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SINGLE-MACHINE LOT SCHEDULING PROBLEM FOR DETERIORATING ITEMS WITH NEGATIVE EXPONENTIAL DETERIORATION RATE

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Abstract. Determining production-inventory level is one of the most important and challenging problems in a manufacturing system. *Economic Batch Quantity* (EBQ) is the simplest and the most employed inventory model in this field. Here, a single-source production lot-sizing model for deteriorating products with negative exponential deterioration rate is developed. In this manufacturing system, we assume that all items are manufactured using a single source. In this paper the optimal values for cycle length and lot size are derived in a way that the total cost (consists of machine setup cost, manufacturing cost, carrying and disposal costs of the deteriorating items) is minimized. Consequently, the developed model for the proposed problem is formulated. According to the derived mathematical model, we could not provide a closed form optimal solution, but we show that there is a unique optimal solution for the cycle length. At first, the upper and lower bounds of the optimal solution are determined, then using a Bi-section method (root-finding method) the problem is solved. To demonstrate the applicability of this method, we solve a simple example for a manufacturing system with three products. Finally, results of ten experiment that obtain from this method compare to answers from Newton-Raphson method, then it has been shown that this method has great effectiveness and less calculation efforts for the single-machine problem with deteriorating items.

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1. Introduction

The Economic Batch Quantity (EBQ) model is one of the most famous, the simplest and the most utilized inventory model. First, the EOQ model was introduced by Harris in February 1913. The main focus of his model was the balancing of the carrying against the fixed costs in order to optimize the total cost [3]. Five years later, Taft developed the Harris's model for manufacturing system, and introduced the first economic manufacturing quantity (EMQ) model [20]. In the late 20th century, Erlenkotter [8,9] rediscovered the Harris's paper and provided a brief history of the EOQ's early life. Both EOQ and EMQ inventory control models are developed under some restrictive assumptions *e.g.*, non-perishability of goods. However, these assumptions are not always true *e.g.*, in food industry and so forth. One of the first investigations on the inventory models regarding perishable products was presented by Whitin [35]. He developed an inventory model considering the

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fashioned goods. After that, Hadley and Whitin [12] presented an inventory system for those products which are outdated after a specific time. Ghare and Schrader [10] extended lot size system for perishable items with negative exponential distribution. There are some important studies that consider the deteriorating items such as [2, 5–7, 11, 15, 17–19, 24, 27, 33, 34].

Considering multiple items for the manufacturing organization is another extension of the classical inventory models [4]. In 1957, Eilon presented the first EMQ model for a manufacturing system with several products [26]. Johnson and Montgomery [16] studied a single-machine manufacturing system under a production constraint for all items. After that, Haji and Mansuri [14] extended Johnson and Montgomery's [16] study with consideration of budgetary constraint. Then, Haji *et al.* [13] proposed an EMQ model for a manufacturing system wherein multiple products were produced on a single-machine. Some features of single-machine production systems are: savings on initial capital by buying only one machine, saving the production space floor, and integrating the production process (postponement).

Taleizadeh *et al.* [29] developed the Haji *et al.*'s [13] study with consideration of a defective manufacturing system. In their problem, the scrapped items are produced. Also, they assumed that the shortage is allowed and partially backordered. Afterwards, Taleizadeh *et al.* [28] recommended a closed-from solution to minimize expected total cost for a defective single-machine system with immediate repair process. Pasandideh *et al.* [25] presented a single-machine nonlinear model for nonconforming items including scrap and repair, where repairable items are categorized into several groups based on defective rate. After that, Nobil *et al.* [21] extended the single-machine scheduling problem and proposed multi-machine lot-sizing scheduling with defective items. Then, they solved the problem by a hybrid genetic algorithm (GA). Taleizadeh *et al.* [31] introduced an EMQ model for a defective manufacturing system with rework process, partial backordering and breakdown in production process for preventive maintenance. At same year, Nobil *et al.* [22] considered a two-level supply chain composed of one vendor and one buyer in order to investigate the impact of the single-machine EBQ problem on the total cost of the supply chain (SC) network. Recently, Nobil *et al.* [23] extended the Pasandideh *et al.* [25] study by considering the non-zero setup times for rework processes. Finally, these studies conducted in the field of single-machine multi-items lot-sizing inventory and their contributions are shown in Table 1.

As it can be seen in Table 1, all performed studies did not consider perishable items for case of multi-product. Here, we extend a single-machine inventory model for perishable products with a negative exponential deterioration rate. Although we could not derive a closed form solution, but both lower and upper bounds of the optimal solution are obtained, and a nonlinear solution procedure, bi-section method, was employed to obtain the near optimal solution.

The rest of this paper is categorized as follows, the proposed single-machine EMQ model is developed and formulated in Section 2. Section 3 represents solution procedure and Section 4 studies a numerical example and its sensitivity analysis. Finally, the future research directions and conclusion are presented in Section 5.

2. Problem statement

The aim of this research is to find the acceptable (near optimal or optimal) value of cycle length so that the total inventory cost is minimized. In this problem, the constant manufacturing rate of the item i, (P_i), is assumed to be greater than to the constant demand rate of item i, (D_i). Mathematically speaking, $P_i > D_i$ or $P_i - D_i > 0$. Additionally, the assumptions of the proposed single-machine lot-sizing inventory-production model are as follows:

- Shortages are not permitted.
- All products are manufactured using a single source and the manufacturing period length for all products is the same, *i.e.* $T = T_i = T_1 = T_2 = \cdots = T_n$.
- Both production and demand rates are known and constant.
- The machine setup time to produce each item is negligible.
- Deterioration rate follows a negative exponential distribution with parameter $\alpha_i, \alpha_i < 1$.

TABLE 1. A brief review of the Single-machine multi-item lot scheduling problems.

Study	Defective			Shortage								
	Rework items	Scrap items	Deteriorating items	Fully backordering	Partial backordering	Multi-shipments	Construction cost	Allocation decisions	Maintenance policy	Multi-level SC	Multi-machine	Solution method
Johnson and Montgomery [16]												Derivative
Haji *et al.* [13]	✓											Derivative
Taleizadeh *et al.* [29]	✓	✓			✓							Derivative
Taleizadeh *et al.* [28]	✓											Derivative
Taleizadeh *et al.* [30]	✓	✓		✓								Derivative
Pasandideh *et al.* [25]	✓	✓		✓			✓					Derivative
Nobil *et al.* [21]		✓						✓			✓	Hybrid GA with Derivative
Taleizadeh *et al.* [31]	✓	✓			✓				✓			Derivative
Nobil *et al.* [22]						✓				✓		Hybrid integer GA
Nobil *et al.* [20]	✓	✓		✓			✓					Derivative
This research			✓									Bi-section

Moreover the problem is modeled using the following notations ($i = 1, 2, \ldots, n$).

n	Number of items
P_i	Manufacturing rate of ith item
D_i	Demand rate of ith item
β_i	$\beta_i = 1 - \frac{D_i}{P_i} \geq 0$
α_i	Deteriorating rate of ith item
$h_i^1(t)$	The ith item inventory level during manufacturing period; $0 \leq t_i \leq t_i^1$
$h_i^2(t)$	The ith item inventory level during down-time period; $t_i^1 \leq t_i \leq T$
S_i	Machine's setup cost of producing ith item ($/setup)
c_i	Production cost of ith item per item ($/unit)
v_i	Disposal cost of ith deteriorating item ($/scrap unit)
I_i	Inventory carrying cost of ith item ($/unit/unit time)
Q_i	lot size of the ith item (decision variables)
T	Period length (a decision variable)
N	Number of cycles per year $N = \frac{1}{T}$.

Manufacturing of the ith item starts at $t_i = 0$ and continues up to $t_i = t_i^1$. Afterwards, at $t_i = t_i^1$ production stops and consumption period begins until the inventory level reaches to zero (see Fig. 1). So, during the production period (*i.e.* $(0,t_i^1)$) system faces production, consumption and deterioration simultaneously. Moreover, during down-time period (*i.e.* $(t_i^1.T)$) the system faces consumption and deterioration. Thus, the following differential equation expresses the inventory level of the ith item during the up time manufacturing period:

$$\frac{\mathrm{d}h_i^1(t_i)}{\mathrm{d}t_i} = (P_i - D_i) - \alpha_i h_i^1(t_i), \qquad 0 \leq t_i \leq t_i^1. \tag{2.1}$$

Using initial condition ($h_i^1(0) = 0$) inventory level of the production period is expressed by:

$$h_i^1(t_i) = \frac{(P_i - D_i)}{\alpha_i}\left(1 - \mathrm{e}^{-\alpha_i t_i}\right), \qquad 0 \leq t_i \leq t_i^1. \tag{2.2}$$

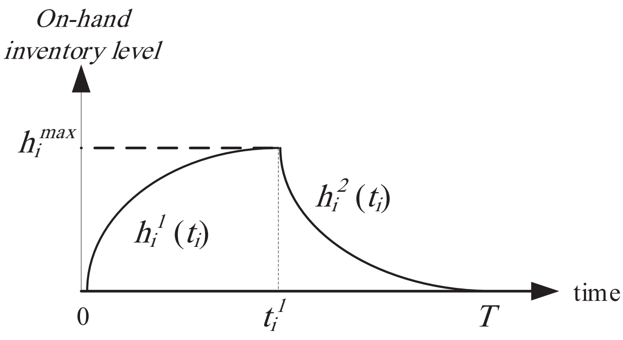

FIGURE 1. Cycle length for on-hand inventory of the ith deteriorating item.

Also, for the down-time period,

$$\frac{\mathrm{d}h_i^2(t_i)}{\mathrm{d}t_i} = -D_i - \alpha_i h_i^2\,(t_i)\,, \qquad t_i^1 \le t_i \le T. \tag{2.3}$$

Using ending condition ($h_i^2\,(T) = 0$) the inventory level of the down-time period is obtained by:

$$h_i^2(t_i) = \frac{D_i}{\alpha_i}\left(\mathrm{e}^{\alpha_i(T-t_i)} - 1\right), \qquad t_i^1 \le t_i \le T. \tag{2.4}$$

Moreover, $h_i^1\left(t_i^1\right) = h_i^2\left(t_i^1\right)$, so:

$$\frac{(P_i - D_i)}{\alpha_i}\left(1 - \mathrm{e}^{-\alpha_i t_i^1}\right) = \frac{D_i}{\alpha_i}\left(\mathrm{e}^{\alpha_i\left(T-t_i^1\right)} - 1\right). \tag{2.5}$$

Therefore, t_i^1 can be expressed as:

$$t_i^1 = \frac{1}{\alpha_i}\ln\left(\frac{D_i}{P_i}\mathrm{e}^{\alpha_i T} + \beta_i\right). \tag{2.6}$$

Moreover, the total inventory cost is:

$$\begin{aligned}\text{total cost (TC)} &= \text{annual setup cost (TSC)} + \text{annual manufacturing cost (TPC)} \\ &\quad + \text{annual disposal cost (TDC)} + \text{annual inventory carrying cost (TIC)}\end{aligned} \tag{2.7}$$

(1) *Annual setup cost*
Total setup cost of the ith item equals $\frac{S_i}{T}$. So, the setup cost of all the items is calculated as:

$$\mathrm{TAC} = \sum_{i=1}^{n}\frac{S_i}{T}\cdot \tag{2.8}$$

(2) *Annual manufacturing cost*
The manufacturing rate and uptime of the ith item are P_i and t_i^1, respectively. So, the related manufacturing lot size equals:

$$\text{manufacturing lot size} = P_i t_i^1. \tag{2.9}$$

Therefore, the total manufacturing cost according to the production cost per unit of the ith product per period (c_i) can be expressed as:

$$\mathrm{TPC} = \sum_{i=1}^{n}\frac{c_i P_i t_i^1}{T}\cdot \tag{2.10}$$

(3) *Annual disposal cost*
Based on Figure 1, the amount of deteriorated products of the ith type is calculated by:

$$(P_i - D_i)\, t_i^1 - D_i\left(T - t_i^1\right). \tag{2.11}$$

Therefore, the total deterioration cost is as follows:

$$\mathrm{TDC} = \sum_{i=1}^{n} \frac{v_i\left[(P_i - D_i)\, t_i^1 - D_i\left(T - t_i^1\right)\right]}{T} = \sum_{i=1}^{n}\left(\frac{v_i P_i t_i^1}{T} - v_i D_i\right). \tag{2.12}$$

(4) *Total inventory carrying cost*
From Figure 1, the holding cost of the ith product regarding its carrying cost is calculated as:

$$\frac{I_i}{T}\left(\int_{o}^{t_i^1} h_i^1\left(t_i\right) \mathrm{d}t_i + \int_{t_i^1}^{T} h_i^2\left(t_i\right) \mathrm{d}t_i\right). \tag{2.13}$$

From equations (2.2) and (2.4), we have:

$$\begin{aligned}
&\frac{I_i}{T}\Bigg(\int_{o}^{t_i^1} \frac{(P_i - D_i)}{\alpha_i}\left(1 - \mathrm{e}^{-\alpha_i t_i}\right)\mathrm{d}t_i + \int_{t_i^1}^{T} \frac{D_i}{\alpha_i}\left(\mathrm{e}^{\alpha_i\left(T - t_i\right)} - 1\right)\mathrm{d}t_i\Bigg) \\
&= \frac{I_i}{\alpha_i^2 T}\left\{(P_i - D_i)\left(\alpha_i t_i^1 - 1 + \mathrm{e}^{-\alpha_i t_i^1}\right) + D_i\left(\mathrm{e}^{\alpha_i\left(T - t_i^1\right)} - \alpha_i\left(T - t_i^1\right) - 1\right)\right\} \\
&= \frac{I_i}{\alpha_i T}\left(P_i t_i^1 - D_i T\right).
\end{aligned}$$

Substituting equation (2.5) in equation (2.13), the inventory carrying cost of the ith item is:

$$\frac{I_i}{\alpha_i T}\left(P_i t_i^1 - D_i T\right). \tag{2.14}$$

Therefore, the inventory carrying cost of all items is determined as:

$$\mathrm{TIC} = \sum_{i=1}^{n} \frac{I_i}{\alpha_i T}\left(P_i t_i^1 - D_i T\right). \tag{2.15}$$

Inserting equations (2.8), (2.10), (2.12), and (2.15) into equation (2.7) results in total cost presented in equation (2.16).

$$\mathrm{TC} = \mathrm{TSC} + \mathrm{TPC} + \mathrm{TDC} + \mathrm{TIC} = \sum_{i=1}^{n}\left\{\frac{S_i}{T} + \frac{c_i P_i t_i^1}{T} + \frac{v_i P_i t_i^1}{T} - v_i D_i + \frac{I_i}{\alpha_i T}\left(P_i t_i^1 - D_i T\right)\right\} \tag{2.16}$$

3. Solution procedure

The decision variable of the proposed inventory model is the common cycle length. So, the optimal cycle length should be derived such that the total inventory cost shown in equation (2.16) is optimized. Therefore, the optimal period length can be derived using the derivatives method. Indeed setting the partial derivative of equation (2.16) with respect to period length equal to zero gives;

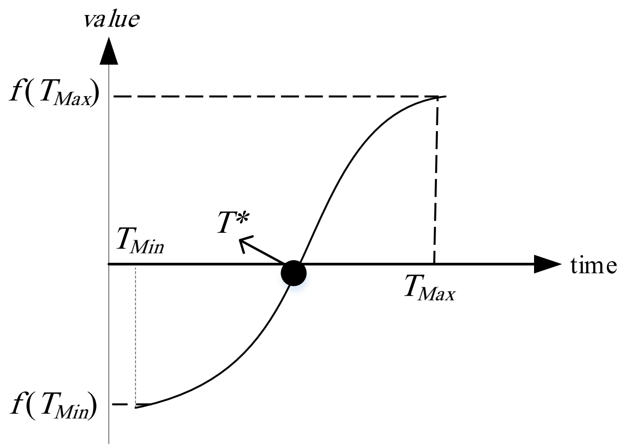

FIGURE 2. The graph of the root of the function (2.16).

$$\sum_{i=1}^{n} \frac{P_i\left(c_i\alpha_i + v_i\alpha_i + I_i\right)}{\alpha_i}\left(T\frac{\mathrm{d}t_i^1}{\mathrm{d}T} - t_i^1\right) - \sum_{i=1}^{n} S_i = 0. \tag{3.1}$$

Equation (3.1) is the optimality condition of equation (2.16). So,

$$f\left(T\right) = \sum_{i=1}^{n} \frac{P_i\left(c_i\alpha_i + v_i\alpha_i + I_i\right)}{\alpha_i}\left(T\frac{\mathrm{d}t_i^1}{\mathrm{d}T} - t_i^1\right) - \sum_{i=1}^{n} S_i. \tag{3.2}$$

Therefore, we can write:

$$f'\left(T\right) = \frac{\mathrm{d}f(T)}{\mathrm{d}T} = \sum_{i=1}^{n} \frac{P_i\left(c_i\alpha_i + v_i\alpha_i + I_i\right)}{\alpha_i}\left(\frac{\mathrm{d}^2t_i^1}{\mathrm{d}^2T}\right) \tag{3.3}$$

where,

$$\frac{\mathrm{d}^2t_i^1}{\mathrm{d}^2T} = \frac{D_iP_i\beta_i\mathrm{e}^{\alpha_iT}}{\left(D_i\mathrm{e}^{\alpha_iT} + P_i\beta_i\right)^2} > 0; \qquad i = 1, 2, \ldots, n. \tag{3.4}$$

Using equation (3.4), we have $f'(T) = \sum_{i=1}^{n} \frac{P_i(c_i\alpha_i + v_i\alpha_i + I_i)}{\alpha_i}\left(\frac{\mathrm{d}^2t_i^1}{\mathrm{d}^2T}\right) > 0$. Thus, $f\left(T\right)$ is increasing on $(0, \infty)$ as well. Since:

$$f\left(0\right) = -\sum_{i=1}^{n} S_i < 0 \tag{3.5}$$

and,

$$\lim_{T\to\infty} f\left(T\right) = \infty > 0 \tag{3.6}$$

we have:

$$\frac{\mathrm{dTC}(T)}{\mathrm{d}T} = \begin{cases} f\left(T\right) < 0. & \text{if} \quad T \in (0, T^*) \\ f\left(T\right) = 0. & \text{if} \quad T = T^* \\ f\left(T\right) > 0. & \text{if} \quad T \in (T^*, \infty) \end{cases}. \tag{3.7}$$

Since the optimal time cannot be expressed as a closed form, see root diagram proposed in Figure 2. Based on the equation (3.7), we employ a well-known method called Bi-section. This method is a simple and robust procedure that finds roots by bisecting intervals repeatedly to find a sub-interval containing a root. This procedure finds the optimum solution by searching point to point, so the upper and lower bounds of the optimal solution are needed. In this study, the lower bound of the optimal cycle length is computed as follows:

$$f(T) = \sum_{i=1}^{n} \frac{P_i\left(c_i\alpha_i + v_i\alpha_i + I_i\right)}{\alpha_i}\left(T\frac{\mathrm{d}t_i^1}{\mathrm{d}T} - t_i^1\right) - \sum_{i=1}^{n} S_i$$
$$< \sum_{i=1}^{n} \frac{P_i\left(c_i\alpha_i + v_i\alpha_i + I_i\right)}{\alpha_i}(T) - \sum_{i=1}^{n} S_i. \quad (3.8)$$

So,

$$\sum_{i=1}^{n} \frac{P_i\left(c_i\alpha_i + v_i\alpha_i + I_i\right)}{\alpha_i}(T) - \sum_{i=1}^{n} S_i = 0. \quad (3.9)$$

Let,

$$T_{\text{Min}} = \frac{\sum_{i=1}^{n} S_i}{\sum_{i=1}^{n} \frac{P_i(c_i\alpha_i + v_i\alpha_i + I_i)}{\alpha_i}}. \quad (3.10)$$

Moreover, the upper bound of the cycle length cannot be expressed as a closed form. As a result, we consider the smallest integer that its related $f(T)$ is greater than zero as the upper bound. So, the upper bound can be obtained from the following procedure: (T_{Max} procedure)

1. Set $j = 1$.
2. Set $T_{\text{Max}} = j$, calculate $f(T_{\text{Max}})$ according to equation (3.2) and go to *Step* 3.
3. If $f(T_{\text{Max}}) > 0$, then go to *Step* 5, else set $j = j+1$ and go to *Step* 2.
4. Show the value of T_{Max}.

Now, based on the lower (T_{Min}) and upper (T_{Max}) bounds of the cycle length, we propose a hybrid approach employing Bi-section method as follows:

1. T_{Min} is calculated using equation (3.10) and T_{Max} determined by T_{Max} procedure. Thereafter, compute $f(T_{\text{Min}})$ and $f(T_{\text{Max}})$ based on the equation (3.2), then go to *Step* 2.
2. Compute the midpoint of the ($T_{\text{Min}}.T_{\text{Max}}$), *i.e.* m, using $m = \frac{T_{\text{Max}} + T_{\text{min}}}{2}$. Then, calculate $f(m)$ by equation (3.2). Then, go to *Step* 3.
3. If $|f(m)| < \varepsilon$ where ε is acceptable error defined by user, go to *Step* 5, else go to *Step* 4.
4. If $f(m) > 0$, then $T_{\text{Max}} = m$, else set $T_{\text{Min}} = m$. Then, go to *Step* 2.
5. Let $T^* = m$, calculate Q_i^* and TC_i^* using equations (2.9) and (2.16) respectively and terminate the procedure.

4. Computational study and sensitivity measurement

The employed parameters in this numerical example are shown in Tables 2 and 3. Also, ε is equal to 0.0001. We performed steps of proposed solution procedure for this numerical example to obtain optimal solution.

TABLE 2. Example.

General data				
Item	P_*i(units/year)	D_i(units/year)	$\beta_i = 1 - \frac{D_i}{P_i}$	α_i
1	2000	1000	0.5	0.01
2	3000	1500	0.5	0.02
3	4000	2000	0.5	0.03

TABLE 3. Example continued.

Item	c_i($/ unit)	v_i ($/rework unit)	S_i($/ setup)	I_i($/unit/unit time)
1	20	7	1000	2
2	18	9	1500	3
3	16	8	1300	4

TABLE 4. Solutions for each iteration of the proposed algorithm.

Iter (j)	T_{Min}	$f(T_{\text{Min}})$	$m(j)$	$f(m_j)$	T_{Max}	$f(T_{\text{Max}})$
1	0.002353912863927	–3799.976	0.501176956431964	–2731.257	1	454.661
2	0.501176956431964	–2731.257	0.750588478215982	–1402.912	1	454.661
3	0.750588478215982	–1402.912	0.875294239107991	–540.273	1	454.661
⋮	⋮		⋮		⋮	
23	0.945055662825067	–0.001342	0.945055781753748	–0.000382	0.945055900682429	0.000570
24	0.945055781753748	–0.000382	0.945055841218089	0.000092	0.945055900682429	0.000570

TABLE 5. Values of Q_i^* and TC*.

	1	2	3
Q_i^*	947.288657905653	1424.28213914104	1903.50818916205
TC*		87042.0436587025	

TABLE 6. The sensitivity analysis.

	% In change				
		T_{Min}	T_{Max}	T^*	TC*
Initial	0	0	0	0	0
P_i	40	–28.5714285714187	0	–11.9782699809848	1.24717572116698
	–40	66.6666666666892	+100	74.7521511817443	–3.92854311599222
D_i	40	0	+100	9.47371083608061	35.5188531958653
	–40	0	+100	8.74726364068090	–37.0615477629997
α_i	40	32.4206085699958	0	–2.83291347991889	0.269563734205412
	–40	–36.3572648823780	0	3.09733311615932	–0.277704910432716
c_i	40	–3.76743631521188	0	–1.93984989194744	36.4870594283114
	–40	4.07444019428046	0	2.05972218766752	–36.4907484609302
v_i	40	–1.77666004136111	0	–0.949929560780985	0.0886041392622797
	–40	1.84211633092188	0	0.977793391604097	–0.0894621131106956
I_i	40	–25.5266799938387	0	–13.6389305593052	1.45908418115332
	–40	52.1520578071207	+100	23.1661536850676	–1.73770802747757
S_i	40	40.0000000000187	+100	18.3232784765361	1.69272862523786
	–40	–39.9999999999919	0	–22.5414351170800	–2.08252679300697

TABLE 7. Initial data of the proposed problems.

$P_i \sim U(2000, 8000); D_i \sim U(1000, 4000); \alpha_i \sim U(0.010, 0.20); c_i \sim U(10, 100); v_i \sim U(5, 40);$ $S_i \sim U(1000, 6000); h_i \sim U(2, 20)$

Notes. U: Uniform distribution.

TABLE 8. Comparison between Bi-section and Newton-Raphson methods by measuring total cost and computational time measures.

Size n	Bi-section		Newton-Raphson	
	TC ($)	CPU (s)	TC ($)	CPU (s)
1	21197.02	0.014	21198.89	0.013
3	87042.04	0.015	87119.65	0.013
4	92191.28	0.016	92191.28	0.014
6	87727.74	0.022	87812.32	0.019
8	102931.88	0.029	10300.01	0.026
10	110921.00	0.032	110921.00	0.029
12	142901.27	0.034	143629.19	0.031
14	162938.09	0.037	163888.52	0.034
16	219471.77	0.042	221938.31	0.038
18	250181.89	0.047	251983.44	0.044

Notes. TC: total profit, CPU: computational time.

Eventually, solutions for each iteration and the final solution are proposed in Table 4. As can be seen in Table 4, the final solution (optimal) cycle length equals 0.945055839863656. Based on $T^* = 0.945055839863656$, Q_i^* and TC^* are represented in Table 5.

In order to perform the sensitivity analysis of the results with respect to the parameters, we increased or decreased parameters presented in Table 6, about 40% fixing other parameters of the problem. As can be seen in Table 6, the demand rate and the production cost have great impacts on the total inventory cost. Meanwhile, the manufacturing rate, the carrying cost, and the machine setup cost have great impacts on the optimal cycle length. Additionally, the manufacturing rate, the deterioration rate, the holding cost, and the machine setup cost have great impacts on the lower bound of the cycle length.

4.1. Numerical comparison

In this subsection, we solve ten proposed instances of the proposed problem with Bi-section and Newton-Raphson methods, and compare the results of these methods with one another. The input data of these instances is generated randomly from Table 7. Newton-Raphson method is suitable for solving non-linear programming problems with no constraint [1]. For this method, we call the objective function and its first derivation $f(T_j)$ and $f'(T_j)$, respectively. So the value of the variable T in step $j+1$ is obtained by equation (4.1).

$$T_{j+1} = T_j - \frac{f(T_j)}{f\prime(T_j)} \tag{4.1}$$

where,

$$f(T_j) = \sum_{i=1}^{n} \left\{ \frac{S_i}{T_j} + \frac{c_i P_i t_i^1}{T_j} + \frac{v_i P_i t_i^1}{T_j} - v_i D_i + \frac{I_i}{\alpha_i T_j}\left(P_i t_i^1 - D_i T\right) \right\} \tag{4.2}$$

$$f'(T_j) = \sum_{i-1}^{n} \frac{P_i\left(c_i\alpha_i + v_i\alpha_i + I_i\right)}{\alpha_i T_j^2} \left(T_j \frac{\mathrm{d}t_i^1}{\mathrm{d}T_j} - \frac{1}{\alpha_i} \ln\left(\frac{D_i}{P_i} \mathrm{e}^{\alpha_i T} + \beta_i \right) \right) - \sum_{i=1}^{n} \frac{S_i}{T_j^2}. \tag{4.3}$$

This method is repeated while the value of the objective function in a step is less than or equal to the solutions of its neighbors ($f(T_{j-1}) \geq f(T_j)$ and $f(T_{j+1}) \geq f(T_j)$). The results of ten instances solved by Bi-section and Newton-Raphson methods are shown in Table 8.

Based on Table 8, Bi-section and Newton-Raphson methods obtain a high quality solution in a very short running time for problems with different sizes. Since, these times are less than 1 second, these two methods are not different in obtaining final solutions. But the objective functions obtained by Bi-section method are better (less or equal) than the solutions of Newton-Raphson method. So, Bi-section method improves solution quality without increasing computational efforts.

5. Conclusion and suggestions

In this research, a single-machine lot-sizing inventory problem was derived for deteriorating items with negative exponential rate. In the proposed production system all products are manufactured by a unique source, so products share the same manufacturing cycle length. To find the optimal common cycle length, both upper and lower bounds are determined. Therefore, a root-finding method (Bi-section) is applied to solve the proposed non-linear programming problem. Then, to demonstrate the effectiveness of Bi-section method, we solved ten different instances of the proposed problem with Bi-section and Newton-Raphson methods, and compared the results of these methods to one other. At the end, it has been shown that Bi-section method has great effectiveness, efficiency and less computational efforts for this single-machine problem with deteriorating items.

For future research directions, the maximum available warehouse space, purchase discounts for deteriorating items, uncertainties in some parameters such as demand or deterioration rates, and the defective production systems including rework and scraps can be explored to extend this work. Also, this mathematical model can be extended by assuming shortage as a combination of lost-sale and backordering, and marketing decisions *e.g.*, advertisement impact on customer demands.

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