RAIRO-Oper. Res. 52 (2018) 849–893
https://doi.org/10.1051/ro/2018002

RAIRO Operations Research
www.rairo-ro.org

CONSTRAINED INTEGRATED INVENTORY MODEL FOR MULTI-ITEM UNDER MIXTURE OF DISTRIBUTIONS

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Abstract. When the demand of different customers are not identical during the lead time, then one cannot use only a single distribution to describe the demand during that lead time. Hence, in this paper we have studied a mixture of normal distributions and a mixture of distribution free for several products under vendor-buyer integrated approach (coordination between both parties). Many integrated inventory models have proved that the integrated total cost is minimum when compared to sum of the total cost of the individuals. The inventory is continuously reviewed by the buyer and next order is placed when the inventory reaches some level called reorder level. The buyer has limited warehouse space capacity and also limited budget to purchase all products. The lead time of receiving all products from the vendor is a variable which is controlled by adding crashing cost. Shortages are allowed for all products and a fraction of shortages will be backordered and the remaining are lost. A mathematical model is developed and a solution procedure is employed in this study to obtain optimum order quantities, lead time and number of shipments in which the integrated total cost function attains its minimum subject to the floor space constraint and budget constraint. The expected integrated cost function is non-linear mixed integer with inequality constraints. Therefore, the proposed model have been solved by using Lagrangian multiplier technique. Finally numerical examples and sensitivity analysis were performed to illustrate the effectiveness of the proposed model.

Mathematics Subject Classification. 90B05, 90C30, 78M50

Received April 13, 2017. Accepted December 21, 2017.

1. Introduction

In the lead time, the demand of the different customers may not be identical, then we cannot use a single distribution (such as Ben-Daya and Hariga [5], Dey and Giri [11], Huang [21]) to describe the lead time demand. The demand of above mentioned works follow a normal distribution, *i.e.*, the demand have unique mean and standard deviation. Several buyer may not have the identical demand and their mean demand also need not be identical in the practical situation. To tackle this scenario, in this paper a mixture of distribution is proposed. The mean of the mixture distribution $\mu_* = p\mu_1 + (1-p)\mu_2$ where μ_1 and μ_2 are the mean of the probability distribution functions F_1 and F_2 respectively and $0 \le p \le 1$ is called the weight of the component distribution, *i.e.*, for different values of p one can get different means. In particular, if $p = 0$ or 1, then this mixture distribution

Keywords and phrases: Constrained non-linear programming problem, inventory control, Lagrangian multiplier, mixture distributions.

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becomes ordinary probability distribution function. Therefore in this paper a mixture of distribution approach have been investigated to describe the lead time demand. Wu and Tsai [40] addressed the mixture of normal distribution in the inventory control theory. Here the lead time consists of n mutually independent components, Economic Order Quantity (EOQ) and lead time are the decision variables. Further, the authors proved that the both order quantity and total cost of complete backorder case is less than the order quantity and total cost of completely lost sale case respectively. The n mutually independent lead time components will be explain later in a detailed manner. Lee *et al.* [26] assumed that the demand of the lead time follows mixture of distribution free approach and the problem have been extended to mixture of normal distribution model as a special case of mixture of distribution free approach. Lee *et al.* [27] dealt the same approach of Lee *et al.* [26] but it differs only in lead time crashing cost, *i.e.*, the negative exponential lead time crashing cost is implemented instead of usual n components lead time crashing cost. Here the lead time can be reduced by adding crashing cost and the crashing cost is related to the lead time by a function of the form $R(L) = \alpha e^{-\gamma L}$ where α and γ are constants. Recently Annadurai [1] examined the mixture of distribution free approach for single item, and the author reduced the ordering cost by adding the logarithmic investment function introduced by Porteus [32]. Annadurai [1] have incorporated the n mutually independent component lead times.

The controllable lead time becomes a prominent issue and its control leads to many benefits. In fact, lead time usually consists of the following components as in Tersine [39] order preparation, order transit, supplier lead time, delivery time, and set-up time. Ben-Daya Raouf [4] dealt with controllable lead time in which the demand is assumed to follow a normal distribution. Ouyang *et al.* [29] extended Ben-Daya and Raouf [4] model by incorporating controllable time, allowing shortages and adding the stock-out costs. In addition, the total amount of stock-out is considered as a mixture of backorders and lost sales during the stock-out period. Ouyang *et al.* [30] developed lead time reduction inventory models under various crashing cost function and practical situations. Recently Hossain *et al.* [20] investigated a generalized lead time distribution model for completely backordering and late delivery cost. Braglia *et al.* [8] studied the completely backorder case in continuous review policy with Gaussian lead-time demand. In inventory control, under most of situations, unsatisfied demands are either completely backordered or completely lost. However, in some real inventory systems, it is more reasonable to assume that some of the excess demands are backordered and the rest is lost.

Integrated strategy gets more attention in last three decades. Because the integrated total cost is minimum compared to the sum of the total cost of the individuals. Firstly, Goyal [15] proved the integrated approach among the single supplier and single customer. After that many researchers like Banarjee [2], Ha [18], Ben Daya [4], Rabbani *et al.* [36] and Güler *et al.* [17] have employed in the synchronization of buyer and vendor. Goyal [16] proved that the integrated approach, the different lot size shipment policies gives a better result when compared with equal lot sized model. Hill [19] proposed a more general batch and shipping policy involving the successive shipment size of the first m shipments increases by a fixed factor and remaining shipments would be equal sized. Recently, Hsien-Jen [23] considered the integrated single vendor single buyer inventory model for defective items. Simultaneously the author reduced the defective percentage by implementing logarithmic investment function in a distribution free approach. Dey and Giri [11] considered the buyer vendor coordination in which the buyer keeps the inventories in two bins such as good items and defective items, also the holding costs of defective and non-defective items are not same. The authors assumed that the holding cost of defective items is less than the holding cost of the non-defective items. Similar to Hsien-Jen [23] the logarithmic investment, power investment functions are incorporated by Dey and Giri [11] to reduce the defective percentage. Braglia *et al.* [9] developed a periodic-review Joint-replenishment problem with ordering cost reduction and controllable lead times are taking into the account. Braglia *et al.* [10] studied the distribution free model in Joint-Replenishment Problem with stochastic demands and assuming that the ordering cost is controllable. Priyan and Manivannan [35] also employed the integrated approach in which the defective inventories are incorporated into the model. The defectiveness is assumed to follow a fuzziness and the quality inspection error also taking into the account. Feng *et al.* [13] investigated a capacity constraints integrated inventory model for single vendor and multiple buyers *via* transshipment.

Most of the inventory literature deals only for single item. However in the buyer's requirement may consist several items. So that, we have developed a multi-item inventory model. If we take a multi-item problem, the natural questions arrives is, whether this model has any constraints like inventory constraints, floor space constraint, budget constraint, warehouse constraints or order constraint. In this model we have studied two types of constraints; one is, buyer's warehouse constraint and another one is buyer's budget constraint. We have to optimize the total cost function within some limitations. Benton [6] investigated multiple item for multiple supplier with resource limitations. Also examined an efficient Heuristic programming procedure for evaluating alternative discount schedules. Ben-Daya and Raouf [3] considered multi-item inventory model under the budget and floor space constraints and the demand is assumed to follows a uniform distribution. Bhattacharya [7] developed a multi product inventory model for defective items. Taleizadeh *et al.* [38] investigated very rigorous assumptions, that is multi buyer multi-vendor multi-item multi-constraint problem. The authors solved this non-linear programming problem by using Genetic Algorithm (GA) and Harmony Search Algorithm (HSA). Finally, the authors compared the optimal solutions obtained from both GA and HSA, and they reveal the optimal solutions of HSA is better than the optimal solutions of GA. Pal *et al.* [31] developed multiple supplier a manufacturer and multiple retailer with multi-item, and assuming the manufacturer produces a finished item by the combination of certain percentage of various types of raw materials. Huang and Lin [22] addressed an integrated model for multi-item in which they determined delivery route and truck loads. The objective is to minimize the total travel length by using Ant Colony Optimization (ACO) algorithm. Priyan and Uthayakumar [33] attempted the mathematical model in which the quality inspection error is encountered. Some limitations to the buyer's side also studied. They are space and budget constraints. The authors determined the optimal solutions by using simple Lagrangian multiplier technique. Priyan and Uthayakumar [34] developed the integrated inventory system for multi-item with permissible delay in payment and considered both budget and floor space constraint. In our proposed model we employ the same Lagrangian multiplier technique to solve the non-linear objective function. Rabbani *et al.* [37] addressed a multi-item problem subject to multi constraints such as storage space, time period and budget constraints under the Vendor Managed Inventory (VMI) policy. Two algorithms, namely simulated annealing and tabu search have been used to minimize the cost function and determining the batch sizes. A multi-item economic production quantity (EPQ) model proposed by Kangi *et al.* [25] dealt the fact that production systems are often not perfect and developing the model by assuming the delivery due date, storage capacity and order frequency are an integral part of many real-world inventory systems.

The remainder of this paper is organized as follows. Motivation of the proposed model is provided in Section 2. In Section 3 Notations and Assumptions are given. The mathematical model is developed in Section 4. Mixture of normal distribution model for multi-item and mixture of distribution free approach for multi-item is examined in Sections 4.1 and 4.2 respectively. For both models, numerical examples and sensitivity analysis are given to illustrated the effectiveness of the proposed result in Section 5. Further, managerial implication is also given in the same section. Finally, conclusion of this study is summarized in Section 6.

2. Motivation to the model

In the previous section we look a detailed literature survey about integrated approach, inventory model of multi-item, inventory control under some constraints, probabilistic environment, mixture of distributions. There are many literature available in the integrated inventory model for multi-item under the stochastic environment. Among the stochastic situations there is no literature discussed about the mixture of distributions for multi-item under some resource limitations. The comparison of present stochastic model with some other existing literatures are tabulated in Table 1. So that in this paper our aim is to develop a mathematical model of constrained integrated inventory system consisting of several products under the mixture of distribution model. In this mixture of distribution model, two cases namely mixture of distribution free approach and mixture of normal distribution approach have been discussed. Free distribution approach is nothing but only the first and second moments (and hence, mean and variance are also known and finite) and we could not predict the distribution of the lead time demand. Another purpose of this paper is to model a partial backorder policy which we observed in practice, where firms buy similar products from competitors to cover shortages. Since the demand is stochastic, the actual shortages during the lead time is not known priori.

TABLE 1. A comparison of the present stochastic model with some existing literatures.

Reference	Distribution	Constraints	Integrated approach	Multi item
Annadurai [1]	Mixture	Service level	No	No
Ben-Daya [3]	Uniform	Budget and floor	No	Yes
Dey [11]	Normal	No	Yes	No
Haung [21]	Normal	No	Yes	No
Lin [23]	Distribution free	No	Yes	No
Priyan [34]	Normal	Budget and floor	Yes	Yes
Priyan [33]	Normal	Budget and floor	Yes	No
Taleizadeh [38]	Uniform	Multi constraints	Yes	Yes
Lee [40]	Mixture	No	No	No
Lee [26]	Mixture	No	No	No
Lee [27]	Mixture	No	No	No
Present model	Mixture	Budget and floor	Yes	Yes

3. Notations and Assumptions

To develop the mathematical model, let us introduce the following notations and assumptions.

3.1. Notations

For products $i = 1, 2, 3, \ldots, M$ the following notations will be used in this paper. Throughout the manuscript index i denotes the parameter of ith product.

3.1.1. Parameters

M	Number of products
D_i	Demand rate of the buyer for the ith product
P_i	Production rate of the vendor for the ith product
A_i	Ordering cost per order incurred by the buyer for the ith product
B_i	Set-up cost per set-up incurred by the vendor for the ith product
h_{bi}	Holding cost of the ith product per unit for the buyer
h_{vi}	Holding cost of the ith product per unit for the vendor
π_i	Fixed penalty cost per unit short for the ith product
π_{i0}	Marginal profit for the ith product (*i.e.*, penalty cost of lost demand of ith product)
β_i	Fraction of the shortage that will be backordered per shipment of the ith product, $0 \leq \beta_i \leq 1$
r_i	Reorder level of the ith product
pc_i	Unit purchase cost of the ith product incurred by the buyer
f_i	Space occupied per unit of the ith product in the buyer's warehouse (square feet/unit)
W	Maximum available storage space for all product in the buyer warehouse
Ω	Maximum available budget for all product incurred by the buyer

3.1.2. Decision variables

Q_i	Lot size of the ith product
n	Number of shipments for all products
L	Length of the deterministic lead time for all products
α, γ	Lagrangian multiplier

3.1.3. Random variables

X_i	The lead time demand with the mixture of distributions for the ith product as in [12]

3.1.4. Functions and Operators

$f(x_i)$	Probability density function of mixture distribution random variable X_i
$E(\cdot)$	Mathematical expectation
$C(L)$	Lead time crashing cost function
x^+	Maximum of x and 0, (*i.e.*, $x^+ = \max\{x, 0\}$)

3.1.5. Performance measure

ETC^N	Expected total cost of the system under mixture of normal distribution
ETC^U	Expected total cost of the system under mixture of distribution free approach

3.2. Assumptions

1. The buyer and vendor belongs to different corporate entities and enthusiastic to have the collaboration inventory system. Thus, both parties agree to minimize the cost under integrated strategy.
2. The buyer use the continuous review policy for all products and the lot size Q_i is placed whenever the inventory level falls to the reorder point r_i.
3. For all products, the lead time L consists of m mutually independent components. The jth component has a normal duration b_j, minimum duration a_j, and crashing cost per unit time c_j such that $c_1 \le c_2 \le \cdots \le c_m$. The components of lead time are crashed one at a time starting from the first component because it has the minimum unit crashing cost, and then the second component, and so on. Let $L_0 = \sum_{j=1}^{m} b_j$, and L_j be the length of the lead time with components $1, 2, \ldots, j$ crashed to their minimum duration, then L_j can be expressed as $L_j = L_0 - \sum_{f=1}^{j} (b_f - a_f)$, $j = 1, 2, \ldots, m$; and for all products, the lead time crashing cost per cycle $C(L)$ is given by $C(L) = c_j(L_{j-1} - L) + \sum_{f=1}^{j-1} c_f(b_f - a_f)$, $L \in [L_j, L_{j-1}]$.
4. During the stock-out period, a fraction β_i of the demand will be backordered, and the remaining fraction $(1 - \beta_i)$ will be lost.
5. The reorder level of the ith product r_i = (expected demand during lead time of the ith product) + (safety stock of the ith product) and safety stock of the ith product = $k_i\times$ standard deviation of the lead time demand of the ith product. (*i.e.*, $r_i = \mu_{i*}L + k_i\sigma_{i*}\sqrt{L}$.) where $\mu_{i*} = p_i\mu_{i1} + (1 - p_i)\mu_{i2}$, $\sigma_{i*} = \sigma_i\sqrt{1 + p_i(1 - p_i)\epsilon_i^2}$, $\mu_{i1} = \mu_{i*} + (1 - p_i)\epsilon_i\sigma_i/\sqrt{L}$, $\mu_{i2} = \mu_{i*} - p_i\epsilon_i\sigma_i/\sqrt{L}$ and k_i is the safety factor satisfies the relation $P(X_i > r_i) = 1 - p_iF(r_{i1}) - (1 - p_i)F(r_{i2}) = q_i$, where q_i represents the allowable stock-out probability during the lead time L and F represent the cumulative distribution of the standard normal random variable, $r_{i1} = k_i\sqrt{1 + \epsilon_i^2 p_i(1 - p_i)} - \epsilon_i(1 - p_i)$ and $r_{i2} = k_i\sqrt{1 + \epsilon_i^2 p_i(1 - p_i)} + \epsilon_i p_i$, $\forall$ $i = 1, 2, \ldots, M$.
6. The buyer has two types of limitations namely budget constraint and warehouse constraint. So that, the buyer receives only limited quantities for all M items.
7. An infinite time horizon is considered.

4. Model development

Consider a two echelon inventory model consisting of M products in which the buyer not only has a limited warehouse capacity of W for all products, but also the total amount of purchasing for all product is less than or equal to the maximum available budget Ω. The buyer places an order of size nQ_i for ith product of non-defective items. The vendor produces these nQ_i items in one set-up and transfers it into equal size Q_i for n times. The set-up cost per set-up for the ith product is B_i. The vendor produces ith product in a finite production rate P_i. Since the demand rate of ith product is D_i, so that, the consumption time of Q_i quantity is Q_i/D_i. Therefore the time for consuming all nQ_i items is nQ_i/D_i. On the other hand, we assume that the integrated production inventory model allows shortages with partial backorders. The fraction β_i of shortages was backordered on the next replenishment. Inventory is continuously reviewed. Replenishment is being made whenever the inventory level is falling to the reorder point r_i. That is, the new ordered is placed whenever the inventory reaches the reorder point. The expected shortages at the end of the cycle for the ith product is $E(X_i - r_i)^+$. Since β_i is the backordered ratio, so that, the expected amount of backorder per cycle is $\beta_i E(X_i - r_i)^+$ and the remaining $(1-\beta_i)E(X_i - r_i)^+$ is lost. The buyer's inventory pattern for the ith product is depicted in Figure 1. Hence the expected stock-out cost per unit time is

$$\frac{D_i}{Q_i}[\pi_i + \pi_{i0}(1-\beta_i)]E(X_i - r_i)^+. \tag{4.1}$$

The expected net inventory level just before the order arrives is $r_i - D_i L + (1-\beta_i)E(X_i - r_i)^+$ and the expected net inventory level at the beginning of the cycle is $Q_i + r_i - D_i L + (1-\beta_i)E(X_i - r_i)^+$. Hence the average inventory at any time is

$$\frac{Q_i}{2} + r_i - D_i L + (1-\beta_i)E(X_i - r_i)^+. \tag{4.2}$$

Therefore the buyer's expected average holding cost is

$$h_{bi}\left\{\frac{Q_i}{2} + r_i - D_i L + (1-\beta_i)E(X_i - r_i)^+\right\}. \tag{4.3}$$

The buyer has to pay ordering cost A_i for each order and lead time reduction crashing cost $C(L)$ for each shipments as in assumption (3). Hence, the buyer's average total ordering cost and lead time crashing cost per unit time is $\frac{D_i}{Q_i}[A_i + C(L)]$. Therefore the buyer's average total cost for the ith product is sum of set-up cost, holding cost, shortage cost and lost sale cost. *i.e.*,

$$\frac{D_i}{Q_i}[A_i + C(L)] + h_{bi}\left\{\frac{Q_i}{2} + r_i - D_i L + (1-\beta_i)E(X_i - r_i)^+\right\} + \frac{D_i}{Q_i}[\pi_i + \pi_{i0}(1-\beta_i)]E(X_i - r_i)^+.$$

Thus the buyer's total expected cost for all M products is

$$\sum_{i=1}^{M}\left\{\frac{D_i}{Q_i}\left[A_i + C(L) + (\pi_i + \pi_{i0}(1-\beta_i))\,E(X_i - r_i)^+\right] + h_{bi}\left[\frac{Q_i}{2} + r_i - D_i L + (1-\beta_i)E(X_i - r_i)^+\right]\right\}. \tag{4.4}$$

Next we form the vendor's total cost. For each production run, the set-up cost per unit time is $\frac{B_i D_i}{nQ_i}$. The vendor produces the items and delivers it to the buyer. The next delivery will be made after $\frac{Q_i}{D_i}$ units of time and this process continues until the vendor's inventory level reaches zero. Applying a similar approach of [24], one can get the vendor holding area as

$$\begin{aligned}
&= (\text{Area of vendor accumulation} - \text{Area of buyer accumulation in Fig. 2}) \div nQ_i/D_i \\
&= \frac{\left[nQ_i\{Q_i/P_i + (n-1)Q_i/D_i\} - \frac{1}{2}n^2Q_i^2/P_i\right] - \frac{Q_i^2}{D_i}[n(n-1)/2]}{nQ_i/D_i} \\
&= \frac{Q}{2}\left[n\left(1 - \frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right].
\end{aligned}$$

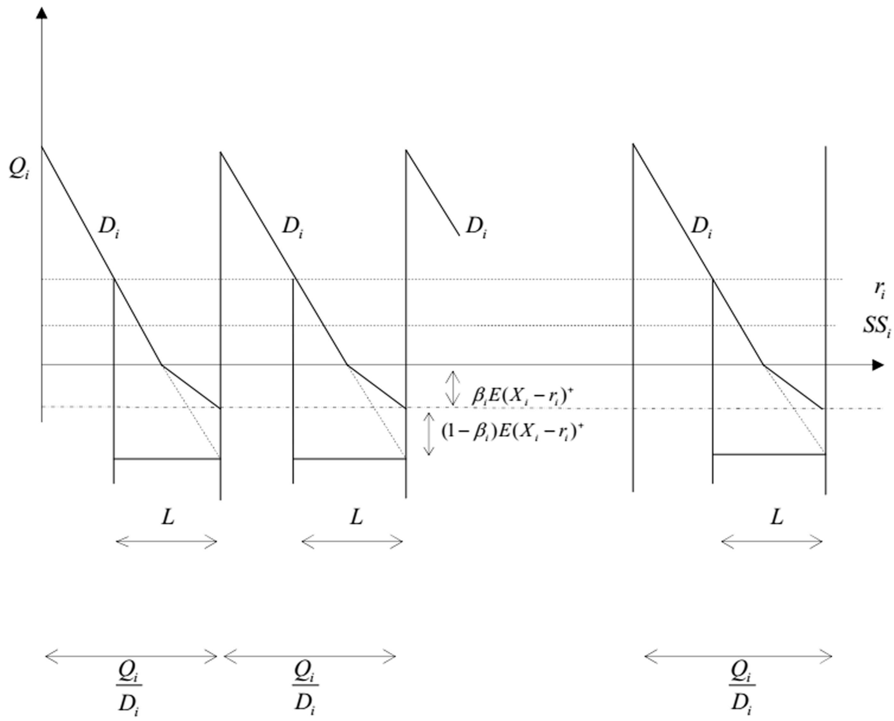

FIGURE 1. Inventory pattern of the buyer for the ith product.

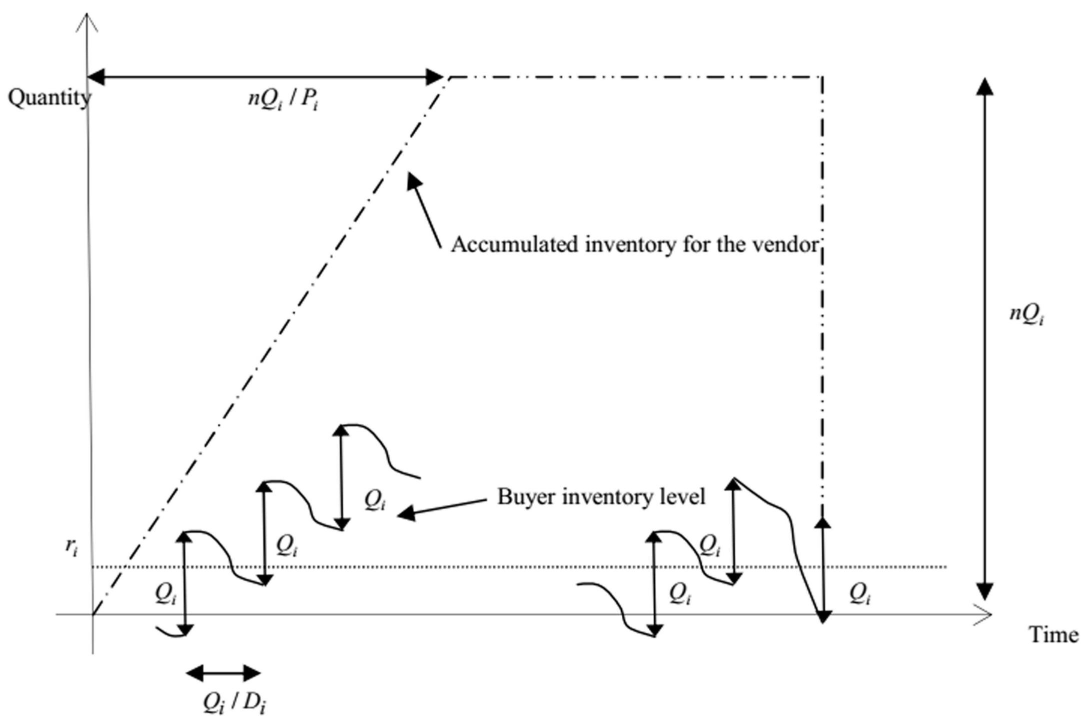

FIGURE 2. Inventory pattern of the buyer and vendor for the ith product.

The vendor's expected holding cost per unit time is

$$h_{vi}\frac{Q}{2}\left[n\left(1-\frac{D_i}{P_i}\right)-1+\frac{2D_i}{P_i}\right]. \tag{4.5}$$

The vendor's expected annual total cost per unit time for all products, which is composed of set-up cost, holding cost are expressed by

$$\sum_{i=1}^{M}\left\{\frac{B_iD_i}{nQ_i}+h_{vi}\frac{Q}{2}\left[n\left(1-\frac{D_i}{P_i}\right)-1+\frac{2D_i}{P_i}\right]\right\}. \tag{4.6}$$

Hence the integrated expected total cost ETC per unit time for all products can be expressed as sum of the buyer's total cost and vendor's total cost as in equations (4.4) and (4.6) respectively.

$$\begin{aligned} ETC &= \sum_{i=1}^{M}\left\{\frac{D_i}{Q_i}\left[A_i+C(L)+\left[\pi_i+\pi_{i0}(1-\beta_i)\right]E(X_i-r_i)^+\right]+h_{bi}\left[\frac{Q_i}{2}+r_i-D_iL+(1-\beta_i)E(X_i-r_i)^+\right]\right\}\\ &\quad+\sum_{i=1}^{M}\left\{\frac{B_iD_i}{nQ_i}+h_{vi}\frac{Q}{2}\left[n\left(1-\frac{D_i}{P_i}\right)-1+\frac{2D_i}{P_i}\right]\right\},\\ &=\sum_{i=1}^{M}\left\{\frac{D_i}{Q_i}\left[A_i+B_i/n+C(L)+\left[\pi_i+\pi_{i0}(1-\beta_i)\right]E(X_i-r_i)^+\right]+h_{bi}\left[\frac{Q_i}{2}+r_i-D_iL+(1-\beta_i)E(X_i-r_i)^+\right]\right.\\ &\quad\left.+h_{vi}\frac{Q}{2}\left[n\left(1-\frac{D_i}{P_i}\right)-1+\frac{2D_i}{P_i}\right]\right\}. \end{aligned} \tag{4.7}$$

Since our aim is to determine the lot size Q_i, lead time L and the number of shipments n for all products in the system such that the integrated expected total cost in the supply chain attains minimum subject to the constraints. The constraints are

(i) The warehouse constraint $\sum_{i=1}^{M} f_iQ_i \leq W$

(ii) The budget constraint $\sum_{i=1}^{M} pc_iQ_i \leq \Omega$.

The model is to minimize the integrated total cost in equation (4.7) subject to both constraints.

$$\begin{aligned} \text{Minimize}\quad ETC = &\sum_{i=1}^{M}\left\{\frac{D_i}{Q_i}\left[A_i+\frac{B_i}{n}+C(L)+\left(\pi_i+\pi_{i0}(1-\beta_i)\right)E(X_i-r_i)^+\right]\right.\\ &\left.+h_{bi}\left[\frac{Q_i}{2}+r_i-D_iL+(1-\beta_i)E(X_i-r_i)^+\right]+h_{vi}\frac{Q}{2}\left[n\left(1-\frac{D_i}{P_i}\right)-1+\frac{2D_i}{P_i}\right]\right\}, \end{aligned} \tag{4.8}$$

subject to $\sum_{i=1}^{M} f_iQ_i \leq W$ and $\sum_{i=1}^{M} pc_iQ_i \leq \Omega$. When the demand of the different customers do not have identical in the lead time, then we cannot use a single distribution to determine the demand of the lead time. So that, in this paper the demand of the lead time follows a mixture distribution have been assumed. Two types of mixture distributions, such as mixture of normal distribution and mixture of distribution free approach will be investigated. Moreover, mixture of normal distribution is a special case of mixture of distribution free approach when the distribution function is known.

4.1. Mixture of normal distribution approach

For the lead time L and assume that the demand of the lead time X_i follows the mixture of normal distributions with the probability density function is given by

$$f(x_i) = p_i \frac{1}{\sqrt{2\pi}\sigma_i\sqrt{L}} \exp\left(-\frac{(x_i-\mu_{i1}L)^2}{2\sigma_i^2 L}\right) + (1-p_i)\frac{1}{\sqrt{2\pi}\sigma_i\sqrt{L}} \exp\left(-\frac{(x_i-\mu_{i2}L)^2}{2\sigma_i^2 L}\right), \tag{4.9}$$

where $\mu_{i1}-\mu_{i2} = \epsilon_i\sigma_i/\sqrt{L}$ or $\mu_{i1}L-\mu_{i2}L = \epsilon_i\sigma_i\sqrt{L}$, $\epsilon_i > 0$, $-\infty < x_i < \infty$, $0 \le p_i \le 1$, $\sigma_i > 0$, $\epsilon_i \in \mathbb{R}$ (see [1, 12, 26, 27]).

Then the expected shortage at the end of the cycle is given by

$$E(X_i-r_i)^+ = \int_{r_i}^{\infty}(x_i-r_i)f(x_i)\mathrm{d}x_i = \sigma_i\sqrt{L}\Psi(r_{i1},r_{i2},p_i), \tag{4.10}$$

where, $\Psi(r_{i1},r_{i2},p_i) = p_i\left\{\phi(r_{i1})-r_{i1}(1-F(r_{i1}))\right\} + (1-p_i)\left\{\phi(r_{i2})-r_{i2}(1-F(r_{i2}))\right\}.$

The expected net inventory level just before the order arrives is

$$\begin{aligned}
&E[(X_i-r_i)^- I_{(0<x<r)}] - \beta E(X_i-r_i)^+\\
&= \sigma_i\sqrt{L}\left\{p_i\left[r_{i1}F\left(\frac{\mu_{i1}\sqrt{L}}{\sigma_i}\right)-\phi\left(\frac{\mu_{i1}\sqrt{L}}{\sigma_i}\right)\right] + (1-p_i)\left[r_{i2}F\left(\frac{\mu_{i2}\sqrt{L}}{\sigma_i}\right)-\phi\left(\frac{\mu_{i2}\sqrt{L}}{\sigma_i}\right)\right]\right\}\\
&\quad + (1-\beta_i)E(X_i-r_i)^+,\\
&= \sigma_i\sqrt{L}\left\{p_i\left[r_{i1}F\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i}+(1-p_i)\epsilon_i\right)-\phi\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i}(1-p_i)\epsilon_i\right)\right]\right.\\
&\quad \left. + (1-p_i)\left[r_{i2}F\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i}-p_i\epsilon_i\right)-\phi\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i}-p_i\epsilon_i\right)\right]\right\} + (1-\beta_i)E(X_i-r_i)^+,
\end{aligned} \tag{4.11}$$

where, $Y^- = \begin{cases}-Y, & Y<0,\\ 0, & Y>0,\end{cases}$ and $I_{(0<x<r)} = \begin{cases}1, & 0<x<r,\\ 0, & \text{otherwise},\end{cases}$
and the expected net inventory level at the beginning of the cycle is

$$\begin{aligned}
&Q_i + \sigma_i\sqrt{L}\left\{p_i\left[r_{i1}F\left(\frac{\mu_{i1}\sqrt{L}}{\sigma_i}\right)-\phi\left(\frac{\mu_{i1}\sqrt{L}}{\sigma_i}\right)\right] + (1-p_i)\left[r_{i2}F\left(\frac{\mu_{i2}\sqrt{L}}{\sigma_i}\right)-\phi\left(\frac{\mu_{i2}\sqrt{L}}{\sigma_i}\right)\right]\right\}\\
&\quad + (1-\beta_i)E(X_i-r_i)^+,\\
&= Q_i + \sigma_i\sqrt{L}\left\{p_i\left[r_{i1}F\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i}+(1-p_i)\epsilon_i\right)-\phi\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i}+(1-p_i)\epsilon_i\right)\right]\right.\\
&\quad \left. + (1-p_i)\left[r_{i2}F\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i}-p_i\epsilon_i\right)-\phi\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i}-p_i\epsilon_i\right)\right]\right\} + (1-\beta_i)E(X_i-r_i)^+.
\end{aligned} \tag{4.12}$$

Therefore, the expected buyer's total cost per unit time is

$$\begin{aligned}
&h_{bi}\left\{\frac{Q_i}{2} + \sigma_i\sqrt{L}\left\{p_i\left[r_{i1}F\left(\frac{\mu_{i1}\sqrt{L}}{\sigma_i}\right)-\phi\left(\frac{\mu_{i1}\sqrt{L}}{\sigma_i}\right)\right] + (1-p_i)\left[r_{i2}F\left(\frac{\mu_{i2}\sqrt{L}}{\sigma_i}\right)-\phi\left(\frac{\mu_{i2}\sqrt{L}}{\sigma_i}\right)\right]\right\}\right\}\\
&\quad + h_{bi}(1-\beta_i)E(X_i-r_i)^+,
\end{aligned}$$

$$= h_{bi}\left\{\frac{Q_i}{2} + \sigma_i\sqrt{L}\left\{p_i\left[r_{i1}F\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} + (1-p_i)\epsilon_i\right) - \phi\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} + (1-p_i)\epsilon_i\right)\right]\right.\right.$$
$$\left.\left. + (1-p_i)\left[r_{i2}F\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} - p_i\epsilon_i\right) - \phi\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} - p_i\epsilon_i\right)\right]\right\}\right\} + h_{bi}(1-\beta_i)E(X_i - r_i)^+. \tag{4.13}$$

By using equation (4.10) and (4.13) the integrated expected total cost in equation (4.8) becomes

$$\text{Minimize}\quad ETC^N = \sum_{i=1}^{M}\left\{\frac{D_i}{Q_i}\left[A_i + B_i/n + C(L) + (\pi_i + \pi_{i0}(1-\beta_i))\,\sigma_i\sqrt{L}\Psi(r_{i1}, r_{i2}, p_i)\right]\right.$$
$$+\ h_{bi}\left\{\frac{Q_i}{2} + \sigma_i\sqrt{L}\left\{p_i\left[r_{i1}F\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} + (1-p_i)\epsilon_i\right) - \phi\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i}(1-p_i)\epsilon_i\right)\right]\right.\right.$$
$$\left.\left. +\ (1-p_i)\left[r_{i2}F\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} - p_i\epsilon_i\right) - \phi\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} - p_i\epsilon_i\right)\right]\right\}\right\}$$
$$\left. + h_{bi}(1-\beta_i)\sigma_i\sqrt{L}\Psi(r_{i1}, r_{i2}, p_i) + h_{vi}\frac{Q_i}{2}\left[n\left(1 - \frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right]\right\}. \tag{4.14}$$

We denote the expected total cost of mixture of normal distributions model by ETC^N. Equation (4.14) is the integrated total expected cost for all M products in the mixture normal distribution model. Now our aim is to minimize the expected total cost in equation (4.14) subject to the constraints

$$\sum_{i=1}^{M} f_i Q_i \le W \quad \text{and} \quad \sum_{i=1}^{M} pc_i Q_i \le \Omega.$$

4.1.1. Solution procedure

The problem in equation (4.14) is a constrained mixed integer non-linear programming model. Therefore we present a simple Lagrangian multiplier technique similar to [33, 34] to solve the given problem. The detailed solution approach of the non-linear problem will be discussed in the following cases.

Case 1. In this case, we temporarily ignore the constraints $\sum_{i=1}^{M} f_i Q_i \le W$ and $\sum_{i=1}^{M} pc_i Q_i \le \Omega$ then determine the optimal solutions of Q_i, L and n which minimize the integrated expected total cost ETC^N. For fixed Q and $L \in [L_j, L_{j-1}]$, ETC^N is convex in n, which indicates that $n = n^*$ which satisfies the following relation

$$ETC^N(Q, L, n^*-1) > ETC^N(Q, L, n^*) \le ETC^N(Q, L, n^*+1) \quad \text{where} \quad Q = (Q_1, Q_2, \ldots, Q_M),$$
$$\text{because}\quad \frac{\partial}{\partial n}ETC^N = \sum_{i=1}^{M}\left\{-\frac{B_i D_i}{n^2 Q_i} + h_{vi}\frac{Q_i}{2}\left(1 - \frac{D_i}{P_i}\right)\right\} \tag{4.15}$$
$$\text{and}\quad \frac{\partial^2}{\partial n^2}ETC^N = \sum_{i=1}^{M}\frac{2B_i D_i}{n^3 Q_i} > 0, \quad \text{for all}\quad n > 0. \tag{4.16}$$

Now, for fixed n, taking partial derivatives of ETC^N with respect to Q_i and L we get

$$\begin{aligned}
\frac{\partial}{\partial Q_i}ETC^N &= \left[A_i + \frac{B_i}{n} + C(L) + (\pi_i + \pi_{i0}(1-\beta_i))\sigma_i\sqrt{L}\Psi(r_{i1}, r_{i2}, p_i)\right]\left[-\frac{D_i}{Q_i^2}\right] \\
&\quad + \frac{h_{bi}}{2} + \frac{h_{vi}}{2}\left[n\left(1 - \frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right], \qquad (4.17)\\
\frac{\partial}{\partial L}ETC^N &= \sum_{i=1}^{M}\Bigg\{-c_j\frac{D_j}{Q_i} + (\pi_i + \pi_{i0}(1-\beta_i))\frac{\sigma_i\Psi(r_{i1}, r_{i2}, p_i)D_i}{2\sqrt{L}Q_i} + h_{bi}\frac{(1-\beta_i)\sigma_i\Psi(r_{i1}, r_{i2}, p_i)}{2\sqrt{L}} \\
&\quad + \frac{h_{bi}\sigma_i}{2\sqrt{L}}\Bigg\{p_i\left[r_{i1}F\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} + (1-p_i)\epsilon_i\right) - \phi\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} + (1-p_i)\epsilon_i\right)\right] \\
&\quad + (1-p_i)\left[r_{i2}F\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} - p_i\epsilon_i\right) - \phi\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} - p_i\epsilon_i\right)\right]\Bigg\} \\
&\quad + \frac{h_{bi}}{2}\mu_{i*}\Bigg[p_i\left(r_{i1} + \frac{\mu_{i*}\sqrt{L}}{\sigma_i} + (1-p_i)\epsilon_i\right)\phi\left(\frac{\mu_{i*\sqrt{L}}}{\sigma_i} + (1-p_i)\epsilon_i\right) \\
&\quad + (1-p_i)\left(r_{i2} + \frac{\mu_{i*}\sqrt{L}}{\sigma_i} - p_i\epsilon_i\right)\phi\left(\frac{\mu_{i*\sqrt{L}}}{\sigma_i} - p_i\epsilon_i\right)\Bigg]\Bigg\}, \qquad (4.18)
\end{aligned}$$

respectively. It is clear that for given r_{i1}, r_{i2} and p_i we have $\Psi(r_{i1}, r_{i2}, p_i) > 0$, for all $i = 1, 2, \ldots, M$. For fixed $L \in [L_j, L_{j-1}]$ and n equation (4.14) is convex in Q_i, since

$$\frac{\partial^2}{\partial Q_i^2}ETC = \left[A_i + \frac{B_i}{n} + (\pi_i + \pi_{i0}(1-\beta_i))\sigma_i\sqrt{L}\Psi(r_{i1}, r_{i2}, p_i)\right]\left[\frac{2D_i}{Q_i^3}\right] > 0 \quad \forall\ Q_i > 0. \qquad (4.19)$$

However for fixed (Q, n), the equation (4.14) is concave in $L \in [L_j, L_{j-1}]$, because

$$\begin{aligned}
\frac{\partial^2}{\partial L^2}ETC = \sum_{i=1}^{M}\Bigg\{&-(\pi_i + \pi_{i0}(1-\beta_i))\frac{\sigma_i\Psi(r_{i1}, r_{i2}, p_i)D_i}{4Q_iL^{\frac{3}{2}}} - h_{bi}\frac{(1-\beta_i)\sigma_i\Psi(r_{i1}, r_{i2}, p_i)}{4L^{\frac{3}{2}}} \\
&- h_{bi}\frac{\sigma_i}{4L^{\frac{3}{2}}}\Bigg\{p_i\left[r_{i1}F\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} + (1-p_i)\epsilon_i\right) - \phi\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} + (1-p_i)\epsilon_i\right)\right] \\
&+ (1-p_i)\left[r_{i2}F\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} - p_i\epsilon_i\right) - \phi\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} - p_i\epsilon_i\right)\right]\Bigg\} \\
&- h_{bi}\frac{\mu_{i*}}{4L}\Bigg\{p_i\phi\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} + (1-p_i)\epsilon_i\right)\Bigg[\frac{\mu_{i*}\sqrt{L}}{\sigma_i}\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} + (1-p_i)\epsilon_i\right)\left(r_{i1} + \frac{\mu_{i*}\sqrt{L}}{\sigma_i} + (1-p_i)\epsilon_i\right) \\
&- \left(r_{i1} + \frac{2\mu_{i*}\sqrt{L}}{\sigma_i} + (1-p_i)\epsilon_i\right)\Bigg] + (1-p_i)\phi\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} - p_i\epsilon_i\right)\Bigg[\frac{\mu_{i*}\sqrt{L}}{\sigma_i}\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} - p_i\epsilon_i\right) \\
&\left(r_{i2} + \frac{\mu_{i*}\sqrt{L}}{\sigma_i} - p_i\epsilon_i\right) - \left(r_{i2} + \frac{2\mu_{i*}\sqrt{L}}{\sigma_i} - p_i\epsilon_i\right)\Bigg]\Bigg\}\Bigg\} < 0, \qquad (4.20)
\end{aligned}$$

if $\frac{\mu_{i*}\sqrt{L}}{\sigma_i} - p_i\epsilon_i > \sqrt{2}$, $\forall\ i = 1, 2, \ldots, M$. Therefore, for fixed Q and n, the minimum ETC^N will occur at the end points of the interval $[L_j, L_{j-1}]$. Equating (4.17) to zero and solve it with respect to Q_i, we get

$$Q_i = \left[\frac{\left[A_i + \frac{B_i}{n} + C(L) + (\pi_i + \pi_{i0}(1-\beta_i))\sigma_i\sqrt{L}\Psi(r_{i1}, r_{i2}, p_i)\right] D_i}{\frac{h_{bi}}{2} + \frac{h_{vi}}{2}\left[n\left(1 - \frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right]}\right]^{\frac{1}{2}}, \quad L \in [L_j, L_{j-1}]. \tag{4.21}$$

For fixed n and $L \in [L_j, L_{j-1}]$, both the constraints $\sum_{i=1}^M f_iQ_i \le W$ and $\sum_{i=1}^M pc_iQ_i \le \Omega$ are ignored, then the equation (4.21) gives optimal values of Q_i. The following iterative algorithm have been developed to find the optimal values of Q_i, L and n.

Algorithm 4.1.

Step 1 Set $n = 1$.
Step 2 For each L_j, $j = 0, 1, 2, \ldots, m$ preform Steps (2.1) and (2.2).
Step 2.1 Determine the corresponding Q_i $(i = 1, 2, \ldots, M)$ from equation (4.21).
Step 2.2 Compute the corresponding $ETC^N(Q, L_j, n)$ from equation (4.14) where $Q = (Q_1, Q_2, \ldots, Q_M)$.
Step 3 Find $\min\limits_{j=0,1,2,\ldots,m} ETC^N(Q, L_j, n)$
Step 4 Set $ETC(Q^*_{(n)}, L^*_{(n)}, n) = \min\limits_{j=0,1,2,\ldots,m} ETC^N(Q, L_j, n)$ and $(Q^*_{(n)}, L^*_{(n)})$ is the optimal solution for fixed n.
Step 5 Set n by $n+1$ and repeat the Steps 2 to 4, to get $ETC(Q^*_{(n)}, L^*_{(n)}, n)$.
Step 6 If $ETC(Q^*_{(n)}, L^*_{(n)}, n) < ETC(Q^*_{(n-1)}, L^*_{(n-1)}, n-1)$ then go to Step 5. Otherwise go to Step 7.
Step 7 Set $(Q^*, L^*, n^*) = (Q^*_{(n-1)}, L^*_{(n-1)}, n-1)$ and (Q^*, L^*, n^*) is the optimal solutions. Stop the algorithm.

Case 2. In this case, we consider only buyer's floor space constraint and ignore budget constraint. To solve this problem, apply the Lagrangian multiplier α:

$$\begin{aligned}\text{Minimize}\quad ETC^N = \sum_{i=1}^M \Bigg\{ &\frac{D_i}{Q_i}\left[A_i + \frac{B_i}{n} + C(L) + (\pi_i + \pi_{i0}(1-\beta_i))\,\sigma_i\sqrt{L}\Psi(r_{i1}, r_{i2}, p_i)\right] \\ &+ h_{bi}\left\{\frac{Q_i}{2} + \sigma_i\sqrt{L}\left\{p_i\left[r_{i1}F\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} + (1-p_i)\epsilon_i\right) - \phi\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} + (1-p_i)\epsilon_i\right)\right]\right.\right. \\ &\left.\left. + (1-p_i)\left[r_{i2}F\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} - p_i\epsilon_i\right) - \phi\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} - p_i\epsilon_i\right)\right]\right\}\right\} \\ &+ h_{bi}(1-\beta_i)\sigma_i\sqrt{L}\Psi(r_{i1}, r_{i2}, p_i) + h_{vi}\frac{Q_i}{2}\left[n\left(1 - \frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right] + \alpha(f_iQ_i - W)\Bigg\}.\end{aligned} \tag{4.22}$$

For fixed n and $L \in [L_j, L_{j-1}]$, the optimal Q_i can be determined by solving the equation $\frac{\partial}{\partial Q_i}ETC^N = 0$ and $\frac{\partial}{\partial \alpha}ETC^N = 0$. *i.e.*,

$$Q_i = \left[\frac{\left[A_i + \frac{B_i}{n} + C(L) + (\pi_i + \pi_{i0}(1-\beta_i))\sigma_i\sqrt{L}\Psi(r_{i1}, r_{i2}, p_i)\right] D_i}{\frac{h_{bi}}{2} + \frac{h_{vi}}{2}\left[n\left(1 - \frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right] + \alpha f_i}\right]^{\frac{1}{2}}, \quad L \in [L_j, L_{j-1}], \tag{4.23}$$

and α can be determined by solving the equation (4.24)

$$\sum_{i=1}^{M} f_i \left[\sqrt{\frac{\left[A_i + \frac{B_i}{n} + C(L) + (\pi_i + \pi_{i0}(1-\beta_i))\sigma_i\sqrt{L}\Psi(r_{i1}, r_{i2}, p_i)\right] D_i}{\frac{h_{bi}}{2} + \frac{h_{vi}}{2}\left[n\left(1 - \frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right] + \alpha f_i}}\right] - W = 0. \tag{4.24}$$

Similar to Case 1, one can prove the equation (4.22) is convex with respect to Q_i and n and concave with respect to $L \in [L_j, L_{j-1}]$. Here we developed an iterative algorithm to find optimal solutions for Q_i,L_i, n and α.

Algorithm 4.2.

Step 1 Set $n = 1$.
Step 2 For each L_j, $j = 0, 1, 2, \ldots, m$ preform Steps (2.1) and (2.3).
  Step 2.1 Calculate α from equation (4.24).
  Step 2.2 Determine the corresponding Q_i $(i = 1, 2, \ldots, M)$ from equation (4.23).
  Step 2.3 Compute the corresponding $ETC^N(Q, L_j, n, \alpha)$ from equation (4.22) where $Q = (Q_1, Q_2, \ldots, Q_M)$.
Step 3 Find $\min\limits_{j=0,1,2,\ldots,m} ETC^N(Q, L_j, n, \alpha)$.
Step 4 Set $ETC(Q^*_{(n)}, L^*_{(n)}, n, \alpha^*_{(n)}) = \min\limits_{j=0,1,2,\ldots,m} ETC^N(Q, L_j, n, \alpha)$ and $(Q^*_{(n)}, L^*_{(n)}, \alpha^*_{(n)})$ is the optimal solution for fixed n.
Step 5 Set n by $n+1$ and repeat the Steps 2 to 4, to get $ETC(Q^*_{(n)}, L^*_{(n)}, n, \alpha^*)$.
Step 6 If $ETC(Q^*_{(n)}, L^*_{(n)}, n, \alpha^*_{(n)}) < ETC(Q^*_{(n-1)}, L^*_{(n-1)}, n-1, \alpha^*_{(n)})$ then go to Step 5. Otherwise go to Step 7.
Step 7 Set $(Q^*, L^*, n^*, \alpha^*) = (Q^*_{(n-1)}, L^*_{(n-1)}, n-1, \alpha^*_{(n)})$ and $(Q^*, L^*, n^*, \alpha^*)$ is the optimal solutions. Stop the algorithm.

Case 3. In this case, consider the buyer's budget constraint and ignore floor space constraint. By adding Lagrangian multiplier γ, the problem have been solved

$$\begin{aligned}
\text{Minimize} \quad ETC^N = \sum_{i=1}^{M} \Bigg\{ & \frac{D_i}{Q_i}\left[A_i + \frac{B_i}{n} + C(L) + (\pi_i + \pi_{i0}(1-\beta_i))\,\sigma_i\sqrt{L}\Psi(r_{i1}, r_{i2}, p_i)\right] \\
& + h_{bi}\left\{\frac{Q_i}{2} + \sigma_i\sqrt{L}\left\{p_i\left[r_{i1}F\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} + (1-p_i)\epsilon_i\right) - \phi\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i}(1-p_i)\epsilon_i\right)\right]\right.\right. \\
& \left.\left. +(1-p_i)\left[r_{i2}F\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} - p_i\epsilon_i\right) - \phi\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} - p_i\epsilon_i\right)\right]\right\}\right\} \\
& + h_{bi}(1-\beta_i)\sigma_i\sqrt{L}\Psi(r_{i1}, r_{i2}, p_i) + h_{vi}\frac{Q_i}{2}\left[n\left(1 - \frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right] + \gamma(pc_iQ_i - \Omega)\Bigg\}.
\end{aligned} \tag{4.25}$$

For fixed n and $L \in [L_j, L_{j-1}]$, the optimal Q_i can be determined by solving the equation $\frac{\partial}{\partial Q_i} ETC^N = 0$ and $\frac{\partial}{\partial \gamma} ETC^N = 0$. *e.g.*,

$$Q_i = \left[\frac{\left[A_i + \frac{B_i}{n} + C(L) + (\pi_i + \pi_{i0}(1-\beta_i))\sigma_i\sqrt{L}\Psi(r_{i1}, r_{i2}, p_i)\right] D_i}{\frac{h_{bi}}{2} + \frac{h_{vi}}{2}\left[n\left(1 - \frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right] + \gamma pc_i}\right]^{\frac{1}{2}}, \quad L \in [L_j, L_{j-1}], \tag{4.26}$$

and γ can be determined by solving the equation (4.27)

$$\sum_{i=1}^{M} pc_i \left[\sqrt{\frac{\left[A_i + \frac{B_i}{n} + C(L) + (\pi_i + \pi_{i0}(1-\beta_i))\sigma_i\sqrt{L}\Psi(r_{i1}, r_{i2}, p_i)\right] D_i}{\frac{h_{bi}}{2} + \frac{h_{vi}}{2}\left[n\left(1 - \frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right] + \gamma pc_i}} \right] - \Omega = 0. \tag{4.27}$$

Similar to Case 1, we can prove equation (4.25) is convex with respect to Q_i and n and concave with respect to $L \in [L_j, L_{j-1}]$. Here we developed an iterative algorithm to find optimal solutions for Q_i,L_i,n and γ.

Algorithm 4.3.

Step 1 Set $n = 1$.
Step 2 For each L_j, $j = 0, 1, 2, \ldots, m$ perform Steps (2.1) and (2.3).
Step 2.1 Calculate γ from equation (4.27).
Step 2.2 Determine the corresponding Q_i $(i = 1, 2, \ldots, M)$ from equation (4.26).
Step 2.3 Compute the corresponding $ETC^N(Q, L_j, n, \gamma)$ from equation (4.25) where $Q = (Q_1, Q_2, \ldots, Q_M)$.
Step 3 Find $\min\limits_{j=0,1,2,\ldots,m} ETC^N(Q, L_j, n, \gamma)$.
Step 4 Set $ETC(Q^*_{(n)}, L^*_{(n)}, n, \gamma^*_{(n)}) = \min\limits_{j=0,1,2,\ldots,m} ETC^N(Q, L_j, n, \gamma)$ and $(Q^*_{(n)}, L^*_{(n)}, \gamma^*_{(n)})$ is the optimal solution for fixed n.
Step 5 Set n by $n + 1$ and repeat the Steps 2 to 4, to get $ETC(Q^*_{(n)}, L^*_{(n)}, n, \gamma^*)$.
Step 6 If $ETC(Q^*_{(n)}, L^*_{(n)}, n, \gamma^*_{(n)}) < ETC(Q^*_{(n-1)}, L^*_{(n-1)}, n-1, \gamma^*_{(n)})$ then go to Step 5. Otherwise go to Step 7.
Step 7 Set $(Q^*, L^*, n^*, \gamma^*) = (Q^*_{(n-1)}, L^*_{(n-1)}, n-1, \gamma^*_{(n-1)})$ and $(Q^*, L^*, n^*, \gamma^*)$ is the optimal solutions. Stop the algorithm.

Case 4. In this case, we consider both the buyer's budget constraint and floor space constraint. To solve this problem, we add Lagrangian multipliers α and γ in the objective function as follows:

$$\begin{aligned} \text{Minimize} \quad ETC^N = &\sum_{i=1}^{M} \Bigg\{ \frac{D_i}{Q_i}\left[A_i + \frac{B_i}{n} + C(L) + (\pi_i + \pi_{i0}(1-\beta_i))\,\sigma_i\sqrt{L}\Psi(r_{i1}, r_{i2}, p_i)\right] \\ &+ h_{bi}\left\{\frac{Q_i}{2} + \sigma_i\sqrt{L}\left\{p_i\left[r_{i1}F\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} + (1-p_i)\epsilon_i\right) - \phi\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i}(1-p_i)\epsilon_i\right)\right]\right.\right. \\ &\left.\left. + (1-p_i)\left[r_{i2}F\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} - p_i\epsilon_i\right) - \phi\left(\frac{\mu_{i*}\sqrt{L}}{\sigma_i} - p_i\epsilon_i\right)\right]\right\}\right\} + h_{bi}(1-\beta_i)\sigma_i\sqrt{L}\Psi(r_{i1}, r_{i2}, p_i) \\ &+ h_{vi}\frac{Q_i}{2}\left[n\left(1 - \frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right] + \alpha(f_iQ_i - W) + \gamma(pc_iQ_i - \Omega)\Bigg\}. \end{aligned} \tag{4.28}$$

For fixed n and $L \in [L_j, L_{j-1}]$, the optimal solutions can be determined by solving the equation $\frac{\partial}{\partial Q_i}ETC^N = 0$, $\frac{\partial}{\partial \alpha}ETC^N = 0$ and $\frac{\partial}{\partial \gamma}ETC^N = 0$. *i.e.*,

$$Q_i = \left[\frac{\left[A_i + \frac{B_i}{n} + C(L) + (\pi_i + \pi_{i0}(1-\beta_i))\sigma_i\sqrt{L}\Psi(r_{i1}, r_{i2}, p_i)\right] D_i}{\frac{h_{bi}}{2} + \frac{h_{vi}}{2}\left[n\left(1 - \frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right] + \alpha f_i + \gamma pc_i}\right]^{\frac{1}{2}}, \quad L \in [L_j, L_{j-1}], \tag{4.29}$$

$$\sum_{i=1}^{M} f_i \left[\sqrt{\frac{\left[A_i+\frac{B_i}{n}+C(L)+(\pi_i+\pi_{i0}(1-\beta_i))\sigma_i\sqrt{L}\Psi(r_{i1},r_{i2},p_i)\right]D_i}{\frac{h_{bi}}{2}+\frac{h_{vi}}{2}\left[n\left(1-\frac{D_i}{P_i}\right)-1+\frac{2D_i}{P_i}\right]+\alpha f_i+\gamma pc_i}}\right]-W=0, \tag{4.30}$$

$$\sum_{i=1}^{M} pc_i \left[\sqrt{\frac{\left[A_i+\frac{B_i}{n}+C(L)+(\pi_i+\pi_{i0}(1-\beta_i))\sigma_i\sqrt{L}\Psi(r_{i1},r_{i2},p_i)\right]D_i}{\frac{h_{bi}}{2}+\frac{h_{vi}}{2}\left[n\left(1-\frac{D_i}{P_i}\right)-1+\frac{2D_i}{P_i}\right]+\alpha f_i+\gamma pc_i}}\right]-\Omega=0. \tag{4.31}$$

Similar to Case 1, one can prove equation (4.28) is convex with respect to Q_i and n and concave with respect to $L \in [L_j, L_{j-1}]$. An iterative algorithm have been developed to find optimal solutions for Q_i,L_i,n, α and γ.

Algorithm 4.4.

Step 1 Set $n = 1$.
Step 2 For each L_j, $j = 0, 1, 2, \ldots, m$ preform Steps (2.1) and (2.3).
Step 2.1 Calculate α and γ from solving equations (4.30) and (4.31).
Step 2.2 Determine the corresponding Q_i $(i = 1, 2, \ldots, M)$ from equation (4.29).
Step 2.3 Compute the corresponding $ETC^N(Q, L_j, n, \alpha, \gamma)$ from equation (4.28) where $Q = (Q_1, Q_2, \ldots, Q_M)$.
Step 3 Find $\min\limits_{j=0,1,2,\ldots,m} ETC^N(Q, L_j, n, \alpha, \gamma)$.
Step 4 Set $ETC(Q^*_{(n)}, L^*_{(n)}, n, \alpha^*_{(n)}, \gamma^*_{(n)}) = \min\limits_{j=0,1,2,\ldots,m} ETC^N(Q, L_j, n, \alpha, \gamma)$ and $(Q^*_{(n)}, L^*_{(n)}, \alpha^*_{(n)}, \gamma^*_{(n)})$ is the optimal solution for fixed n.
Step 5 Set n by $n + 1$ and repeat the Steps 2 to 4, to get $ETC(Q^*_{(n)}, L^*_{(n)}, n, \alpha^*_{(n)}, \gamma^*_{(n)})$.
Step 6 If $ETC(Q^*_{(n)}, L^*_{(n)}, n, \alpha^*_{(n)}, \gamma^*_{(n)}) < ETC(Q^*_{(n-1)}, L^*_{(n-1)}, n-1, \alpha^*_{(n-1)}, \gamma^*_{(n-1)})$ then go to Step 5. Otherwise go to Step 7.
Step 7 Set $(Q^*, L^*, n^*, \alpha^*, \gamma^*) = (Q^*_{(n-1)}, L^*_{(n-1)}, n-1, \alpha^*_{(n-1)}, \gamma^*_{(n-1)})$ and $(Q^*, L^*, n^*, \alpha^*, \gamma^*)$ is the optimal solutions. Stop the algorithm.

4.1.2. Main computational procedure

By using above four cases one can determine the optimal solutions with in the given limitations (constraints). The computational steps are as follows and the flow chart is given in Figures 3–5:

Step 1 Ignoring both the constraints, we find the optimal values using Algorithm 4.1. If Q_i satisfy both constraints, then the obtained values of Q_i, L and n are optimal solutions and go to Step 5.
Step 2 Else optimize the cost function subject to floor space constraint and ignore budget constraint. That is, determine the optimal values using Algorithm 4.2. If Q_i satisfies the budget constraint, then the obtained values of Q_i, L, α and n are optimal solutions and go to Step 5.
Step 3 Else optimize the cost function subject to budget constraint and ignore space constraint. That is, determine the optimal values using Algorithm 4.3. If Q_i satisfies the space constraint, then the obtained values of Q_i, L, γ and n are optimal solutions and go to Step 5.
Step 4 If none of the above three steps then, both constraints are active. Now optimize the cost function subject to both constraints such as floor space and budget. That is, determine the optimal values using Algorithm 4.4 and the optimal solutions Q_i, L, α, γ and n has been found such that the integrated expected total cost is minimum and go to Step 5.
Step 5 Stop.

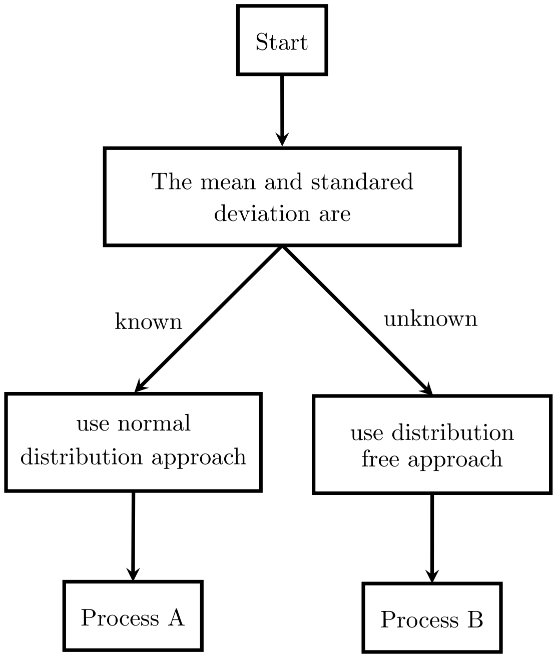

FIGURE 3. Flowchart of the solution procedure.

4.2. The mixture of distribution free model

Let $\mathfrak{R}$ be the set of all single cumulative distribution functions with a finite mean and variance. We assume that the demand of the lead time X_i has the mixture of cumulative distribution function F_* with finite mean $\mu_{i*}L$ and standard deviation σ_{i*} where $F_* = p_iF_1 + (1-p_i)F_2$, F_1 has a finite mean $\mu_{i1}L$, standard deviation $\sigma_i\sqrt{L}$, F_2 has finite mean $\mu_{i2}L$, standard deviation $\sigma_i\sqrt{L}$, $\mu_{i1} - \mu_{i2} = \epsilon_i\sigma_i/\sqrt{L}$, $\epsilon_i \in \mathbb{R}$, $F_1, F_2 \in \mathfrak{R}$ and $0 \le p_i \le 1$ $\forall\ i = 1, 2, , \ldots, M$. Since the expected shortages at the end of the cycle for the ith product is $E(X_i - r_i)^+$. By using the same procedure in Section 4.1, the expected total cost of the mixture of distribution free will become

$$ETC^F = \sum_{i=1}^{M} \left\{ \frac{D_i}{Q_i} \left[A_i + \frac{B_i}{n} + C(L) + [\pi_i + \pi_{i0}(1-\beta_i)]\, E(X_i - r_i)^+ \right] + h_{vi}\frac{Q}{2}\left[n\left(1 - \frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i} \right] \right.$$
$$\left. + h_{bi}\left[\frac{Q_i}{2} + r_i - D_iL + (1-\beta_i)E(X_i - r_i)^+ \right] \right\}. \tag{4.32}$$

The superscript F denote the mixture of distribution free approach. The minimax approach for this model is to find the most unfavorable cumulative distribution functions F_1 and F_2 then to minimize the total expected annual cost over (Q, n, L). *i.e.*, our problem is to solve

$$\min_{Q,L,n}\ \max_{F_1,F_2\in\mathfrak{R}} ETC^F(Q, L, n). \tag{4.33}$$

The above problem is solved through the following proposition. The proof of the proposition is given in [14].

Proposition 4.5. *For each $F \in \mathfrak{R}$,*

$$E(X - r)^+ \le \frac{1}{2}\left\{ \sqrt{\sigma^2 L + (r - \mu L)^2} - (r - \mu L) \right\}. \tag{4.34}$$

Process A

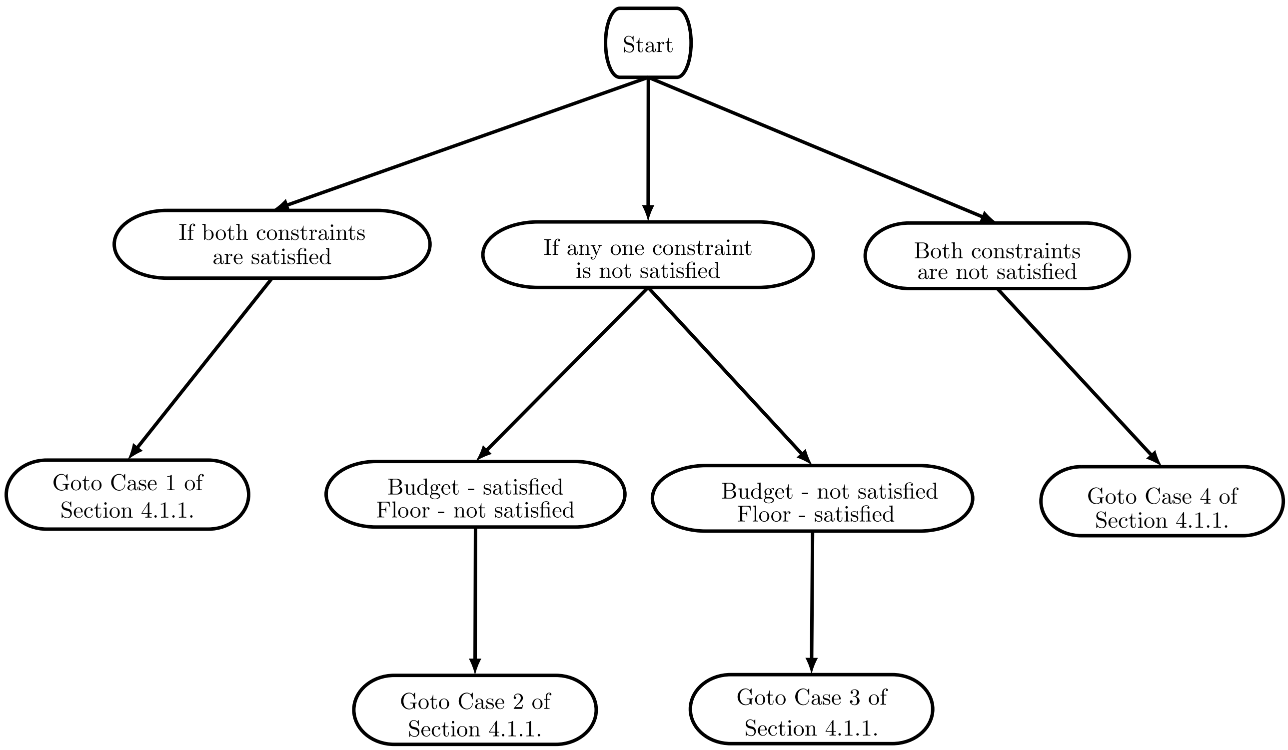

FIGURE 4. Flowchart of Process A.

Moreover, the upper bound of (4.34) is tight. In other words we can always find a distribution in which the above bound is satisfied with equality for every r.

Using inequality (4.34) for F_1 and F_2, we obtain

$$\begin{aligned}
E(X_i - r_i)^+ &= \int_{r_i}^{\infty} (x_i - r_i) dF_*(x) \\
&= p_i \int_{r_i}^{\infty} (x_i - r_i) dF_1(x) + (1-p_i) \int_{r_i}^{\infty} (x_i - r_i) dF_2(x), \\
&\le \frac{p_i}{2} \left\{ \sqrt{\sigma_i^2 L + (r_i - \mu_{i1} L)^2} - (r_i - \mu_{i1} L) \right\} + \frac{1-p_i}{2} \left\{ \sqrt{\sigma_i^2 L + (r_i - \mu_{i2} L)^2} - (r_i - \mu_{i2} L) \right\}, \\
&= -\frac{r_i - \mu_{i*} L}{2} + \frac{p_i}{2} \sqrt{\sigma_i^2 L + [(1-p_i)\epsilon_i \sigma_i \sqrt{L} + \mu_{i*} L - r_i]^2} \\
&\quad + \frac{1-p_i}{2} \sqrt{\sigma_i^2 L + [-p_i \epsilon_i \sigma_i \sqrt{L} + \mu_{i*} L - r_i]^2}, \\
&= -\frac{k_i l_i \sigma_i \sqrt{L}}{2} + \frac{p_i}{2} \sigma_i \sqrt{L} \sqrt{1 + [(1-p_i)\epsilon_i - k_i l_i]^2} + \frac{1-p_i}{2} \sigma_i \sqrt{L} \sqrt{1 + (p_i \epsilon_i + k_i l_i)^2}, \qquad (4.35)
\end{aligned}$$

where $l_i = \sqrt{1+p_i(1-p_i)\epsilon_i^2}$. Then the equation (4.33) can be written as

$$\begin{aligned}\min_{Q,L,n} ETC^U =& \sum_{i=1}^{M}\left\{\left[A_i+\frac{B_i}{n}+C(L)+[\pi_i+\pi_{i0}(1-\beta_i)]\left\{\frac{p_i}{2}\sigma_i\sqrt{L}\sqrt{1+[(1-p_i)\epsilon_i-k_il_i]^2}\right.\right.\right.\\&\left.\left.-\frac{k_il_i\sigma_i\sqrt{L}}{2}+\frac{1-p_i}{2}\sigma_i\sqrt{L}\sqrt{1+(p_i\epsilon_i+k_il_i)^2}\right\}\right]\frac{D_i}{Q_i}+h_{vi}\frac{Q}{2}\left[n\left(1-\frac{D_i}{P_i}\right)-1+\frac{2D_i}{P_i}\right]\\&+h_{bi}\left\{\frac{Q_i}{2}+r_i-\mu_{i*}L+(1-\beta_i)\left\{-\frac{k_il_i\sigma_i\sqrt{L}}{2}+\frac{p_i}{2}\sigma_i\sqrt{L}\sqrt{1+[(1-p_i)\epsilon_i-k_il_i]^2}\right.\right.\\&\left.\left.\left.+\frac{1-p_i}{2}\sigma_i\sqrt{L}\sqrt{1+(p_i\epsilon_i+k_il_i)^2}\right\}\right\}\right\}. \qquad (4.36)\end{aligned}$$

Here ETC^U denote $\max\limits_{F_1,F_2\in\Re} ETC^F(Q,L,n)$. For our convenient equation (4.36) can be rewritten as

$$\begin{aligned}\min_{Q,L,n} ETC^U =& \sum_{i=1}^{M}\left\{\left[A_i+\frac{B_i}{n}+C(L)\right]\frac{D_i}{Q_i}+h_{bi}\left[\frac{Q_i}{2}+r_i-\mu_{i*}L\right]+h_{vi}\frac{Q}{2}\left[n\left(1-\frac{D_i}{P_i}\right)-1+\frac{2D_i}{P_i}\right]\right.\\&+\left[[\pi_i+\pi_{i0}(1-\beta_i)]\frac{D_i}{Q_i}+h_{bi}(1-\beta_i)\right]\left\{-\frac{k_il_i\sigma_i\sqrt{L}}{2}+\frac{p_i}{2}\sigma_i\sqrt{L}\sqrt{1+[(1-p_i)\epsilon_i-k_il_i]^2}\right.\\&\left.\left.+\frac{1-p_i}{2}\sigma_i\sqrt{L}\sqrt{1+(p_i\epsilon_i+k_il_i)^2}\right\}\right\}. \qquad (4.37)\end{aligned}$$

Equation (4.37) is the expected integrated total cost for all M products in the mixture of distribution free model. Now we have to optimize (minimize) the equation (4.37) subject to the constraints $\sum_{i=1}^{M} f_iQ_i \leq W$ and $\sum_{i=1}^{M} pc_iQ_i \leq \Omega$.

4.2.1. Solution procedure

The same Lagrangian multiplier technique which used in Section 4.1.1 is utilized in this solution procedure.
Case 1. In this case, we temporarily ignore the constraints $\sum_{i=1}^{M} f_iQ_i \leq W$ and $\sum_{i=1}^{M} pc_iQ_i \leq \Omega$ then determine the optimal solutions of Q_i ,L and n which minimize the integrated expected total cost ETC^U. For fixed Q_i and $L\in[L_j,L_{j-1}]$ the ETC^U is convex in n, which indicates that there is a $n=n^*$ which satisfies the following relation

$$ETC^U(Q,L,n^*-1) > ETC^U(Q,L,n^*) \leq ETC^U(Q,L,n^*+1) \quad \text{where} \quad Q=(Q_1,Q_2,\ldots,Q_M).$$

$$\text{Because} \quad \frac{\partial}{\partial n}ETC^U = \sum_{i=1}^{M}\left\{-\frac{B_iD_i}{n^2Q_i}+h_{vi}\frac{Q_i}{2}\left(1-\frac{D_i}{P_i}\right)\right\}, \qquad (4.38)$$

$$\text{and} \quad \frac{\partial^2}{\partial n^2}ETC^U = \sum_{i=1}^{M}\frac{2B_iD_i}{n^3Q_i} > 0, \quad \text{for all} \quad n\in\mathbb{N}. \qquad (4.39)$$

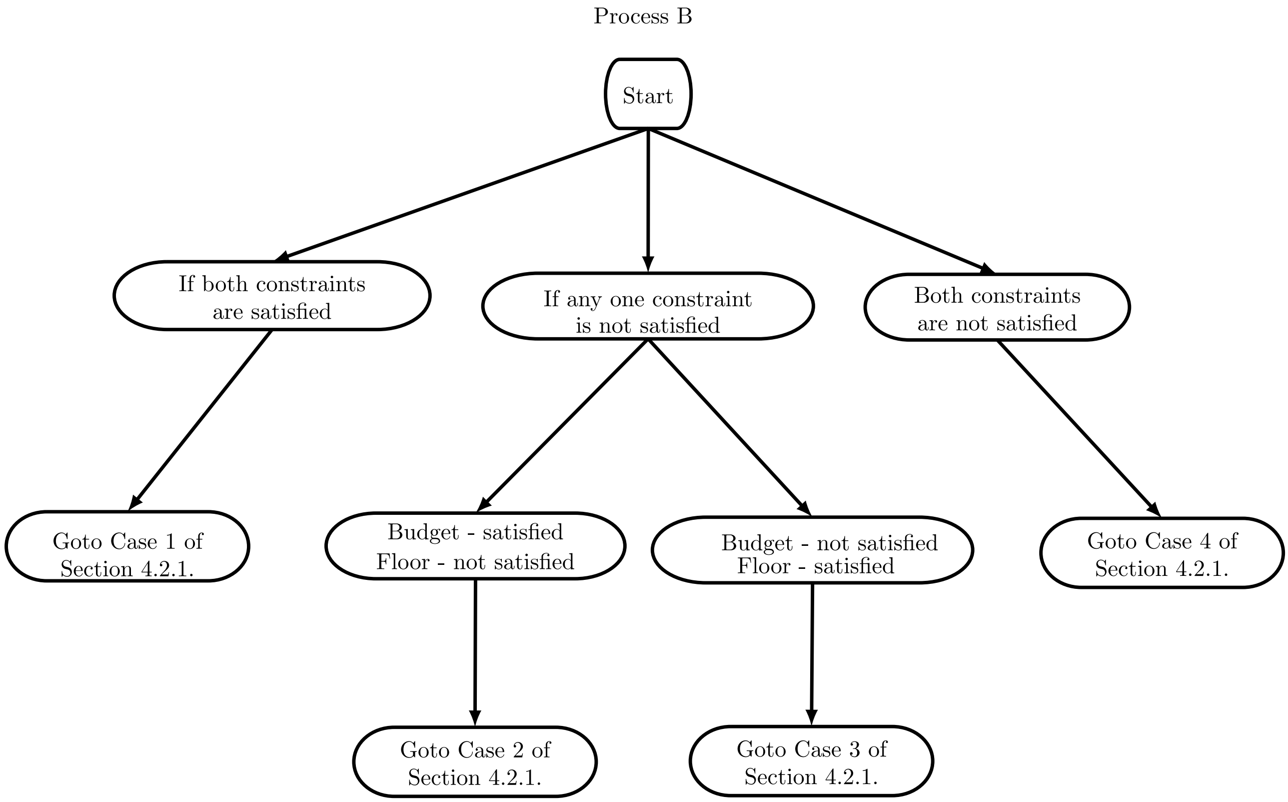

FIGURE 5. Flowchart of Process B.

TABLE 2. Summary for the values of r_1, r_2 and k.

p	r_1	r_2	k
0.0	0.14161	0.84161	0.84161
0.1	0.22806	0.92806	0.83974
0.2	0.31245	1.01245	0.84013
0.3	0.39370	1.09370	0.84147
0.4	0.47102	1.17102	0.84284
0.5	0.54394	1.24394	0.84376
0.6	0.61226	1.21226	0.84401
0.7	0.67595	1.37595	0.84361
0.8	0.73524	1.43524	0.84282
0.9	0.79036	1.49036	0.84199
1.0	0.84161	1.54161	0.84161

TABLE 3. Parameters of the buyer.

Product (i)	D_i	A_i	h_{bi}	σ_i	f_i	π_i	π_{i0}
1	600	200	25	7	4	50	150
2	1000	300	35	8	6	50	150
3	800	250	30	7.5	5.5	50	150

TABLE 4. Parameters of the vendor.

Product (i)	P_i	B_i	h_{vi}	pc_i
1	2000	1500	20	500
2	2500	1650	30	600
3	2300	1600	25	400

TABLE 5. Lead time component with data.

Lead time component j	Normal duration b_j (days)	Minimum duration cost a_j (days)	Unit crashing c_j (\$/days)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

Now, for fixed n, taking partial derivatives of ETC^U with respect to Q_i and L we get,

$$\frac{\partial}{\partial Q_i}ETC^U = \left[A_i + \frac{B_i}{n} + C(L)\right]\left(-\frac{D_i}{Q_i^2}\right) + \frac{h_{bi}}{2} + \frac{h_{vi}}{2}\left[n\left(1-\frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right] + [\pi_i + \pi_{i0}(1-\beta_i)]\left(-\frac{D_i}{Q_i^2}\right)$$
$$\times\left\{-\frac{k_i l_i \sigma_i\sqrt{L}}{2} + \frac{p_i}{2}\sigma_i\sqrt{L}\sqrt{1+[(1-p_i)\epsilon_i - k_i l_i]^2} + \frac{1-p_i}{2}\sigma_i\sqrt{L}\sqrt{1+(p_i\epsilon_i + k_i l_i)^2}\right\}, \quad (4.40)$$

$$\frac{\partial}{\partial L}ETC^U = \sum_{i=1}^{M}\left\{-c_j\frac{D_i}{Q_i} + \frac{\sigma_i l_i k_i h_{bi}}{2\sqrt{L}} + \left[[\pi_i + \pi_{i0}(1-\beta_i)]\frac{D_i}{Q_i} + h_{bi}(1-\beta_i)\right]\right.$$
$$\left.\times\left\{-\frac{k_i l_i \sigma_i}{4\sqrt{L}} + \frac{p_i}{4\sqrt{L}}\sigma_i\sqrt{1+[(1-p_i)\epsilon_i - k_i l_i]^2} + \frac{1-p_i}{4\sqrt{L}}\sigma_i\sqrt{1+(p_i\epsilon_i + k_i l_i)^2}\right\}\right\},$$
$$= \sum_{i=1}^{M}\left\{-c_j\frac{D_i}{Q_i} + \frac{\sigma_i l_i k_i h_{bi}}{2\sqrt{L}} + \frac{1}{4L}\left[[\pi_i + \pi_{i0}(1-\beta_i)]\frac{D_i}{Q_i} + h_{bi}(1-\beta_i)\right]\right.$$
$$\left. - \times\left[-k_i\sigma_i l_i\sqrt{L} + p_i\sigma_i\sqrt{L}\sqrt{1+[(1-p_i)\epsilon_i - k_i l_i]^2} + (1-p_i)\sigma_i\sqrt{L}\sqrt{1+(p_i\epsilon_i + k_i l_i)^2}\right]\right\}. \quad (4.41)$$

For fixed n and $L \in [L_j, L_{j-1}]$, ETC^U is convex in Q_i, since

$$\frac{\partial^2}{\partial Q_i^2}ETC^U = \left[A_i + \frac{B_i}{n} + C(L)\right]\frac{2D_i}{Q_i^3} + [\pi_i + \pi_{i0}(1-\beta_i)]\left(\frac{2D_i}{Q_i^3}\right)\left\{-\frac{k_i l_i \sigma_i\sqrt{L}}{2}\right.$$
$$\left. + \frac{p_i}{2}\sigma_i\sqrt{L}\sqrt{1+[(1-p_i)\epsilon_i - k_i l_i]^2} + \frac{1-p_i}{2}\sigma_i\sqrt{L}\sqrt{1+(p_i\epsilon_i + k_i l_i)^2}\right\} > 0. \quad (4.42)$$

Because $-\frac{k_i l_i \sigma_i\sqrt{L}}{2} + \frac{p_i}{2}\sigma_i\sqrt{L}\sqrt{1+[(1-p_i)\epsilon_i - k_i l_i]^2} + \frac{1-p_i}{2}\sigma_i\sqrt{L}\sqrt{1+(p_i\epsilon_i + k_i l_i)^2}$ is the upper bound of the

TABLE 6. Summarized lead time data.

Lead time (week)	$C(L)$
8	0
6	5.6
4	22.4
3	57.4

$E(X_i - r_i)^+$. So it should be positive. However for fixed (Q, n), ETC^U is concave in $L \in [L_j, L_{j-1}]$, because

$$\begin{aligned}\frac{\partial^2}{\partial L^2} ETC^U &= \sum_{i=1}^{M} \left\{ -\frac{\sigma_i l_i k_i}{4L^{\frac{3}{2}}} h_{bi} - \frac{1}{4L^2} \left[[\pi_i + \pi_{i0}(1-\beta_i)] \frac{D_i}{Q_i} + h_{bi}(1-\beta_i) \right] \right. \\ &\quad \times \left[-k_i \sigma_i l_i \sqrt{L} + p_i \sigma_i \sqrt{L} \sqrt{1 + [(1-p_i)\epsilon_i - k_i l_i]^2} + (1-p_i)\sigma_i \sqrt{L} \sqrt{1 + (p_i \epsilon_i + k_i l_i)^2} \right] \\ &\quad + \frac{1}{4L} \left[[\pi_i + \pi_{i0}(1-\beta_i)] \frac{D_i}{Q_i} + h_{bi}(1-\beta_i) \right] \left[-\frac{k_i l_i \sigma_i \sqrt{L}}{2} + \frac{p_i}{2} \sigma_i \sqrt{L} \sqrt{1 + [(1-p_i)\epsilon_i - k_i l_i]^2} \right. \\ &\quad \left. \left. + \frac{1-p_i}{2} \sigma_i \sqrt{L} \sqrt{1 + (p_i \epsilon_i + k_i l_i)^2} \right] \right\}, \\ &= \sum_{i=1}^{M} \left\{ -\frac{\sigma_i l_i k_i}{4L^{\frac{3}{2}}} h_{bi} - \frac{1}{8L^2} \left[[\pi_i + \pi_{i0}(1-\beta_i)] \frac{D_i}{Q_i} + h_{bi}(1-\beta_i) \right] \left[p_i \sigma_i \sqrt{L} \sqrt{1 + [(1-p_i)\epsilon_i - k_i l_i]^2} \right. \right. \\ &\quad \left. \left. - k_i \sigma_i l_i \sqrt{L} + (1-p_i)\sigma_i \sqrt{L} \sqrt{1 + (p_i \epsilon_i + k_i l_i)^2} \right] \right\} < 0. \end{aligned} \tag{4.43}$$

Therefore, for fixed Q, n, the minimum ETC^U will occur at the end points of the interval $[L_j, L_{j-1}]$. Equating (4.40) to zero, we can determine Q_i, as follows

$$Q_i = \sqrt{\frac{\begin{array}{c} D_i \left[A_i + \frac{B_i}{n} + C(L) + (\pi_i + \pi_{i0}(1-\beta_i)) \right] \\ \times \left\{ -\frac{k_i l_i \sigma_i \sqrt{L}}{2} + \frac{p_i}{2} \sigma_i \sqrt{L} \sqrt{1 + [(1-p_i)\epsilon_i - k_i l_i]^2} + \frac{1-p_i}{2} \sigma_i \sqrt{L} \sqrt{1 + (p_i \epsilon_i + k_i l_i)^2} \right\} \end{array}}{\frac{h_{bi}}{2} + \frac{h_{vi}}{2} \left[n \left(1 - \frac{D_i}{P_i} \right) - 1 + \frac{2D_i}{P_i} \right]}}. \tag{4.44}$$

For fixed n and $L \in [L_j, L_{j-1}]$, when both constraints $\sum_{i=1}^{M} f_i Q_i \leq W$ and $\sum_{i=1}^{M} pc_i Q_i \leq \Omega$ are ignored, equation (4.44) gives optimal values of Q_i such that ETC^U is minimum. Then, the similar solution procedure proposed in Algorithm 4.1 can be performed to obtain the optimal solutions of (Q_i, L, n).

Case 2. In this case, consider the buyer's floor space constraint and ignore budget constraint. To solve this problem, we add Lagrangian multiplier α:

$$\begin{aligned}\min_{Q,L,n} ETC^U &= \sum_{i=1}^{M} \left\{ \left[A_i + \frac{B_i}{n} + C(L) \right] \frac{D_i}{Q_i} + h_{bi} \left[\frac{Q_i}{2} + r_i - \mu_{i*} L \right] + h_{vi} \frac{Q}{2} \left[n \left(1 - \frac{D_i}{P_i} \right) - 1 + \frac{2D_i}{P_i} \right] \right. \\ &\quad + \left[[\pi_i + \pi_{i0}(1-\beta_i)] \frac{D_i}{Q_i} + h_{bi}(1-\beta_i) \right] \left\{ -\frac{k_i l_i \sigma_i \sqrt{L}}{2} + \frac{p_i}{2} \sigma_i \sqrt{L} \sqrt{1 + [(1-p_i)\epsilon_i - k_i l_i]^2} \right. \\ &\quad \left. \left. + \frac{1-p_i}{2} \sigma_i \sqrt{L} \sqrt{1 + (p_i \epsilon_i + k_i l_i)^2} \right\} + \alpha (f_i Q_i - W) \right\}. \end{aligned} \tag{4.45}$$

For fixed n and $L \in [L_j, L_{j-1}]$, the optimal Q_i and α can be determined by solving the equation $\frac{\partial}{\partial Q_i} ETC^U = 0$ and $\frac{\partial}{\partial \alpha} ETC^U = 0$. *i.e.*,

$$Q_i = \sqrt{\frac{D_i\left[A_i + \frac{B_i}{n} + C(L) + (\pi_i + \pi_{i0}(1-\beta_i))\right] \times \left\{-\frac{k_i l_i \sigma_i \sqrt{L}}{2} + \frac{p_i}{2}\sigma_i\sqrt{L}\sqrt{1+[(1-p_i)\epsilon_i - k_i l_i]^2} + \frac{1-p_i}{2}\sigma_i\sqrt{L}\sqrt{1+(p_i\epsilon_i + k_i l_i)^2}\right\}}{\frac{h_{bi}}{2} + \frac{h_{vi}}{2}\left[n\left(1-\frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right] + \alpha f_i}}. \tag{4.46}$$

e.g.

$$\sum_{i=1}^{M} f_i \left[\sqrt{\frac{D_i\left[A_i + \frac{B_i}{n} + C(L) + (\pi_i + \pi_{i0}(1-\beta_i))\right] \times \left\{-\frac{k_i l_i \sigma_i \sqrt{L}}{2} + \frac{p_i}{2}\sigma_i\sqrt{L}\sqrt{1+[(1-p_i)\epsilon_i - k_i l_i]^2} + \frac{1-p_i}{2}\sigma_i\sqrt{L}\sqrt{1+(p_i\epsilon_i + k_i l_i)^2}\right\}}{\frac{h_{bi}}{2} + \frac{h_{vi}}{2}\left[n\left(1-\frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right] + \alpha f_i}}\right] - W = 0. \tag{4.47}$$

Similar to Case 1, in this section one can prove equation (4.45) is convex with respect to Q_i and n, and concave with respect to $L \in [L_j, L_{j-1}]$. Then, the similar solution procedure which is proposed in Algorithm 4.2 can be performed to obtain the optimal solution of (Q_i, L, n) and α.

Case 3. In this case, we consider the buyer's budget constraint and ignore floor space constraint. To solve this problem, we add Lagrangian multiplier γ:

$$\begin{aligned}\min_{Q,L,n} ETC^U = \sum_{i=1}^{M}\Bigg\{&\left[A_i + \frac{B_i}{n} + C(L)\right]\frac{D_i}{Q_i} + h_{bi}\left[\frac{Q_i}{2} + r_i - \mu_{i*}L\right] + h_{vi}\frac{Q}{2}\left[n\left(1-\frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right] \\ &+ \left[[\pi_i + \pi_{i0}(1-\beta_i)]\frac{D_i}{Q_i} + h_{bi}(1-\beta_i)\right]\Bigg\{-\frac{k_i l_i \sigma_i\sqrt{L}}{2} + \frac{p_i}{2}\sigma_i\sqrt{L}\sqrt{1+[(1-p_i)\epsilon_i - k_i l_i]^2} \\ &+ \frac{1-p_i}{2}\sigma_i\sqrt{L}\sqrt{1+(p_i\epsilon_i + k_i l_i)^2}\Bigg\} + \gamma(pc_iQ_i - \Omega)\Bigg\}. \end{aligned} \tag{4.48}$$

For fixed n and $L \in [L_j, L_{j-1}]$, the optimal Q_i and γ can be determined by solving the equation $\frac{\partial}{\partial Q_i} ETC^U = 0$ and $\frac{\partial}{\partial \gamma} ETC^U = 0$. i.e.,

$$Q_i = \sqrt{\frac{D_i\left[A_i + \frac{B_i}{n} + C(L) + (\pi_i + \pi_{i0}(1-\beta_i))\right] \times \left\{-\frac{k_i l_i \sigma_i \sqrt{L}}{2} + \frac{p_i}{2}\sigma_i\sqrt{L}\sqrt{1+[(1-p_i)\epsilon_i - k_i l_i]^2} + \frac{1-p_i}{2}\sigma_i\sqrt{L}\sqrt{1+(p_i\epsilon_i + k_i l_i)^2}\right\}}{\frac{h_{bi}}{2} + \frac{h_{vi}}{2}\left[n\left(1-\frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right] + \gamma pc_i}}. \tag{4.49}$$

$$\sum_{i=1}^{M} f_i \left[\sqrt{\frac{D_i\left[A_i + \frac{B_i}{n} + C(L) + (\pi_i + \pi_{i0}(1-\beta_i))\right] \times \left\{-\frac{k_i l_i \sigma_i \sqrt{L}}{2} + \frac{p_i}{2}\sigma_i\sqrt{L}\sqrt{1+[(1-p_i)\epsilon_i - k_i l_i]^2} + \frac{1-p_i}{2}\sigma_i\sqrt{L}\sqrt{1+(p_i\epsilon_i + k_i l_i)^2}\right\}}{\frac{h_{bi}}{2} + \frac{h_{vi}}{2}\left[n\left(1-\frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right] + \gamma pc_i}}\right] - \Omega = 0. \tag{4.50}$$

Similar to Case 1, in this section we can prove equation (4.48) is convex with respect to Q_i and n, and concave with respect to $L \in [L_j, L_{j-1}]$. Then, the similar solution procedure which is proposed in Algorithm 4.3 can be performed to obtain the optimal solution of (Q_i, L, n) and γ.

Case 4. In this case, we consider both buyer's budget constraint and floor space constraint. To solve this problem, we add Lagrangian multiplier α and γ:

$$\begin{aligned}\min_{Q,L,n} ETC^U = \sum_{i=1}^{M} \Bigg\{ & \left[A_i + \frac{B_i}{n} + C(L)\right]\frac{D_i}{Q_i} + h_{bi}\left[\frac{Q_i}{2} + r_i - \mu_{i*}L\right] + h_{vi}\frac{Q}{2}\left[n\left(1-\frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right] \\ & + \left[[\pi_i + \pi_{i0}(1-\beta_i)]\frac{D_i}{Q_i} + h_{bi}(1-\beta_i)\right]\Bigg\{-\frac{k_i l_i \sigma_i\sqrt{L}}{2} + \frac{p_i}{2}\sigma_i\sqrt{L}\sqrt{1+[(1-p_i)\epsilon_i - k_i l_i]^2} \\ & + \frac{1-p_i}{2}\sigma_i\sqrt{L}\sqrt{1+(p_i\epsilon_i + k_i l_i)^2}\Bigg\} + \alpha(f_iQ_i - W) + \gamma(pc_iQ_i - \Omega)\Bigg\}. \end{aligned} \tag{4.51}$$

For fixed n and $L \in [L_j, L_{j-1}]$, the optimal Q_i, α and γ can be determined by solving the equation $\frac{\partial}{\partial Q_i}ETC^U = 0$, $\frac{\partial}{\partial \alpha}ETC^U = 0$ and $\frac{\partial}{\partial \gamma}ETC^U = 0$. *e.g.*,

$$Q_i = \sqrt{\frac{D_i\left[A_i + \frac{B_i}{n} + C(L) + (\pi_i + \pi_{i0}(1-\beta_i))\right] \times \left\{-\frac{k_il_i\sigma_i\sqrt{L}}{2} + \frac{p_i}{2}\sigma_i\sqrt{L}\sqrt{1+[(1-p_i)\epsilon_i - k_il_i]^2} + \frac{1-p_i}{2}\sigma_i\sqrt{L}\sqrt{1+(p_i\epsilon_i+k_il_i)^2}\right\}}{\frac{h_{bi}}{2} + \frac{h_{vi}}{2}\left[n\left(1-\frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right] + \alpha f_i + \gamma pc_i}}. \tag{4.52}$$

$$\sum_{i=1}^{M} f_i\left[\sqrt{\frac{D_i\left[A_i + \frac{B_i}{n} + C(L) + (\pi_i + \pi_{i0}(1-\beta_i))\right] \times \left\{-\frac{k_il_i\sigma_i\sqrt{L}}{2} + \frac{p_i}{2}\sigma_i\sqrt{L}\sqrt{1+[(1-p_i)\epsilon_i - k_il_i]^2} + \frac{1-p_i}{2}\sigma_i\sqrt{L}\sqrt{1+(p_i\epsilon_i+k_il_i)^2}\right\}}{\frac{h_{bi}}{2} + \frac{h_{vi}}{2}\left[n\left(1-\frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right] + \alpha f_i + \gamma pc_i}}\right] - \Omega = 0. \tag{4.53}$$

$$\sum_{i=1}^{M} pc_i\left[\sqrt{\frac{D_i\left[A_i + \frac{B_i}{n} + C(L) + (\pi_i + \pi_{i0}(1-\beta_i))\right] \times \left\{-\frac{k_il_i\sigma_i\sqrt{L}}{2} + \frac{p_i}{2}\sigma_i\sqrt{L}\sqrt{1+[(1-p_i)\epsilon_i - k_il_i]^2} + \frac{1-p_i}{2}\sigma_i\sqrt{L}\sqrt{1+(p_i\epsilon_i+k_il_i)^2}\right\}}{\frac{h_{bi}}{2} + \frac{h_{vi}}{2}\left[n\left(1-\frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right] + \alpha f_i + \gamma pc_i}}\right] - W = 0. \tag{4.54}$$

Similar to Case 1, we can prove equation (4.51) is convex with respect to Q_i and n, and concave with respect to $L \in [L_j, L_{j-1}]$. Then, the similar solution procedure which is proposed in Algorithm 4.4 can be performed to obtain the optimal solutions of (Q_i, L, n) and (α, γ).

5. Numerical example

The proposed model is illustrated through some numerical examples. The solutions for this examples are obtained by using the computer MATLAB software. The computational effort and time are small for the proposed algorithm and it is simple to implement.

Example 5.1. To avoid huge number of parameters we fix $\beta = \beta_i$, $p = p_i$, $q = q_i$, $r_{i1} = r_1$, $r_{i2} = r_2$, $k = k_i$, $\epsilon_i = \epsilon$, $\forall i = 1, 2, \ldots, M$. In this example, we consider a two-echelon supply chain inventory problem for three products, that is $M = 3$ and the demand is assumed to follow a mixture of normal distribution. The identical parameters of the three products are $W = 3000$ square feet, $\Omega = 3$ lakhs, $\epsilon = 0.7$, $p = 0$, $\beta = 1$, $q = 0.2$ (in this situation, the value of k can be found using the relation $P(X_i > r_i) = 1 - pF(r_1) - (1-p)F(r_2) = q$ where $r_1 = k\sqrt{1 + \epsilon^2 p(1-p)} - \epsilon(1-p)$ and $r_2 = k\sqrt{1 + \epsilon^2 p(1-p)} + \epsilon p$ and also we tabulated the values of r_1, r_2 and k for given p in Table 2) as in [40]. Other parameters of the buyer and vendor is shown in Tables 3 and 4. In addition the lead time data is summarized in Tables 5 and 6. The lead time consisting of three components.

5.1. Solution procedure and analysis of numerical results

In this section we are going to see the way of getting optimal solutions. According to Algorithm 4.1 ignore both budget and floor space constraint, determine the solution.

1. Set $n = 1$, $p = 0$ and $\beta = 1$
   (a) Put $L = 8$, then we obtain $Q = (Q_1, Q_2, Q_3) = (265, 297, 285)$ using (4.21). We obtain the corresponding total cost ETC^N as 34835 by putting the values $Q = (Q_1, Q_2, Q_3)$ in (4.14).
   (b) Put $L = 6$, then we obtain $Q = (Q_1, Q_2, Q_3) = (264, 296, 285)$ using (4.21). We obtain the corresponding total cost ETC^N as 34530 by putting the values $Q = (Q_1, Q_2, Q_3)$ in (4.14).
   (c) Put $L = 4$, then we obtain $Q = (Q_1, Q_2, Q_3) = (264, 296, 284)$ using (4.21). We obtain the corresponding total cost ETC^N as 34254 by putting the values $Q = (Q_1, Q_2, Q_3)$ in (4.14).
   (d) Put $L = 3$, then we obtain $Q = (Q_1, Q_2, Q_3) = (266, 298, 286)$ using (4.21). We obtain the corresponding total cost ETC^N as 34301 by putting the values $Q = (Q_1, Q_2, Q_3)$ in (4.14). By examine the solutions for all lead times, we see that the optimal solution for the given $n = 1$ is $Q = (Q_1, Q_2, Q_3) = (264, 296, 284)$ and the corresponding $ETC^N = 34254$.
2. Set $n = 2$, $p = 0$ and $\beta = 1$
   (a) Put $L = 8$, then we obtain $Q = (Q_1, Q_2, Q_3) = (168, 196, 184)$ using (4.21). We obtain the corresponding total cost ETC^N as 32079 by putting the values $Q = (Q_1, Q_2, Q_3)$ in (4.14).
   (b) Put $L = 6$, then we obtain $Q = (Q_1, Q_2, Q_3) = (167, 195, 184)$ using (4.21). We obtain the corresponding total cost ETC^N as 31727 by putting the values $Q = (Q_1, Q_2, Q_3)$ in (4.14).
   (c) Put $L = 4$, then we obtain $Q = (Q_1, Q_2, Q_3) = (167, 195, 183)$ using (4.21). We obtain the corresponding total cost ETC^N as 31442 by putting the values $Q = (Q_1, Q_2, Q_3)$ in (4.14).
   (d) Put $L = 3$, then we obtain $Q = (Q_1, Q_2, Q_3) = (169, 197, 185)$ using (4.21). We obtain the corresponding total cost ETC^N as 31597 by putting the values $Q = (Q_1, Q_2, Q_3)$ in (4.14). By examine the solutions for all lead times, we see that the optimal solution for the given $n = 2$ is $Q = (Q_1, Q_2, Q_3) = (167, 195, 183)$ and the corresponding $ETC^N = 31442$.
3. Set $n = 3$, $p = 0$ and $\beta = 1$
   (a) Put $L = 8$, then we obtain $Q = (Q_1, Q_2, Q_3) = (128, 153, 142)$ using (4.21). We obtain the corresponding total cost ETC^N as 32066 by putting the values $Q = (Q_1, Q_2, Q_3)$ in (4.14).
   (b) Put $L = 6$, then we obtain $Q = (Q_1, Q_2, Q_3) = (128, 152, 141)$ using (4.21). We obtain the corresponding total cost ETC^N as 31674 by putting the values $Q = (Q_1, Q_2, Q_3)$ in (4.14).
   (c) Put $L = 4$, then we obtain $Q = (Q_1, Q_2, Q_3) = (127, 152, 141)$ using (4.21). We obtain the corresponding total cost ETC^N as 31381 by putting the values $Q = (Q_1, Q_2, Q_3)$ in (4.14).
   (d) Put $L = 3$, then we obtain $Q = (Q_1, Q_2, Q_3) = (130, 154, 143)$ using (4.21). We obtain the corresponding total cost ETC^N as 31626 by putting the values $Q = (Q_1, Q_2, Q_3)$ in (4.14). By examine the solutions for all lead times, we see that the optimal solution for the given $n = 3$ is $Q = (Q_1, Q_2, Q_3) = (127, 152, 141)$ and the corresponding $ETC^N = 31381$.
4. Set $n = 4$, $p = 0$ and $\beta = 1$
   (a) Put $L = 8$, then we obtain $Q = (Q_1, Q_2, Q_3) = (106, 128, 118)$ using (4.21). We obtain the corresponding total cost ETC^N as 32761 by putting the values $Q = (Q_1, Q_2, Q_3)$ in (4.14).

(b) Put $L = 6$, then we obtain $Q = (Q_1, Q_2, Q_3) = (105, 128, 117)$ using (4.21). We obtain the corresponding total cost ETC^N as 32335 by putting the values $Q = (Q_1, Q_2, Q_3)$ in (4.14).
(c) Put $L = 4$, then we obtain $Q = (Q_1, Q_2, Q_3) = (105, 128, 118)$ using (4.21). We obtain the corresponding total cost ETC^N as 32034 by putting the values $Q = (Q_1, Q_2, Q_3)$ in (4.14).
(d) Put $L = 3$, then we obtain $Q = (Q_1, Q_2, Q_3) = (107, 130, 119)$ using (4.21). We obtain the corresponding total cost ETC^N as 32359 by putting the values $Q = (Q_1, Q_2, Q_3)$ in (4.14). By examine the solutions for all lead times, we see that the optimal solution for the given $n = 4$ is $Q = (Q_1, Q_2, Q_3) = (105, 128, 118)$ and the corresponding $ETC^N = 32034$.

These numerical values are tabulated in Table 7 and the graphical representation is depicted in Figure 6. From Table 7 least total cost occur when $n = 3$ and $L = 4$, the order quantities are $Q = (Q_1, Q_2, Q_3) = (127, 152, 141)$ and their corresponding total cost is 31381.

$$ETC^N(Q, L, 2) > ETC^N(Q, L, 3) \leq \ ETC^N(Q, L, 4), \quad \text{where} \quad Q = (127, 152, 141).$$

Now we consider the budget and floor space constraints. Then

$$f_1Q_1 + f_2Q_2 + f_3Q_3 = 2196 < 3000,$$
$$pc_1Q_1 + pc_2Q_2 + pc_3Q_3 = 211100 < 3 \text{ lakhs}.$$

The optimal solution does not affect the constraints, it satisfies both floor space and budget constraints. Suppose that the optimal solution are not satisfied by the constraints then we move to the Algorithms 4.2–4.4 which depends on the order quantities. The sensitivity analysis is performed by changing the parameter β for fixed p. Optimal lead times, order quantities and total expected cost for different backorder ratio for fixed p is tabulated from the Tables 8–18. For given p and q the values of r_1 and r_2 can be found in Table 2 as in [40]. And the graphical representation of the optimal solutions for different backorder ratios is depicted in Figures 7–17. In Tables 8–18 and Tables 20–30 the optimal solutions are satisfied by the budget constraint for this numerical

Table 7. Optimal solutions Example 5.1 when $p = 0$ and $\beta = 1.0$.

n	L	Q	ETC^N
1	8	(265, 297, 285)	34835
	6	(264, 296, 285)	34530
	4	(264, 296, 284)	34254
	3	(266, 298, 286)	34301
2	8	(168, 196, 184)	32079
	6	(167, 195, 184)	31727
	4	(167, 195, 183)	31442
	3	(169, 197, 185)	31597
3	8	(128, 153, 142)	32066
	6	(128, 152, 141)	31674
	4	(127, 152, 141)	31381
	3	(130, 154, 143)	31626
4	8	(106, 128, 118)	32761
	6	(105, 128, 117)	32335
	4	(105, 128, 118)	32034
	3	(107, 130, 119)	32359

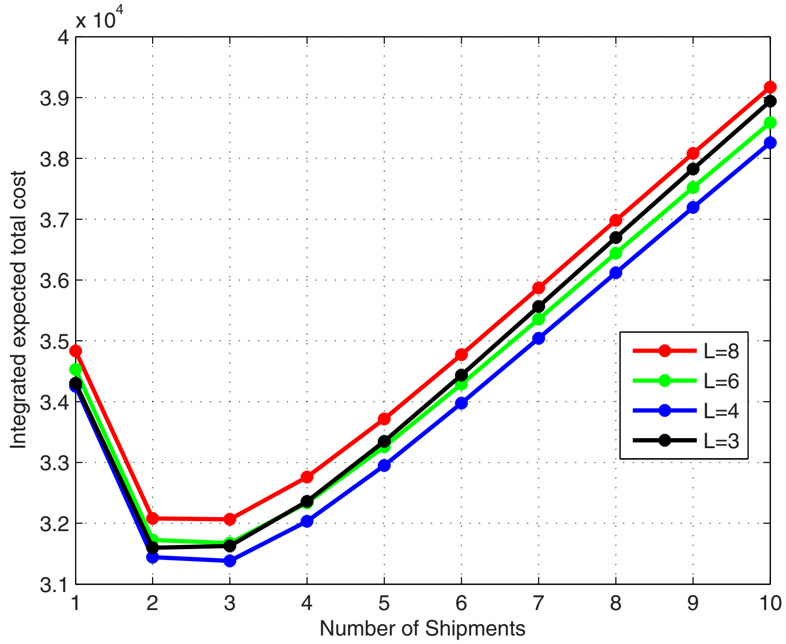

FIGURE 6. Graphical representation of total costs of Example refex1 when $p = 0$ and $\beta = 1.0$.

example. So we need not to implement the Lagrangian multiplier γ. Therefore, we avoid the separate column for γ.

Example 5.2. In this example, we assume that the lead time demand follows a mixture of distribution free and use the same parameter as in Example 5.1 except $k = 3$. The same solution procedure and analysis of numerical results implemented in this section. The optimal solutions of Example 5.2 is tabulated in Table 19. According to this, the minimum total cost occur when $n = 3$, $Q = (Q_1, Q_2, Q_3) = (128, 152, 142)$, the lead time is $L = 3$ and the minimum total cost is 33834. The graphical representation is depicted in Figure 18. The sensitivity analysis is performed by changing the parameters β for fixed p. Optimal lead times, order quantities and total expected cost for different backorder ratio for fixed p is tabulated in Tables 20–30. The graphical representation of the optimal solutions for different backorder ratios is depicted in Figures 19–29.

5.2. Managerial implications

1. It is clear that if $\sigma_i = 0$, then the optimal safety stock is $k_i\sigma_i\sqrt{L} = 0$, hence the equation (4.8) is reduced to a deterministic case. If $\sigma_i > 0$ and $\min\left\{\frac{\mu_{i1}\sqrt{L}}{\sigma_i}, \frac{\mu_{i2}\sqrt{L}}{\sigma_i}\right\} \geq 3.9$ or $\min\left\{\frac{\mu_{i*}\sqrt{L}}{\sigma_i} + (1-p_i)\epsilon_i, \frac{\mu_{i*}\sqrt{L}}{\sigma_i} - p_i\epsilon_i\right\} \geq 3.9$ then according the properties of normal distribution we obtain $F\left(\frac{\mu_{ij}\sqrt{L}}{\sigma_i}\right) \to 1$ and $\phi\left(\frac{\mu_{ij}\sqrt{L}}{\sigma_i}\right) \to 0$, $j = 1, 2$. Hence equation (4.14) can be reduced to

$$ETC^N = \sum_{i=1}^{M}\left\{\frac{D_i}{Q_i}\left[A_i + \frac{B_i}{n} + C(L) + (\pi_i + \pi_{i0}(1-\beta_i))\,\sigma_i\sqrt{L}\Psi(r_{i1}, r_{i2}, p_i)\right]\right.$$
$$\left. + h_{bi}\left\{\frac{Q_i}{2} + \sigma_i\sqrt{L}(p_i r_{i1} + (1-p_i)r_{i2})\right\} + h_{vi}\frac{Q_i}{2}\left[n\left(1 - \frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right]\right\}. \quad (5.1)$$

Table 8. Sensitivity analysis of β when $p = 0.0$.

β	α	L	n	Q	ETC^N
0.0	1.1	8	2	(176, 204, 191)	30166
0.1	0.9	8	2	(176, 205, 192)	30902
0.2	0.8	8	2	(176, 204, 191)	31036
0.3	0.6	8	2	(176, 204, 192)	31771
0.4	0.5	8	2	(176, 204, 191)	31905
0.5	0.1	3	2	(176, 204, 192)	32448
0.6	0.0	3	2	(175, 204, 191)	32774
0.7	0.0	4	2	(173, 201, 189)	32464
0.8	0.0	4	2	(171, 199, 187)	32127
0.9	0.0	4	2	(169, 197, 185)	31786
1.0	0.0	4	2	(167, 195, 183)	31442

Table 9. Sensitivity analysis of β when $p = 0.1$.

β	α	L	n	Q	ETC^N
0.0	1.1	8	2	(177, 205, 192)	30408
0.1	1.0	8	2	(176, 204, 190)	30526
0.2	0.8	8	2	(177, 205, 192)	31245
0.3	0.7	8	2	(176, 204, 191)	31363
0.4	0.5	8	2	(176, 205, 192)	32082
0.5	0.4	8	2	(175, 204, 191)	32200
0.6	0.0	3	2	(176, 204, 192)	32853
0.7	0.0	3	2	(174, 203, 191)	32555
0.8	0.0	4	2	(171, 200, 188)	32209
0.9	0.0	4	2	(169, 197, 186)	31857
1.0	0.0	4	2	(168, 195, 184)	31502

Table 10. Sensitivity analysis of β when $p = 0.2$.

β	α	L	n	Q	ETC^N
0.0	1.2	8	2	(176, 204, 191)	29956
0.1	1.0	8	2	(177, 205, 191)	30666
0.2	0.9	8	2	(176, 204, 191)	30775
0.3	0.7	8	2	(176, 205, 191)	31485
0.4	0.5	8	2	(177, 205, 192)	32195
0.5	0.4	8	2	(176, 204, 192)	32304
0.6	0.0	3	2	(176, 205, 193)	32910
0.7	0.0	3	2	(175, 204, 192)	33122
0.8	0.0	4	2	(172, 200, 188)	32263
0.9	0.0	4	2	(170, 198, 186)	31905
1.0	0.0	4	2	(168, 195, 184)	31543

TABLE 11. Sensitivity analysis of β when $p = 0.3$.

β	α	L	n	Q	ETC^N
0.0	1.2	8	2	(177, 205, 191)	30118
0.1	1.0	8	2	(177, 206, 192)	30817
0.2	0.9	8	2	(176, 205, 191)	30915
0.3	0.7	8	2	(177, 205, 192)	31614
0.4	0.6	8	2	(176, 204, 191)	31713
0.5	0.3	6	2	(176, 205, 192)	32304
0.6	0.3	8	2	(175, 204, 191)	32510
0.7	0.0	4	2	(173, 202, 190)	32679
0.8	0.0	4	2	(172, 200, 188)	32318
0.9	0.0	4	2	(170, 198, 186)	31953
1.0	0.0	4	3	(128, 153, 142)	31547

TABLE 12. Sensitivity analysis of β when $p = 0.4$.

β	α	L	n	Q	ETC^N
0.0	1.2	8	2	(177, 205, 191)	30041
0.1	1.0	8	2	(177, 205, 192)	30747
0.2	0.9	8	2	(176, 204, 191)	30852
0.3	0.7	8	2	(176, 205, 192)	31558
0.4	0.6	8	2	(176, 204, 191)	31663
0.5	0.4	8	2	(176, 205, 192)	32370
0.6	0.0	3	2	(176, 205, 193)	32948
0.7	0.0	3	2	(175, 203, 191)	32644
0.8	0.0	4	2	(172, 200, 188)	32302
0.9	0.0	4	2	(170, 198, 186)	31942
1.0	0.0	4	3	(128, 153, 142)	31536

TABLE 13. Sensitivity analysis of β when $p = 0.5$.

β	α	L	n	Q	ETC^N
0.0	1.2	8	2	(176, 204, 191)	30014
0.1	1.0	8	2	(177, 205, 191)	30723
0.2	0.9	8	2	(176, 204, 191)	30830
0.3	0.7	8	2	(176, 205, 192)	31539
0.4	0.4	6	2	(176, 204, 191)	32081
0.5	0.4	8	2	(176, 204, 192)	32355
0.6	0.0	3	2	(176, 205, 193)	32941
0.7	0.0	3	2	(176, 203, 191)	32638
0.8	0.0	4	2	(172, 200, 188)	32297
0.9	0.0	4	2	(170, 198, 186)	31938
1.0	0.0	4	3	(130, 153, 142)	31532

Table 14. Sensitivity analysis of β when $p = 0.6$.

β	α	L	n	Q	ETC^N
0.0	1.2	8	2	(177, 205, 191)	30114
0.1	1.1	8	2	(177, 205, 192)	30208
0.2	0.9	8	2	(176, 204, 191)	30903
0.3	0.7	8	2	(176, 205, 192)	31598
0.4	0.6	8	2	(176, 204, 191)	31692
0.5	0.4	8	2	(176, 205, 192)	32387
0.6	0.3	8	2	(176, 205, 193)	32480
0.7	0.0	3	2	(175, 203, 191)	32641
0.8	0.0	4	2	(172, 200, 188)	32291
0.9	0.0	4	2	(170, 198, 186)	31923
1.0	0.0	4	3	(128, 153, 142)	31517

Table 15. Sensitivity analysis of β when $p = 0.7$.

β	α	L	n	Q	ETC^N
0.0	1.1	8	2	(177, 205, 192)	30484
0.1	1.0	8	2	(176, 204, 191)	30600
0.2	0.8	8	2	(177, 205, 192)	31318
0.3	0.7	8	2	(176, 204, 191)	31434
0.4	0.5	8	2	(176, 205, 192)	32151
0.5	0.0	8	2	(175, 204, 191)	32268
0.6	0.0	3	2	(176, 204, 192)	32893
0.7	0.0	3	2	(174, 203, 191)	32595
0.8	0.0	4	2	(171, 200, 188)	32253
0.9	0.0	4	2	(170, 198, 186)	31900
1.0	0.0	4	3	(128, 152, 142)	31495

Table 16. Sensitivity analysis of β when $p = 0.8$.

β	α	L	n	Q	ETC^N
0.0	1.1	8	2	(177, 205, 191)	30387
0.1	1.0	8	2	(176, 204, 191)	30509
0.2	0.8	8	2	(177, 205, 192)	31232
0.3	0.7	8	2	(176, 204, 191)	31354
0.4	0.5	8	2	(176, 205, 192)	32078
0.5	0.1	3	2	(176, 204, 192)	32547
0.6	0.0	3	2	(176, 204, 192)	32855
0.7	0.0	3	2	(174, 203, 191)	32560
0.8	0.0	4	2	(171, 200, 188)	32217
0.9	0.0	4	2	(169, 197, 186)	31868
1.0	0.0	4	3	(128, 152, 141)	31462

TABLE 17. Sensitivity analysis of β when $p = 0.9$.

β	α	L	n	Q	ETC^N
0.0	1.1	8	2	(177, 205, 191)	30283
0.1	0.9	8	2	(177, 205, 192)	31012
0.2	0.8	8	2	(176, 204, 191)	31140
0.3	0.6	8	2	(177, 205, 192)	31869
0.4	0.5	8	2	(176, 204, 191)	31997
0.5	0.1	3	2	(176, 204, 192)	32501
0.6	0.0	3	2	(176, 204, 192)	32812
0.7	0.0	4	2	(173, 201, 189)	32516
0.8	0.0	4	2	(171, 199, 187)	32175
0.9	0.0	4	2	(169, 197, 185)	31830
1.0	0.0	4	3	(128, 152, 141)	31425

TABLE 18. Sensitivity analysis of β when $p = 1.0$.

β	α	L	n	Q	ETC^N
0.0	1.1	8	2	(176, 204, 191)	30166
0.1	0.9	8	2	(177, 205, 192)	30902
0.2	0.8	8	2	(176, 204, 191)	31036
0.3	0.6	8	2	(176, 205, 192)	31771
0.4	0.5	8	2	(176, 204, 191)	31905
0.5	0.1	3	2	(176, 204, 192)	32448
0.6	0.0	3	2	(176, 204, 192)	32763
0.7	0.0	4	2	(173, 201, 189)	32464
0.8	0.0	4	2	(171, 199, 187)	32127
0.9	0.0	4	2	(170, 197, 185)	31786
1.0	0.0	4	3	(128, 152, 141)	31381

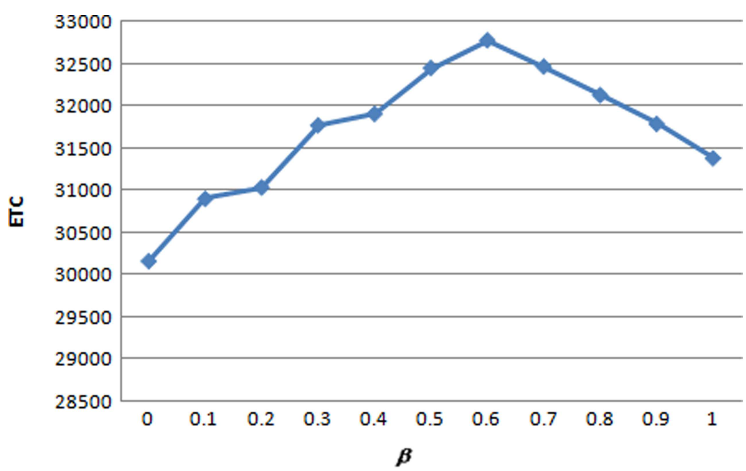

FIGURE 7. Effect of β on the ETC^N when $p = 0.0$.

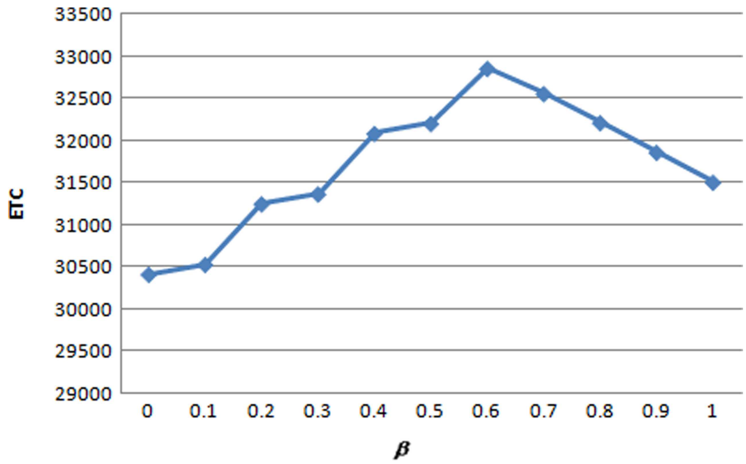

FIGURE 8. Effect of β on the ETC^N when $p = 0.1$.

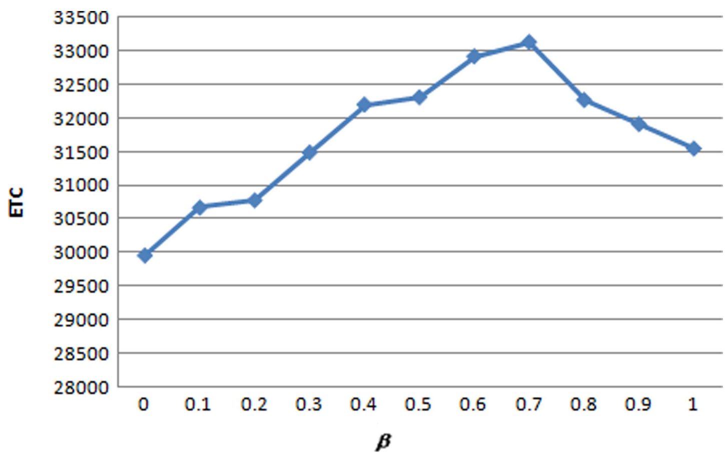

FIGURE 9. Effect of β on the ETC^N when $p = 0.2$.

2. Let $M = 1$ and consider only on the buyer's perspective and let $p = 1$ (or 0); then r_1 (or r_2) $= k$. Hence, equation (5.1) reduces to the result of [29]. Further, when r_1 or $r_2 = k > 3.9$, we obtain $\Psi(r_1, r_2, p) \to 0$. Hence equation (5.1) reduces to the result of [4].
3. Note that if $\beta_i = 1$, $\forall\ i = 1, 2, \ldots, M$ then equation (4.8) reduces to the total of the complete backorder case, hence (4.21) becomes

$$Q_i = \left[\frac{\left[A_i + \frac{B_i}{n} + C(L) + \pi_i \sigma_i \sqrt{L} \Psi(r_{i1}, r_{i2}, p_i) \right] D_i}{\frac{h_{bi}}{2} + \frac{h_{vi}}{2} \left[n \left(1 - \frac{D_i}{P_i} \right) - 1 + \frac{2D_i}{P_i} \right]} \right]^{\frac{1}{2}}, \quad L \in [L_j, L_{j-1}] \tag{5.2}$$

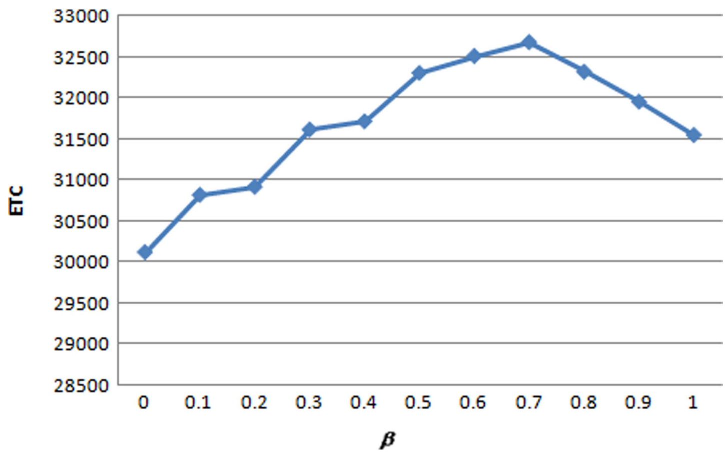

FIGURE 10. Effect of β on the ETC^N when $p = 0.3$.

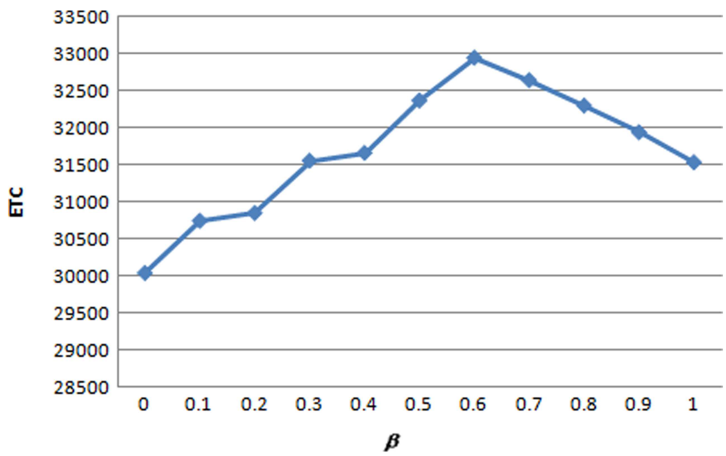

FIGURE 11. Effect of β on the ETC^N when $p = 0.4$.

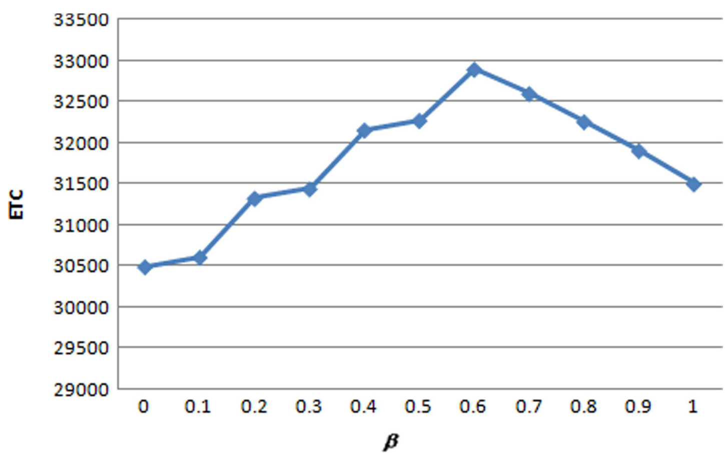

FIGURE 12. Effect of β on the ETC^N when $p = 0.5$.

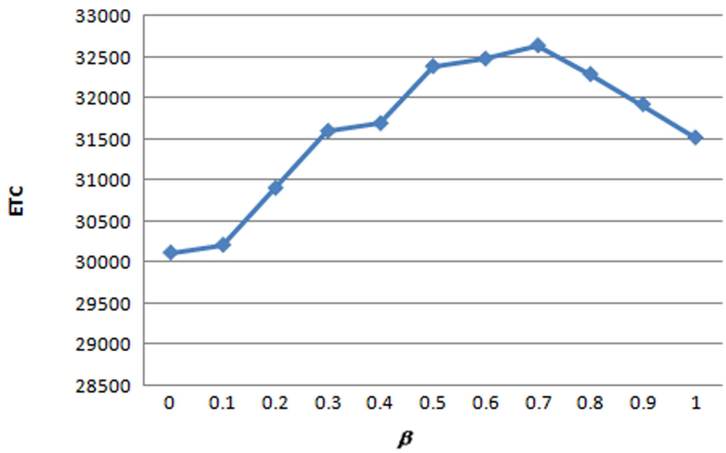

FIGURE 13. Effect of β on the ETC^N when $p = 0.6$.

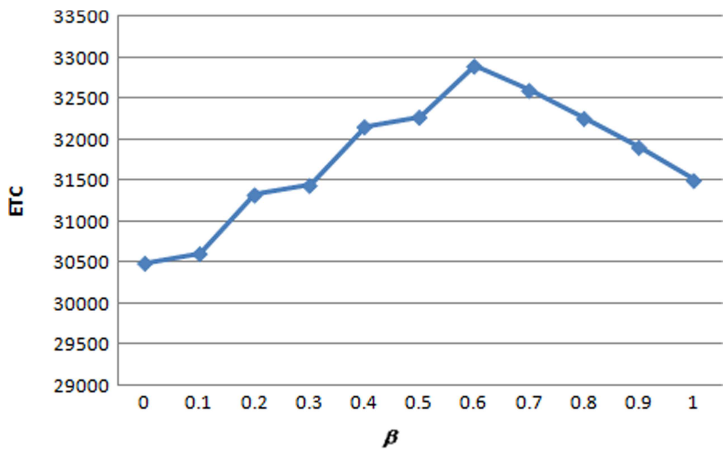

FIGURE 14. Effect of β on the ETC^N when $p = 0.7$.

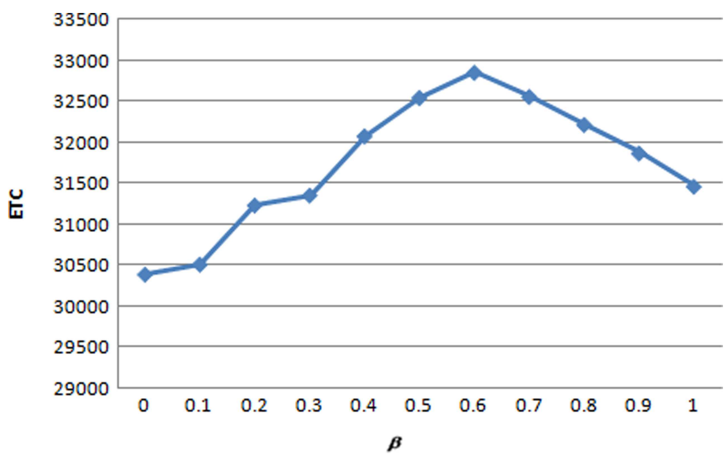

FIGURE 15. Effect of β on the ETC^N when $p = 0.8$.

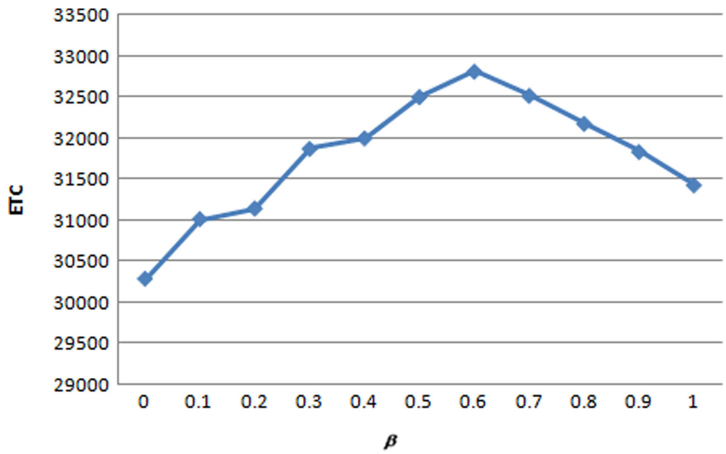

FIGURE 16. Effect of β on the ETC^N when $p = 0.9$.

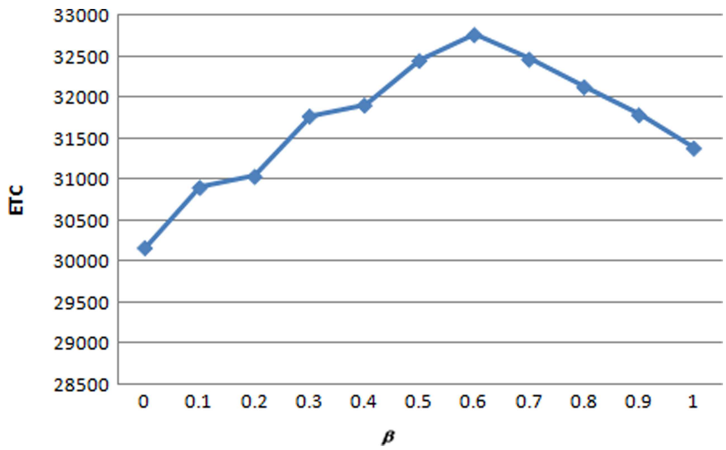

FIGURE 17. Effect of β on the ETC^N when $p = 1.0$.

and if $\beta_i = 0$, $\forall\ i = 1, 2, \ldots, M$ then equation (4.8) reduces to the total of the completely lost sale case, hence equation (4.21) becomes

$$Q_i = \left[\frac{\left[A_i + \frac{B_i}{n} + C(L) + (\pi_i + \pi_{i0})\sigma_i \sqrt{L}\Psi(r_{i1}, r_{i2}, p_i)\right] D_i}{\frac{h_{bi}}{2} + \frac{h_{vi}}{2}\left[n\left(1 - \frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right]} \right]^{\frac{1}{2}}, \quad L \in [L_j, L_{j-1}]. \tag{5.3}$$

Hence, in order to compare the equations (5.2) and (5.3) and their corresponding total costs are tabulated in Tables 8–18 and Tables 20–30. In both mixture of normal distribution case and mixture of free distribution case, β increases both total cost and order quantities, but at one stage β increases the total cost and order quantities reduces. This happens because we are using Lagrangian multiplier α. While considering this problem without constraints for single item, then the buyer's total cost of this proposed model becomes similar to [40]. That is $Q_{\beta=0} > Q_{\beta=1}$ and $EAC_{\beta=1} < EAC_{\beta=0}$.

TABLE 19. Optimal solutions of Example 5.2 when $\beta = 1.0$ and $p = 0.0$.

n	L	Q	ETC^U
1	8	(263, 295, 283)	38710
	6	(262, 294, 283)	37886
	4	(262, 294, 283)	36995
	3	(264, 296, 285)	36675
2	8	(166, 193, 182)	35805
	6	(165, 193, 181)	34952
	4	(166, 193, 182)	34075
	3	(168, 195, 184)	33880
3	8	(126, 151, 140)	35666
	6	(126, 150, 139)	34789
	4	(126, 150, 139)	33924
	3	(128, 152, 142)	33834
4	8	(104, 126, 116)	36251
	6	(103, 126, 115)	35353
	4	(104, 126, 116)	34499
	3	(106, 128, 118)	34501

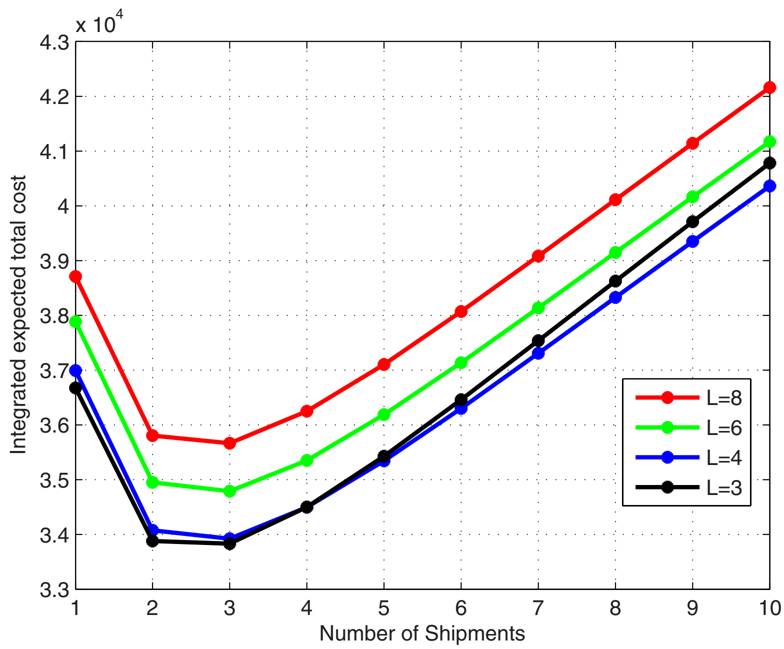

FIGURE 18. Graphical representation of total costs of Example 5.2 when $\beta = 1.0$ and $p = 0.0$.

TABLE 20. Sensitivity analysis of β when $p = 0.0$.

β	α	L	n	Q	ETC^U
0.0	0.2	3	2	(176, 204, 192)	34792
0.1	0.1	3	2	(177, 205, 193)	35185
0.2	0.1	3	2	(175, 204, 192)	34976
0.3	0.0	3	2	(176, 204, 192)	35370
0.4	0.0	3	2	(173, 202, 190)	34556
0.5	0.0	3	2	(172, 200, 189)	34951
0.6	0.0	3	2	(171, 199, 187)	34739
0.7	0.0	3	2	(170, 198, 186)	34527
0.8	0.0	3	2	(169, 197, 185)	34312
0.9	0.0	3	2	(169, 197, 185)	34097
1.0	0.0	3	3	(128, 152, 142)	33834

TABLE 21. Sensitivity analysis of β when $p = 0.1$.

β	α	L	n	Q	ETC^U
0.0	0.2	3	2	(176, 204, 192)	34826
0.1	0.1	3	2	(176, 205, 193)	35223
0.2	0.1	3	2	(175, 203, 191)	35017
0.3	0.0	3	2	(176, 204, 192)	35414
0.4	0.0	3	2	(175, 203, 191)	35208
0.5	0.0	3	2	(173, 202, 190)	35001
0.6	0.0	3	2	(172, 200, 189)	34793
0.7	0.0	3	2	(171, 199, 187)	34583
0.8	0.0	3	3	(130, 155, 144)	34441
0.9	0.0	3	2	(169, 196, 185)	34160
1.0	0.0	3	3	(128, 152, 142)	33897

TABLE 22. Sensitivity analysis of β when $p = 0.2$.

β	α	L	n	Q	ETC^U
0.0	0.2	3	2	(176, 204, 192)	34852
0.1	0.1	3	2	(176, 205, 192)	35251
0.2	0.0	3	2	(177, 205, 193)	35650
0.3	0.0	3	2	(176, 204, 192)	35448
0.4	0.0	3	2	(174, 203, 191)	35244
0.5	0.0	3	2	(173, 201, 190)	35039
0.6	0.0	3	2	(172, 200, 188)	34833
0.7	0.0	3	2	(171, 199, 187)	34626
0.8	0.0	3	2	(170, 198, 186)	34418
0.9	0.0	3	2	(169, 196, 185)	34208
1.0	0.0	3	3	(128, 152, 141)	33945

TABLE 23. Sensitivity analysis of β when $p = 0.3$.

β	α	L	n	Q	ETC^U
0.0	0.2	3	2	(176, 204, 191)	34870
0.1	0.1	3	2	(176, 204, 192)	35271
0.2	0.0	3	2	(177, 205, 193)	35671
0.3	0.0	3	2	(175, 204, 192)	35471
0.4	0.0	3	2	(174, 203, 191)	35269
0.5	0.0	3	2	(173, 201, 190)	35066
0.6	0.0	3	2	(172, 200, 188)	34862
0.7	0.0	3	2	(171, 199, 187)	34656
0.8	0.0	3	2	(170, 198, 186)	34450
0.9	0.0	3	2	(169, 196, 185)	34242
1.0	0.0	3	3	(128, 152, 141)	33979

TABLE 24. Sensitivity analysis of β when $p = 0.4$.

β	α	L	n	Q	ETC^U
0.0	0.2	3	2	(175, 204, 191)	34881
0.1	0.1	3	2	(176, 204, 192)	35282
0.2	0.0	3	2	(176, 205, 193)	35684
0.3	0.0	3	2	(175, 204, 192)	35484
0.4	0.0	3	2	(174, 203, 191)	35284
0.5	0.0	3	2	(173, 201, 189)	35082
0.6	0.0	3	2	(172, 200, 188)	34878
0.7	0.0	3	2	(171, 199, 187)	34674
0.8	0.0	3	2	(170, 198, 186)	34469
0.9	0.0	3	2	(169, 196, 185)	34262
1.0	0.0	3	3	(128, 152, 141)	33999

TABLE 25. Sensitivity analysis of β when $p = 0.5$.

β	α	L	n	Q	ETC^U
0.0	0.2	3	2	(175, 204, 191)	34883
0.1	0.1	3	2	(176, 204, 192)	35285
0.2	0.0	3	2	(176, 205, 193)	35687
0.3	0.0	3	2	(175, 203, 192)	35488
0.4	0.0	3	2	(174, 203, 191)	35288
0.5	0.0	3	2	(173, 201, 189)	35086
0.6	0.0	3	2	(172, 200, 188)	34884
0.7	0.0	3	2	(171, 199, 187)	34680
0.8	0.0	3	2	(170, 198, 186)	34474
0.9	0.0	3	2	(169, 196, 185)	34268
1.0	0.0	3	3	(128, 152, 141)	34005

TABLE 26. Sensitivity analysis of β when $p = 0.6$.

β	α	L	n	Q	ETC^U
0.0	0.2	3	2	(175, 204, 191)	34878
0.1	0.1	3	2	(176, 204, 192)	35280
0.2	0.0	3	2	(176, 205, 193)	35682
0.3	0.0	3	2	(175, 204, 192)	35483
0.4	0.0	3	2	(174, 203, 191)	35282
0.5	0.0	3	2	(173, 201, 189)	35080
0.6	0.0	3	2	(172, 200, 188)	34877
0.7	0.0	3	2	(171, 199, 187)	34673
0.8	0.0	3	2	(170, 198, 186)	34468
0.9	0.0	3	2	(169, 196, 185)	34261
1.0	0.0	3	3	(128, 152, 141)	33998

TABLE 27. Sensitivity analysis of β when $p = 0.7$.

β	α	L	n	Q	ETC^U
0.0	0.2	3	2	(176, 204, 191)	34866
0.1	0.1	3	2	(176, 204, 192)	35267
0.2	0.0	3	2	(177, 205, 193)	35668
0.3	0.0	3	2	(175, 204, 192)	35468
0.4	0.0	3	2	(174, 203, 191)	35266
0.5	0.0	3	2	(173, 201, 189)	35064
0.6	0.0	3	2	(172, 200, 188)	34860
0.7	0.0	3	2	(171, 199, 187)	34655
0.8	0.0	3	2	(170, 198, 186)	34448
0.9	0.0	3	2	(169, 196, 185)	34241
1.0	0.0	3	3	(128, 152, 141)	33978

TABLE 28. Sensitivity analysis of β when $p = 0.8$.

β	α	L	n	Q	ETC^U
0.0	0.2	3	2	(175, 204, 192)	34848
0.1	0.1	3	2	(176, 204, 192)	35247
0.2	0.0	3	2	(177, 205, 193)	35642
0.3	0.0	3	2	(176, 204, 192)	35444
0.4	0.0	3	2	(174, 203, 191)	35241
0.5	0.0	3	2	(173, 201, 190)	35036
0.6	0.0	3	2	(172, 200, 188)	34831
0.7	0.0	3	2	(171, 199, 187)	34624
0.8	0.0	3	2	(170, 198, 186)	34416
0.9	0.0	3	2	(169, 196, 185)	34207
1.0	0.0	3	3	(128, 152, 141)	33944

TABLE 29. Sensitivity analysis of β when $p = 0.9$.

β	α	L	n	Q	ETC^U
0.0	0.2	3	2	(176, 204, 192)	34823
0.1	0.1	3	2	(176, 205, 192)	35219
0.2	0.1	3	2	(175, 203, 191)	35014
0.3	0.0	3	2	(176, 204, 192)	35411
0.4	0.0	3	2	(175, 203, 191)	35206
0.5	0.0	3	2	(173, 202, 190)	34999
0.6	0.0	3	2	(172, 200, 188)	34791
0.7	0.0	3	2	(171, 199, 187)	34581
0.8	0.0	3	2	(170, 198, 186)	34371
0.9	0.0	3	2	(169, 196, 185)	34159
1.0	0.0	3	3	(128, 152, 142)	33896

TABLE 30. Sensitivity analysis of β when $p = 1.0$.

β	α	L	n	Q	ETC^U
0.0	0.2	3	2	(176, 204, 192)	34792
0.1	0.1	3	2	(177, 205, 193)	35185
0.2	0.1	3	2	(175, 204, 192)	34976
0.3	0.0	3	2	(176, 204, 192)	35370
0.4	0.0	3	2	(175, 203, 191)	35161
0.5	0.0	3	2	(174, 202, 190)	34951
0.6	0.0	3	2	(172, 200, 189)	34739
0.7	0.0	3	2	(171, 199, 187)	34527
0.8	0.0	3	2	(170, 198, 186)	34312
0.9	0.0	3	2	(169, 197, 185)	34097
1.0	0.0	3	3	(128, 152, 142)	33834

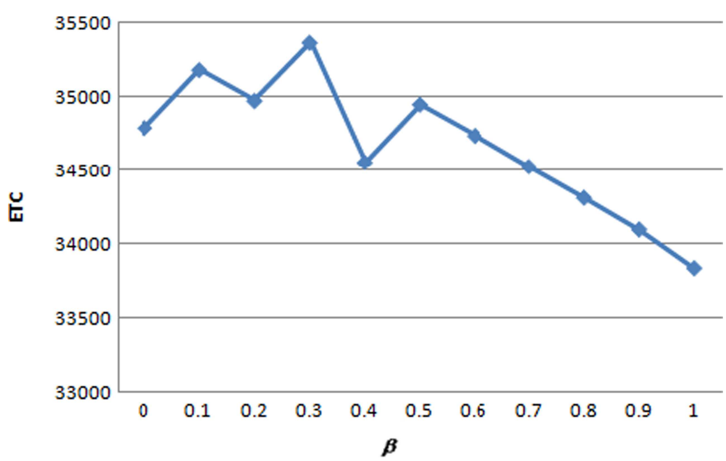

FIGURE 19. Effect of β on the ETC^U when $p = 0.0$.

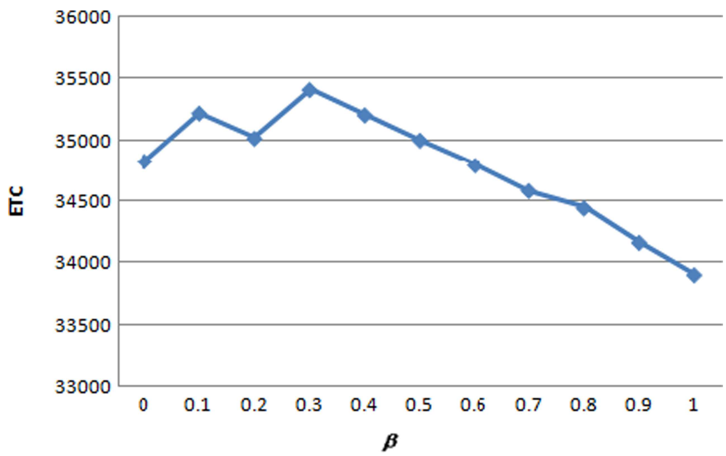

FIGURE 20. Effect of β on the ETC^U when $p = 0.1$.

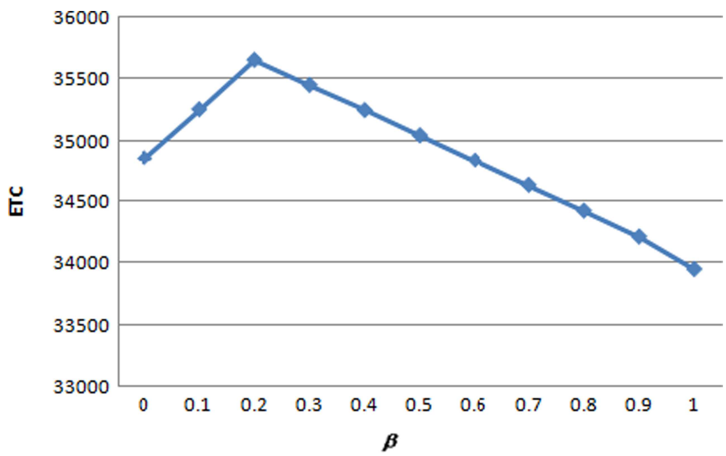

FIGURE 21. Effect of β on the ETC^U when $p = 0.2$.

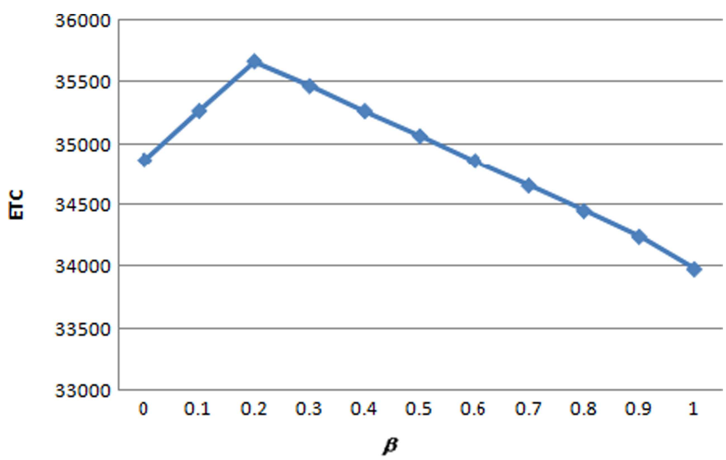

FIGURE 22. Effect of β on the ETC^U when $p = 0.3$.

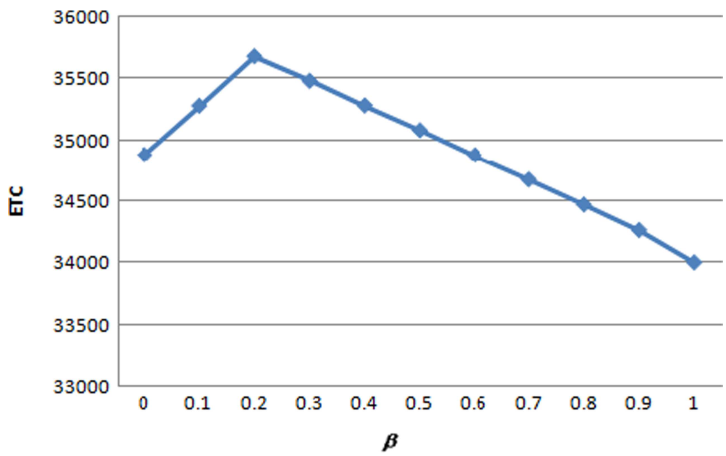

FIGURE 23. Effect of β on the ETC^U when $p = 0.4$.

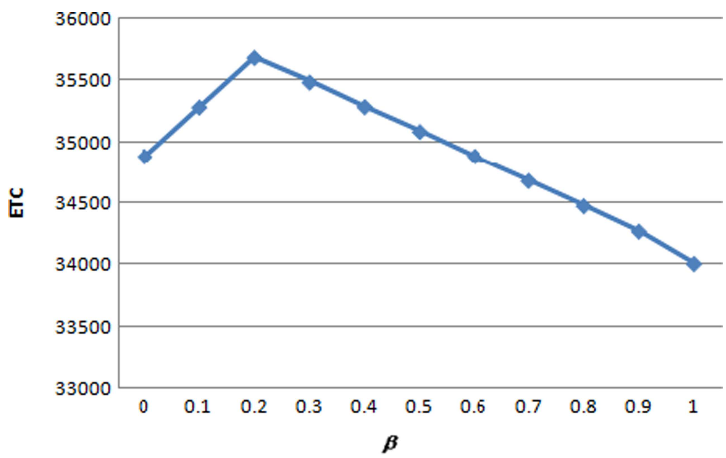

FIGURE 24. Effect of β on the ETC^U when $p = 0.5$.

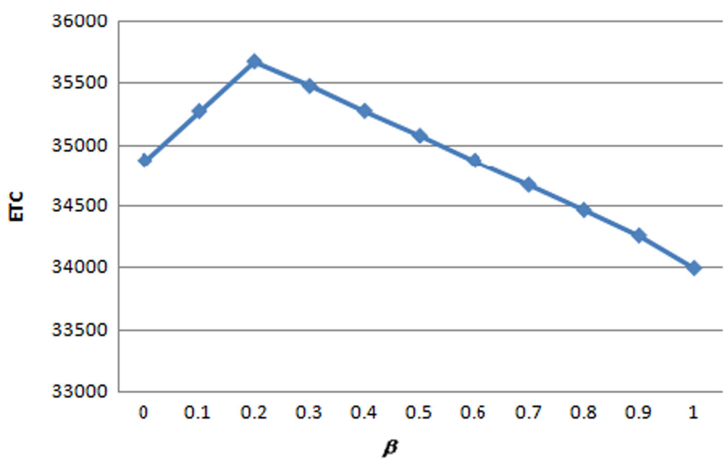

FIGURE 25. Effect of β on the ETC^U when $p = 0.6$.

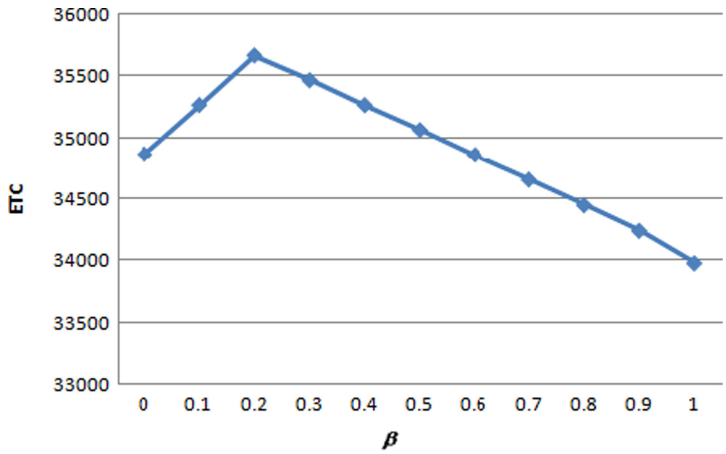

FIGURE 26. Effect of β on the ETC^U when $p = 0.7$.

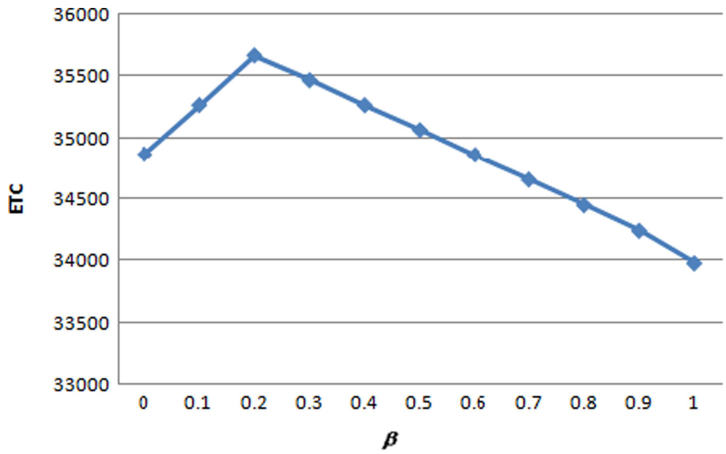

FIGURE 27. Effect of β on the ETC^U when $p = 0.8$.

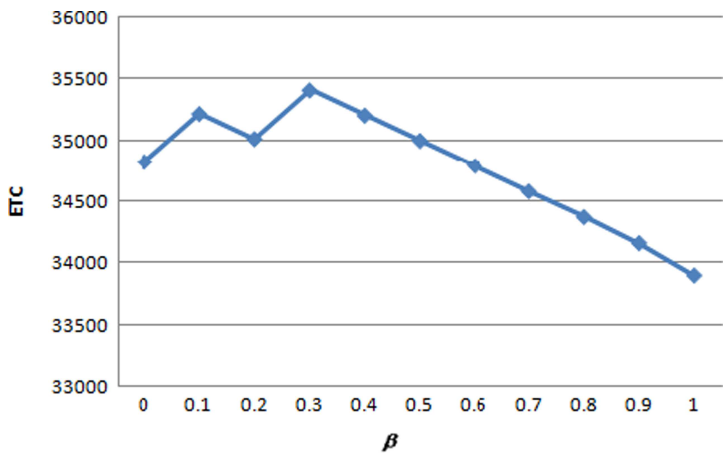

FIGURE 28. Effect of β on the ETC^U when $p = 0.9$.

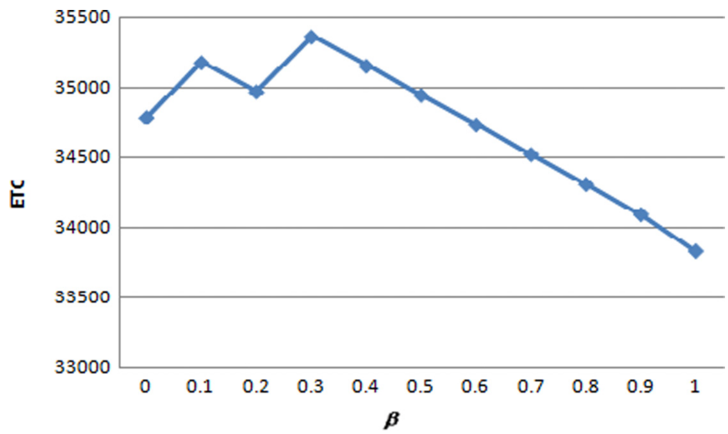

FIGURE 29. Effect of β on the ETC^U when $p = 1.0$.

4. We can easily observed that for fixed $Q = (Q_1, Q_2, Q_3)$, L, n the Lagrangian multiplier α and γ increases, then their corresponding total cost decreases. From Tables 8–18 and Tables 20–30 one can realize that the Lagrangian multiplier α decreases if the backorder ratio β increases.
5. Table 7 shows that when the number of shipments n increase, the order quantity Q_i, $i = 1, 2, 3$ decreases. This is not unexpected, because in practice this fact may occur in the supply chain system.

6. Conclusion

The consumption rate of the several customers are not identical in the lead time, so one cannot use only a single distribution to describe the lead time demand. Hence in this paper the mixture of distribution model for multi-item is developed. An integrated vendor-buyer inventory policy for a continuous review model have been considered, because the integrated total cost is minimum when compared to the total cost of the individuals. The buyer has limited spaces and budget constraints, and within this constraints the buyer and vendor try to minimize their costs and the procurement lead time is assumed as n mutually independent components. The model allows shortages and they are backordered partially. We have divided the paper into two cases of demand patterns: (i) mixture of normal distribution (ii) mixture of distribution free approach. In each case we utilize the Lagrangian multiplier technique to optimize the cost function. Since the objective function is highly non-linear, so we have presented an algorithm to find the optimal solutions such as Q_i, L and n. Sensitivity analysis is performed for the different values of backorder ratio β and p. The graphical interpretation of the sensitivity analysis is also presented. Managerial implications are also given according to the results of the sensitivity analysis.

In future research, it would be interesting to deal with different constraints like ordering constraints, inventory constraints etc., This work can be extended by incorporating deterministic demand patterns such as stock-depend demand or price-depend demand.

Acknowledgements. The work of authors are supported by Department of Science and Technology-Science and Engineering Research Board (DST – SERB), Government of India, New Delhi, under the grant number DST-SERB/SR/S4/MS: 814/13-Dated 24.04.2014.

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