

ROBUST STOCK AND BOND ALLOCATION WITH END-OF-HORIZON EFFECTS

MICHAEL DZIECICHOWICZ¹ AND AURÉLIE C. THIELE^{2,*}

Abstract. We propose an approach to portfolio management over a finite time horizon that (i) does not require the precise knowledge of the underlying probability distributions, instead relying on *range forecasts* for the stock returns, and (ii) allows the fund manager to capture the degree of the investor’s risk aversion through a single, intuitive parameter called the *budget of uncertainty*. This budget represents the worst-case number of time periods with poor performance that the investor is willing to plan for. An application of this setting is target-date funds for pension fund management. We describe an efficient procedure to compute the dynamic allocation between (riskless) bonds and (riskier) stocks at each time period, and we illustrate the risk-to-time-horizon tradeoff on optimal allocation tables, which can easily be provided to fund participants to help them select their strategy. The proposed approach refines rules implemented by practitioners and provides an intuitive framework to incorporate risk in applications with end of horizon effects. In contrast with existing literature providing robust fund management approaches to mathematically sophisticated finance professionals, our goal is to provide a simple framework for less quantitative fund participants who seek to understand how stock return uncertainty and planned retirement date affect the optimal stock-vs-bond allocation in their portfolio. We extend our procedure to the case when the investor’s wealth is penalized for falling short of performance benchmarks across the time horizon. We also discuss the case where the manager can invest in multiple stocks. Numerical results are provided.

Mathematics Subject Classification. 90–XX.

Received February 24, 2016. Accepted September 7, 2017.

1. MOTIVATION

Target-date funds, which shift their stock-bond allocation toward a more conservative mix as the target date (often the investor’s expected retirement date) grows near, now have about \$700 billion in total assets and represent about 20% of the defined-contribution retirement market, according to Bary [1]. According to the head of the retirement group at BlackRock, “target-date funds are one of the most important and successful innovations of the past 20 years”; further, the target-date market is expected to keep increasing at a brisk pace and Vanguard, the largest manager of defined-contribution assets, estimates that 80% of new plan entrants

Keywords. Target-date funds; decision-making under uncertainty; stock-bond allocation mix.

¹ Senior Manager, Ernst & Young, 5 Times Square, New York, NY 10036, USA.

² Associate Professor, Department of Engineering Management, Information and Systems, Southern Methodist University, 3145 Dyer St Rm 313, Dallas TX 75205, USA.

*Corresponding author: athiele@smu.edu

will be entirely in target-date funds by 2018 [1]. A key concern is that such funds currently have too much equity exposure around the target retirement date. As an example, in 2008, short-dated funds with a 2010 target date dropped 20% to 25%, hurting workers nearing retirement [1]. There is therefore a need for novel approaches that better protect workers' retirement portfolio against risk, especially while keeping into account the nearing retirement date. The purpose of the present paper is to provide such an approach, based on robust optimization (RO). RO optimizes the worst-case objective over a "reasonable" convex set of possible values for the uncertainty parameters, centered at their nominal values. The appeal of robust optimization is twofold: (i) it inherently protects the portfolio against risk, because it is a worst-case optimization approach, and (ii) it only requires a small amount of information about the stock returns – in particular, it requires no probabilistic knowledge on those returns – thus making its use more practical for fund participants with a modest analytical background who seek to protect themselves against the excessive equity exposure mentioned above as they near retirement.

A variety of analytical approaches are currently implemented in pension fund management. While pension funds such as TIAA-CREF let its members pick a fund based on their estimated year of retirement (<http://www.tiaa-cref.org/>), such "one-size-fits-all" rules of thumb, however, fail to capture the investor's risk preferences. A more sophisticated pension fund strategy is bucket planning or time-segmented asset allocation, which requires continual rebalancing if a consistent risk level is to be maintained, with the resulting taxes and transaction costs (Evensky *et al.* [15]). At the other end of the spectrum of available techniques are quantitative models to pension fund management, which have traditionally incorporated the following features (Geyer *et al.* [20]): (i) a probabilistic description of the random returns, (ii) a utility-based description of the investor's risk preferences, and (iii) solution techniques based on dynamic programming, multi-stage stochastic programming or related mathematical models.

Dynamic programming is for instance used in Vigna and Haberman [36] to find an optimal investment strategy in a defined contribution pension scheme, based on a series of interim targets and a target at retirement. The authors highlight the large variability, for target-date funds, of the level of pension achieved at retirement in the case of a variable annuity conversion rate. Haberman and Vigna [23] focus on the optimal investment strategy in a defined contribution pension scheme when the fund is invested in a high-risk and low-risk assets whose returns are correlated, and consider three downside risk measures to quantify the risk faced by the retiree: the probability of failing the target, the mean shortfall and Value-at-Risk. Deelstra *et al.* [13] investigate optimal investment strategies in order to maximize the expected utility function of the terminal wealth under the constraint that it must exceed the minimum guarantee, under the assumption that the utility function exhibits constant relative risk aversion.

Consigli and Dempster [12] apply multi-stage stochastic programming to asset-liability management for a pension fund problem where the liabilities refer to five different classes of pension contracts, with the objective of maximizing wealth at the end of the horizon. Yet, in many cases, multi-stage stochastic programming requires access to high-performance computing resources to solve the large-scale mathematical problems that arise (see for instance Gondzio and Kouwenberg [21]); when such resources are not available, the investor is faced with tractability issues. Kouwenberg [28] in particular investigates scenario generation methods for asset liability management applied to a Dutch pension fund, using randomly sampled event trees and event trees fitting the mean and the covariance of the return distribution. Using rolling horizon simulations, the author argues that performance of multi-stage stochastic programming can be improved drastically by choosing an appropriate scenario generation method. Kilianova and Pflug [27] apply multi-stage stochastic optimization to the case where members of a pension fund can switch periodically between funds with different risk profiles. Their objective is to minimize a multi-period average value-at-risk. They illustrate their method on the pension system of the Slovak Republic. Bogentoft *et al.* [10] consider Conditional Value-at-Risk as a risk measure and sample-path simulation of the liabilities in asset-liability management for a pension fund. Sharpe [34] provides an in-depth discussion of the issues related to budgeting and monitoring pension fund risk. Mulvey *et al.* [29] present a case study in asset-liability management for Towers Perrin–Tillinghast. Pension fund management is reviewed in Muralidhar [30].

A drawback of the stochastic programming framework is that random returns over long time horizons are difficult to estimate accurately. Therefore, significant computing power could be spent finding a solution to the stochastic programming problem, without any guarantee that the model considered will capture the real-life problem accurately. As another drawback, individual investors who lack either access to computational resources or mathematical sophistication are unable to implement such approaches, but must nonetheless protect their portfolio against downside risk.

In the remainder of this paper, we propose a middle-ground approach that incorporates uncertainty and risk aversion through simple allocation rules and does not require sophisticated optimization software. Our focus is on helping individual investors with some mathematical familiarity, rather than quantitative professionals, understand how uncertainty affects their optimal portfolio allocation for various time horizons, *i.e.*, various times-to-retirement, under simple assumptions on uncertainty. We hope thus to demystify the portfolio allocation process and provide actionable insights for such fund participants. Investors will gain deeper insights into their strategy, while the structured, parametric nature of the rules makes them easy to implement from a pension fund's perspective. The approach relies on a non-probabilistic model of randomness first proposed by Soyster [35]; uncertain parameters are modeled using range forecasts and the decision-maker solves the problem for the worst-case instance of the coefficients. Because it is unlikely that parameters will all be equal to their worst-case value, Soyster's approach was deemed too conservative for practical implementation until the mid-1990s, when Ben–Tal and Nemirovski [3] imposed a bound on the total deviation of the parameters from their nominal values and derived tractable reformulations. The reader is referred to Gabrel *et al.* [19] for an extensive literature review on robust optimization.

Robust multiperiod portfolio optimization has been the focus of substantial research efforts since 2000, when Ben–Tal *et al.* [2] developed and analyzed the first robust counterpart to multi-stage portfolio problems. Pachamanova [31] investigate robust optimization problems in which the uncertainty sets are polyhedral, applied to single period portfolio optimization, multiperiod portfolio management and credit risk estimation. For multiperiod portfolio management, the author shows that the robust formulations only require a modest amount of information about the future asset returns, and can incorporate transaction costs and taxes while preserving computational tractability. Fabozzi *et al.* [16] provide a book treatment of robust portfolio optimization and management, including classical theory and extensions, robust parameter estimation, optimization techniques and robust asset allocation. Bertsimas and Pachamanova [7] consider robust optimization approaches to multiperiod portfolio selection in the presence of transaction costs that are linear and computationally efficient. Pachamanova *et al.* [32] investigate further robust approaches to pension fund asset liability management (ALM) under uncertainty, first presenting a scenario-based ALM model for pension funds and then presenting robust ALM models with a discussion of symmetric and asymmetric uncertainty sets as well as input selection.

Two works are most relevant to the present approach. Iyengar and Ma [25] devise a robust optimization approach to pension fund management. They provide worst-case performance guarantees on future contributions, investigate the computational complexity of the optimization problem to find the contribution schedule – showing it scales well with the number of assets in the fund's portfolio and the number of trading dates – and compute the worst case contribution rate for a given pension fund management strategy. Gülpinar and Pachamanova [22] provide a robust optimization approach to asset-liability management under time-varying investment opportunities. They derive a tractable robust counterpart of the ALM problem when future uncertainties are represented by ellipsoidal sets, as in [3], which have the advantage over [8] of not requiring auxiliary variables but lead to robust counterparts of slightly increased complexity compared to the nominal problem; further, those sets can be easily generated from statistical factor models. They also model the time-varying features of asset returns and interest rates through a vector-autoregressive process, and benchmark their approach against a stochastic programming formulation. Their approach, leading to second-order cone problem formulations, provides a comprehensive and powerful framework for robust ALM for finance professionals but we feel it is outside the reach of the less mathematically sophisticated individual investors whom the present paper targets. Thus, we hope our paper can be of value to practitioners who want to take an active role in the management of their own portfolio but are not operations research specialists.

Contributions.

- We model uncertainty on the stock returns through range forecasts and capture the investor’s risk aversion through a “budget of uncertainty” parameter, which represents the number of future time periods with poor performance over the time horizon, as defined by the risk-averse investor.
- In the case of stock returns that are identically distributed over time, we provide iterative formulas to compute the best allocation between stocks and (riskless) bonds without resorting to optimization software; hence, the approach remains highly tractable for large time horizons and can be implemented easily. This will help the investor understand how the degree of his risk aversion affects his optimal allocation.
- We provide guidelines to select the budget of uncertainty and perform extensive numerical experiments using the publicly available strategy of a pension fund as benchmark. The key output of the method is a set of two-dimensional tables quantifying the impact of the budget of uncertainty and retirement time on the allocation; in practice, those tables should be provided to the investors to help them select the strategy that best matches their attitude towards risk.
- In the case of time-varying stock returns, we explain how to compute the optimal allocation efficiently and study the problem from the perspective of a fund manager who is penalized when the portfolio return falls below performance benchmarks.
- We perform numerical experiments to illustrate the performance of our approach, and assess the value of solving this more complex problem.

Outline. The remainder of the paper is structured as follows. Section 2 provides the problem setup. Section 3 considers the special case when the returns are identically distributed over time. We extend the approach to auto-regressive AR(1) dynamics in Section 4. We discuss the extension to multiple assets in Section 5 and conclude in Section 6.

2. PROBLEM SETUP

A pension fund manager seeks to allocate an investor’s contribution between one risky asset (stock or index fund) and one riskless asset (bond), knowing that the investor is planning to retire in T time periods, *e.g.* years. A key focus is the optimal tradeoff between the stock and the bond as a function of the time to retirement and the investor’s degree of risk aversion.

2.1. Modeling of the returns

2.1.1. Riskless asset

The riskless rate of return is assumed known, equal to r_f at each time period.

2.1.2. Stock return dynamics

The rate of return of the stock (or index fund) is random, but follows a first-order autoregressive process, or AR(1), so that the realization of the stock return in the next time period depends on the value of the return now.

Let subscript t represent the number of time periods *left to go* until the end of the time horizon; in other words, t represents the current time period and $t+1$ represents the previous (earlier) one. We refer the reader to Pandit and Wu [33] for a comprehensive introduction to AR(1) processes and time series analysis. The random return S_t can be written as:

$$S_t = (1 - \varphi)\mu + \varphi S_{t+1} + \sigma \epsilon_t, \quad (2.1)$$

where φ , μ , and σ are fixed scalars and the ϵ_t are independent, identically distributed (i.i.d.) zero-mean random variables. The expected dynamics of the AR(1) process are summarized by its autocorrelation parameter, φ . If $\varphi = 0$, then the distribution of S_t does not depend on S_{t+1} and the S_t are independent, identically distributed random variables with mean μ . If $|\varphi| < 1$, then the process is said to be stationary and the past realizations

matter more when φ is closer to 1. If $\varphi = 1$, then the process is called a “unit-root process” and does not depend on μ . If $\varphi > 1$, then the process is divergent, in the sense that it never reverts back to a long-run value.

2.1.3. Modeling of uncertainty

The volatility of the stock process is captured by the standard deviation σ and the random variable ϵ_t . Probability distributions for the random variables ϵ_t are difficult to estimate accurately, especially years ahead of time. Here, we adopt a robust optimization approach, where we model the random variables as uncertain parameters belonging to known intervals (a technique pioneered by Bertsimas and Sim [8]), and optimize the worst-case value of the portfolio, where the worst case is defined below. We assume that the ϵ_t belong to a known range, *i.e.*, $\epsilon_t \in [-c, c]$ for some $c > 0$. (While the approach extends to the case where $\epsilon_t \in [-c_t, c_t]$ for all t , we take c_t constant to reflect that the ϵ_t are i.i.d. In addition, our results still hold if the range is not symmetric around zero, because only its lower part $[-c, 0]$ is incorporated into our approach).

The return at each time period, then, belongs to a confidence interval $[\bar{S}_t - \sigma c, \bar{S}_t + \sigma c]$, also called *range forecast*, where $\bar{S}_t = (1 - \varphi)\mu + \varphi S_{t+1}$ for all t . Short sales are not allowed and there is no transaction cost. Note that because we cannot short sell stocks, the worst case at a given time period occurs when stock prices decrease and $\epsilon_t = -c$. We refer to \bar{S}_t as the nominal (or mean) return and $\bar{S}_t - \sigma c$ as the worst-case return.

In line with classical robust optimization theory, our approach relies on the range $[\bar{S}_t - \sigma c, \bar{S}_t]$ of returns below or equal to the nominal value, *i.e.*, the risk-averse investor discards upside risk. Such built-in risk aversion, where we do not count on better-than-expected returns to make up for poor performance at other time periods, makes the approach particularly well-suited to pension funds and retirement planning. In a dynamic setting, the risk-averse investor selects an optimal allocation at each time period. Note that:

- If $r_f > \bar{S}_t$, the investor will put all his money in bonds at time t ,
- If $\bar{S}_t - \sigma c > r_f$, the investor will put all his money in stocks at time t .
- The investor’s allocation at time t will be a mix of the stock and bond when $\bar{S}_t - \sigma c < r_f < \bar{S}_t$.

We describe the investor’s risk preferences through a parameter τ , called the *budget of uncertainty*, which takes integer values between 0 and T . τ represents the number of time periods with poor performance that the investor wants to protect his portfolio against. Introducing the notation:

$$S_t = \bar{S}_t - \sigma c \epsilon_t$$

with $\epsilon_t \in \{0, 1\}$ for all t , we connect τ to the allowable deviations of the stock returns from its nominal values, when the manager has T time periods left in the decision-making process and a budget of uncertainty τ , through the constraint:

$$\sum_{t=1}^T \epsilon_t \leq \tau.$$

In the extreme cases,

- a risk-neutral investor will select $\tau = 0$, thus replacing the random stock returns by their nominal values (since only $\epsilon_t = 0$ for all t is feasible).
- a very conservative investor will plan for the possibility that the stock underperform the riskless asset in every single time period (making $\epsilon_t = 1$ for all t feasible) and thus select $\tau = T$.

Most investors will choose a value of τ strictly between 0 and T , in order to achieve an acceptable tradeoff between risk and return.

2.2. Mathematical programming formulations

In this section we first provide the robust optimization formulation, and then the classical multi-stage stochastic programming formulation for easy comparison.

2.2.1. Robust optimization formulation

Here, we assume that the investor solves his problem at once for all the time periods considered, and resolves his problem in subsequent time periods, every time for all the remaining time periods, in an open-loop fashion, in line with Bertsimas and Thiele [4]. While closed-loop optimization would be a more realistic framework, open-loop optimization provides a good start to investigate the potential of the proposed approach. At each time period, the decision maker has two types of decision to take: (i) whether to use one unit of the budget of uncertainty now to protect his portfolio against the stock's worst-case return, or use it later, and (ii) how to select the stock-bond mix for that time period.

We use the following notation:

- $x_{\tau t}$ the optimal *fraction* of wealth in the risky asset with t time periods left and budget of uncertainty τ ,
- $y_{\tau t}$ the optimal *fraction* of wealth in the riskless asset with t time periods left and budget of uncertainty τ ,
- $\omega_{\tau t}$ the optimal rate of growth of the investor's portfolio with t time periods left and budget of uncertainty τ ,
- W_0 the initial amount of wealth,
- $W_{\tau t}$ the optimal wealth with t time periods left and budget of uncertainty τ .

The feasible set for the stock-bond mix at each time period is defined as:

$$\mathcal{X} = \{(x, y) | x + y = 1, x, y \geq 0\}$$

and the feasible set for the uncertainty with t time periods left is defined as:

$$\mathcal{E}_t^{\tau}(\epsilon_{-t}) = \left\{ \epsilon_t | \epsilon_t \leq \tau - \sum_{s=t+1}^T \epsilon_s, \epsilon_t \in \{0, 1\} \forall t \right\}$$

where ϵ_{-t} , by an abuse of notation, refers to the vector of scaled deviations for the time periods earlier than time t . (ϵ_{-T} is empty. Note that we do not need to include uncertainty in later time periods since at time t we only need to know whether we have enough budget left to possibly use one unit now or not).

The robust optimization problem with T time periods left in the time horizon and a budget of uncertainty τ can therefore be formulated as:

$$\begin{aligned} W_{\tau T}^{Rob} = W_0 & \max_{(x_T, y_T) \in \mathcal{X}} \min_{\epsilon_T \in \mathcal{E}_T^{\tau}(\epsilon_{-T})} [(\bar{S}_T - \sigma c \epsilon_T) x_T + r_f y_T] \cdot \\ & \max_{(x_{T-1}, y_{T-1}) \in \mathcal{X}} \min_{\epsilon_{T-1} \in \mathcal{E}_{T-1}^{\tau}(\epsilon_{-(T-1)})} [(\bar{S}_{T-1} - \sigma c \epsilon_{T-1}) x_{T-1} + r_f y_{T-1}] \cdot \\ & \dots \max_{(x_1, y_1) \in \mathcal{X}} \min_{\epsilon_1 \in \mathcal{E}_1^{\tau}(\epsilon_{-1})} [(\bar{S}_1 - \sigma c \epsilon_1) x_1 + r_f y_1] \end{aligned}$$

Note that we have $W_{\tau t}^{Rob} = W_0 \omega_{\tau t}$ so that, if $\tau \geq 1$, $\omega_{\tau t}$ is obtained by comparing the portfolio's return if we use one unit of the budget of uncertainty now, planning for the stock yielding its worst-case return now, or if we use all the budget later (for the remaining $t - 1$ time periods), planning for the stock yielding its nominal return now:

$$\omega_{\tau t} = \max_{(x_t, y_t) \in \mathcal{X}} \min \left\{ \omega_{\tau-1, t-1} [(\bar{S}_t - \sigma c) x_t + r_f y_t], \omega_{\tau, t-1} [\bar{S}_t x_t + r_f y_t] \right\}. \quad (2.2)$$

(If $\tau = 0$, $\omega_{0t} = \omega_{0, t-1} \bar{S}_t$ for all t under the natural assumption $\bar{S}_t > r_f$).

In contrast with Bertsimas and Sim [8], it does not seem possible to derive probabilistic guarantees for this problem because (i) we impose to use either one unit of the budget or none at each time period, instead of considering continuous variables as in [8], and (ii) the objective involves a product over time periods in the objective definition, so that even using logarithms to create sums would lead to a nonlinear objective, while [8]

uses the additive, linear structure of the function subject to uncertainty and the symmetry of the uncertain parameters around their nominal values to obtain probabilistic guarantees.

Above, the stock only takes two possible values at each time period: nominal or worst-case. We can achieve a finer description of uncertainty by discretizing the range forecast for the stock price, so that the worst-case stock price is achieved when a given number N of uncertainty tokens are used, rather than simply one. We would then optimize over the number of uncertainty tokens (between 0 and N) that the manager should protect his portfolio against in the present time period, keeping the rest of the uncertainty budget to protect his portfolio in subsequent time periods. This requires more evaluations of the objective function but does not conceptually change the problem. Another approach would be to consider shorter time intervals, for instance, months or quarters rather than years, and allow rebalancing at those time periods, so that the yearly stock returns can take 2^p values if the year is divided into p intervals and the approach described in the present paper would be immediately applicable. Decreasing the time step leads to an increasingly better approximation of the real-life process by the binomial tree. (Hull [24])

While we focus here on the simple model with two possible realizations of the stock return to investigate the potential of our approach, the more complex model with more than two realization at each time period would remain tractable in the case of one stock and one bond. Extending the more complex model to multiple assets leads to exponential growth in the number of cases to consider and more sophisticated techniques would have to be used to keep it tractable in that case.

2.2.2. Stochastic programming formulation

For completeness, we also provide a stochastic programming formulation for the problem.

Since the robust optimization approach is a conservative one, taking into account the decision-maker's aversion to ambiguity, we may also wish to incorporate risk aversion in stochastic programming to better compare the two approaches. A possible way to do that is to use a utility function (an increasing, concave function in wealth, first presented by von Neumann and Morgenstern [37]) at the last time period. This is in line with for instance the work of Deelstra *et al.* [13] in the context of pension fund management and Birge and Louveaux [9] for stochastic programming applied more broadly to portfolio management with a wealth goal at the end of the time horizon. Because stochastic programming algorithms build upon the problem's linear structure, utility functions used in that context are piecewise linear (concave) functions, as in [9], which allow the manager to retain an overall linear structure for the problem. Here we will model the utility of the terminal wealth as $E[U(W)] = E[\min_{k=1,\dots,K}(a^k W + b^k)]$. (The special case $K = 1$ with $a^1 = 1$ and $b^1 = 0$ recovers the case where the manager maximizes his average wealth).

We use the following notation for the stochastic programming model:

- \mathcal{N} the finite set of event nodes,
- \mathcal{T} the set of terminal nodes,
- π_n the probability of being at node $n \in \mathcal{T}$ at the end of the time horizon,
- $p(n)$ the parent of node n for any $n \in \mathcal{N}$,
- 0 the root node of the tree,
- S_n the stock return for the time period between nodes $p(n)$ and n , $n \in \mathcal{N}/\{0\}$,
- W_0 the initial budget,
- \tilde{x}_n the amount invested in the stock, in dollars,
- \tilde{y}_n the amount invested in the bond, in dollars.

The formulation becomes, assuming no transaction cost:

$$\begin{aligned}
 W_{\tau T}^{Stoc} &= \max \sum_{n \in \mathcal{T}} \pi_n z_n \\
 \text{s.t. } z_n &\leq a^k (S_n \tilde{x}_{p(n)} + r_f \tilde{y}_{p(n)}) + b^k, \quad \forall k, \forall n \in \mathcal{T} \\
 \tilde{x}_n + \tilde{y}_n &= S_n \tilde{x}_{p(n)} + r_f \tilde{y}_{p(n)}, \quad \forall n \in \mathcal{S}/\{0\} \\
 \tilde{x}_0 + \tilde{y}_0 &= W_0, \quad \tilde{x}_n, \tilde{y}_n \geq 0, \quad \forall n \in \mathcal{S}.
 \end{aligned}$$

A simple description of randomness with two possible outcomes at each time period leads to 2^T terminal nodes. ($T = 35$ leads to over 34 billion terminal nodes, specifically, 34, 359, 738, 368 nodes). This is why stochastic programming is often used in conjunction with sample path generation in practice, or why some practitioners prefer to generate more scenarios (finer descriptions of uncertainty) for random variables in the early time periods of the tree only and fewer for later time periods, or even replace later random variables with their average (Cariño *et al.* [11]). In addition, while the manager in the robust optimization approach only needs to specify the budget of uncertainty τ to capture his aversion to uncertainty, the manager in the stochastic programming approach must specify the utility function to use at the last time period, which is notoriously difficult to estimate (Bertsimas *et al.* [6]). This issue is further complicated by the need to approximate this poorly understood utility into a piecewise linear function to ultimately linearize the formulation. Hence, the fund manager keen on applying stochastic programming must make many assumptions on the underlying parameters (utility function) or random variables (stock return), and then make further simplifying assumptions to keep the problem tractable, such as which piecewise linear approximation to use, or how to approximate the stochastic process by a tractable event tree (at which time periods to consider multiple stock returns, how many possible realizations at those nodes).

Further, in practice, stochastic programming formulations are rarely solved as generic linear problems. Instead, recourse functions – representing the optimal objective value for given portfolio allocations at earlier time periods – are approximated as piecewise linear functions, with such pieces being generated only as needed during the algorithm. The decision maker must then decide at each step whether to keep improving the approximation for a given candidate solution or re-run the master problem to find a better candidate solution. Improvements on the approximation can also be done through a variety of methods, generating a single cut for the expected wealth function or multiple cuts for each of the return scenario at the present time period. This is known in the literature as Benders decomposition or L-Shaped method and falls within the class of delayed constraint generation algorithms (Kall and Wallace [26]).

In the remainder of the paper, we argue that robust optimization shows promise to capture uncertainty and risk at an level appropriate for the less sophisticated investor who must take an active involvement in his retirement portfolio.

3. THE OPTIMAL SOLUTION TO THE BASIC PROBLEM

In this section, we assume that the nominal return of the risky asset \bar{S}_t is constant over the time horizon, denoted \bar{S} . This corresponds to the case $\varphi = 0$ in the AR(1) model. Such a situation arises for instance when the manager does not have enough information about the returns in future time periods to justify having the distributions vary with time. We define $\hat{S} = \sigma c$ for notational convenience.

3.1. One-stage and Two-stage problems

Assume $\bar{S} - \hat{S} < r_f < \bar{S}$ to avoid the trivial case where the manager invests everything in either the stock or the bond.

3.1.1. One-stage problem

When there is only one time period left, the decision maker solves:

$$\omega_{\tau 1} = \max_{x, y} \min_{\epsilon_1 \leq \tau, \epsilon_1 \in \{0, 1\}} \left[(\bar{S} - \hat{S} \epsilon_1) x + r_f y \right].$$

$$\text{s.t. } x + y = 1,$$

$$x, y \geq 0.$$

Theorem 3.1 follows immediately from the model setup. Therefore, we state it without proof.

Theorem 3.1 (One-stage problem).

$\tau = 0$: The investor plans for return $\bar{S} > r_f$. Therefore, $x_{01} = 1$ and $\omega_{01} = \bar{S}$.

$\tau = 1$: The investor plans for return $\bar{S} - \hat{S} < r_f$. Therefore, $x_{11} = 0$ and $\omega_{11} = r_f$.

3.1.2. Two-stage problem

The robust optimization problem is:

$$\omega_{\tau 2} = \max_{(x_2, y_2) \in \mathcal{X}} \min_{\epsilon_2 \in \mathcal{E}_2^{\tau}(\epsilon_{-2})} \left[(\bar{S}_2 - \hat{S} \epsilon_2) x_2 + r_f y_2 \right] \cdot \max_{(x_1, y_1) \in \mathcal{X}} \min_{\epsilon_1 \in \mathcal{E}_1^{\tau}(\epsilon_{-1})} \left[(\bar{S}_1 - \hat{S} \epsilon_1) x_1 + r_f y_1 \right]$$

i.e.:

$$\omega_{\tau 2} = \max_{(x_2, y_2) \in \mathcal{X}} \min_{\substack{\epsilon_2 \\ \text{s.t. } \epsilon_2 \leq \tau, \epsilon_2 \in \{0, 1\}}} \left[(\bar{S}_2 - \hat{S} \epsilon_2) x_2 + r_f y_2 \right] \cdot \max_{(x_1, y_1) \in \mathcal{X}} \min_{\substack{\epsilon_1 \\ \text{s.t. } \epsilon_1 \leq \tau - \epsilon_2, \epsilon_1 \in \{0, 1\}}} \left[(\bar{S}_1 - \hat{S} \epsilon_1) x_1 + r_f y_1 \right]$$

In Theorem 3.2, the focus is on the optimal allocation in the first time period (two time periods left), since the strategy in the second time period (one time period left) is given by Theorem 3.1.

Theorem 3.2 (Two-stage problem).

$\tau = 0$: The manager only invests in the stock. Hence, $x_{02} = 1$ and $\omega_{02} = \bar{S}^2$.

$\tau = 1$: The optimal fraction invested in stocks at time 1 and rate of growth of the portfolio are given by:

$$x_{12} = \left[1 + \frac{\bar{S}}{r_f} \cdot \frac{r_f - (\bar{S} - \hat{S})}{\bar{S} - r_f} \right]^{-1}, \quad (3.1)$$

and:

$$\omega_{12} = r_f [(\bar{S} - r_f) x_{12} + r_f]. \quad (3.2)$$

$\tau = 2$: The manager only invests in the bond. Hence, $x_{22} = 0$ and $\omega_{22} = r_f^2$.

Proof. The cases for $\tau = 0$ and $\tau = 2$ are immediate. For $\tau = 1$, we have two possible outcomes: either the stock has low return at time 1 and nominal return at time 2, or nominal return at time 1 and low return at time 2. Injecting the optimal solution at time 2 given by Theorem 3.1, we obtain the following linear programming problem:

$$\begin{aligned} & \max \omega_{12} \\ & \text{s.t. } \omega_{12} \leq r_f [\bar{S} x_2 + r_f y_2], \\ & \omega_{12} \leq \bar{S} [(\bar{S} - \hat{S}) x_2 + r_f y_2], \\ & x_2 + y_2 = 1, \\ & x_2, y_2 \geq 0. \end{aligned}$$

or equivalently:

$$\max_{0 \leq x_2 \leq 1} \min \left\{ r_f [\bar{S} x_2 + r_f (1 - x_2)], \bar{S} [(\bar{S} - \hat{S}) x_2 + r_f (1 - x_2)] \right\}.$$

The slope of the objective changes sign (recall that $\bar{S} - \hat{S} < r_f < \bar{S}$) for x_2 s.t.:

$$r_f [\bar{S} x_2 + r_f (1 - x_2)] = \bar{S} [(\bar{S} - \hat{S}) x_2 + r_f (1 - x_2)]. \quad (3.3)$$

It is straightforward to check that the solution of equation (3.3) falls indeed within $[0, 1]$. Equations (3.1) and (3.2) follow immediately. \square

Remark 3.3. The fraction of wealth invested in the risky asset is a convex, decreasing function of the uncertainty \hat{S} on the stock return, with \hat{S} in $[\bar{S} - r_f, \bar{S}]$. The manager never invests less than $\frac{\bar{S} - r_f}{2\bar{S} - r_f}$ in stocks in the presence of uncertainty. This bound is achieved when $\hat{S} = \bar{S}$. The approach can easily be modified to incorporate an upper bound on the fraction invested in stocks.

3.2. The general case

We now provide a recursive formula for $x_{\tau t}$ and $\omega_{\tau t}$ (with $0 \leq \tau \leq T, 1 \leq t \leq T$), which allows the fund manager to capture risk over large time horizons without having to solve an optimization problem. For instance, the optimal allocation can be computed using a simple spreadsheet for 40 time periods, *e.g.* years, or more without any difficulty, while a stochastic programming approach with two outcomes per year would generate $2^{40} \approx 10^{12}$ scenarios, and hence would raise significant tractability issues.

Before stating the theorem, we observe that increasing the budget of uncertainty decreases terminal wealth, *i.e.*, $\omega_{\tau t} \leq \omega_{\tau-1, t}, \forall \tau, 0 \leq \tau \leq t-1, \forall t, 1 \leq t \leq T$. This is because increasing the budget of uncertainty by 1 while keeping the time horizon constant forces one observation of the stock return to decrease from its nominal value \bar{S} to its worst-case value $\bar{S} - \hat{S}$.

From equation (2.2), we have when the nominal stock price does not vary with time:

$$\omega_{\tau t} = \max_{(x_t, y_t) \in \mathcal{X}} \min \left\{ \omega_{\tau-1, t-1} \left[(\bar{S} - \hat{S}) x_t + r_f y_t \right], \omega_{\tau, t-1} \left[\bar{S} x_t + r_f y_t \right] \right\}. \quad (3.4)$$

Theorem 3.4 (Optimal allocation and terminal wealth).

(Boundary conditions) For all $t, 1 \leq t \leq T$, we have: $x_{0t} = 1, \omega_{0t} = \bar{S}^t$, and $x_{tt} = 0, \omega_{tt} = r_f^t$.

(Recursive equations) For all $t, 1 \leq t \leq T$ and $\tau, 1 \leq \tau \leq t-1$, the optimal fraction invested in stocks at time 1 and terminal wealth are given by:

$$x_{\tau t} = \min \left\{ \frac{(\omega_{\tau-1, t-1} - \omega_{\tau, t-1}) r_f}{\omega_{\tau, t-1} (\bar{S} - r_f) + \omega_{\tau-1, t-1} (r_f - (\bar{S} - \hat{S}))}, 1 \right\}, \quad (3.5)$$

and:

$$\omega_{\tau t} = \omega_{\tau-1, t-1} [r_f - (r_f - (\bar{S} - \hat{S})) x_{\tau t}]. \quad (3.6)$$

Proof. The boundary conditions are immediate, because for $\tau = 0$ the fund manager solves the nominal problem, for which the stock outperforms the bond at each time period ($\bar{S} > r_f$), and for $\tau = t$ the fund manager solves the worst-case problem, for which the bond outperforms the stock at each time period ($r_f > \bar{S} - \hat{S}$). To derive the recursive equations (3.5) and (3.6), we reformulate the portfolio management problem at t and τ given in equation (3.4) as:

$$\max_{0 \leq x_t \leq 1} \min \left\{ \omega_{\tau-1, t-1} [(\bar{S} - \hat{S}) x_t + r_f (1 - x_t)], \omega_{\tau, t-1} [\bar{S} x_t + r_f (1 - x_t)] \right\}. \quad (3.7)$$

The unconstrained maximum of Problem (3.7) is achieved at $(\bar{S} - \hat{S} < r_f < \bar{S})$:

$$x_{\tau t} = \frac{(\omega_{\tau-1, t-1} - \omega_{\tau, t-1}) r_f}{\omega_{\tau, t-1} (\bar{S} - r_f) + \omega_{\tau-1, t-1} (r_f - (\bar{S} - \hat{S}))}. \quad (3.8)$$

Since $\omega_{\tau, t-1} \leq \omega_{\tau-1, t-1}, x_{\tau t} \geq 0$ for all t and τ , which yields equation (3.5) as the optimal solution to the constrained problem. Equation (3.6) follows by injecting equation (3.8) into the objective of problem (3.7). \square

Remark 3.5.

- The upper bound of 1 in equation (3.5) can be replaced by any upper bound between 0 and 1 as specified by the pension fund manager.
- The fund manager will invest everything in the stock if and only if $(\bar{S} - \hat{S}) \omega_{\tau-1, t-1} > \bar{S} \omega_{\tau, t-1}$. This corresponds to a case where the manager has a higher terminal wealth if he invests everything in the stock at time 1 and the stock underperforms, than if the stock performs as expected and he keeps the budget of uncertainty for later.

- If $x_{\tau t} < 1$, equation (3.8) can be reformulated as:

$$x_{\tau t} = \left[1 - \frac{\bar{S}}{r_f} + \frac{\hat{S}}{r_f} \left(1 - \frac{\omega_{\tau, t-1}}{\omega_{\tau-1, t-1}} \right)^{-1} \right]^{-1}. \quad (3.9)$$

Therefore, the entire impact of the time horizon and of the investor's risk preferences is captured through a *single parameter*, the ratio $\frac{\omega_{\tau, t-1}}{\omega_{\tau-1, t-1}}$, which measures the change in the optimal rate of growth when the budget of uncertainty is increased by 1 unit, with the time horizon kept constant.

3.3. Numerical experiments

Here, we illustrate the proposed methodology on several examples and compare the performance of the robust approach with that of retirement funds, using numerical data on the TIAA-CREF website <http://www.tiaa-cref.org/performance/retirement/index.html>.

The purpose of these numerical experiments is to gain insights into the benefits and drawbacks of the proposed approach compared to the benchmark, and to provide guidelines to select the budget of uncertainty. While the budget of uncertainty can also be provided by the pension fund participant to reflect the number of time periods he wants to protect himself against, these guidelines are particularly useful from the fund manager's perspective.

We will make the following observations from our numerical experiments.

- The portfolio resulting from the proposed model, in the specific way described in Section 3.3.2 where the budget of uncertainty is selected according to a linear rule in the remaining time horizon, exhibits higher Sharpe ratio than the industry benchmark.
- The portfolio resulting from the proposed model often, but not always, performs better than the industry benchmark according to other metrics such as mean, median, certain quantiles and/or standard deviation.
- Investors with a low degree of risk aversion (as defined precisely in Section 3.3.2) have portfolios that have both higher median and lower bottom decile (higher downside risk) than the industry benchmark, while investors with a high degree of risk aversion have portfolios that have little volatility, showing smaller median and higher bottom decile.
- The range of the portfolio return for the highest degree of risk aversion decreases as the time horizon increases, leading to a zero-volatility portfolio when $T = 35$.
- The degree of risk aversion for which the proposed portfolio switches from having lower bottom decile and higher median return than the industry benchmark to higher bottom decile and lower median return decreases as the time horizon increases.

3.3.1. Setup and Allocations

The composition of the TIAA-CREF Lifecycle funds is summarized in Table 1. YTD refers to Year-to-Date. Time horizon is in years.

We take $W_0 = 1$, $r_f = 1.05$, $\bar{S} = 1.1$ and compute the tables of (i) the optimal fractions invested in the stock and of (ii) the increase in wealth as functions of the time horizon T and the degree of risk aversion τ for several values of \hat{S} . These tables allow us to find an approximate value of τ for each fund to match the stock and bond allocation in the benchmark, shown in Table 1. To avoid repetition of similar results, we only present here the tables for $\hat{S} = 0.11$, $\hat{S} = 0.22$, and $\hat{S} = 0.6$ (Tabs. 2, 3, and 4, respectively). Also, the fractions were computed for each year; however, to keep the table relatively small, we only show the results for T increasing in five-year increments. Note that we always have $\tau \leq T$ by definition of τ . We indicate in bold the allocation that matches most closely the one used in the TIAA-CREF retirement funds; if the Lifetime fund allocation falls about halfway between two numbers in the table, we highlight those two numbers.

3.3.2. Linear rule

In this section we address how to select the budget of uncertainty as a function of the time horizon. Having such a rule allows for a consistent way of adjusting the budget as time to retirement draws near. The two

TABLE 1. Benchmark: Allocations in the TIAA-CREF funds.

Time horizon	YTD	Bonds	Stocks
5	8.4%	54.7%	45.3%
10	9.4%	46.1 %	53.9 %
15	10.3%	41.2%	58.8%
20	10.8%	36.1%	63.9%
25	11.7%	31.2%	68.8 %
30	12.5%	26.2%	73.8%
35	13.1%	21.2%	78.8%

TABLE 2. Fraction invested in the stock for $\hat{S} = 0.11$.

$\tau \downarrow / T \rightarrow$	5	10	15	20	25	30	35
0	100%	100%	100%	100%	100%	100%	100%
1	89.9%	99.4%	99.9%	99.9%	100%	100%	100%
2	59.4%	95.3%	99.6%	99.9%	100%	100%	100%
3	23.6%	83.1%	97.7%	99.8%	99.9%	100%	100%
4	4.1%	61.6%	92.1%	98.8%	99.9%	99.9%	100%
5	0%	36.4%	80.7%	96.2%	99.4%	99.9%	100%
6	—	16.1%	63.4%	90.2%	98.1%	99.7%	99.9%
7	—	4.9%	43.2%	79.7%	94.9%	99.0%	99.8%
8	—	0.9%	24.8%	65.0%	88.9%	97.3%	99.5%
9	—	0%	11.6%	47.8%	79.3%	93.9%	98.6%
10	—	0%	4.3%	31.2%	66.3%	88.0%	96.7%
11	—	—	1.2%	17.7%	51.3%	79.2%	93.2%
12	—	—	0.2%	8.5%	36.2%	67.5%	87.5%
13	—	—	0%	3.4%	23.0%	54.1%	79.2%
14	—	—	0%	1.1%	12.9%	40.2%	68.6%
15	—	—	0%	0.3%	6.4%	27.5%	56.4%
16	—	—	—	0%	2.7%	17.2%	43.6%
17	—	—	—	0%	1.0 %	9.6%	31.5%
18	—	—	—	0%	0.3%	4.8%	21.1%
19	—	—	—	0%	0%	2.1%	13.0%
20	—	—	—	0%	0%	0.8%	7.3%
21	—	—	—	—	0%	0.3%	3.7%
22	—	—	—	—	0%	0%	1.7%
23	—	—	—	—	0%	0%	0.7%
24	—	—	—	—	0%	0%	0.2%
25	—	—	—	—	0%	0%	0%
26–30	—	—	—	—	—	0%	0%
31–35	—	—	—	—	—	—	0%

main choices, as discussed in Dziecichowicz *et al.* [14], are to select either a budget linear in the remaining time horizon or a budget concave in the remaining time horizon. The advantage of a budget linear in the remaining time horizon, which we call the *linear rule*, is that there is only one parameter to select, namely the coefficient of proportionality. A budget concave in the remaining time horizon captures that it is more likely that all uncertain returns will be equal to their worst case when there are only a few time periods left, rather than when there are many time periods left; however, the concave structure must then be explicitly defined. Here, we pick the structure that matches most closely that of real-life pension funds.

TABLE 3. Fraction invested in the stock for $\widehat{S} = 0.22$.

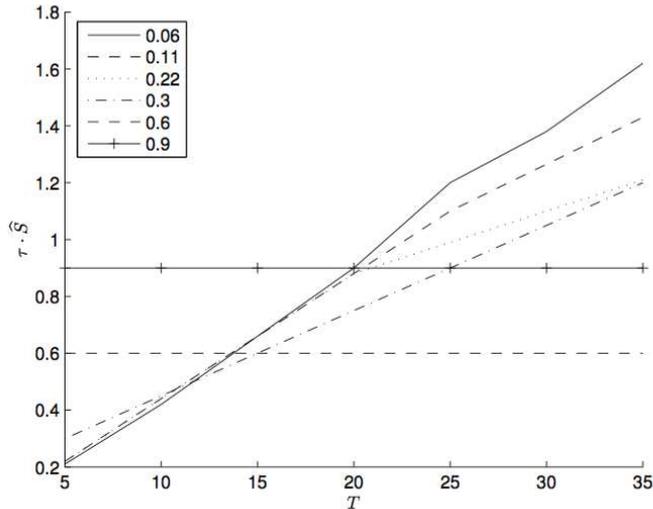
$\tau \downarrow / T \rightarrow$	5	10	15	20	25	30	35
0	100%	100%	100%	100%	100%	100%	100%
1	61.3%	87.2%	95.6%	98.5%	99.5%	99.8%	99.9%
2	21.0%	59.4%	81.4%	91.9%	96.6%	98.6%	99.4%
3	3.7%	30.4%	58.8%	77.9%	88.9%	94.7%	97.6%
4	0.3%	11.4%	35.2%	58.5%	75.6%	86.6%	93.0%
5	0%	3.1%	17.2%	38.3%	58.4%	74.0%	84.7%
6	—	0.6%	6.8%	21.5%	40.5%	58.4%	72.7%
7	—	0.1%	2.1%	10.3%	24.9%	42.2%	58.4%
8	—	0%	0.5%	4.2%	13.5%	27.6%	43.5%
9	—	0%	0.1%	1.4%	6.4%	16.3%	29.9%
10	—	0%	0%	0.4%	2.6%	8.6%	18.8%
11	—	—	0%	0.1%	0.9%	4.0%	10.7%
12	—	—	0%	0%	0.3%	1.7%	5.5%
13	—	—	0%	0%	0.1%	0.6%	2.6%
14	—	—	0%	0%	0%	0.2%	1.1%
15	—	—	0%	0%	0%	0.1%	0.4%
16	—	—	—	0%	0%	0%	0.1%
17–20	—	—	—	0%	0%	0%	0%
21–25	—	—	—	—	0%	0%	0%
26–30	—	—	—	—	—	0%	0%
31–35	—	—	—	—	—	—	0%

TABLE 4. Fraction invested in the stock for $\widehat{S} = 0.6$.

$\tau \downarrow / T \rightarrow$	5	10	15	20	25	30	35
0	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
1	27.0%	48.0%	61.6%	70.9%	77.5%	82.4%	86.1%
2	3.4%	14.7%	27.2%	38.7%	48.6%	56.9%	63.7%
3	0.2%	2.9%	8.8%	16.6%	25.1%	33.3%	41.1%
4	0.0%	0.4%	2.1%	5.6%	10.6%	16.5%	23.0%
5	0.0%	0.0%	0.4%	1.5%	3.6%	6.9%	11.1%
6	—	0.0%	0.1%	0.3%	1.0%	2.4%	4.5%
7	—	0.0%	0.0%	0.1%	0.2%	0.7%	1.6%
8	—	0.0%	0.0%	0.0%	0.0%	0.2%	0.5%
9	—	0.0%	0.0%	0.0%	0.0%	0.0%	0.1%
10	—	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
11–15	—	—	0.0%	0.0%	0.0%	0.0%	0.0%
16–20	—	—	—	0.0%	0.0%	0.0%	0.0%
21–25	—	—	—	—	0.0%	0.0%	0.0%
26–30	—	—	—	—	—	0.0%	0.0%
31–35	—	—	—	—	—	—	0.0%

Figure 1 suggests that $\tau \cdot \widehat{S}$, computed using (the bold numbers in) Tables 2–4 above as proxy for the real-life pension funds, is approximately linear in T . Using this observation, let the rule for selecting τ be:

$$\tau = \min\left(\frac{\pi}{\widehat{S}}, 1\right) T, \tag{3.10}$$

FIGURE 1. Evolution of $\tau \cdot \widehat{S}$ as a function of T .

where $\pi \in [0, 1]$ measures risk aversion. Then, we invest the following fraction of portfolio in the stock:

$$x_{\tau T} = x_{\lfloor \tau \rfloor T} + (\tau - \lfloor \tau \rfloor)(x_{\lceil \tau \rceil T} - x_{\lfloor \tau \rfloor T}), \quad (3.11)$$

i.e., the fraction allocated is a convex combination of those for $\lfloor \tau \rfloor$ and $\lceil \tau \rceil$.

3.3.3. Comparison between linear rule and industry benchmark

For each value of \widehat{S} , we run 10 000 simulations for i.i.d stock returns obeying a Normal distribution with mean \widehat{S} and standard deviation $\widehat{S}/2$. (Our observations hold for other distributions as well; the corresponding results have been omitted as they are similar to those in the Normal case). For each simulation run, we track the return that the linear and industry rules would yield over a one-year span for each value of T . We only track the one-period return here because we have assumed that the range forecasts for the returns over time are the same, so the relative performance of each rule is simply amplified as the time horizon increases.

Figure 2 shows how the first five deciles (from bottom to top) of terminal wealth change across π for select values of T when $\widehat{S} = 0.22$. The curves show the performance of the linear rule, while the dotted lines show the performance of the industry policy. Notice that as T gets larger, the value of π for which the two strategies cross over gets smaller. (After performing simulations for all specified values of \widehat{S} , we find that the performances of the decision-maker and industry cross over for $\pi \in [0.02, 0.06]$). This means that the policy obtained for a given π might be outperformed by the industry rule when the time horizon is short, but will outperform the industry rule when the time horizon becomes longer. For low values of π (low risk aversion), the curves obtained for the linear rule are more spread out, indicating a higher median (return) but also a higher first decile (risk). As π increases, the curves converge; the bigger the time horizon, the earlier the convergence. This means that the allocation generated by the robust approach when π is large enough (the decision-maker is risk-averse enough) has no volatility. Its median is lower than the median of the industry rule (lower return) but its first decile is higher (lower risk).

Table 5 shows the fraction invested in the stock for $\pi = 0.04$ and T increasing by increments of 5. It also shows the industry ratio for comparison purposes. The linear rule is more aggressive than the industry rule for all values of \widehat{S} given shorter time horizons (specifically, $T = 5, 10$ and 15). For example, in the extreme case when $T = 5$ and $\widehat{S} = 0.9$, the linear rule suggests investing 81.91% of wealth in stock (which corresponds to a very small τ) as opposed to 45.3% suggested by the industry rule. For the 10 000 runs with these parameters

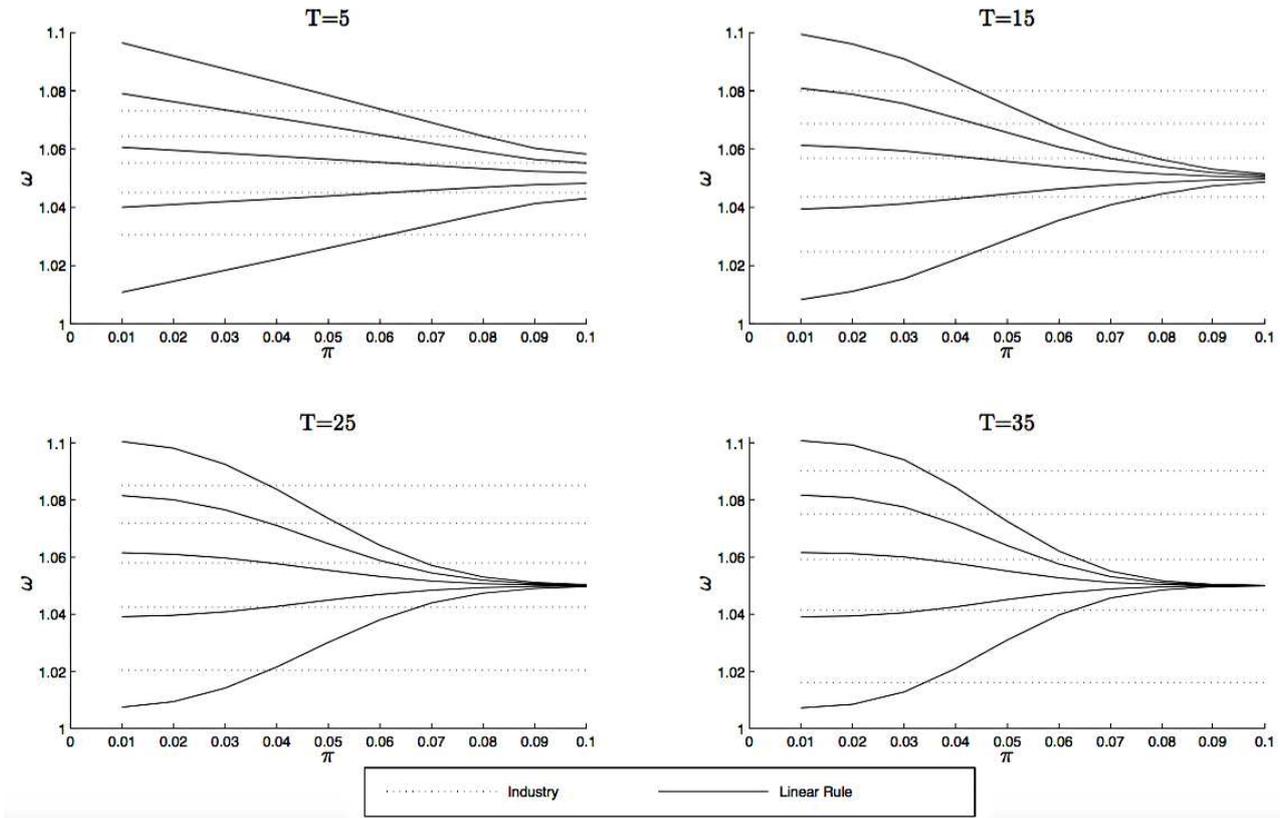


FIGURE 2. First five deciles of terminal wealth across π , $T = \{5, 15, 25, 35\}$, $\hat{S} = 0.22$.

TABLE 5. Fraction invested in the stocks, $\pi = 0.04$.

T	industry	\hat{S}					
		0.06	0.11	0.22	0.3	0.6	0.9
5	45.3	72.95	64.94	64.83	65.76	75.66	81.91
10	53.9	85.33	69.42	64.46	64.30	65.34	70.90
15	58.8	92.07	72.86	64.93	63.47	61.56	63.83
20	63.9	94.52	75.69	65.57	62.91	60.16	59.15
25	68.8	96.57	78.10	66.23	63.03	58.24	56.94
30	73.8	97.98	79.97	66.89	63.17	56.86	55.44
35	78.8	98.55	81.46	67.52	63.09	56.19	53.69

set, we find the linear rule to have a mean return of 1.0946 and standard deviation 0.2443. The industry rule has lower mean return 1.0747 but lower standard deviation 0.1351. Also notice that for longer T , there is a gradual shift towards a strategy more conservative than the industry rule for large \hat{S} . Tables for other values of π are similar and omitted here.

TABLE 6. W_T performance comparison for normal distribution, $\pi = 0.04$.

\widehat{S}	Mean (μ)		Stdev (σ)		Sharpe		10 th %		90 th %	
	I	L	I	L	I	L	I	L	I	L
0.06	108.16	109.55	1.39	1.88	2.27	2.42	106.45	107.17	110.00	111.99
0.11	108.17	108.74	2.41	2.75	1.32	1.36	105.18	105.22	111.30	112.26
0.22	108.21	108.34	4.75	4.83	0.68	0.69	102.31	102.14	114.23	114.53
0.3	108.17	108.19	6.47	6.38	0.49	0.50	100.05	99.96	116.35	116.32
0.6	108.22	108.13	12.87	12.47	0.25	0.25	92.14	92.32	124.56	124.01
0.9	108.14	108.15	19.36	19.17	0.16	0.16	83.78	83.94	132.47	132.40

Table 6 summarizes the performance of the allocation used by the industry rule (I) as well as those generated by the linear rule (L), when the stock obeys a Normal distribution. All numbers are given in dollars, where the initial budget was \$100.

The linear rule generally performs better than the industry rule. For $\widehat{S} = 0.06$ and 0.11, the linear rule results in higher average wealth (μ), lower downside risk (10th%), and higher upside potential (90th%), at the expense of having higher standard deviation. For example, when $\widehat{S} = 0.06$, the fund mean rate of return of the linear rule increased by $(0.0955 - 0.0816)/0.0816 = 0.1703$ or 17.03% while standard deviation increased by $(1.88 - 1.39)/1.39 = 0.3525$ or 35.25%.

For all \widehat{S} values, the linear rule has a higher Sharpe ratio ($= (\mu - r_f)/\sigma$), representing risk-adjusted return.

4. THE MODEL WITH AUTOREGRESSIVE DYNAMICS

We now extend the approach above to time-varying stock prices.

4.1. Model

In this section, we consider the problem of optimizing a portfolio with stock returns that exhibit AR(1) dynamics. Recall from equation (2.1) that:

$$S_t = (1 - \varphi)\mu + \varphi S_{t+1} + \sigma \varepsilon_t, \quad (4.1)$$

and our investor assumes that the worst-case stock return occurs when $\varepsilon_t = -c$. We refer to \overline{S}_t , defined by $\overline{S}_t = (1 - \varphi)\mu + \varphi S_{t+1}$ as the nominal return and $\overline{S}_t - \widehat{S}$ as the worst-case return. The optimal $\omega_{\tau t}(S_{t+1})$ is found by solving:

$$\max_{0 \leq x_t \leq 1} \min \left\{ \omega_{\tau-1, t-1}(\overline{S}_t - \widehat{S})[(\overline{S}_t - \widehat{S})x_t + r_f(1 - x_t)], \omega_{\tau, t-1}(\overline{S}_t)[\overline{S}_t x_t + r_f(1 - x_t)] \right\}, \quad (4.2)$$

which follows directly from the setup in Section 3.2 above, except that future returns $\omega_{\tau, t-1}$ and $\omega_{\tau-1, t-1}$ are a function of \overline{S}_t and $\overline{S}_t - \widehat{S}$ respectively, and $\omega_{\tau t}$ is indexed by S_{t+1} .

Theorem 4.1 (Optimal allocation and terminal wealth).

(Boundary conditions for $\tau = 0$ and $\tau = t$) Let $\omega_{00}(S_{t+1}) = 1$. For $t, 1 \leq t \leq T - 1$:

- If $\overline{S}_t > r_f$: $x_{0t}(S_{t+1}) = 1$ and $\omega_{0t}(S_{t+1}) = \overline{S}_t \cdot \omega_{0, t-1}(\overline{S}_t)$. Otherwise $x_{0t}(S_{t+1}) = 0$, $\omega_{0t}(S_{t+1}) = r_f \cdot \omega_{0, t-1}(\overline{S}_t)$.
- If $\overline{S}_t - \widehat{S} > r_f$: $x_{tt}(S_{t+1}) = 1$ and $\omega_{tt}(S_{t+1}) = (\overline{S}_t - \widehat{S}) \cdot \omega_{t-1, t-1}(\overline{S}_t - \widehat{S})$. Otherwise $x_{tt}(S_{t+1}) = 0$ and $\omega_{tt}(S_{t+1}) = r_f \cdot \omega_{t-1, t-1}(\overline{S}_t - \widehat{S})$.

(Boundary conditions for τ , $1 \leq \tau \leq t-1$) For t , $1 \leq t \leq T-1$:

- If $\bar{S}_t - \hat{S} > r_f$, $x_{\tau t}(S_{t+1}) = 1$ and $\omega_{\tau t}(S_{t+1}) = (\bar{S}_t - \hat{S}) \cdot \omega_{\tau-1, t-1}(\bar{S}_t - \hat{S})$.
- If $\bar{S}_t < r_f$, $x_{\tau t}(S_{t+1}) = 0$ and $\omega_{\tau t}(S_{t+1}) = r_f \cdot \omega_{\tau-1, t-1}(\bar{S}_t)$.

(Recursive equations) For all t , $1 \leq t \leq T-1$ and τ , $1 \leq \tau \leq t-1$, if $\bar{S}_t - \hat{S} \leq r_f \leq \bar{S}_t$, the optimal fraction invested in stocks at time t and terminal wealth are:

$$x_{\tau t}(S_{t+1}) = \min \left\{ \frac{[\omega_{\tau-1, t-1}(\bar{S}_t - \hat{S}) - \omega_{\tau, t-1}(\bar{S}_t)]r_f}{\omega_{\tau, t-1}(\bar{S}_t)(\bar{S}_t - r_f) + \omega_{\tau-1, t-1}(\bar{S}_t - \hat{S})(r_f - (\bar{S}_t - \hat{S}))}, 1 \right\},$$

and:

$$\omega_{\tau t}(S_{t+1}) = [r_f - (r_f - (\bar{S}_t - \hat{S}))x_{\tau t}] \cdot \omega_{\tau-1, t-1}(\bar{S}_t - \hat{S}).$$

Proof. This is an immediate extension of the proof for Theorem 3.4 to the case with autoregressive dynamics. \square

Remark 4.2. We can no longer find the optimal $x_{\tau t}(S_{t+1})$ in closed form because the optimal rates of return, e.g., $\omega_{\tau, t-1}(\bar{S}_t)$, are no longer available in closed form.

We implement an approximation technique over a grid representing state space (stock price) versus time to keep the recursion tractable. The time horizon is divided into T equally spaced time intervals of length 1, while the state space is divided into M equally spaced subintervals of user defined length, *step*. The upper and lower bounds of the grid are S^{\max} and S^{\min} , respectively.

With S_{t+1} (the stock return when $t+1$ time periods remain) given, the *nominal* stock return with t periods remaining is given by:

$$S_t^0 = \bar{S}_t \approx \left[\frac{(1-\varphi)\mu + \varphi S_{t+1}}{\text{step}} \right] \cdot \text{step}. \quad (4.3)$$

Similarly, the *worst-case* stock return, achieved when the manager uses one unit of his budget at time t , is given by:

$$S_t^1 = \bar{S}_t - \hat{S} \approx \left[\frac{(1-\varphi)\mu - \hat{S} + \varphi S_{t+1}}{\text{step}} \right] \cdot \text{step}. \quad (4.4)$$

Thus, equation (4.2) becomes:

$$\max_{0 \leq x_t \leq 1} \min \left\{ \omega_{\tau-1, t-1}(S_t^1)[(\bar{S}_t - \hat{S})x_t + r_f(1-x_t)], \omega_{\tau, t-1}(S_t^0)[\bar{S}_t x_t + r_f(1-x_t)] \right\}.$$

4.2. Example

In these numerical experiments, we focus on comparing optimal allocation and portfolio return in the constant-expected-return case and in the AR(1) case, referred to as constant and time-varying cases for short. We will observe that:

- For a rather low value of the expected return \bar{S} , the fund participant invests more, respectively less, in the stock under the assumption of AR(1) dynamics than under the constant-expected-return case when the degree of risk aversion is low, respectively high. Optimal portfolio return is higher in the AR(1) case.
- For a rather high value of the expected return \bar{S} , the fund participant invests significantly more in the stock under the constant-expected-return assumption, for all values of risk aversion. In fact, in the constant case, he invests solely in the stock for most values of risk aversion. Optimal portfolio return is higher in the constant case; however, the lack of the diversification in the portfolio in the constant case is concerning.

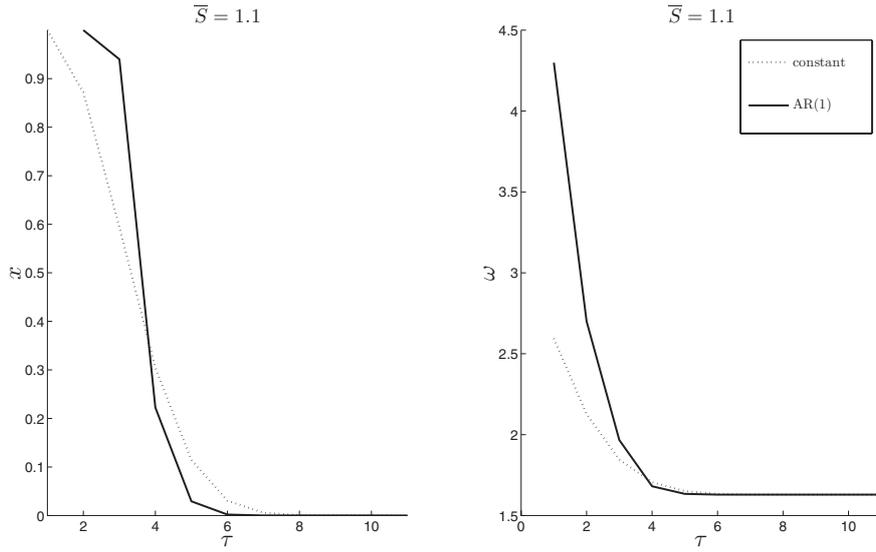


FIGURE 3. Comparison of optimal $x_{\tau T}(\bar{S})$ and $\omega_{\tau T}(\bar{S})$ for constant and time-varying cases when $\bar{S} = 1.1$.

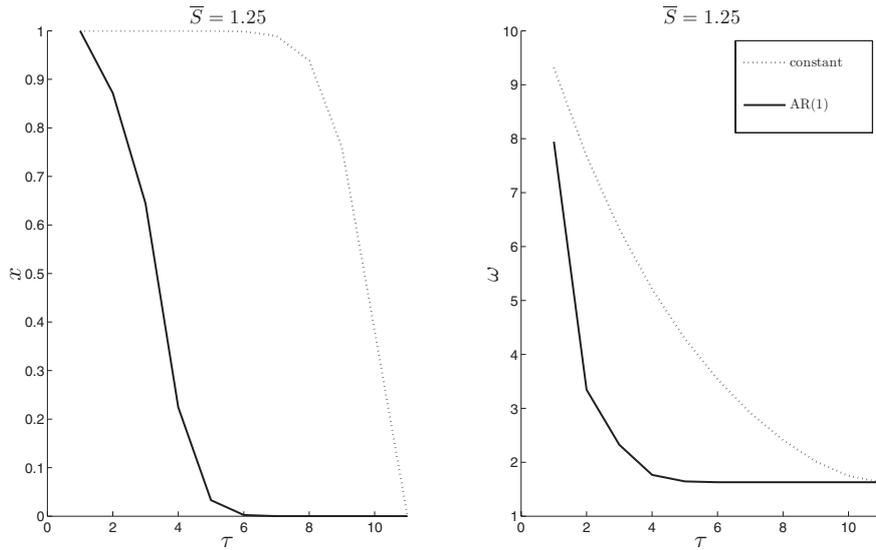


FIGURE 4. Comparison of optimal $x_{\tau T}(\bar{S})$ and $\omega_{\tau T}(\bar{S})$ for constant and time-varying cases when $\bar{S} = 1.25$.

First, we set an appropriate interval for our state space, $[S^{\min}, S^{\max}]$ for all t , where $S^{\min} = r_f$ and $S^{\max} = r_f + \hat{S}$. We address boundary issues on the grid by setting all $S_t < r_f$ equal to r_f (the lowest point on the grid) and set all $S_t > r_f + \sigma c$ equal to $r_f + \sigma c$ (the highest point).

Let $T = 10$, $\mu = 1.12$, $\varphi = 0.9$, $\sigma = 0.11$, $step = 0.001$, $c = 2$, $r_f = 1.05$, $S^{\min} = r_f$, and $S^{\max} = r_f + \sigma c = 1.27$. Recall that our results will vary with each possible starting state on the grid. For the sake of brevity, we provide results for a current value of the return near S^{\min} and a value near S^{\max} , namely, 1.1 in the first case and 1.25 in the second.

Figures 3 and 4 show the optimal values of $x_{\tau T}(S_{t+1})$ and $\omega_{\tau T}(S_{t+1})$ as a function of the budget τ in the constant and time-varying cases. They also illustrate how the initial stock price will affect the anticipated total wealth at the end of the time horizon. For $\bar{S} = 1.1$, the investor has higher expected wealth for moderate τ when he picks his allocation assuming stock returns obey an AR(1) process, despite an allocation similar to that in the constant return case. For higher values of τ ($\tau \geq 4$), the expected wealths in both cases are equal. For $\bar{S} = 1.25$, the investor who assumes constant stock returns returns a higher rate, by investing far more into the stock, which in turn yields a much riskier position. This suggests that the time-varying approach is desirable from a risk management perspective.

4.3. Extension to a tracking penalty

4.3.1. Setup

In practice, a fund manager might want to monitor his fund's performance relative to some benchmark threshold as time passes. The fund manager is subject to a unit penalty k when the realized wealth falls below a threshold θ_t at any time t , because he is then at higher risk of falling short of reaching his objective at the end of the investment horizon.

We introduce the quantities $\omega_{\tau t}^0(S_{t+1})$ and $\omega_{\tau t}^1(S_{t+1})$ to model the effect of a penalty resulting from the decision to wait before using one unit of uncertainty and the decision to use one unit now, respectively. The objective is:

$$\max_{0 \leq x_t \leq 1} \min \left\{ \omega_{\tau t}^1(S_{t+1}) - k \cdot \max \{0, \theta_t - \omega_{\tau t}^1(S_{t+1})\}, \right. \\ \left. \omega_{\tau t}^0(S_{t+1}) - k \cdot \max \{0, \theta_t - \omega_{\tau t}^0(S_{t+1})\} \right\}. \quad (4.5)$$

Hence, the problem can be written as:

$$\max \min \left\{ \omega_{\tau t}^1(S_{t+1}), (1+k)\omega_{\tau t}^1(S_{t+1}) - k\theta_t, \omega_{\tau t}^0(S_{t+1}), (1+k)\omega_{\tau t}^0(S_{t+1}) - k\theta_t \right\} \\ \text{s.t. } \omega_{\tau t}^1(S_{t+1}) = \omega_{\tau-1,t-1}(S_t^1) \left[(\bar{S}_t - \hat{S})x_t(S_{t+1}) + r_f y_t(S_{t+1}) \right], \\ \omega_{\tau t}^0(S_{t+1}) = \omega_{\tau,t-1}(S_t^0) \left[\bar{S}_t x_t(S_{t+1}) + r_f y_t(S_{t+1}) \right], \\ x_t(S_{t+1}) + y_t(S_{t+1}) = 1, \\ x_t(S_{t+1}), y_t(S_{t+1}) \geq 0,$$

where S_t^0 and S_t^1 are defined in equations (4.3) and (4.4).

This is a piecewise linear, convex problem because the rates of return $\omega_{\tau t}^0$ and $\omega_{\tau t}^1$ are linear in the allocation $x_{\tau t}$. Note that the case without a tracking penalty can be incorporated in this model by setting $k = 0$. In this framework, the objective remains piecewise linear in the decision variables but has now four pieces. The first two pieces have a slope proportional to $\bar{S}_t - \hat{S} - r_f$ and the other two have a slope proportional to $\bar{S}_t - r_f$.

If $\bar{S}_t \leq r_f$, $x_t(S_{t+1}) = 0$, and if $\bar{S}_t \geq r_f + \hat{S}$, $x_t(S_{t+1}) = 1$. If $r_f < \bar{S}_t < r_f + \hat{S}$, the optimal solution will not be at the bounds of the interval. Specifically, it will fall within one of the four cases below.

Case 1. $\omega_{\tau t}^1(S_{t+1}) = \omega_{\tau t}^0(S_{t+1})$ at optimality (which is equivalent to the problem with no penalty): The optimal $x_{\tau t}(S_{t+1})$ is given in Theorem 4.1. In this case, the stock price falls in a range such that the overall return to the investor will not fall below the threshold, both in the case where he uses a unit of uncertainty and in the case where he does not. It follows that the optimal rate of return is given by $\omega_{\tau t}(S_{t+1})$, where $\omega_{\tau t}(S_{t+1})$ is also given by Theorem 4.1.

Case 2. $(1+k)\omega_{\tau t}^1(S_{t+1}) - k\theta_t = \omega_{\tau t}^0(S_{t+1})$ at optimality: the overall rate of return (over the whole time horizon) falls below the threshold when the decision-maker uses a unit of uncertainty to protect the portfolio, but not when he does not. The optimal $x_{\tau t}(S_{t+1})$ is given by:

$$\max \left\{ 0, \min \left\{ \frac{((1+k)\omega_{\tau-1,t-1}(S_t^1) - \omega_{\tau,t-1}(S_t^0))r_f - k\theta_t}{\omega_{\tau,t-1}(S_t^0)[\bar{S}_t - r_f] + (1+k)\omega_{\tau-1,t-1}(S_t^1)[r_f - (\bar{S}_t - \hat{S})]}, 1 \right\} \right\},$$

and $\omega_{\tau t}(S_{t+1}) = \omega_{\tau t}^0(S_{t+1}) = \omega_{\tau, t-1}(S_t^0)[r_f - (r_f - \bar{S}_t)x_{\tau t}(S_{t+1})]$.

Case 3. $\omega_{\tau t}^1(S_{t+1}) = (1+k)\omega_{\tau t}^0(S_{t+1}) - k\theta_t$: the overall rate of return (over the whole time horizon) falls below the threshold when the decision-maker does not use a unit of uncertainty to protect the portfolio, but not when he does. (This can arise when the decision of not using a unit of budget uncertainty is suboptimal, resulting in lower rates of return). The optimal $x_{\tau t}(S_{t+1})$ is given by:

$$\max \left\{ 0, \min \left\{ \frac{(\omega_{\tau-1, t-1}(S_t^1) - (1+k)\omega_{\tau, t-1}(S_t^0))r_f + k\theta_t}{(1+k)\omega_{\tau, t-1}(S_t^0)[\bar{S}_t - r_f] + \omega_{\tau-1, t-1}(S_t^1)[r_f - (\bar{S}_t - \hat{S})]}, 1 \right\} \right\},$$

and $\omega_{\tau t}(S_{t+1}) = \omega_{\tau t}^1(S_{t+1}) = \omega_{\tau-1, t-1}(S_t^1)[r_f - (r_f - (\bar{S}_{t-1} - \hat{S}))x_{\tau t}(S_{t+1})]$.

Case 4. $(1+k)\omega_{\tau t}^1(S_{t+1}) - k\theta_t = (1+k)\omega_{\tau t}^0(S_{t+1}) - k\theta_t$: the penalty is computed in both cases. In this case, finding the optimal $x_{\tau t}(S_{t+1})$ is equivalent to Case 1, and $\omega_{\tau t}(S_{t+1}) = (1+k)x_{\tau t}(S_{t+1}) - k\theta_t$.

Figures 5 and 6 illustrate how $x_{\tau T}(S_{t+1})$ changes with state $i \in [0, \dots, S^{\max}/step]$ for various τ and k values.

The x-axis represents state i , such that $\bar{S}_T^{(i)} = r_f + i \cdot step$. We observe the following.

For $\tau = 0$, $x_{\tau T}(S_{t+1}) = 1$, $\forall i$ and for $\tau = 10 = T$, $x_{\tau T}(S_{t+1}) = 0$, $\forall i$.

For $\tau = 1$ (low budget of uncertainty), the investor assumes little uncertainty in the stock price over the time horizon, and is willing to hold an aggressive position in the stock. At $k = 0$, the investor is willing to invest everything in stock up to some $\bar{S}_T^{(i)}$. Beyond this $\bar{S}_T^{(i)}$, the mean reversion to μ of the autoregressive process encourages the investor to be more conservative. This, coupled with our approach of restricting the maximum possible state space on the grid of stock prices explain the gradual decrease in allocation as $\bar{S}_T^{(i)}$ continues to increase. As k increases, the investor must change his strategy to balance the tradeoff between the risk of incurring a penalty for low returns (penalty risk), and investing in more stock at the risk of a downward turn in stock price resulting in a bigger loss than a more conservative competitor (allocation risk).

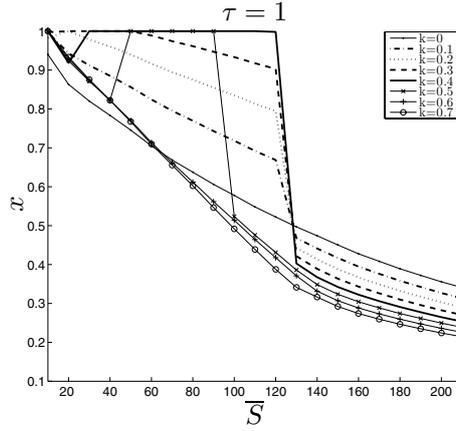
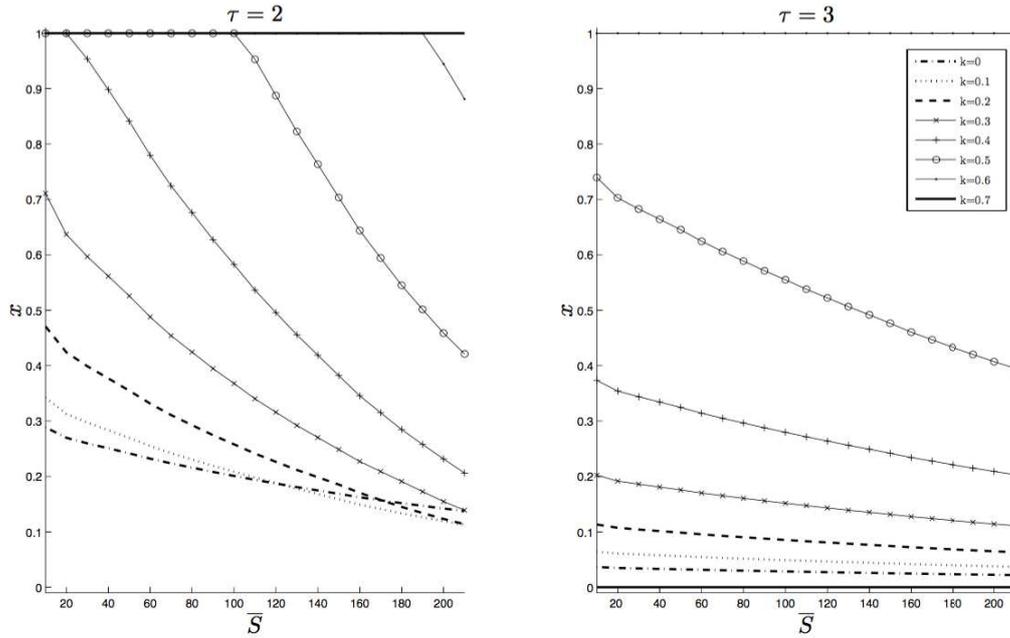
For low values of k , the investor is more willing to take on penalty risk than allocation risk because the penalty amount is small. Therefore, he invests conservatively to avoid the high chance of low stock return (this is illustrated in the leftmost values for $\bar{S}_T^{(i)}$ in Fig. 5 where the curves initially decrease). As $\bar{S}_T^{(i)}$ increases, however, the investor finds it best to take his chances investing completely in the stock because the chance of low stock return and the chance of penalty is small (this explains the jumps up to $x = 1$). Then, as $\bar{S}_T^{(i)}$ continues to increase beyond μ , the investor becomes more conservative because of the autoregressive property assumed to hold (this explains the jumps down). For high values of k , the investor is again more willing to take on allocation risk to outweigh penalty risk. In other words, he invests more aggressively with the hope that the higher returns are enough to raise his performance beyond the penalty threshold. Eventually, for high enough values of k , the penalty risk is too high to be offset by taking on allocation risk so the investor stays conservative in his strategy (and thus there are no jumps in Fig. 5 for large k).

For τ , $2 \leq \tau \leq 5$, the investor finds it optimal to increase his allocation in stock as k increases while allocation risk is smaller than penalty risk. Beyond some k value, the penalty risk becomes too high and the investor only invests in the bond. Figure 6 shows this for $\tau = 2$ and $\tau = 3$. For $\tau = 2$, the investor takes on the most allocation risk for $k = 0.7$ (where penalty risk is also highest). However, for $\tau = 3$, the investor invests everything in the bond for $k = 0.7$ because both penalty risk and allocation risk of the stock are too high in this case. For $\tau = 4$ and $\tau = 5$, the same observations hold but the amount invested in stock falls as τ increases.

For τ , $6 \leq \tau \leq T - 1$, the uncertainty in the stock price, and thus, the allocation risk leads the investor to invest everything in the bond for all values of k . The worst-case stock price is too low to motivate the investor to invest in it.

4.3.2. Comparison with and without tracking penalty

In this section, we present a comparison of the tracking penalty and no tracking penalty cases. We make the following observations.


 FIGURE 5. Change in optimal $x_{\tau t}(S_T)$ across $\bar{S}_T^{(i)}$ for $\tau = 1$.

 FIGURE 6. Change in optimal $x_{\tau t}(S_T)$ across $\bar{S}_T^{(i)}$ for $\tau = 2$ and $\tau = 3$.

- The investor who allocates his portfolio under AR(1) assumptions obtains a portfolio with lower left tail risk, measured by the lowest two deciles of portfolio return, both in the cases with and without penalty.
- The difference in performance across deciles widens when the expected return \bar{S} increases.
- The portfolio in the AR(1) case has a higher Sharpe ratio than in the constant case.
- The return distribution in the case with penalty has similar shape to the return distribution in the case without a penalty, but is shifted down.

Let $T = 10$, $\mu = 1.12$, $\varphi = 0.9$, $\sigma = 0.11$, $step = 0.001$, $c = 2$, $r_f = 1.05$, $S^{\min} = r_f$, and $S^{\max} = r_f + \sigma c = 1.27$ as before, and now let $\theta_t = 1.055^t$ and $k = 0.1$. This means that the manager has a performance benchmark of

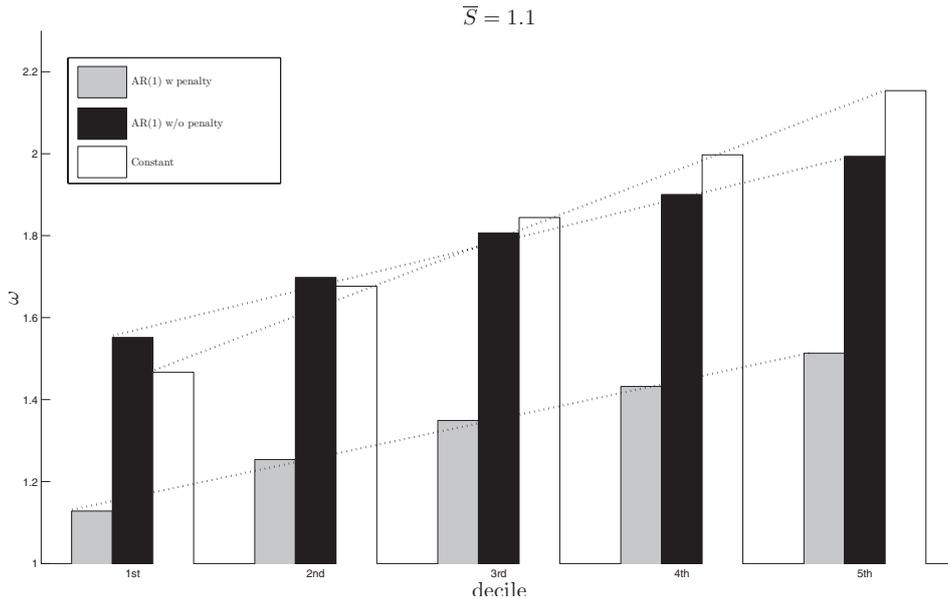


FIGURE 7. Comparison of optimal $\omega_{\tau T}(1.1)$ for constant and time-varying case.

returning $\omega_{\tau t}(S_{t+1}) > \theta_t$ at all times t , and he is penalized $k\%$ of the amount by which he underperforms, as expressed in equation (4.5). Again, let $S_{t+1} = \bar{S} = 1.1$ in the first case and $\bar{S} = 1.25$ in the second.

To test our observations, we simulate 10 000 Normally distributed stock prices and compare the performance of three investors who act under constant, and AR(1) dynamic stock price assumptions both with and without a tracking penalty. First, find τ using the linear rule presented in equation (3.10), then find the corresponding $x_{\tau t}(\bar{S})$, $\forall t$, using equation (3.11). In this example, $\pi = 0.04$ as in the numerical example for the constant stock case. Figure 7 below compares the lowest five deciles of $\omega_{\tau T}(\bar{S})$ for $\bar{S} = 1.1$, which illustrates downside performance risk. We find that the investor who allocates his portfolio under more conservative AR(1) assumptions performs consistently better regarding the low tail risk of the histogram (lowest two deciles). For the higher deciles, the investor who assumes constant stock returns obtains a higher rate. Increasing \bar{S} causes the difference in performance to widen across deciles. For $\bar{S} = 1.25$ (and higher values of \bar{S} in general) the level of risk taken on by the investor who assumes constant returns increases with \bar{S} .

As expected, higher volatility also implies more extreme upside and downside returns (in other words the dotted line connecting the first and fifth decile in Fig. 7 becomes steeper). Comparing excess returns above the riskless rate per unit of risk (Sharpe ratio), the investor who assumes constant returns 121.7% of initial wealth on average over 10 years (or consistently about 8.3% per year) with standard deviation of 65.5%, or a Sharpe ratio of 0.90 ($= \frac{121.7 - (1.05)^{10}}{65.5}$).

The investor who assumes AR(1) dynamics returns 99.9% on average, with standard deviation 36.2%, or a Sharpe ratio of 1.02, which is 13.6% higher than in the constant case. Thus, our approach appeals to a risk-averse investor, and the benefit of using a more complex autoregressive model is justified by its performance. Figure 7 also shows that the investor who faces a penalty will return a similar distribution of wealth as the investor who does not face a penalty, but the actual return is shifted down. Further, we obtain almost identical distributions in the constant return case with penalty and the AR(1) case with penalty, which implies that the summary of performance above also holds for the case with a penalty. (In fact, in this case, the investor using an AR(1) model tends to outperform in the third decile as well in our simulations).

Recall that in an AR(1) setting, the autoregressive dynamics exhibit more variability than its non autoregressive counterpart for a given σ . This implies that the simulated stock prices used in the example above exhibit lower volatility than what the investor assumes in the AR(1) case. To ensure that this volatility mismatch does not explain the performance differences, we also simulate stock prices with higher values of σ and find similar results. Similar results also hold for other stock price distributions, *e.g.* lognormal and triangular.

5. EXTENSION TO MULTIPLE ASSETS

5.1. The model with AR(1) dynamics

In practice, fund managers can invest in several risky assets, in order to diversify their portfolio and decrease their exposure to risk. In this section, we describe how our approach can be extended to this setting. We only show below the case when \bar{S}_t exhibits AR(1) dynamics, since the case of a constant \bar{S}_t is a special case.

The main difference from the single-risky-asset case is that at each time period, the decision-maker must now decide how many units of his budget he should use now, and how these units should be allocated among the risky assets.

Correlation among risky assets can be incorporated using a factor model such as the famous three-factor Fama-French factor model (Fama and French [17], Fama and French [18]). Assume a m -factor model such that the i -th stock return, $i = 1, \dots, n$, can be modeled as:

$$S_t^{(i)} = \bar{S}_t^i + c \sum_{j=1}^m L_{ij} \epsilon_{jt}.$$

Here, because stock returns may be negatively correlated and a factor affects multiple stocks, we consider $\epsilon_{jt} \in \{-1, 0, 1\}$ for all j and t .

Once the fund manager has decided to protect his portfolio against the underperformance of $\Gamma_t \leq \tau$ factors at time t , he must decide how to allocate wealth among the risky assets to maximize his portfolio return under these circumstances. The fact that Γ_t factors underperform in the current time period is captured by the constraint: $\sum_{j=1}^m |\epsilon_{jt}| = \Gamma_t$. In the remainder of this section, we denote by $\mathcal{E}_t(\Gamma_t)$ the discrete set of feasible factor realizations in the current time period for our robust optimization framework:

$$\mathcal{E}_t = \left\{ \epsilon \mid \sum_{j=1}^m |\epsilon_{jt}| = \Gamma_t, \forall t, \epsilon_{jt} \in \{-1, 0, 1\}, \forall j, \forall t \right\}.$$

We do not allow the investor to use a unit of uncertainty on stock i when $\bar{S}_t^{(i)} < r_f$, or in other words, when the nominal stock price has fallen below the r_f rate. By doing this, we ensure the budget of uncertainty is spent consistently for our analysis. For example, in the case $n = 10$, $T = 2$, and $\tau = 5$, such that the investor must allocate a budget of 5 units of uncertainty over 10 stocks across 2 time periods. If we did not limit the spending of uncertainty, it is possible that for $t = 2$, all of the nominal stock prices fall below r_f . If the investor is allowed to allocate uncertainty in this case, then he would use all 5 units now, still return r_f over the current time period, and make his allocation decision at $t = 1$ with certainty (or $\tau = 0$). Thus, he has gained confidence in the realization of the stock prices while not losing any wealth. This is unrealistic and results in inflated expected wealth. (Note that our assumptions on \bar{S} and \hat{S} in the constant case do not allow this to happen). We propose that in the case the nominal stock price of any stock is below r_f , the investor is not allowed to allocate uncertainty on this stock at the current time. In the case above, then, all 5 units would then carry over to $t = 1$. If there are not at least 5 stocks above r_f at $t = 1$, then the investor will use the units on whichever stocks are above r_f , and the rest of the uncertainty can be dropped to zero because the investor is already in the worst case. Note that in this setting, the maximum budget of uncertainty is no longer T but mT since we have m uncertain factors at each time periods.

In Theorem 5.1 below, the boldface parameters represent the vector of stock returns at a given $t + 1$. We find the optimal stock allocations $x_{\tau t}^{(i)}(\mathbf{S}_{t+1})$, and the optimal portfolio value $\omega_{\tau t}(\mathbf{S}_{t+1})$:

Theorem 5.1 (Boundary equations).

Solve for $\tau = 0$, for all t , starting with $t = 1$:

$$\begin{aligned} & \max \omega_{0t}(\mathbf{S}_{t+1}) \\ & \text{s.t. } \omega_{0t} \leq \omega_{0,t-1}(\mathbf{S}_{t+1}) \cdot \left(r_f + \sum_{i=1}^n (\bar{S}_t^{(i)} - r_f) x_t^{(i)} \right), \\ & \quad \sum_{i=1}^n x_t^{(i)}(\mathbf{S}_{t+1}) \leq 1, \quad x_t^{(i)}(\mathbf{S}_{t+1}) \geq 0, \quad \forall i. \end{aligned}$$

and for $\tau = mt$, for all t , starting with $t = 1$:

$$\begin{aligned} & \max \omega_{mt,t}(\mathbf{S}_{t+1}) \\ & \text{s.t. } \omega_{mt,t} \leq \omega_{m(t-1),t-1}(\mathbf{S}_{t+1}) \cdot \left(r_f + \sum_{i=1}^n (\bar{S}_t^{(i)} - r_f) x_t^{(i)} - c \sum_{j=1}^m z_{jt} \right), \\ & \quad -z_{jt} \leq \sum_{i=1}^n L_{ij} x_t^{(i)} \leq z_{jt}, \quad \forall j, \\ & \quad \sum_{i=1}^n x_t^{(i)}(\mathbf{S}_{t+1}) \leq 1, \quad x_t^{(i)}(\mathbf{S}_{t+1}) \geq 0, \quad \forall i. \end{aligned}$$

(Recursive equations) For all t , $1 \leq t \leq T$ and τ , $1 \leq \tau \leq mt - 1$, the optimal allocation $x_{\tau t}^{(i)}(\mathbf{S}_{t+1})$, $\forall i$, and rate of growth $\omega_{\tau t}(\mathbf{S}_{t+1})$ are obtained by solving the following linear programming problem:

$$\begin{aligned} & \max \omega_{\tau t}(\mathbf{S}_{t+1}) \\ & \text{s.t. } \omega_{\tau t} \leq \omega_{\tau - \Gamma_t, t-1}(\mathbf{S}_{t+1}) \cdot \left(r_f + \sum_{i=1}^n (\bar{S}_t^{(i)} - r_f) x_t^{(i)} + c \sum_{i=1}^n \sum_{j=1}^m L_{ij} \epsilon_{jt} x_t^{(i)} \right), \\ & \quad \forall \Gamma_t = 0, \dots, \min(\tau, n), \quad \forall \epsilon_t \in \mathcal{E}_t(\Gamma_t), \\ & \quad \sum_{i=1}^n x_t^{(i)}(\mathbf{S}_{t+1}) \leq 1, \quad x_t^{(i)}(\mathbf{S}_{t+1}) \geq 0, \quad \forall i. \end{aligned}$$

Proof. Is similar to Theorem 4.1, while incorporating that we choose how many units Γ_t of the budget of uncertainty to use at time t , and enumerating the possible ways to use it through the finite (although perhaps large) set $\mathcal{E}_t(\Gamma_t)$. For the boundary equation with $\tau = mt$, we linearize $\left| \sum_{i=1}^n L_{ij} x_t^{(i)} \right|$ by introducing a new variable z_{jt} and adding the appropriate constraints as in [8]. \square

In practice, the fund manager will implement a delayed constraint generation algorithm to only use elements of the set $\mathcal{E}_t(\Gamma_t)$ as needed for all t and keep the problem formulation tractable.

The extension to the multiple-assets case with penalty follows a similar approach and is omitted here.

5.2. Numerical experiments

Here, we benchmark the approach presented in Theorem 5.1 in the multiple assets case against a simple benchmark where the bond-stock portfolio mix as a function of the time horizon is given in Table 1 and the manager invests in each stock in equal amounts. For instance with 3 stocks and a time horizon of 15 years, the fund manager invests 41.2% in bonds and $58.8/3 = 19.6\%$ in each stock. We will observe that the performances

TABLE 7. Data for problem with multiple stocks.

Parameters	Stock 1	Stock 2	Stock 3
μ	1.03	1.05	1.07
φ	0.5	0.25	0
σ	0.07	0.08	0.09
S_0	1.1	1.08	1.09

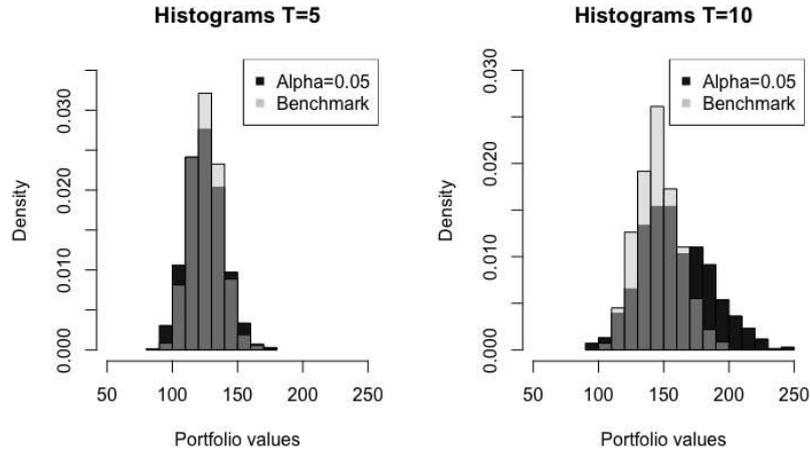


FIGURE 8. Histograms for $T = 5$ (left) and $T = 10$ (right).

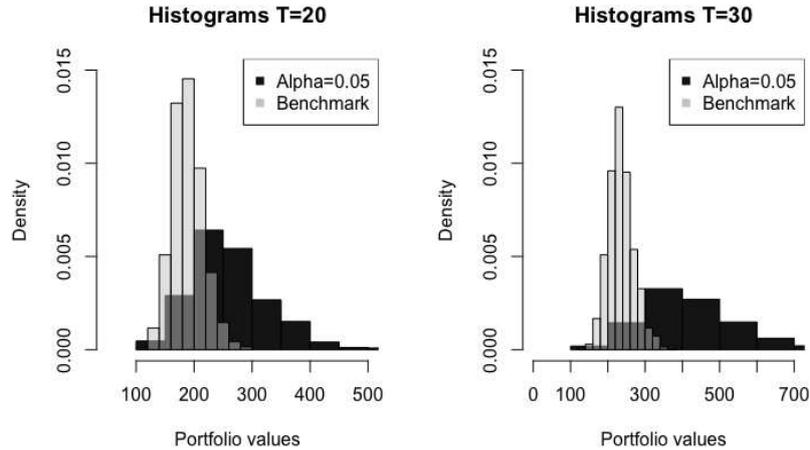


FIGURE 9. Histograms for $T = 20$ (left) and $T = 30$ (right).

of the proposed approach and of the benchmark are comparable for short time horizons, while our approach outperforms the benchmark in terms of left tail of the return as the time horizon increases, while leading to a slight worsening of the right tail of the distribution.

We take $step = 0.001$, $c = 2$, $r_f = 1.01$ and use a grid to compute the optimal allocations based on the current stock returns. We take $T = 35$, $W_0 = \$100$. We consider a linear rule to select τ , where τ is taken

proportional to the number of uncertain parameters:

$$\tau = \alpha \cdot nT$$

where α varies from 0 to 0.95 in steps of 0.05 and $n = 3$ stocks. The numerical values used for the 3 stock dynamics, which are assumed to be independent, are listed in Table 7.

We perform 10 000 simulation runs of 35 time periods (years) each and plot the histograms of the portfolio values (benchmark and the robust optimization approaches for the various values of α) from $T = 5$ to $T = 35$ in steps of 5 years. Due to space constraints we only show the histograms for $T = 5$, $T = 10$, $T = 20$ and $T = 30$ here with $\alpha = 0.05$ against the benchmark. Figures 8 and 9 show overlaying histograms with the dark shade showing our approach when not overlaid, the light shade showing the benchmark when not overlaid and the grey shade showing the area of overlay for the two approaches. We choose the value of $\alpha = 0.05$ (meaning that τ is about 15% the time horizon T) because it provides the best practical performance of our algorithm in this numerical setting. We observe that for short time horizons the benchmark and the robust optimization approach perform similarly, while for increasing time horizons the robust optimization approach leads to a very slightly bigger left tail that is compensated by a much larger right tail of portfolio values, suggesting the potential superiority of our approach when α is chosen appropriately.

Future research directions include developing robust adaptive policies, building upon Bertsimas *et al.* [5]. A possible idea would be to design policies where the budget of uncertainty τ would be updated (used) based on realizations of the stock returns in earlier time periods.

6. CONCLUSIONS

We have presented a simple, intuitive approach to compute the optimal portfolio fraction invested in stocks that captures stock market volatility, length of time horizon and investors' risk preferences. It extends and refines the allocation rules traditionally used in retirement planning. The key contribution of this paper is to provide tools and techniques for fund participants who do not have the necessary expertise to implement the sophisticated robust multiperiod portfolio optimization approaches available in the literature and seek an alternative decision-making model under uncertainty to gain insights into the optimal stock and bond allocation of their portfolio as their retirement draws nearer. We hope that the proposed framework will help them develop a better understanding of appropriate allocations and take a more informed, active role in managing their own retirement portfolio and ensuring their future financial health. For the case with one risky asset, the optimal strategy does not require the use of optimization software. The parametric structure of the decision rule makes it easy for mathematically unsophisticated investors to understand and implement it. The multiple asset case can be solved using linear programming techniques. We investigate constant returns and AR(1) dynamics, and also consider an extension to a tracking penalty.

We make the following observations from our numerical experiments. (i) For the basic problem, we investigate how the optimal allocation in stock varies as a function of the budget of uncertainty for a given time horizon. We see that the allocation in stocks remains high for budgets of uncertainty of small size, decreases rapidly to values close to zero for budgets of moderate size, and then slowly converges to zero for budgets of large size. We select a budget of uncertainty linear in the remaining time horizon and compare the portfolio resulting from our model with that obtained by an industry benchmark. Our experiments indicate our method performs better in terms of Sharpe ratio, and also often, but not always, in terms of other metrics such as mean, median, quantiles. The linear rule is more aggressive than the industry rule given shorter time horizons, while for longer time horizons, the proposed model leads to more conservative allocations for large uncertainty. (ii) In the case of a tracking penalty, we illustrate the tradeoff between incurring a penalty for low returns and the risk of portfolio underperformance due to overly aggressive allocation compared to a more conservative competitor. (iii) We also compare the performance of three investors who act under constant expected return and AR(1) dynamic stock price assumptions both with and without a tracking penalty. We find that the investor who allocates his portfolio under the AR(1) assumptions performs consistently better regarding the low tail of the histogram

(lower deciles). The investor who assumes constant expected stock returns obtains a higher rate for the higher deciles. Our experiments also suggest that the investor who assumes AR(1) dynamics obtains a higher Sharpe ratio. (iv) Finally, our numerical experiments in the multiple assets case suggest that our proposed method and the benchmark perform similarly for short time horizons, while for longer time horizons, our approach has a larger right tail of portfolio returns, thus showing promise in robust multiperiod portfolio allocation for retirement planning.

Acknowledgements. The authors thank Zhongzhong Zeng for her preliminary work regarding Section 2.1 and part of Section 2.2 of this paper as well as three anonymous referees for their constructive comments that have helped substantially improve the paper.

REFERENCES

- [1] A. Bary, Target–date funds take over *Barron's*. July 5 issue (2014).
- [2] A. Ben–Tal, T. Margalit and A. Nemirovski, Robust modeling of multi-stage portfolio problems, in: *High Performance Optimization*. Springer, New York, NY (2000) 303–328.
- [3] A. Ben–Tal and A. Nemirovski, Robust convex optimization. *Math. Oper. Res.* **23** (1998) 769–805.
- [4] D. Bertsimas and A. Thiele, A robust optimization approach to inventory theory. *Oper. Res.* **54** (2006a) 150–168.
- [5] D. Bertsimas, D. Iancu and P. Parrilo, A hierarchy of near-optimal policies for multistage adaptive optimization. *IEEE Trans. Automatic Control* **56** (2011) 2809–2824.
- [6] D. Bertsimas, G. Lauprete and A. Samarov, Shortfall as a risk measure: properties, optimization and applications. *J. Econ. Dynamics Control* **28** 1353–1381.
- [7] D. Bertsimas and D. Pachamanova, Robust multiperiod portfolio management in the presence of transaction costs. *Comput. Oper. Res.* (2008) **35** 3–17.
- [8] D. Bertsimas and M. Sim, Price of Robustness. *Oper. Res.* **52** (2004) 35–53.
- [9] J. Birge and F. Louveaux, *Introduction to Stochastic Programming*, 2nd edition. Springer, New York, NY (2011).
- [10] E. Bogentoft and H.E. Romeijn and S. Uryasev, Asset/liability management for pension funds using CVaR constraints. *J. Risk Finance* **3** (1999) 57–71.
- [11] D.R. Cariño, T. Kent and D.H. Myers, C. Stacy, M. Sylvanus, A.L. Turner, K. Watanabe, W.T. Ziemba, The Russel-Yasuda Kasai model: An asset-liability model for a Japanese insurance company using multistage stochastic programming. *Interfaces* **24** (1994) 29–49.
- [12] G. Consigli and M.A.H. Dempster, Dynamic stochastic programming for asset-liability management. *Ann. Oper. Res.* **81** (1998) 131–162.
- [13] G. Deelstra, M. Grasselli and P.F. Koehl, Optimal investment strategies in the presence of a minimum guarantee. *Insurance: Math. Econom.* **33** (2003) 189–207.
- [14] M. Dziecichowicz, D. Caro and A. Thiele, Robust timing of markdowns. *Ann. Oper. Res.* **235** (2015) 203–231.
- [15] H. Evensky, S.M. Horan and T.R. Robinson, *The new wealth management: The Financial Advisor's Guide to Managing and Investing Client Assets*. Wiley, New York, NY (2011).
- [16] F.J. Fabozzi, P.N. Kolm, D. Pachamanova and S. Focardi, *Robust Portfolio Optimization and Management*. Wiley, New York, NY (2007).
- [17] E.F. Fama and K.R. French, The cross-section of expected stock returns. *J. Finance* **47** (1992) 427–465.
- [18] E.F. Fama and K.R. French, Common risk factors in the returns on stocks and bonds. *J. Financial Econom.* **33** (1993) 3–56.
- [19] V. Gabrel, C. Murat and A.Thiele, Recent advances in robust optimization: an overview. *Eur. J. Oper. Res.* **235** (2014) 471–483.
- [20] A. Geyer, W. Herold, K. Kontriner and W. Ziemba, The InnoVest Austrian Pension Fund Financial Planning Model InnoALM. *Oper. Res.* **56** (2008) 797–810.
- [21] J. Gondzio and R. Kouwenberg, High-Performance Computing for Asset-Liability Management. *Oper. Res.* **49** (2011) 879–891.
- [22] N. Gülpinar and D. Pachamanova, A robust optimization approach to asset-liability management under time-varying investment opportunities. *J. Bank. Finance* **37** (2013) 2031–2041.
- [23] S. Haberman and E. Vigna, Optimal investment strategies and risk measures in defined contribution pension schemes. *Insurance: Math. Econom.* **31** (2002) 35–69.
- [24] J. Hull, *Options, Futures and Other Derivatives*, 9th Ed. Pearson, London (2014).
- [25] G. Iyengar and A.K.C. Ma, A robust optimization approach to pension fund management *J. Asset Manag.* **11** (2010) 163–177.
- [26] P. Kall and S. Wallace, *Stochastic Program*. Wiley, New York (1994).
- [27] S. Kilianova and G. Pflug, Optimal pension fund management under multi-period risk minimization. *Ann. Oper. Res.* **166** (2009) 261–270.
- [28] R. Kouwenberg, Scenario generation and stochastic programming models for asset liability management. *Eur. J. Oper. Res.* **134** (2001) 279–292.
- [29] J. Mulvey, G. Gould and C. Morgan, An asset and liability management system for Towers Perrin – Tillinghast. *Interfaces* **30** (2000) 96–114.

- [30] A. Muralidhar, Innovations in pension fund management. Stanford University Press. Stanford, CA (2001).
- [31] D. Pachamanova, *A robust optimization approach to finance*. Ph.D. dissertation, Massachusetts Institute of Technology, Cambridge, MA, USA (2002).
- [32] D. Pachamanova, N. Gülpinar and E. Çanakoğlu, Robust approaches to pension fund asset liability management under uncertainty, in *Optimal financial decision making under uncertainty*. Springer, New York, NY (2017).
- [33] S. Pandit and S. Wu, Time series and system analysis, with applications. John Wiley and Sons, Inc., New York, NY (1983).
- [34] W. Sharpe, Budgeting and monitoring pension fund risk. *Financial Anal. J.* **58** (2002) 74–86.
- [35] A. Soyster, Convex programming with set-inclusive constraints and applications to inexact linear programming. *Oper. Res.* **21** (1973) 1153–1157.
- [36] E. Vigna and S. Haberman, Optimal investment strategy for defined contribution pension schemes. *Insurance: Mathe. Econom.* **28** (2001) 233–262.
- [37] J. von Neumann and O. Morgenstern, Theory of games and economic behavior. Princeton University Press. Princeton, NJ (1953).