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A NEWTON METHOD FOR CAPTURING PARETO OPTIMAL SOLUTIONS OF FUZZY MULTIOBJECTIVE OPTIMIZATION PROBLEMS

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Abstract. In this study, a Newton method is developed to obtain (weak) Pareto optimal solutions of an unconstrained multiobjective optimization problem (MOP) with fuzzy objective functions. For this purpose, the generalized Hukuhara differentiability of fuzzy vector functions and fuzzy max-order relation on the set of fuzzy vectors are employed. It is assumed that the objective functions of the fuzzy MOP are twice continuously generalized Hukuhara differentiable. Under this assumption, the relationship between weakly Pareto optimal solutions of a fuzzy MOP and critical points of the related crisp problem is discussed. Numerical examples are provided to demonstrate the efficiency of the proposed methodology. Finally, the convergence analysis of the method under investigation is discussed.

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1. Introduction

Many real decision situations can not be characterized by a single measure of their performance, but rather by multiobjectives, often conflicting among themselves. This fact has led to the growth of the field of multiobjective optimization. Multiobjective optimization has been applied in many areas of science, including engineering, economics, medicine and transportation. From a large amount of publications in the field of multiobjective optimization, we refer to [1,2,8,10,11,24,25,27], and references therein.

In the crisp optimization problems, all the coefficients of the mathematical formulation appear with exact real numbers. However, since decision makers often face problems with vague and incomplete information, most of the realistic optimization problems must be characterized with inexact (or fuzzy) parameters. The optimization problems with fuzzy parameters are called fuzzy optimization problems. Although there are many methods for solving crisp optimization problems, these methods can not be easily applied to fuzzy optimization problems. The first attempt for solving fuzzy optimization problems was done by Bellman and Zadeh [6]. Over the past years, many scholars have applied fuzzy set theory in optimization problems. We refer to [4,13,21–23,28,29,33,41] that have been done in this direction.

Zimmermann [40] utilized fuzzified constraints and objective functions for multiobjective linear programming problems. Afterwards, a lot of papers and books dealing with fuzzy multiobjective optimization were published.

Keywords. Fuzzy multiobjective problem, Newton method, Pareto optimal solution, Generalized Hukuhara differentiability, Critical point.

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As an example, for solving a multiobjective optimization problem with fuzzy coefficients, Wu [35,36] transformed the problem into a vector optimization problem by applying embedding theorem and a suitable scalarization problem. Moreover, the Karush-Kuhn-Tucker optimality conditions for multiobjective optimization problems with fuzzy-valued and interval-valued objective functions were considered in [16, 37, 38]. Furthermore, several applications and different methods to solve a given fuzzy (linear) multiobjective optimization problem can be found in [3, 18–20, 32, 34, 39].

Newton method for solving a fuzzy single objective optimization problem was proposed by Pirzada and Pathak [26]. In the developed approach, they utilized the notation of Hukuhara differentiability of fuzzy-valued functions and max-ordering relation defined on the set of fuzzy numbers. Thereafter, Chalco-Cano *et al.* [7] utilized the concept of generalized Hukuhara differentiability of fuzzy functions and resolved some difficulties of the Pirzada and Pathak's method [26]. More recently, Ghaznavi and Hoseinpoor [12] obtained a quasi-Newton approach for solving fuzzy optimization problems. They extended the BFGS method for fuzzy optimization problems. Moreover, Ghosh [14, 15] proposed (quasi-)Newton methods for finding efficient solutions of optimization problems with interval-valued objective functions.

Now, in this paper, motivated by the works of [7, 9, 12, 26], we propose a Newton method for solving an unconstrained fuzzy multiobjective optimization problem. In fact, we generalize the idea given in [7] to fuzzy multiobjective optimization problems. The objective functions of the fuzzy MOP are fuzzy-valued. Using the concept of gH-differentiable fuzzy vector function, we find the (weak) Pareto optimal solutions of a fuzzy MOP specified by the fuzzy max-order relation defined on the set of fuzzy numbers. We propose a Newton-method for solving a fuzzy multiobjective optimization problem. In the proposed algorithm, we assume that the objective functions of the fuzzy MOP are twice continuously generalized Hukuhara differentiable.

The remainder of the paper is demonstrated in the following sequence: In Section 2, we recall some necessary terminologies and concepts related to fuzzy set theory that are used throughout the paper. In Section 3 we present some results on gH-differentiability of fuzzy vector functions. We explain some results in fuzzy multiobjective optimization problem in Section 4. The proposed Newton method is given in Section 5. To demonstrate efficiency of the proposed methodology, some illustrative examples are provided in Section 6. Finally, the conclusions and some suggestions for future research are given in Section 7.

2. Preliminaries and basic definitions

In this section, we review some of the basic definitions and terminologies of fuzzy set theory which are used in the remaining parts of the paper.

Definition 2.1. [4] Let $\mathbb{R}$ be the set of real numbers and A be a subset of $\mathbb{R}$. Then, the corresponding indicator function of A is given by (Fig. 1)

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

Definition 2.2. [4] Let $X \subseteq \mathbb{R}$ be the universe whose generic element be denoted by x. A fuzzy subset u in X is a function $u : X \to [0, 1]$. A fuzzy set u is characterized by its membership function $\mu_u : X \to [0, 1]$, which associates with each x in X, a real number $\mu_u(x)$ in $[0, 1]$ (Fig. 1).

Definition 2.3. [41] Let u be a fuzzy set in X and $\alpha \in [0, 1]$. The α-cut or α-level of the fuzzy set u is the crisp set $[u]^\alpha$ given by $[u]^\alpha = \{x \in X : \mu_u(x) \geq \alpha\}$ (Fig. 2).

Definition 2.4. [41] Let u be a fuzzy set in X. Then, the support of u denoted by $S(u)$, is the crisp set given by $S(u) = \{x \in X : \mu_u(x) > 0\}$.

Definition 2.5. [4] (Subset) A fuzzy set A is a subset of a fuzzy set B, or A is contained in B if $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$. This is denoted as $A \subseteq B$ (Fig. 3).

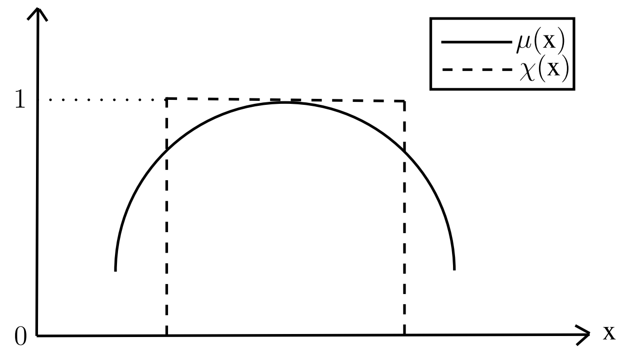

FIGURE 1. Indicator function and membership function.

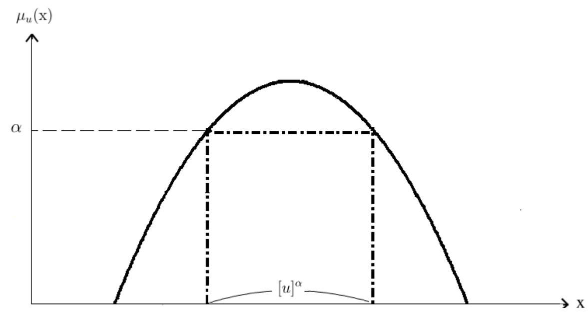

FIGURE 2. α-level set.

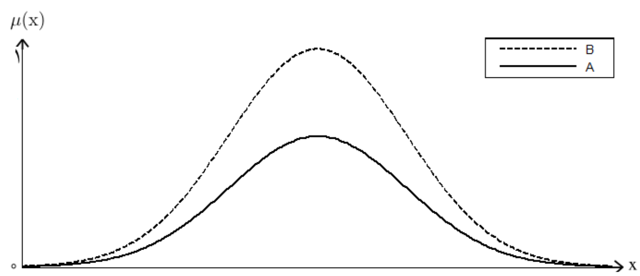

FIGURE 3. Subset of a fuzzy set ($A \subset B$).

Definition 2.6. [41] (Standard complement) The standard complement of a fuzzy set A is another fuzzy set, denoted by A^c whose membership function is defined as $\mu_{A'}(x) = 1 - \mu_A(x)$ for all $x \in X$ (Fig. 4).

Definition 2.7. [4] (Standard union) The standard union of two fuzzy sets A and B is a fuzzy set C whose membership function is given by

$$\mu_C(x) = \max\left(\mu_A(x), \mu_B(x)\right),$$

for all $x \in X$ (Fig. 5).

Definition 2.8. [4] (Standard intersection) The standard intersection of two fuzzy sets A and B is a fuzzy set D whose membership function is given by

$$\mu_D(x) = \min\left(\mu_A(x), \mu_B(x)\right),$$

for all $x \in X$ (Fig. 5).

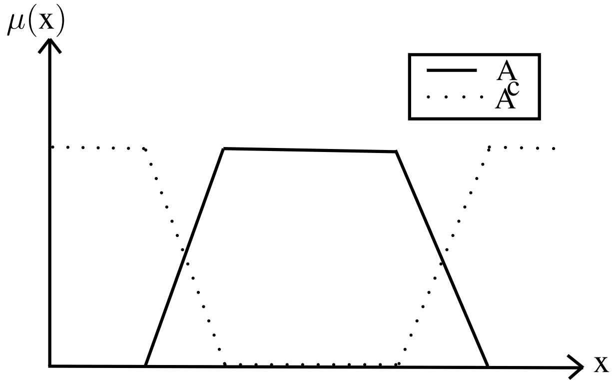

FIGURE 4. Standard complement.

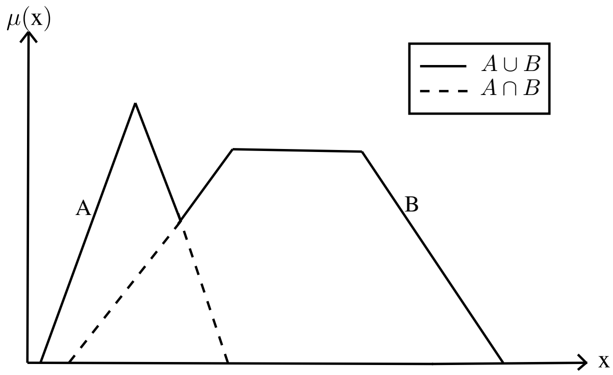

FIGURE 5. Standard union and standard intersection.

Definition 2.9. [41] (Convex fuzzy set) A fuzzy set A in $\mathbb{R}^n$ is called a convex fuzzy set if its $\alpha-$cuts A_α are (crisp) convex sets for all $\alpha \in (0, 1]$.

Definition 2.10. [4] (Bounded fuzzy set) A fuzzy set A in $\mathbb{R}^n$ is called a bounded fuzzy set if its $\alpha-$cuts A_α are (crisp) bounded sets for all $\alpha \in (0, 1]$.

A fuzzy set A in $\mathbb{R}^n$ which is both bounded and convex is called bounded convex fuzzy set. The following result gives an equivalent definition of a convex fuzzy set.

Theorem 2.11. [41] *A fuzzy set A in $\mathbb{R}^n$ is a convex fuzzy set if and only if for all $x_1, x_2 \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$,*

$$\mu_A\left(\lambda x_1 + (1-\lambda)x_2\right) \geq \min\left(\mu_A(x_1), \mu_A(x_2)\right).$$

Definition 2.12. [41] A fuzzy set u in $\mathbb{R}$ is called a fuzzy number if it satisfies the following conditions:

(i) u is normal,
(ii) $[u]^\alpha$ is a closed interval for every $\alpha \in [0, 1]$,
(iii) the support of u is bounded.

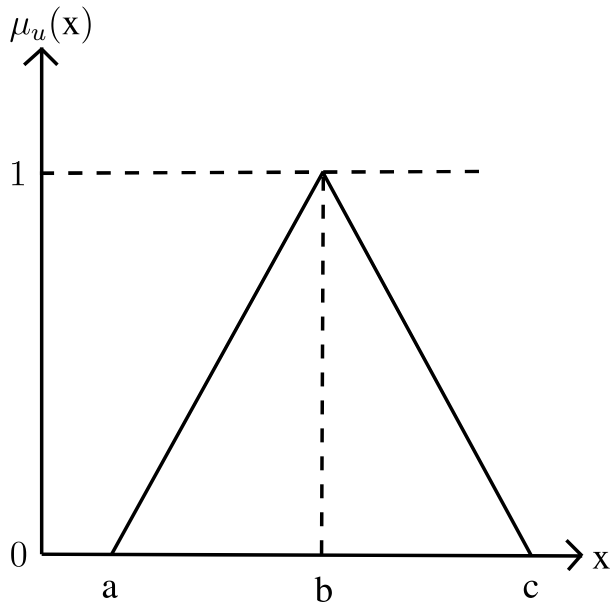

Figure 6. Triangular fuzzy number.

The set of all fuzzy numbers on $\mathbb{R}$ is denoted by $F(\mathbb{R})$. By definition of fuzzy numbers, we can see that $[u]^\alpha$ is a compact interval in $\mathbb{R}$, for all $\alpha \in [0,1]$, and therefore the $\alpha-$level sets of a fuzzy number u are denoted by $[u]^\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$, where $\underline{u}_\alpha, \overline{u}_\alpha \in \mathbb{R}$ for all $\alpha \in [0,1]$.

Let u and v be two fuzzy numbers. Using their α-level sets, the addition and scalar multiplication in $F(\mathbb{R})$ are defined as follows, respectively:

$$[u+v]^\alpha = [\underline{u}_\alpha + \underline{v}_\alpha, \overline{u}_\alpha + \overline{v}_\alpha], \tag{2.1}$$

and

$$[\lambda u]^\alpha = [\min\{\lambda \underline{u}_\alpha, \lambda \overline{u}_\alpha\}, \max\{\lambda \underline{u}_\alpha, \lambda \overline{u}_\alpha\}], \tag{2.2}$$

where $\lambda \in \mathbb{R}$ and $\alpha \in [0,1]$.

Definition 2.13. [41] A fuzzy number u is called a triangular fuzzy number, if its membership function is given by (Fig. 6):

$$\mu_u(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \le x \le b, \\ \dfrac{c-x}{c-b}, & b \le x \le c, \\ 0, & \text{otherwise}. \end{cases} \tag{2.3}$$

A triangular fuzzy number is shown by $u = (a,b,c)$. The $\alpha-$level set of a triangular fuzzy number $u = (a,b,c)$ is given by:

$$\begin{aligned} [u]^\alpha = \{x : \mu_u(x) \ge \alpha\} &= \left\{x : \frac{x-a}{b-a} \ge \alpha, \quad \frac{c-x}{c-b} \ge \alpha\right\} \\ &= [(1-\alpha)a + \alpha b, (1-\alpha)c + \alpha b], \ \forall \alpha \in [0,1]. \end{aligned}$$

Definition 2.14. [38] Let u and v be two fuzzy numbers in $F(\mathbb{R})$. Hence $[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]$ and $[v]^\alpha = [\underline{v}^\alpha, \overline{v}^\alpha]$ are two intervals in $\mathbb{R}$, for all $\alpha \in [0,1]$. We define

$$u \preceq v \leftrightarrow [u]^\alpha \preceq [v]^\alpha, \forall\alpha \in [0,1] \leftrightarrow \underline{u}^\alpha \leq \underline{v}^\alpha \text{ and } \overline{u}^\alpha \leq \overline{v}^\alpha, \forall\alpha \in [0,1]$$

and

$$u \prec v \leftrightarrow u \preceq v \text{ and } u \neq v \leftrightarrow [u]^\alpha \preceq [v]^\alpha \ \forall\alpha \in [0,1], \text{ and } \exists\alpha^* \in [0,1] \text{ s.t. } \underline{u}^{\alpha^*} < \underline{v}^{\alpha^*} \text{ or } \overline{u}^{\alpha^*} < \overline{v}^{\alpha^*}.$$

We say that $\mathbf{x} = (x^1, x^2, \ldots, x^m)$ is a fuzzy vector if each x^i, for $i = 1, \ldots, m$ is a fuzzy number. Let $\mathbf{x} = (x^1, x^2, \ldots, x^m)$ and $\mathbf{y} = (y^1, y^2, \ldots, y^m)$ be two fuzzy vectors. We write $\mathbf{x} \preceq \mathbf{y}$ if and only if $x^i \preceq y^i$ for all $i = 1, 2, \ldots, m$ and write $\mathbf{x} \prec \mathbf{y}$ if and only if $x^i \preceq y^i$ for all $i = 1, 2, \ldots, m$ and $x^j \prec y^j$ for at least one index $j \in \{1, 2, \ldots, m\}$.

Definition 2.15. [30] Let $u, v \in F(\mathbb{R})$. The fuzzy number w is the generalized Hukuhara difference (gH-difference) of u and v, if it exists such that

$$u \ominus_{gh} v = w \Longleftrightarrow \begin{cases} \text{(i) } u = v + w & \text{or} \\ \text{(ii) } v = u + (-1)w. \end{cases}$$

It is obvious that (i) and (ii) are both valid if and only if w is a crisp number.

If $u \ominus_{gh} v$ exists then, in terms of α-levels, we have

$$[u \ominus_{gh} v]^\alpha = [u]^\alpha \ominus_{gh} [v]^\alpha = \left[\min\left\{\underline{u}_\alpha - \underline{v}_\alpha, \overline{u}_\alpha - \overline{v}_\alpha\right\}, \max\left\{\underline{u}_\alpha - \underline{v}_\alpha, \overline{u}_\alpha - \overline{v}_\alpha\right\}\right],$$

$\forall\alpha \in [0,1]$, where $[u]^\alpha \ominus_{gh} [v]^\alpha$ denotes the gH-difference between two intervals (see [30,31]).

Proposition 2.16. [37] *Let* $u, v \in F(\mathbb{R})$. *Then we have*

(i) $u \oplus v \in F(\mathbb{R})$ *and*

$$(u \oplus v)_\alpha = [\underline{u} + \underline{v}, \bar{u} + \bar{v}],$$

(ii) $u \ominus v \in F(\mathbb{R})$ *and*

$$(u \ominus v)_\alpha = [\underline{u} - \bar{v}, \bar{u} - \underline{v}],$$

(iii) $u \otimes v \in F(\mathbb{R})$ *and*

$$(u \otimes v)_\alpha = u_\alpha \times v_\alpha = [\min\{\underline{u}_\alpha\underline{v}_\alpha, \underline{u}_\alpha\bar{v}_\alpha, \bar{u}_\alpha\underline{v}_\alpha, \bar{u}_\alpha\bar{v}_\alpha\}, \max\{\underline{u}_\alpha\underline{v}_\alpha, \underline{u}_\alpha\bar{v}_\alpha, \bar{u}_\alpha\underline{v}_\alpha, \bar{u}_\alpha\bar{v}_\alpha\}].$$

Definition 2.17. [7] Let X be an open subset of $\mathbb{R}^n$ and $F(\mathbb{R})$ display the set of all fuzzy numbers. A function $f : X \longrightarrow F(\mathbb{R})$ is called a fuzzy function defined on X. For each $\alpha \in [0,1]$, associated to f, the family of interval-valued functions $f_\alpha : X \to X_c$ is defined by $f_\alpha(x) = [f(x)]^\alpha$, where X_c is the family of all bounded closed intervals in $\mathbb{R}$. For any $\alpha \in [0,1]$, we denote

$$f_\alpha(x) = [f(x)]^\alpha = \left[\underline{f}^\alpha(x), \overline{f}^\alpha(x)\right].$$

Here, for each $\alpha \in [0,1]$, the endpoint functions $\underline{f}^\alpha, \overline{f}^\alpha : X \longrightarrow \mathbb{R}$ are called upper and lower functions of $f_\alpha(x)$, respectively.

3. Differentiability of fuzzy vectors

Let X be an open subset of $\mathbb{R}^n$ and $f_i : X \longrightarrow F(\mathbb{R}), \forall i = 1, 2, \ldots, m$ be fuzzy functions defined on X. Moreover, let $F^m(\mathbb{R}) = F(\mathbb{R})\times\ldots\times F(\mathbb{R})$ (m times). A function $F : X \to F^m(\mathbb{R})$ with $F(x) = (f_1(x), \ldots, f_m(x))$ is called a fuzzy vector function. Let X_c be the family of all bounded closed intervals in $\mathbb{R}$. Corresponding to each fuzzy function $f_i : X \to F(\mathbb{R}), \forall i = 1, \ldots, m$ we define the family of interval-valued functions $f_{i\alpha} : X \to X_c$ presented by $f_{i\alpha}(x) = [f_i(x)]^\alpha$. For any $\alpha \in [0, 1]$, $f_{i\alpha}(x)$ is denoted by $f_{i\alpha}(x) = [f_i(x)]^\alpha = \left[\underline{f}_i^\alpha(x), \overline{f}_i^\alpha(x)\right], \forall i = 1, 2, \ldots, m$. Here, the endpoint functions $\underline{f}_i^\alpha, \overline{f}_i^\alpha : X \longrightarrow \mathbb{R}$ are called upper and lower functions of $f_{i\alpha}(x), \forall i = 1, \ldots, m$, respectively. Therefore the α-level set of fuzzy vector function F, denoted by $[F]^\alpha$, is defined by

$$\begin{aligned}&F_\alpha : X \longrightarrow X_c^m \Rightarrow F_\alpha(x) = [F(x)]^\alpha = ([f_1(x)]^\alpha, [f_2(x)]^\alpha, \ldots, [f_m(x)]^\alpha) \Rightarrow\\ &F_\alpha(x) = \left[\underline{F}^\alpha, \overline{F}^\alpha\right] = \left(\left[\underline{f_1}^\alpha(x), \overline{f_1}^\alpha(x)\right], \left[\underline{f_2}^\alpha(x), \overline{f_2}^\alpha(x)\right], \ldots, \left[\underline{f_m}^\alpha(x), \overline{f_m}^\alpha(x)\right]\right).\end{aligned}$$

Definition 3.1. [5] Let $X \subset \mathbb{R}$ and $f : X \longrightarrow F(\mathbb{R})$ be a fuzzy function. Also, assume that $x_0 \in X$ and h is such that $x_0 + h \in X$. The generalized Hukuhara derivative (gH-derivative) of f at x_0 is defined as

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) \ominus_{gh} f(x_0)}{h}\cdot \tag{3.1}$$

If $f'(x_0) \in F(\mathbb{R})$ satisfying (3.1) exists, we say that f is generalized Hukuhara differentiable (gH-differentiable) at x_0. If f is gH-differentiable at any $x \in X$, we say that f is gH-differentiable over X.

Now, we present the concept of gH-differentiability of a fuzzy vector function.

Definition 3.2. The generalized Hukuhara derivative (gH-derivative) of a fuzzy vector function $F : X \subset \mathbb{R} \longrightarrow F^m(\mathbb{R})$ is defined by

$$F'(x_0) = (f_1'(x_0), f_2'(x_0), \ldots, f_m'(x_0)).$$

In fact, F is gH-differentiable at x_0 if and only if $f_i, \forall i = 1, \ldots, m$ are gH-differentiable at x_0.

Definition 3.3. [31] Let X be an open set in $\mathbb{R}$. An interval-valued function $f : X \to X_c$ is gH-differentiable at $x_0 \in X$, if (3.1) exists with respect to the limit in the metric space (X_c, H), where the difference is given by the gH-difference between intervals.

Theorem 3.4. [7] *Let $f : X \longrightarrow F(\mathbb{R})$ be a fuzzy function. If f is gH-differentiable, the interval-valued function $f_\alpha : X \longrightarrow X_c$ is gH-differentiable for each $\alpha \in [0, 1]$. Moreover*

$$f_\alpha'(x) = [f'(x)]^\alpha = \left[\underline{f}'^\alpha(x), \overline{f}'^\alpha(x)\right].$$

Corollary 3.5. *Let $F : X \longrightarrow F^m(\mathbb{R})$ be a fuzzy vector function. If F is gH-differentiable, the interval-valued vector function $F_\alpha : X \longrightarrow X_c^m$ is gH-differentiable for each $\alpha \in [0, 1]$. Moreover*

$$F_\alpha'(x) = [F'(x)]^\alpha = ([f_1'(x)]^\alpha, [f_2'(x)]^\alpha, \ldots, [f_m'(x)]^\alpha).$$

Proof. The proof follows from Theorem 3.4 and Definitions 3.2 and 3.3. □

Definition 3.6. [7] Let f be a fuzzy function defined on $X \subseteq \mathbb{R}^n$ and let $x^0 = (x_1^0, x_2^0, \ldots, x_n^0)$ be a fixed element of X. Consider the fuzzy function $h(x_i) = f(x_1^0, \ldots, x_{i-1}^0, x_i, x_{i+1}^0, \ldots, x_n^0)$. If h is gH-differentiable at x_i^0, we say that f has the ith partial gH-derivative at x^0 (denoted by $\frac{\partial f}{\partial x_i}(x^0)$) and $\frac{\partial f}{\partial x_i}(x^0) = (h')(x_i^0)$.

Definition 3.7. Let $F = (f_1, f_2, \dots, f_m)$ be a fuzzy vector function defined on $X \subseteq \mathbb{R}^n$ and let $x^0 = (x_1^0, x_2^0, \dots, x_n^0)$ be a fixed element of X. Then, we say that the fuzzy vector function F has the ith partial gH-derivative at x^0, if and only if f_j, for each $j = 1, 2, \dots, m$, has ith partial gH-derivative at x^0 and is denoted by

$$\left(\frac{\partial F}{\partial x_i}\right)(x^0) = \left(\frac{\partial f_1}{\partial x_i}(x^0), \frac{\partial f_2}{\partial x_i}(x^0), \dots, \frac{\partial f_m}{\partial x_i}(x^0)\right).$$

Definition 3.8. Let F be a fuzzy vector function defined on $X \subseteq \mathbb{R}^n$ and let $x^0 = (x_1^0, x_2^0, \dots, x_n^0) \in X$ be a fixed element of X. We say that F is gH-differentiable at x^0 if all the partial gH-derivatives $\frac{\partial F}{\partial x_1}(x^0), \frac{\partial F}{\partial x_2}(x^0), \dots, \frac{\partial F}{\partial x_n}(x^0)$ exist in some neighborhood of x^0 and are continuous at x^0.

Now, we can provide the following theorem.

Theorem 3.9. *Let $F : X \longrightarrow F^m(\mathbb{R})$ be a fuzzy vector function. If F is gH-differentiable at $x^0 \in X$, for each $\alpha \in [0, 1]$ the real-valued vector function $\underline{F}^\alpha + \overline{F}^\alpha : X \longrightarrow \mathbb{R}^m$ is differentiable at x^0. Moreover,*

$$\begin{aligned}\frac{\partial \underline{F}^\alpha}{\partial x_i}(x^0) + \frac{\partial \overline{F}^\alpha}{\partial x_i}(x^0) &= \frac{\partial(\underline{F}^\alpha + \overline{F}^\alpha)}{\partial x_i}(x^0)\\ &= \left(\frac{\partial(\underline{f_1}^\alpha + \overline{f_1}^\alpha)}{\partial x_i}(x^0), \frac{\partial(\underline{f_2}^\alpha + \overline{f_2}^\alpha)}{\partial x_i}(x^0), \dots, \frac{\partial(\underline{f_m}^\alpha + \overline{f_m}^\alpha)}{\partial x_i}(x^0)\right).\end{aligned}$$

Proof. The proof follows from Proposition 1 in [7]. □

Definition 3.10. [7] Let $f : X \longrightarrow F(\mathbb{R})$ be a fuzzy function. The gradient of f at x^0, denoted by $\tilde{\nabla} f(x^0)$, is defined by

$$\tilde{\nabla} f(x^0) = \left(\left(\frac{\partial f}{\partial x_1}\right)(x_0), \left(\frac{\partial f}{\partial x_2}\right)(x_0), \dots, \left(\frac{\partial f}{\partial x_n}\right)(x_0)\right).$$

The α-level set of $\tilde{\nabla} f(x^0)$ is defined and denoted by

$$[\tilde{\nabla} f(x^0)]^\alpha = \left(\left[\frac{\partial f}{\partial x_1}\right]^\alpha (x_0), \left[\frac{\partial f}{\partial x_2}\right]^\alpha (x_0), \dots, \left[\frac{\partial f}{\partial x_n}\right]^\alpha (x_0)\right),$$

where

$$\left[\frac{\partial f}{\partial x_i}\right]^\alpha = \left[\frac{\partial \underline{f}^\alpha}{\partial x_i}, \frac{\partial \overline{f}^\alpha}{\partial x_i}\right].$$

Definition 3.11. Let $F : X \longrightarrow F^m(\mathbb{R})$ be a fuzzy vector function. The Jacobian of F at x^0, denoted by $\tilde{D}F(x^0)$, is defined as:

$$\tilde{D}F(x^0) = \left(\tilde{\nabla} f_1(x^0), \tilde{\nabla} f_2(x^0), \dots, \tilde{\nabla} f_m(x^0)\right)^T.$$

Definition 3.12. [7] Let $f : X \longrightarrow F(\mathbb{R})$ be a fuzzy function. If gradient of f, $\tilde{\nabla} f$, is itself gH-differentiable at x^0, that is, for each i, the function $\frac{\partial f}{\partial x_i} : X \longrightarrow F(\mathbb{R})$ is gH-differentiable at x^0, we say that f is twice gH-differentiable at x^0. The gH-partial derivative of $\frac{\partial f}{\partial x_i}$ is denoted by

$$\tilde{\mathcal{D}}_{ij}^2 f(x^0) = \frac{\partial^2 f}{\partial x_i x_j}(x^0), \qquad i \neq j,$$

and

$$\tilde{\mathcal{D}}_{ii}^2 f(x^0) = \frac{\partial^2 f}{\partial {x_i}^2}(x^0), \qquad i = j.$$

If for each $i, j = 1, 2, \dots, n$, the cross-partial derivative $\frac{\partial^2 f}{\partial x_i x_j}$ is continuous from X to $F(\mathbb{R})$, we say that f is twice continuously gH-differentiable.

Definition 3.13. The fuzzy vector function $F : X \to F^m(\mathbb{R})$ is twice gH-differentiable at $x^0 \in X$ if and only if f_i for each $i = 1, \ldots, m$ is twice gH-differentiable at x^0. We say that F is twice gH-differentiable on X if f_i, for each $i = 1, \ldots, m$, is twice gH-differentiable at each $x^0 \in X$. Moreover, F is m times gH-differentiable on X, if and only if all of the partial gH-derivatives of order m exist and are continuous.

Theorem 3.14. *Let $F : X \longrightarrow F^m(\mathbb{R})$ be a fuzzy vector function. If F is m-times gH-differentiable at $x^0 \in X$, for each $\alpha \in [0,1]$ the real-valued vector function $\underline{F}^\alpha + \overline{F}^\alpha : X \longrightarrow \mathbb{R}^m$ is m times differentiable at x^0.*

Proof. The proof follows from Theorem 3.9. □

4. Fuzzy multiobjective optimization

We consider the following unconstrained fuzzy multiobjective optimization problem (FMOP):

$$(FMOP)\ \min_{x\in\mathbb{R}^n}\ F(x) = (f_1(x), f_2(x), \ldots, f_m(x)), \tag{4.1}$$

where each objective function $f_i : X \subseteq \mathbb{R}^n \longrightarrow F(\mathbb{R}),\ i = 1, \ldots, m$ is a fuzzy-valued function and $X \subseteq \mathbb{R}^n$ is the domain of F which is assumed to be an open set. In the remainder of the paper we assume that F is twice gH-differentiable. We use the ordering relation defined in Definition 2.14 to investigate a solution concept for FMOP (4.1).

Definition 4.1. [37] Let $x^* \in X$.

(i) x^* is called a Pareto optimal solution of FMOP (4.1) if there exists no $x \in X$ such that $F(x) \prec F(x^*)$, i.e, there exists no $x \in X$ such that $f_i(x) \preceq f_i(x^*)$ for all $i = 1, \ldots, m$ and $f_j(x) \prec f_j(x^*)$ for at least one index j.

(ii) x^* is called a weakly Pareto optimal solution of FMOP (4.1) if there exists no $x \in X$ such that $f_i(x) \prec f_i(x^*)$ for all $i = 1, 2, \ldots, m$.

Definition 4.2. [37] Let $X \subset \mathbb{R}^n$ be an open set. We say that $x^* \in X$ is a locally Pareto optimal solution of FMOP (4.1) if there exists no $x \in N_\epsilon(x^*) \cap X$ such that $F(x) \prec F(x^*)$, where $N_\epsilon(x^*)$ is an ϵ-neighborhood of x^*.

In the following theorem, based on the sum of the endpoints of the objective functions, we present a sufficient condition for a locally Pareto optimal solution of FMOP (4.1). This theorem, generalizes Theorem 4 in [7] from fuzzy single objective to fuzzy multiobjective optimization problems.

Theorem 4.3. *Let $X \subset \mathbb{R}^n$ be an open set and $F : X \longrightarrow F^m(\mathbb{R})$ be a fuzzy vector function. If x^* is a local minimizer of the real-valued function $\underline{f}_i^\alpha + \overline{f}_i^\alpha$, for all $i = 1, \ldots, m$ and for all $\alpha \in [0,1]$ then x^* is a locally Pareto optimal solution of the FMOP (4.1).*

Proof. Suppose that x^* is not a locally Pareto optimal solution of FMOP (4.1). Then, there exists $x \in N_\epsilon(x^*)$ such that $F(x) \prec F(x^*)$. Therefore $f_i(x) \preceq f_i(x^*)\ \ \forall i = 1, \ldots, m$ and $f_j(x) \prec f_j(x^*)$ for at least one index j. Hence, $\underline{f}_j^\alpha(x) \le \underline{f}_j^\alpha(x^*)$ and $\overline{f}_j^\alpha(x) \le \overline{f}_j^\alpha(x^*)$, for all $\alpha \in [0,1]$. Moreover, there exists $\alpha^* \in [0,1]$ such that $\underline{f_j}^{\alpha^*}(x) < \underline{f_j}^{\alpha^*}(x^*)$ or $\overline{f_j}^{\alpha^*}(x) < \overline{f_j}^{\alpha^*}(x^*)$. Therefore,

$$\left(\underline{f}_j^{\alpha^*}(x) + \overline{f}_j^{\alpha^*}(x)\right) < \left(\underline{f}_j^{\alpha^*}(x^*) + \overline{f}_j^{\alpha^*}(x^*)\right).$$

This is a contradiction to local minimality of x^* for $\underline{f}_j^{\alpha^*} + \overline{f}_j^{\alpha^*}$. Therefore, x^* is a local Pareto optimal for FMOP (4.1). □

Now, we present the definition of a critical point.

Definition 4.4. Let $\alpha \in [0,1]$. A point $\overline{x}$ is critical (or stationary) for $\overline{F}^{\alpha} + \underline{F}^{\alpha}$, if

$$R(D(\overline{F}^{\alpha} + \underline{F}^{\alpha})(\overline{x})) \cap (-\mathbb{R}^{m}_{++}) = \emptyset, \tag{4.2}$$

where

$$D((\overline{F}^{\alpha} + \underline{F}^{\alpha})(\overline{x})) = \begin{bmatrix} \frac{\partial(\overline{f}_1^{\alpha}+\underline{f}_1^{\alpha})(\overline{x})}{\partial x_1} & \frac{\partial(\overline{f}_1^{\alpha}+\underline{f}_1^{\alpha})(\overline{x})}{\partial x_2} & \dots & \frac{(\overline{f}_1^{\alpha}+\underline{f}_1^{\alpha})(\overline{x})}{\partial x_n} \\ \frac{\partial(\overline{f}_2^{\alpha}+\underline{f}_2^{\alpha})(\overline{x})}{\partial x_1} & \frac{(\overline{f}_2^{\alpha}+\underline{f}_2^{\alpha})(\overline{x})}{\partial x_2} & \dots & \frac{\partial(\overline{f}_2^{\alpha}+\underline{f}_2^{\alpha})(\overline{x})}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial(\overline{f}_m^{\alpha}+\underline{f}_m^{\alpha})(\overline{x})}{\partial x_1} & \frac{\partial(\overline{f}_m^{\alpha}+\underline{f}_m^{\alpha})(\overline{x})}{\partial x_2} & \dots & \frac{\partial(\overline{f}_m^{\alpha}+\underline{f}_m^{\alpha})(\overline{x})}{\partial x_n} \end{bmatrix},$$

and $R(D)$ is the linear combination of the columns of D. It can be observed that for $m = 1$ (scalar optimization), relation (4.2) reduces to the one given in [7].

Relation (4.2) is equivalent to the following condition:

$$\left(s_1 \frac{\partial(\overline{f}_1^{\alpha} + \underline{f}_1^{\alpha})}{\partial x_1}(\overline{x}) + \ldots + s_n \frac{\partial(\overline{f}_1^{\alpha} + \underline{f}_1^{\alpha})}{\partial x_n}(\overline{x}), \ldots, s_1 \frac{\partial(\overline{f}_m^{\alpha} + \underline{f}_m^{\alpha})}{\partial x_1}(\overline{x}) + \ldots + s_n \frac{\partial(\overline{f}_m^{\alpha} + \underline{f}_m^{\alpha})}{\partial x_n}(\overline{x}) \right)^T \tag{4.3}$$

$$\cap(-\mathbb{R}^{m}_{++}) = \emptyset,$$

where $s = (s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$.

We suppose that at each point x_k, the values of $f_i(x_k)$, $\tilde{\nabla} f_i(x_k)$ and $\tilde{\nabla}^2 f_i(x_k)$, for all $i = 1, 2, \ldots, m$ can be calculated. From gH-differentiability of f_i, $\forall i = 1, \ldots, m$ and according to Theorems 3.9 and 3.14 we can calculate $\bigtriangledown(\underline{f}_i^{\alpha} + \overline{f}_i^{\alpha})(x_k), \forall i = 1, 2, \ldots, m$. Let $\alpha \in [0,1]$. If $\overline{x}$ is critical for $\overline{F}^{\alpha} + \underline{F}^{\alpha}$, from relation (4.3) it can be seen that for all $s \in \mathbb{R}^n$, there is an index $i = i(s) \in \{1, \ldots, m\}$ such that

$$\bigtriangledown \left(\overline{f}_i^{\alpha} + \underline{f}_i^{\alpha}\right)(\overline{x})^T s \geq 0. \tag{4.4}$$

Also, we can observe that if $\overline{x} \in \mathbb{R}^n$ is non-critical, there exists a $s \in \mathbb{R}^n$ so that

$$\bigtriangledown \left(\overline{f}_j^{\alpha} + \underline{f}_j^{\alpha}\right)(\overline{x})^T s < 0, \;\; \forall\, j = 1, \ldots, m.$$

Let $\alpha \in [0,1]$ and $x \in \mathbb{R}^n$ be non-critical. From continuous differentiability of $\overline{f}_j^{\alpha} + \underline{f}_j^{\alpha}, \forall j = 1, \ldots, m$ we have:

$$\lim_{t \to 0} \frac{\left(\overline{f}_j^{\alpha} + \underline{f}_j^{\alpha}\right)(x + ts) - \left(\overline{f}_j^{\alpha} + \underline{f}_j^{\alpha}\right)(x)}{t} = \bigtriangledown \left(\overline{f}_j^{\alpha} + \underline{f}_j^{\alpha}\right)(x)^T s < 0, \quad \forall\, j = 1, \ldots, m.$$

Hence, s is a descent direction of $\overline{F}^{\alpha} + \underline{F}^{\alpha}$ at x, that is, there exists $t_0 > 0$ such that the following inequalities hold for all $t \in (0, t_0]$:

$$\left(\overline{f}_j^{\alpha} + \underline{f}_j^{\alpha}\right)(x + ts) < \left(\overline{f}_j^{\alpha} + \underline{f}_j^{\alpha}\right)(x), \qquad \forall j = 1, \ldots, m. \tag{4.5}$$

Pareto optimality and criticality are related. In the following theorem, under some assumptions, we show that if $\overline{x}$ is a critical point then $\overline{x}$ is weakly Pareto optimal.

Theorem 4.5. *Assume that $F : X \to F^m(\mathbb{R})$ is continuously gH-differentiable. If X is convex, $\overline{F}^\alpha + \underline{F}^\alpha$ is strictly $\mathbb{R}^m$-convex, for all $\alpha \in [0,1]$ (i.e., $\overline{F}^\alpha + \underline{F}^\alpha$ is componentwise strictly convex) and $\overline{x} \in X$ is critical for $\overline{F}^\alpha + \underline{F}^\alpha$, for all $\alpha \in [0,1]$ then $\overline{x}$ is weakly Pareto optimal for FMOP (4.1).*

Proof. Consider $x \in X$ and $\alpha^* \in [0,1]$. Since $\overline{x}$ is critical for $\overline{F}^{\alpha^*} + \underline{F}^{\alpha^*}$, therefore there exists $j_{\alpha^*} \in \{1, 2, \ldots, m\}$ such that (4.4) holds for $s = x - \overline{x}$, that is:

$$\triangledown \left(\overline{f}^{\alpha^*}_{j_{\alpha^*}} + \underline{f}^{\alpha^*}_{j_{\alpha^*}}\right)(\overline{x})^T (x - \overline{x}) \geq 0. \tag{4.6}$$

Now, since $\overline{f}^{\alpha^*}_{j_{\alpha^*}} + \underline{f}^{\alpha^*}_{j_{\alpha^*}}$ is strictly convex, we have:

$$\begin{aligned}\left(\overline{f}^{\alpha^*}_{j_{\alpha^*}} + \underline{f}^{\alpha^*}_{j_{\alpha^*}}\right)(x) &> \left(\overline{f}^{\alpha^*}_{j_{\alpha^*}} + \underline{f}^{\alpha^*}_{j_{\alpha^*}}\right)(\overline{x}) + \triangledown \left(\overline{f}^{\alpha^*}_{j_{\alpha^*}} + \underline{f}^{\alpha^*}_{j_{\alpha^*}}\right)(\overline{x})^T (x - \overline{x}) \\ &\geq \left(\overline{f}^{\alpha^*}_{j_{\alpha^*}} + \underline{f}^{\alpha^*}_{j_{\alpha^*}}\right)(\overline{x}).\end{aligned} \tag{4.7}$$

Therefore, for any $\alpha \in [0,1]$ and each $x \in X$, there exists $j_\alpha \in \{1, 2, \ldots, m\}$ such that

$$\left(\overline{f}^{\alpha}_{j_{\alpha}} + \underline{f}^{\alpha}_{j_{\alpha}}\right)(x) > \left(\overline{f}^{\alpha}_{j_{\alpha}} + \underline{f}^{\alpha}_{j_{\alpha}}\right)(\bar{x}). \tag{4.8}$$

Now, by contradiction, assume that $\overline{x}$ is not a weakly Pareto optimal solution of FMOP (4.1). Therefore, there exists $x \in X$ such that $f_i(x) \prec f_i(\overline{x})$ for all $i = 1, 2, \ldots, m$. Hence, $\overline{f}^\alpha_i(x) \leq \overline{f}^\alpha_i(\overline{x})$ and $\underline{f}^\alpha_i(x) \leq \underline{f}^\alpha_i(\overline{x})$ for all $\alpha \in [0,1]$ and $i = 1, 2, \ldots, m$. Therefore, for any $\alpha \in [0,1]$, we have

$$\overline{f}^\alpha_i(x) + \underline{f}^\alpha_i(x) \leq \overline{f}^\alpha_i(\overline{x}) + \underline{f}^\alpha_i(\overline{x}), \quad \forall\, i = 1, 2, \ldots, m, \tag{4.9}$$

which is a contradiction to (4.8). ☐

In the remainder of the paper, we assume that $\underline{F}^\alpha + \overline{F}^\alpha$ for each $\alpha \in [0,1]$ is $\mathbb{R}^m$-strictly convex, that is $\triangledown^2(\underline{f}^\alpha_j + \overline{f}^\alpha_j)(x),\ \forall j = 1, 2, \ldots, m,\ x \in X$, is positive definite.

5. Newton method

In this section, we will define the Newton direction for the FMOP (4.1). We assume that the objective functions of the FMOP (4.1) are twice continuously gH-differentiable. We define the Newton direction for the unconstrained fuzzy multiobjective optimization problem (4.1) at $x \in X$, as the optimal solution of the following problem and is denoted by $s(x)$:

$$\begin{aligned}&\min \max_{j=1,\ldots,m} \int_0^1 \left(\triangledown \left(\overline{f}^\alpha_j + \underline{f}^\alpha_j\right)(x)^T s + \frac{1}{2} s^T \triangledown^2 \left(\overline{f}^\alpha_j + \underline{f}^\alpha_j\right)(x) s\right) \mathrm{d}\alpha \\ &\text{s.t.} \qquad s \in \mathbb{R}^n.\end{aligned} \tag{5.1}$$

Function $g_j(s) = \triangledown(\overline{f}^\alpha_j + \underline{f}^\alpha_j)(x)^T s + \frac{1}{2} s^T \triangledown^2 (\overline{f}^\alpha_j + \underline{f}^\alpha_j)(x) s$ is strictly convex, for all $j \in \{1, \ldots, m\}$ and all $\alpha \in [0,1]$, since

$$\triangledown^2 g_j(s) = \triangledown^2 \left(\overline{f}^\alpha_j + \underline{f}^\alpha_j\right)(x).$$

Since $\overline{f}^\alpha_j + \underline{f}^\alpha_j$ is strictly convex for all $j = 1, \ldots, m$ and all $\alpha \in [0,1]$, (5.1) has a unique solution. Let $\theta(x)$ be the optimal objective function value for (5.1), hence we have

$$\theta(x) = \min_{s \in \mathbb{R}^n} \max_{j=1,\ldots,m} \int_0^1 \left(\triangledown \left(\overline{f}^\alpha_j + \underline{f}^\alpha_j\right)(x)^T s + \frac{1}{2} s^T \triangledown^2 \left(\overline{f}^\alpha_j + \underline{f}^\alpha_j\right)(x) s\right) \mathrm{d}\alpha, \tag{5.2}$$

and

$$s(x) = arg \min_{s\in\mathbb{R}^n} \max_{j=1,\dots,m} \int_0^1 \left(\triangledown \left(\overline{f}_j^\alpha + \underline{f}_j^\alpha\right)(x)^T s + \frac{1}{2} s^T \triangledown^2 \left(\overline{f}_j^\alpha + \underline{f}_j^\alpha\right)(x) s \right) \mathrm{d}\alpha. \tag{5.3}$$

Next, we state some important lemmas. We prove some properties of function $\theta(x)$ and analyze its relation with criticality. Indeed, for a non-critical point x, due to Lemma 5.2, we have that $\theta(x) < 0$. Algorithm 1 uses the stopping criterion $\theta(\overline{x}) = 0$. So, $\overline{x}$ is critical point for $\overline{F}^\alpha + \underline{F}^\alpha$, for all $\alpha \in [0,1]$ and under Theorem 4.5, $\overline{x}$ is weakly Pareto optimal for FMOP (4.1).

Lemma 5.1. *Consider $\theta(x)$ as defined by (5.2). Then, for any $x \in X$, $\theta(x) \leq 0$.*

Proof. Let $s = 0$, so we have

$$\theta(x) \leq \max_{j=1,\dots,m} \int_0^1 \left(\triangledown(\overline{f}_j^\alpha + \underline{f}_j^\alpha)(x)^T 0 + \frac{1}{2} 0^T \triangledown^2 (\overline{f}_j^\alpha + \underline{f}_j^\alpha)(x) 0 \right) \mathrm{d}\alpha = 0.$$

□

Lemma 5.2. *Consider $\theta(x)$ as defined by (5.2). If x is non-critical for some $\overline{F}^{\alpha_i} + \underline{F}^{\alpha_i}$, $\alpha_i \in [0,1]$ then $\theta(x) < 0$.*

Proof. From Lemma 5.1, we see that $\theta(x) \leq 0$. Consider an arbitrary partition of the interval $[0,1]$ as the form $0 = t_0 < t_1 < \ldots < t_l = 1$, where l tends to ∞. Assume that $\alpha_i \in [t_{i-1}, t_i]$, $\forall i = 1, 2, \ldots, l$. Therefore

$$\begin{aligned}\theta(x) = \min_{s\in\mathbb{R}^n} \max_{j=1,\dots,m} \Big[& \triangledown \left(\overline{f_j}^{\alpha_1} + \underline{f_j}^{\alpha_1}\right)(x)^T s(t_1 - t_0) + \frac{1}{2} s^T \triangledown^2 \left(\overline{f_j}^{\alpha_1} + \underline{f_j}^{\alpha_1}\right)(x) s(t_1 - t_0) \\ & + \ldots + \triangledown \left(\overline{f_j}^{\alpha_i} + \underline{f_j}^{\alpha_i}\right)(x)^T s(t_i - t_{i-1}) + \frac{1}{2} s^T \triangledown^2 (\overline{f_j}^{\alpha_i} + \underline{f_j}^{\alpha_i})(x) s(t_i - t_{i-1}) \\ & + \ldots + \triangledown \left(\overline{f_j}^{\alpha_l} + \underline{f_j}^{\alpha_l}\right)(x)^T s(t_l - t_{l-1}) + \frac{1}{2} s^T \triangledown^2 \left(\overline{f_j}^{\alpha_l} + \underline{f_j}^{\alpha_l}\right)(x) s(t_l - t_{l-1}) \Big].\end{aligned}$$

Now suppose that $\varphi_j^{\alpha_i}(s) = \triangledown(\overline{f_j}^{\alpha_i} + \underline{f_j}^{\alpha_i})(x)^T s(t_i - t_{i-1}) + \frac{1}{2} s^T \triangledown^2 (\overline{f_j}^{\alpha_i} + \underline{f_j}^{\alpha_i})(x) s(t_i - t_{i-1})$, $\forall i = 1, 2, \ldots, l$. Therefore, we have

$$\begin{aligned}\theta(x) &= \min_{s\in\mathbb{R}^n} \max_{j=1,\dots,m} \left[\varphi_j^{\alpha_1}(s) + \ldots + \varphi_j^{\alpha_i}(s) + \ldots + \varphi_j^{\alpha_l}(s)\right] \\ &\leq \min_{s\in\mathbb{R}^n} \max_{j=1,\dots,m} \varphi_j^{\alpha_1}(s) + \ldots + \min_{s\in\mathbb{R}^n} \max_{j=1,\dots,m} \varphi_j^{\alpha_i}(s) + \ldots + \min_{s\in\mathbb{R}^n} \max_{j=1,\dots,m} \varphi_j^{\alpha_l}(s).\end{aligned} \tag{5.4}$$

Moreover

$$\min_{s\in\mathbb{R}^n} \max_{j=1,\dots,m} \varphi_j^{\alpha_i}(s) \leq \max_{j=1,\dots,m} \varphi_j^{\alpha_i}(0) = 0 \leq 0, \qquad \forall i \in \{1, \ldots, l\}. \tag{5.5}$$

Now assume that x is a non-critical point for $\underline{F}^{\alpha_i} + \overline{F}^{\alpha_i}$, for some $\alpha_i \in [t_{i-1}, t_i]$ then there exists $s \in \mathbb{R}^n$ such that the following relationship is established:

$$\triangledown \left(\overline{f}_j^{\alpha_i} + \underline{f_j}^{\alpha_i}\right)(x)^T s < 0, \ \forall j = 1, 2, \ldots, m.$$

Due to this relationship, for $t > 0$ we have:

$$\begin{aligned}\min_{s\in\mathbb{R}^n} \max_{j=1,\dots,m} \varphi_j^{\alpha_i}(ts) &\leq \max_{j=1,\dots,m} \left(\triangledown \left(\overline{f_j}^{\alpha_i} + \underline{f_j}^{\alpha_i}\right)(x)^T ts + \frac{1}{2} ts^T \triangledown^2 \left(\overline{f_j}^{\alpha_i} + \underline{f_j}^{\alpha_i}\right)(x) ts \right) \\ &= t \max_{j=1,\dots,m} \left(\triangledown \left(\overline{f_j}^{\alpha_i} + \underline{f_j}^{\alpha_i}\right)(x)^T s + \frac{1}{2} s^T \triangledown^2 \left(\overline{f_j}^{\alpha_i} + \underline{f_j}^{\alpha_i}\right)(x) ts \right).\end{aligned}$$

Therefore, for $t > 0$ small enough, the right hand side of the above inequality is negative, that is:

$$\min_{s\in\mathbb{R}^n} \max_{j=1,\ldots,m} \varphi_j^{\alpha_i}(ts) < 0. \tag{5.6}$$

Therefore, letting $\varphi(\alpha) = \min_{s\in\mathbb{R}^n} \max\limits_{j=1,\ldots,m} \varphi_j^{\alpha}(s)$, we have $\varphi(\alpha_i) < 0$. But, since $\varphi(\alpha)$ is a continuous function, there exists a neighborhood of α_i such that

$$\varphi(\alpha) < 0,\ \forall\alpha \in [\alpha_i - \varepsilon, \alpha_i + \varepsilon],\ \varepsilon > 0. \tag{5.7}$$

So, if x is non-critical for $\underline{F}^{\alpha_i} + \overline{F}^{\alpha_i}$, for some $\alpha_i \in [t_i, t_{i+1}]$, then from equations (5.4)–(5.7) and by splitting the integral into three separate integrals on the intervals $[0, \alpha_i - \varepsilon], [\alpha_i - \varepsilon, \alpha_i + \varepsilon]$ and $[\alpha_i + \varepsilon, 1]$ we have $\theta(x) < 0$. □

We use the following lemma to choose an appropriate step length t, for the Newton algorithm.

Lemma 5.3. *Let $\theta(x) < 0$. Then, for each $0 < \sigma < 1$ there exists $\bar{t} \in (0, 1]$ such that*

$$x + ts(x) \in X \ \text{and} \ \int_0^1 \left(\left(\overline{f}_j^{\alpha} + \underline{f}_j^{\alpha}\right)(x + ts(x))\right) \mathrm{d}\alpha \leq \int_0^1 \left(\left(\overline{f}_j^{\alpha} + \underline{f}_j^{\alpha}\right)(x) + \sigma t\theta(x)\right) \mathrm{d}\alpha, \tag{5.8}$$

for any $t \in [0, \bar{t}]$ and $j \in \{1, \ldots, m\}$.

Proof. Since X is an open set and $x \in X$, it follows that there exists $0 \leq \bar{t} \leq 1$ such that $x + ts(x) \in X$ for all $t \in [0, \bar{t}]$. Hence, for $t \in [0, \bar{t}]$ we have

$$\left(\overline{f}_j^{\alpha} + \underline{f}_j^{\alpha}\right)(x + ts(x)) = \left(\overline{f}_j^{\alpha} + \underline{f}_j^{\alpha}\right)(x) + t \bigtriangledown \left(\overline{f}_j^{\alpha} + \underline{f}_j^{\alpha}\right)(x)^T s(x) + o_j(t),\ \ j = 1, 2, \ldots, m, \alpha \in [0, 1],$$

where $\lim_{t\to 0^+} o_j(t)/t = 0$. However, $\int_o^1 \bigtriangledown(\overline{f}_j^{\alpha} + \underline{f}_j^{\alpha})(x)^T s(x) \leq \theta(x) = \int_0^1 \theta(x)d\alpha$. Therefore for all $\alpha \in [0, 1]$ and $t \in [0, \bar{t}]$, we conclude that:

$$\int_0^1 \left(\overline{f}_j^{\alpha} + \underline{f}_j^{\alpha}\right)(x + ts(x))\mathrm{d}\alpha \leq \int_0^1 \left(\left(\overline{f}_j^{\alpha} + \underline{f}_j^{\alpha}\right)(x) + t\theta(x) + o_j(t)\right) \mathrm{d}\alpha$$

$$= \int_0^1 \left(\left(\overline{f}_j^{\alpha} + \underline{f}_j^{\alpha}\right)(x) + t\sigma\theta(x) + t[(1 - \sigma)\theta(x) + o_j(t)/t]\right) \mathrm{d}\alpha,\ \forall j = 1, 2, \ldots, m.$$

Since $\theta(x) < 0$, it follows that $\int_0^1 t[(1 - \sigma)\theta(x) + o_j(t)/t]d\alpha \leq 0$, for $t \in [0, \bar{t}]$ small enough. Hence

$$\int_0^1 \left(\left(\overline{f}_j^{\alpha} + \underline{f}_j^{\alpha}\right)(x + ts(x))\right) \mathrm{d}\alpha \leq \int_0^1 \left(\left(\overline{f}_j^{\alpha} + \underline{f}_j^{\alpha}\right)(x) + \sigma t\theta(x)\right) \mathrm{d}\alpha,$$

for all $t \in [0, \bar{t}]$ and $j = 1, 2, \ldots, m$. □

The algorithmic implementation of Newton method for fuzzy multiobjective optimization problems is given in Algorithm 1. In this algorithm we pick $x^{(0)} \in \mathbb{R}^n$ arbitrarily. At each step, by solving the min-max optimization problem (5.1) we obtain the descent direction. Thereafter, we determine the step length by means of Lemma 5.3. Note that, due to Lemma 5.2, if $x^{(k)}$ is a non-critical point, we have $\theta(x^{(k)}) < 0$ and therefore a descent direction. Hence, selecting a step length by using Lemma 5.3 leads to a decrease of the objective function.

The following theorem provides a sufficient condition for local convergence of the Newton Algorithm 1.

Algorithm 1. Newton algorithm for fuzzy multiobjective optimization problems.

Step 0. Initialization: Let $x^{(0)}$ be the initial decision vector chosen from X. Consider $0 < \sigma < 1$, set $k := 0$ and define $J = \{1/2^n \mid n = 0, 1, 2, \ldots\}$.
Step 1. Generation of search direction: Solve the problem (5.1) to obtain $\theta(x^{(k)})$ and $s(x^{(k)})$ as in (5.2) and (5.3).

Step 2. If $\theta(x^{(k)}) = 0$, stop. Else, go to **Step 3**.
Step 3. Choose t_k as the largest $t \in J$ such that it satisfies in the following relationships:

$$x^{(k)} + ts(x^{(k)}) \in X$$

and

$$\int_0^1 \left((\overline{f}_j^\alpha + \underline{f}_j^\alpha)(x + ts(x))\right) \mathrm{d}\alpha \leq \int_0^1 \left((\overline{f}_j^\alpha + \underline{f}_j^\alpha)(x) + \sigma t\theta(x)\right) \mathrm{d}\alpha,$$

$j = 1, \ldots, m$.
Step 4. Define

$$x^{(k+1)} = x^{(k)} + t_k s(x^{(k)})$$

and set $k := k + 1$. Go to **Step 1.**

Theorem 5.4. *Let $\{x^{(k)}\}_k$ be a sequence generated by Algorithm 1. Suppose that $V \subset X$, $0 < \sigma < 1$, $a, b, r, \delta, \epsilon > 0$ and*

(a) $aI \leq \int_0^1 \left(\nabla^2(\overline{f}_j^\alpha + \underline{f}_j^\alpha)(x)\right) \mathrm{d}\alpha \leq bI$, *for all $x \in V$ and all $j = 1, \ldots, m$,*
(b) $\| \int_0^1 \left(\nabla^2(\overline{f}_j^\alpha + \underline{f}_j^\alpha)(x) - \nabla^2(\overline{f}_j^\alpha + \underline{f}_j^\alpha)(y)\right) \mathrm{d}\alpha\| \leq \epsilon$, *for all $x, y \in V$ that $\|x - y\| < \delta$,*
(c) $\epsilon/a \leq 1 - \sigma$,
(d) $N_r(x^{(0)}) \subset V$,
(e) $\|s(x^{(0)})\| \leq \min\{\delta, r(1 - \epsilon/a)\}$.

Then, for all k, we have:

(1) $\|x^{(k)} - x^{(0)}\| \leq \|s(x^{(0)})\| \dfrac{1 - (\epsilon/a)^k}{1 - \epsilon/a}$,
(2) $\|s(x^{(k)})\| \leq \|s(x^{(0)})\|(\epsilon/a)^k$,
(3) $t_k = 1$,
(4) $\|s(x^{(k+1)})\| \leq \|s(x^{(k)})\|(\epsilon/a)$.

Moreover, the sequence $\{x^{(k)}\}_k$ converges to some locally Pareto optimal point $\overline{x} \in \mathbb{R}^n$ with

$$\|\overline{x} - x^{(0)}\| \leq \frac{\|s(x_0)\|}{1 - \epsilon/a} \leq r. \tag{5.9}$$

The convergence rate of $\{x^{(k)}\}_k$ is superlinear.

Proof. The proof follows from the classical proof of the crisp Newton algorithm for multiobjective optimization problems [9]. □

6. Numerical examples

In this section numerical examples are provided to illustrate the mentioned methodology. We note that the objective functions of the fuzzy MOPs are twice continuously gH-differentiable. All examples are executed within MATLAB (R2013a). The implementation employs the termination condition $\theta(x^{(k)}) > -\epsilon$, where $\epsilon > 0$ is a pre-specified tolerance level, in order to stop at the point $x^{(k)}$. It is important to note that (weakly) Pareto optimal solution of an FMOP is not necessarily unique. Therefore, if the user starts with any initial point and runs the algorithm then (s)he may reach at one of these (weakly) Pareto optimal points. If the assumptions of Theorem 4.5 are satisfied, every accumulation point of the sequence $\{x^{(k)}\}$ can be a weakly Pareto optimal or Pareto optimal solution of FMOP (4.1).

Example 6.1. As the first example, consider the following nonlinear unconstrained fuzzy single objective optimization problem, given in [7, 26]:

$$\min_{x\in\mathbb{R}^2} F(x) = f(x),$$

where

$$f(x) = (-1,1,3)x_1^2 + (0,1,2)x_1x_2 + (1,2,4)x_2^2.$$

With the help of fuzzy arithmetics, we obtain

$$\left(\overline{f}^{\alpha} + \underline{f}^{\alpha}\right)(x) = 2x_1^2 + 2x_1x_2 + (5-\alpha)x_2^2.$$

Therefore

$$\int_0^1 \left(\overline{f}^{\alpha} + \underline{f}^{\alpha}\right)(x)\mathrm{d}\alpha = \left(\overline{f} + \underline{f}\right)(x) = 2x_1^2 + 2x_1x_2 + 4.5x_2^2.$$

Then

$$\triangledown\left(\overline{f} + \underline{f}\right)(x) = \begin{bmatrix} 4x_1 + 2x_2 \\ 2x_1 + 9x_2 \end{bmatrix}, \qquad \triangledown^2\left(\overline{f} + \underline{f}\right)(x) = \begin{bmatrix} 4 & 2 \\ 2 & 9 \end{bmatrix}.$$

Consider the initial point $x^{(0)} = (1,2)$ and the stopping condition $\theta(x^{(k)}) \geq -10^{-10}$. A program in MATLAB is written following the Algorithm 1. A weakly Pareto optimal solution of this problem is found after two iterations as $x^{(2)} = (-10^{-8}\times 0.2396, -10^{-8}\times 0.2660) \simeq (0,0)$. The iterations of $x^{(k)}$ are given in Table 1.

TABLE 1. Performance of Newton Algorithm 1 on Example 6.1.

k	$(x^{(k)})^T$	$(\overline{f}+\underline{f})(x^{(k)})$	t_k	$s(x^{(k)})$	$\theta(x^{(k)})$
0	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$	24	1	$(-1.0000, -2.0000)$	-24.0000
1	$\begin{pmatrix} -10^{-7}\times 0.1401 \\ 10^{-7}\times 0.0896 \end{pmatrix}$	$10^{-16}\times 5.0298$	0.0020	$(10^{-5}\times 0.5946, -10^{-5}\times 0.5952)$	$-10^{-12}\times 2.1849$
2	$\begin{pmatrix} -10^{-8}\times 0.2396 \\ -10^{-8}\times 0.2660 \end{pmatrix}$	$10^{-17}\times 5.6067$			

Example 6.2. Consider the following bi-objective nonlinear programming problem with fuzzy parameters:

$$(FMOP)\ \min_{x\in\mathbb{R}^3} F(x) = (f_1(x), f_2(x)),$$

where

$$f_1(x) = \left(0, \frac{1}{2}, 1\right)x_1^2 + \left(0, \frac{1}{2}, 1\right)x_2^2 + \left(0, \frac{1}{2}, 1\right)x_3^2,$$

$$f_2(x) = \left(0, \frac{1}{6}, \frac{1}{3}\right)x_1^2 + \left(0, \frac{1}{6}, \frac{1}{3}\right)x_2^2 + \left(0, \frac{1}{6}, \frac{1}{3}\right)x_3^2 + \left(\frac{-4}{3}, \frac{-2}{3}, 0\right)x_1$$
$$+ \left(\frac{-4}{3}, \frac{-2}{3}, 0\right)x_2 + \left(\frac{-4}{3}, \frac{-2}{3}, 0\right)x_3 + (0,2,4).$$

TABLE 2. Performance of Newton Algorithm 1 on Example 6.2.

k	$x^{(k)}$	$\left((\overline{f_1}+\underline{f_1})(x^{(k)}), (\overline{f_2}+\underline{f_2})(x^{(k)})\right)$	t_k	$\left(s(x^{(k)})\right)^T$	$\theta(x^{(k)})$
0	$(-25, -15, -12)$	$(994.0000, 404.6667)$	1	$\begin{pmatrix}27.0000\\17.0000\\14.0000\end{pmatrix}$	-404.6667
1	$(2.0000, 2.0000, 2.0000)$	$(12.0001, 0.0000)$	$10^{-9}\times 7.4506$	$\begin{pmatrix}-10^{-4}\times 0.7523\\-10^{-4}\times 0.6388\\-10^{-4}\times 0.2453\end{pmatrix}$	$-10^{-7}\times 3.2071$
2	$(2.0000, 2.0000, 2.0000)$	$(12.0001, 0.0000)$	1	$\begin{pmatrix}-10^{-4}\times 0.2430\\10^{-4}\times 0.0253\\10^{-4}\times 0.1039\end{pmatrix}$	$-10^{-10}\times 2.5853$
3	$(2.0000, 2.0000, 2.0000)$	$(12.0000, 0.0000)$	1	$\begin{pmatrix}-10^{-5}\times 0.0955\\-10^{-5}\times 0.1264\\-10^{-5}\times 0.1345\end{pmatrix}$	$-10^{-12}\times 4.4702$
4	$(2.0000, 2.0000, 2.0000)$	$(12.0000, 0.0000)$	1	$\begin{pmatrix}-10^{-6}\times 0.3754\\-10^{-6}\times 0.3712\\-10^{-6}\times 0.3802\end{pmatrix}$	$-10^{-13}\times 5.2942$
5	$(2.0000, 2.0000, 2.0000)$	$(12.0000, 0.0000)$			

With the help of fuzzy arithmetics, we can write

$$\int_0^1 \left(\overline{f_1}^\alpha + \underline{f}_1^\alpha\right)(x)\mathrm{d}\alpha = x_1^2 + x_2^2 + x_3^2,$$
$$\int_0^1 \left(\overline{f_2}^\alpha + \underline{f}_2^\alpha\right)(x)\mathrm{d}\alpha = \frac{1}{3}x_1^2 + \frac{1}{3}x_2^2 + \frac{1}{3}x_3^2 - \frac{4}{3}x_1 - \frac{4}{3}x_2 - \frac{4}{3}x_3 + 4.$$

Then

$$\triangledown\left(\overline{f}_1 + \underline{f}_1\right)(x) = (2x_1, 2x_2, 2x_3), \qquad \triangledown^2\left(\overline{f}_1 + \underline{f}_1\right)(x) = \begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix},$$

and

$$\triangledown\left(\overline{f}_2 + \underline{f}_2\right)(x) = \left(\frac{2}{3}x_1 - \frac{4}{3}, \frac{2}{3}x_2 - \frac{4}{3}, \frac{2}{3}x_3 - \frac{4}{3}\right), \qquad \triangledown^2\left(\overline{f}_2 + \underline{f}_2\right)(x) = \begin{bmatrix}\frac{2}{3}&0&0\\0&\frac{2}{3}&0\\0&0&\frac{2}{3}\end{bmatrix}.$$

Applying Algorithm 1 with initial point $x^{(0)} = (-25, -15, -12)$ and stopping condition $\theta(x^{(k)}) \geq -10^{-12}$, we obtain a weakly Pareto optimal solution of this problem after five iterations as $x^{(5)} = (2.0000, 2.0000, 2.0000)$. The performance of Algorithm 1 is shown in Table 2.

Example 6.3. Consider the following unconstrained nonlinear multiobjective optimization problem with fuzzy parameters:

$$(FMOP)\ \min_{x\in\mathbb{R}^3} F(x) = (f_1(x), f_2(x), f_3(x)),$$

where

$$f_1(x) = \left(\frac{1}{36}, \frac{2}{36}, \frac{3}{36}\right) x_1 + \left(\frac{-3}{36}, \frac{-2}{36}, \frac{-1}{36}\right) + \left(\frac{1}{18}, \frac{1}{9}, \frac{3}{18}\right) x_2 + \left(\frac{-1}{3}, \frac{-2}{9}, \frac{-1}{9}\right)$$

$$+ \left(\frac{1}{12}, \frac{1}{6}, \frac{1}{4}\right) x_3 + \left(-1, \frac{-1}{3}, \frac{-1}{3}\right) + \left(\frac{1}{2}, 1, \frac{3}{2}\right) x_1^2 + \left(\frac{3}{4}, \frac{3}{2}, \frac{9}{4}\right) x_2^2 + \left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right) x_3^2,$$

$$f_2(x) = \mathrm{e}^{\left(\frac{1}{9}, \frac{1}{9}, \frac{1}{3}\right)x_1 + \left(\frac{1}{12}, \frac{1}{6}, \frac{1}{4}\right)x_2 + \left(\frac{1}{12}, \frac{1}{6}, \frac{1}{4}\right)x_3} + \left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right) x_1^2 + \left(\frac{1}{3}, \frac{1}{3}, 1\right) x_2^2 + \left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right) x_3^2,$$

$$f_3(x) = \left(\frac{3}{48}, \frac{1}{8}, \frac{3}{16}\right) \mathrm{e}^{\left(-1, \frac{-1}{3}, \frac{-1}{3}\right)x_1} + \left(\frac{1}{12}, \frac{1}{6}, \frac{1}{4}\right) \mathrm{e}^{\left(\frac{-3}{4}, \frac{-2}{4}, \frac{-1}{4}\right)x_2} + \left(\frac{1}{16}, \frac{1}{8}, \frac{3}{16}\right) \mathrm{e}^{\left(\frac{-3}{2}, \frac{-1}{2}, \frac{1}{2}\right)x_3}.$$

We have

$$\int_0^1 \left(\overline{f_1}^{\alpha} + \underline{f}_1^{\alpha}\right)(x)\mathrm{d}\alpha = \frac{1}{9}(x_1 - 1) + \frac{2}{9}(x_2 - 2) + \frac{3}{9}(x_3 - 3) + 2x_1^2 + 3x_2^2 + x_3^2,$$

$$\int_0^1 \left(\overline{f_2}^{\alpha} + \underline{f}_2^{\alpha}\right)(x)\mathrm{d}\alpha = \mathrm{e}^{\frac{x_1}{3} + \frac{x_2}{3} + \frac{x_3}{3}} + x_1^2 + x_2^2 + x_3^2,$$

$$\int_0^1 \left(\overline{f_3}^{\alpha} + \underline{f_3}^{\alpha}\right)(x)\mathrm{d}\alpha = \frac{1}{4}\mathrm{e}^{-x_1} + \frac{1}{3}\mathrm{e}^{-x_2} + \frac{1}{4}\mathrm{e}^{-x_3}.$$

Also,

$$\triangledown\left(\overline{f}_1 + \underline{f}_1\right)(x) = \begin{bmatrix} \frac{1}{9} + 4x_1 \\ \frac{2}{9} + 6x_2 \\ \frac{3}{9} + 2x_3 \end{bmatrix}, \qquad \triangledown^2\left(\overline{f}_1 + \underline{f}_1\right)(x) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$\triangledown\left(\overline{f}_2 + \underline{f}_2\right)(x) = \begin{bmatrix} \frac{1}{3}\mathrm{e}^{\frac{x_1}{3} + \frac{x_2}{3} + \frac{x_3}{3}} + 2x_1 \\ \frac{1}{3}\mathrm{e}^{\frac{x_1}{3} + \frac{x_2}{3} + \frac{x_3}{3}} + 2x_2 \\ \frac{1}{3}\mathrm{e}^{\frac{x_1}{3} + \frac{x_2}{3} + \frac{x_3}{3}} + 2x_3 \end{bmatrix},$$

$$\triangledown^2\left(\overline{f}_2 + \underline{f}_2\right)(x) = \begin{bmatrix} \frac{1}{9}\mathrm{e}^{\frac{x_1}{3} + \frac{x_2}{3} + \frac{x_3}{3}} + 2 & \frac{1}{9}\mathrm{e}^{\frac{x_1}{3} + \frac{x_2}{3} + \frac{x_3}{3}} & \frac{1}{9}\mathrm{e}^{\frac{x_1}{3} + \frac{x_2}{3} + \frac{x_3}{3}} \\ \frac{1}{9}\mathrm{e}^{\frac{x_1}{3} + \frac{x_2}{3} + \frac{x_3}{3}} & \frac{1}{9}\mathrm{e}^{\frac{x_1}{3} + \frac{x_2}{3} + \frac{x_3}{3}} + 2 & \frac{1}{9}\mathrm{e}^{\frac{x_1}{3} + \frac{x_2}{3} + \frac{x_3}{3}} \\ \frac{1}{9}\mathrm{e}^{\frac{x_1}{3} + \frac{x_2}{3} + \frac{x_3}{3}} & \frac{1}{9}\mathrm{e}^{\frac{x_1}{3} + \frac{x_2}{3} + \frac{x_3}{3}} & \frac{1}{9}\mathrm{e}^{\frac{x_1}{3} + \frac{x_2}{3} + \frac{x_3}{3}} + 2 \end{bmatrix},$$

TABLE 3. Performance of Newton Algorithm 1 on Example 6.3.

k	$x^{(k)}$	$\left((\overline{f_1}+\underline{f_1})(x^{(k)}), (\overline{f_2}+\underline{f_2})(x^{(k)}), (\overline{f_3}+\underline{f_3})(x^{(k)})\right)$	t_k	$\left(s(x^{(k)})\right)^T$	$\theta(x^{(k)})$
0	$(15, 2, 2)$	$(467.2222, 796.0302, 0.0789)$	1	$\begin{pmatrix} -2.3696 \\ 1.0000 \\ 1.0000 \end{pmatrix}$	-0.0395
1	$(12.6304, 3.0000, 3.0000)$	$(356.5707, 675.3115, 0.0290)$	0.5000	$\begin{pmatrix} -1.8304 \\ 0.9999 \\ 0.9998 \end{pmatrix}$	-0.0145
2	$(11.7152, 3.5000, 3.5000)$	$(325.1815, 673.7706, 0.0176)$	0.5000	$\begin{pmatrix} -1.8605 \\ 0.9994 \\ 0.9993 \end{pmatrix}$	-0.0088
3	$(10.7850, 3.9997, 3.9996)$	$(298.4845, 672.2666, 0.0107)$	1	$\begin{pmatrix} -1.8866 \\ 0.9976 \\ 0.9969 \end{pmatrix}$	-0.0053
4	$(8.8984, 4.9973, 4.9965)$	$(260.4549, 672.2654, 0.0040)$	1	$\begin{pmatrix} -1.8524 \\ 0.9572 \\ 0.9430 \end{pmatrix}$	-0.0018
5	$(7.0460, 5.9545, 5.9395)$	$(243.4696, 672.2639, 0.0017)$	1	$\begin{pmatrix} -0.8930 \\ 0.4978 \\ 0.3990 \end{pmatrix}$	$-10^{-4} \times 2.5236$
6	$(6.1530, 6.4524, 6.3385)$	$(243.4693, 672.2637, 0.0015)$	1	$\begin{pmatrix} 0.0667 \\ -0.0307 \\ -0.0359 \end{pmatrix}$	$-10^{-6} \times 1.7724$
7	$(6.2197, 6.4217, 6.3026)$	$(243.4693, 672.2637, 0.0015)$	1	$\begin{pmatrix} 0.0012 \\ -0.0005 \\ -0.0006 \end{pmatrix}$	$-10^{-10} \times 5.1166$
8	$(6.2209, 6.4211, 6.3020)$	$(243.4693, 672.2637, 0.0015)$	1	$\begin{pmatrix} 10^{-6} \times 0.3219 \\ -10^{-6} \times 0.1506 \\ -10^{-6} \times 0.1709 \end{pmatrix}$	$-10^{-17} \times 4.6776$
9	$(6.2209, 6.4211, 6.3020)$	$(243.4693, 672.2637, 0.0015)$			

and

$$\triangledown\left(\overline{f}_3+\underline{f}_3\right)(x)=\begin{bmatrix}\frac{-1}{4}\mathrm{e}^{-x_1}\\ \frac{-1}{3}\mathrm{e}^{-x_2}\\ \frac{-1}{4}\mathrm{e}^{-x_3}\end{bmatrix},\qquad \triangledown^2\left(\overline{f}_3+\underline{f}_3\right)(x)=\begin{bmatrix}\frac{1}{4}\mathrm{e}^{-x_1} & 0 & 0\\ 0 & \frac{1}{3}\mathrm{e}^{-x_2} & 0\\ 0 & 0 & \frac{1}{4}\mathrm{e}^{-x_3}\end{bmatrix}.$$

Consider an initial point $x^{(0)}=(15,2,2)$ and stopping condition $\theta(x^{(k)})\geq -10^{-10}$. A weakly Pareto optimal solution of this problem is found after nine iterations as $x^{(9)}=(6.2209,6.4211,6.3020)$. Table 3 exhibits the iterations of $x^{(k)}$ for this example.

7. Conclusions

Employing the obtained theoretical results, we proposed a Newton algorithm for solving unconstrained fuzzy multiobjective problems. In the proposed algorithmic procedure, we employed a max-min optimization problem to obtain a descent direction. Moreover, we applied an Armijo-like rule to find the step length. We showed that for non-critical points the algorithm decreases the objective function values. The convergence of the proposed algorithm was discussed. The suggested procedure was accompanied by numerical examples to illustrate its efficiency.

However, to apply the proposed Newton algorithm it is necessary that the objective function be twice continuously gH-differentiable. Therefore, proposing a quasi-Newton method that approximates the Hessian matrix in each iteration, can be worth studying. Furthermore, future research can be on the extending the suggested Newton algorithm for solving a fuzzy vector optimization problem. Moreover, finding a Newton procedure for constrained fuzzy multiobjective optimization problems will be considered in forthcoming papers.

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