$. Clearly, g is an SDR5DF of T and by the induction hypothesis we have

$$\gamma^5_{sdR}(T)\leq\omega(g)=\gamma^5_{sdR}(T')+7r+2\leq\frac{5(n-(3r+1))}{2}+7r+3<\frac{5n}{2}\cdot$$

This completes the proof. □

Theorem 3.12. *Let T be a tree of order $n\geq 2$. Then*

$$\gamma^5_{sdR}(T)\geq\frac{5n+4}{3}\cdot$$

Proof. We proceed by induction on n. The cases $n=2$ and $n=3$ follows from Proposition 2.4. Let $n \geq 4$ and let the statement hold for all trees of order less than n. Assume T is a tree of order n. If $\operatorname{diam}(T)=2$, then T is a star and by Observation 1.1 (part (v)) we have $\gamma^5_{sdR}(T)=2n+1>\frac{5n+4}{3}$. If $\operatorname{diam}(T)=3$, then T is a double star $\mathrm{DS}_{p,q}$ for some $q \geq p \geq 1$ and we conclude from Observation 1.1 (part (iii)) that $\gamma_{sdR}(T) \geq 2n+2>\frac{5n+4}{3}$. Assume that $\operatorname{diam}(T) \geq 4$. Let $v_1v_2\dots v_k$ $(k \geq 5)$ be a diametrical path in T such that $\deg_T(v_2)$ is as large as possible and root T at v_k. Let $f=(V_{-1},V_1,V_2,V_3)$ be a $\gamma^5_{sdR}(T)$-function.

If T has a non-support vertex v of degree at least two with $f(v)=-1$, then let $T_1,\dots,T_r$ be the components of $T-v$. Clearly, the function $f_i=f|_{T_i}$ is an SDR5DF of T_i for $i=1,\dots,r$, and we deduce from the induction hypothesis that

$$\gamma_{sdR}(T)=-1+\sum_{i=1}^{r}\omega(f_i) \geq -1+\sum_{i=1}^{r}\frac{5n_i+4}{3}=\frac{5(n-1)+4r}{3}-1 \geq \frac{5n+4}{3}\cdot$$

Henceforth, we suppose any non-support vertex of degree at least two, has positive weight. We conclude from Observation 1.1 that $V_{-1}=\emptyset$. If $\deg_T(v_2) \geq 3$ or $f(v_3) \geq 2$, then let $T'=T-v_1$. It is not hard to see that the function f restricted to T' is an SDR5DF of T' of weight $\omega(f)-2$ and by the induction hypothesis we have

$$\gamma^5_{sdR}(T) \geq \gamma^5_{sdR}(T')+2 \geq \frac{5(n-1)+4}{3}+2>\frac{5n+4}{3}\cdot$$

Assume that $\deg_T(v_2)=2$ and $f(v_3)=1$. It follows from $\deg_T(v_2)=2$ and the choice of diametrical path that any child of v_3 with depth 1, is of degree 2. Also, it follows from $f(v_3)=1$ that v_3 is not a support vertex and so all children of v_3 have degree 2. First let $\deg_T(v_3) \geq 3$. Then the function f restricted to $T-\{v_1,v_2\}$ is an SDR5DF of T' and by the induction hypothesis we have

$$\gamma^5_{sdR}(T) \geq \gamma^5_{sdR}(T-\{v_1,v_2\})+5 \geq \frac{5(n-2)+4}{3}+5>\frac{5n+4}{3}\cdot$$

Now, let $\deg_T(v_3)=2$. Suppose that $T'=T-\{v_1,v_2,v_3\}$. If $f(w)=3$ for each $w \in N[v_4]-\{v_3\}$, then obviously the function f restricted to T' is an SDR5DF of T' and by the induction hypothesis we have $\gamma^5_{sdR}(T) \geq \gamma^5_{sdR}(T')+6 \geq \frac{5(n-3)+4}{3}+6>\frac{5n+4}{3}$. Assume $f(w) \leq 2$ for some $w \in N[v_4]-\{v_3\}$ and define $g: V(T') \to \{-1,1,2,3\}$ by $g(w)=f(w)+1$ and $g(x)=f(x)$ otherwise. Clearly, g is an SDR5DF of T' and by the induction hypothesis we have $\gamma^5_{sdR}(T) \geq \gamma^5_{sdR}(T')+5 \geq \frac{5(n-3)+4}{3}+5=\frac{5n+4}{3}$. This completes the proof. □

3.5. k = 6

First, we establish an upper bound on the signed double Roman 6-domination number of trees and characterize all extreme trees.

Theorem 3.13. *Let T be a tree of order $n \geq 2$. Then*

$$\gamma^6_{sdR}(T) \leq 3n$$

with equality if and only if every vertex of T is either a leaf or a support vertex.

Proof. Clearly, the function $f: V(T) \to \{-1,1,2,3\}$ defined by $f(x)=3$ for each $x \in V(T)$, is an SDR6DF of T of weight $3n$ and hence $\gamma^6_{sdR}(T) \leq 3n$.

If T is a non-trivial tree such that every vertex of T is a leaf or a support vertex, then we deduce from Observation 1.1 (part (vi)) that $\gamma^6_{sdR}(T) \geq 3n$ and so $\gamma^6_{sdR}(T)=3n$.

Conversely, let $\gamma^6_{sdR}(T)=3n$. If T has a non-support vertex of degree at least two, say v, then the function $g: V(T) \to \{-1,1,2,3\}$ defined by $g(v)=1$ and $g(x)=3$ otherwise, is an SDR6DF of T of weight at most $3n-1$ that leads to a contradiction. Thus each vertex of T is either a leaf or a support vertex and the proof is complete. □

Next we present a lower bound on $\gamma_{sdR}^6(T)$.

Theorem 3.14. *For any tree T of order $n \geq 2$, $\gamma_{sdR}^6(T) \geq 2n+2$.*

Proof. The proof is by induction on n. If $n=2$, then clearly $\gamma_{sdR}^6(T)=6=2n+2$, and if $n=3$ then by Proposition 2.5 we have $\gamma_{sdR}^6(T)=9>2n+2$. Let $n \geq 4$ and let the statement hold for all trees of order less than n. Assume T is a tree of order n. If $\mathrm{diam}(T) \leq 3$, then T is a star or a double star and we conclude from Observation 1.1 (part (iv)) that $\gamma_{sdR}^6(T)=3n>2n+2$. Therefore, we suppose that $\mathrm{diam}(T) \geq 4$. Let $v_1v_2\dots v_k$ $(k \geq 5)$ be a diametrical path in T such that $\deg_T(v_2)$ is as large as possible and root T at v_k. Assume $f=(V_{-1},V_1,V_2,V_3)$ is a $\gamma_{sdR}^6(T)$-function. If T has a non-support vertex v of degree at least two with $f(v)=-1$, then as in the proof of Theorem 3.12 we can see that $\gamma_{sdR}^6(T) \geq 2n+2$. Henceforth, we assume that any non-support vertex of degree at least two, has positive weight. Now it follows from Observation 1.1 (part (iv)) that $V_{-1}=\emptyset$.

If $\deg_T(v_2) \geq 3$ or $f(v_3)=3$, then let $T'=T-v_1$. Clearly, the function f, restricted to T' is an SDR6DF of T' of weight $\omega(f)-3$ and by the induction hypothesis we obtain

$$\gamma_{sdR}^6(T) \geq \gamma_{sdR}(T')+3 \geq 2(n-1)+5=2n+3>2n+2.$$

Assume that $\deg_T(v_2)=2$ and $f(v_3) \leq 2$. As in the proof of Theorem 3.12, we can see that all children of v_3 are of depth 1 and degree 2. If $\deg_T(v_3) \geq 3$, then let $T'=T-\{v_1,v_2\}$ and define $g:V(T') \to \{-1,1,2,3\}$ by $g(v_3)=f(v_3)+1$ and $g(x)=f(x)$ otherwise. Clearly, g is an SDR6DF of T' of weight $\omega(f)-5$ and by the induction hypothesis we have

$$\gamma_{sdR}^6(T) \geq \gamma_{sdR}(T')+5 \geq 2(n-2)+2+5>2n+2.$$

Now, let $\deg_T(v_3)=2$. If $f(v_3)=2$, then let $T'=T-\{v_1\}$ and define $g:V(T') \to \{-1,1,2,3\}$ by $g(v_3)=3$ and $g(x)=f(x)$ otherwise. Obviously, g is an SDR6DF of T' of weight $\omega(f)-2$ and by the induction hypothesis we have

$$\gamma_{sdR}^6(T) \geq \gamma_{sdR}(T')+2 \geq 2(n-1)+2+2=2n+2.$$

Let $f(v_3)=1$ and $T'=T-\{v_1,v_2,v_3\}$. If $f(w)=3$ for each $w \in N[v_4]-\{v_3\}$, then obviously the function f, restricted to T' is an SDR6DF of T' and by the induction hypothesis we have $\gamma_{sdR}^6(T) \geq \gamma_{sdR}^6(T')+7 \geq 2(n-3)+2+7>2n+2$. Suppose $f(w) \leq 2$ for some $w \in N[v_4]-\{v_3\}$ and define $g:V(T') \to \{-1,1,2,3\}$ by $g(w)=f(w)+1$ and $g(x)=f(x)$ otherwise. Clearly, g is an SDR6DF of T' and by the induction hypothesis we have $\gamma_{sdR}^6(T) \geq \gamma_{sdR}^6(T')+6 \geq 2(n-3)+2+6=2n+2$. This completes the proof. ☐

We conclude this paper with some open problems.

Problem 1. Characterize the trees attaining the upper bound of Theorem 3.4.
Problem 2. Characterize all trees achieving the lower bound of Theorem 3.12.
Problem 3. Characterize all trees attaining the lower bound of Theorem 3.14.

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