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RECYCLING OF LIFETIME DEPENDENT DETERIORATED PRODUCTS THROUGH DIFFERENT SUPPLY CHAINS

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Abstract. Conservation of natural resources in order to protect the environment, support the economy, and offer a better life to living beings has become an urgent need of modern business so that future generations can survive within available resources and in a healthy environment. Recycling of used products plays an important role in the conservation of natural resources and the development of sustainable business for deteriorating products because the number of these items increases with time, which creates economic loss and environmental pollution. This paper considers the production and cycle time as decision variables to design a forward and reverse supply chain system that produces two different types of products, which are subject to deterioration. Rate of deterioration is time-varying and depends on the maximum lifetime of products. Used products of a forward supply chain are collected and treated as raw materials in a reverse supply chain to produce other products. The system involves three types of inventory stocks, *i.e.*, product 1, 2, and returned inventory. The objective of this research is to minimize total cost per unit time for two types of systems, one in which products of both the supply chains deteriorate and the second in which the products of the first supply chain deteriorate. Kuhn−Tucker method is employed to solve the model and a solution algorithm is proposed to obtain optimal solution. Application of the model is supported with numerical examples and sensitivity analysis. Some managerial insights are provided to help managers while applying the proposed models in real situations. Results of numerical experiments suggest for deteriorating products to plan short replenishment cycles of inventory.

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1. Introduction

1.1. Motivation and background

Assuming the available amount of natural resources as constant, the exponential increase in world population is creating an imbalance between resource availability and consumption. A large amount of used products is being wasted that, if reworked, could fulfil a large part of the world's consumption requirements and generate a good amount of revenue. Instead, this waste is ultimately causing an economic loss. Moreover, today's environment is being polluted by the waste of food products, and carbon emissions in the atmosphere are increasing. There is a

Keywords. Supply chain management, time-varying deterioration; reused products, maximum lifetime, demand dependent production rate, Non-linear programming.

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dire need to conserve natural resources and the environment, avoid landfills, and develop efficient economies to achieve the sustainability objectives, which are mandatory for the survival of future generations. Foreseeing the situation after fifty years from now, the increasing demand of the food will require ten times more resources, which obviously seems impossible within the available limited capacity of natural resources [22].

A sustainable system is the one that can buffer the negative changes in its environment and economy. An essential aspect of green supply chain design is the inclusion of product recycling. Thus, reverse logistics of end-of-use (EOU) or end-of-life (EOL) products is vital in order to fulfill the objectives of sustainable green supply chain [37]. In today's competitive business environment, reverse logistics of EOU and EOL products has been a focus of academia and industry due to government policies, environmental concerns, and worldwide awareness about limited natural resources [38]. Industry managers are well aware of the fact that, to fulfill the increasing demand, consumption of earth's natural resources cannot be avoided and it has become urgent to implement the 3R (reduce, reuse, recycle) policies for EOU and EOL products in existing supply chains that, if overlooked, will deplete important natural resources [20]. A deep insight into the sustainable development, inspired by rapidly depleting natural resources, increasing environmental pollution, and competitive business scope, has urged the adoption of reverse logistics of EOU products [23].

Shelf life products with fixed lifetime, such as food products, deteriorate with time. In the US approximately 15% of products expire due to deterioration before reaching the consumers. It is reported that, throughout the world, due to deterioration, 1.3 billion tons of food are discarded every year, and that amount is approximately 33% of all the food produced for human consumption [18]. The goals of sustainability, *i.e.*, efficient resource utilization and environmental protection, are being violated when such a huge amount of product is wasted due to improper handling. Such shelf life products, when exceeding their useful lifetime, can be recycled and transformed into other products. For example, food waste is processed to prepare animal feed and organic fertilizers.

Inappropriate waste disposal of expired shelf life products creates environmental pollution that is a risk for the health of humans and other living beings. Realizing their social responsibility along with government legislation, corporations have a tendency to dispose of their waste through proper channels. In contrast, it was estimated that half of the collected waste in residential areas is being dumped illegally in vacant lots or is burned openly. These practices are associated with major health hazards for people living in those habitations [2]. In order to control pollution, many governments have mandated that manufacturers are accountable for their industrial waste [36]. Realizing these facts and to fulfill the legal requirements, it has become mandatory to manage and dispose of product waste in a proper way.

Considering the above facts, this study addresses the objectives of sustainability and green supply chain by designing a forward and reverse supply chain system for deteriorating products, where EOU shelf life products of the first supply chain are processed to produce the products of the secondary supply chain. In this system, deteriorated and EOU products, which are non-reworkable (*NRWA*) are disposed of properly, thus providing a channel to conserve natural resources, protect the environment, and acquire economic benefit.

1.2. Literature review

Deterioration in inventory and production models has been considered by many researchers since long ago. The first inventory model on deteriorating inventory was proposed by .Ghare and Schrader [17]. After the pioneer attempt, models for deteriorating products were improved, and many assumptions were relaxed. Sachan [35] developed an economic order quantity (EOQ) model with shortages and time-dependent demand with a constant rate of deterioration. Chang and Dye [7] considered an EOQ model for deteriorating items with time-varying demand and partial backlogging. The same research dimension was further explored by Skouri *et al.* [26]. They proposed an inventory model with a time-dependent rate of deterioration, ramp type demand rate, and partial backlogging. Sett *et al.* [27] proposed a two warehouse inventory model for deteriorating products with quadratic demand and unequal lengths of cycle time. The concept of stochasticity for the rate of deterioration was introduced by Sarkar [28]. He studied an economic production quantity (EPQ) model considering three probabilistic (uniform, triangular, beta) deterioration rates to find the optimum lot size and number of deliveries. Several research models regarding EOQ and EMQ are suggested by the following authors as [29]–[31].

Sarkar [32] introduced the concept of time-varying deterioration rate that depends on the maximum lifetime of the product. Sarkar and Sarkar [33] proposed an inventory model, where products have stock-dependent demand and deteriorate at a time-varying rate. Further, Sarkar and Sarkar [34] proposed an economic manufacturing quantity model for the products deteriorating with a probabilistic rate. Wang *et al.* [39] used the concept of the maximum lifetime to calculate the rate of deterioration. Chen and Teng [9] developed an EOQ model for deteriorating products having maximum lifetime with permissible delay in payments in order to minimize the total cost with an optimal value of the retailer's replenishment cycle. Sarkar *et al.* [40] considered fixed lifetime products, which deteriorated with time-varying rate in a supply chain model, where supplier and retailer adopt trade-credit policy. Goel *et al.* [16] considered a variable quadratic rate of deterioration in a deterministic inventory system with stock-dependent demand and partial backlogging. In their model, they considered the "critical time at which inventory goes zero" as a decision variable. Sarkar *et al.* [41] studied a two-echelon supply chain system focusing on deterioration, setup cost, and system reliability. They concluded that setup cost is directly and deterioration rate is inversely proportional to system reliability. They solved the model to minimize total system cost with an optimum value of system reliability.

Wang and Jiang [42] investigated a system of deteriorating inventory, where the demand depends on selling-price and retailer can adjust the price to increase its sales. Sarkar and Saren [43] discussed a retailer's inventory model for exponentially deteriorating products, where retailer offers partial credit policy to customers and attains full credit policy from manufacturer. Lin *et al.* [44] proposed an inventory model considering deterioration that depends on the maximum lifetime of the product, and the product price is demand-dependent. They solved the problem by maximizing profit at optimal values of preservation cost and replenishment cycle. Pal and Chaudhuri [25] considered an integrated production-inventory system for deteriorating products, lifetime of which is finite and products expire at the end of their lifetime. They assumed that the rate of production is a random variable within a finite range and that the cost of production is a function of lot size and rate of production. Sarkar [45] suggested a supply chain coordination system of single-setup-multi-delivery (SSMD) for fixed lifetime products. Shin *et al.* [46] developed a continuous review inventory model with lead time demand and service level constraints.

The process of reverse logistics includes collection of EOL or EOU products, sorting these products with respect to their quality levels, transformation (reworking, refurbishing, remanufacturing, recycling, material, and energy recovery *etc.*), and waste disposal. Reverse logistics is a big source of alternative material resources, avoid landfills, and provide economic value in order to sustain the environment, resources, and economy. For the last decade, many researchers explored various aspects of reverse logistics. Barry *et al.* [3] discussed the importance of reverse logistics operations in a conceptual frame work. During the decade of 1990s, much work was done by scientists in the field of reverse logistics. The state of the art research articles were summarized by Fleischmann *et al.* [13]. They presented a comprehensive literature review of mathematical models on reverse logistics and a closed-loop supply chain (CLSC) published in scientific journals in order to propose a general framework. They classified the reverse logistics operations into three major categories of distribution planning, production planning, and inventory control. Later, Giri and Sharma [19] proposed a CLSC model in an imperfect production system where imperfect items were reworked. They concluded that profit is higher when an integrated manufacturing approach is considered. Weng and Chen [50] considered a two-echelon supply chain and proved a correlation between rate of return and payment to some customers for returning several used products. They maximized the profit function using a game theoretic approach.

Alamri [1] formulated a reverse logistics model for a joint production system considering deterioration of products. Kim *et al.* [21] considered a partial CLSC system for deteriorating products that are shipped in returnable containers with a stochastic return time. Bouras and Tadj [5] studied three-stock reverse logistics system under deterioration. They supposed that new and remanufactured products had different quality levels and hence, were sold for different prices. Learning effects in a reverse logistics system for deteriorating products have been studied and modeled by Singh and Rathore [48]. They considered time-varying, under the learning effects, and demand-dependent production rate with deterioration in an inflationary environment to design a reverse logistic supply chain model and minimized the total system cost.

Table 1. Authors' contribution to the literature.

Authors	Model formulation	Product deterioration	Sustainability	Product lifetime	Product disposal
Garg *et al.* [20]	CLSC	–	✓	–	✓
Sarkar [28]	EPQ	Random	–	✓	–
Goel *et al.* [16]	SC	Time-varying	–	–	–
Giri and Sharma [19]	CLSC	Imperfect production	–	–	–
Alamri [1]	RL inventory	Time-varying	–	–	–
Kim *et al.* [21]	CLSC	Lot size dependent	–	–	–
Nagurney and Nagurney [24]	SC	–	✓	–	–
Bouras and Tadj [5]	CLSC	Constant	–	–	–
Chung and Wee [6]	CLSC	Constant	✓	–	✓
Devika *et al.* [11]	CLSC	–	✓	–	✓
Jawla and Singh [14]	RL inventory	Time-varying	–	–	✓
This paper	Two SC	Time-varying	✓	✓	✓

Yang *et al.* [51] analyzed a CLSC inventory model using sequential and global optimization for outdated items that are considered as deteriorating items. They concluded that global optimization gives better results than sequential optimization. Chung and Wee [6] developed a CLSC system for short life cycle deteriorating products to highlight green product designs and remanufacturing efforts. They suggested important factors that significantly affect decision making in the inventory control system of green supply chains including new technology evolution, remanufacturing ratios, and system's holding costs. Yang *et al.* [52] developed a centrally optimized CLSC model with benefit sharing considering the decrease in value of outdated products as deterioration. Hedjar *et al.* [15] designed a CLSC with three-stock inventory subject to deterioration considering equal quality level of new and remanufactured items but targeting different market segments. They minimized the total cost using trapezoidal formula and Taylor expansion at optimal values of rate of manufacturing, remanufacturing, and disposal.

Among recent research efforts on sustainability is the work of Nagurney and Nagurney [24]. They designed a sustainable supply chain network to determine manufacturing, storage, and distribution capacities while minimizing the total cost and carbon emissions. Dornfeld [10] studied the essential requirements to adopt green technology and the solution techniques for its effective implementation. Bhanot *et al.* [4] conducted a survey-based study involving researchers and industry professionals to formulate a list of significant enablers and barriers to implement sustainable manufacturing and analyzed the results using statistical methods. Shi [49] measured an environmental performance of a production system by dividing the system into two stages, where first stage is the manufacturing stage while the second stage is the treatment of the pollutants produced during manufacturing. He defined the total environmental efficiency of production system as a sum of environmental efficiency of each stage.

The literature gap addressed by the proposed model is presented in Table 1. (✓) means the field is considered and (–) means the field is not considered by the authors. The terms used are abbreviated as: SC (supply chain), CLSC (closed-loop supply chain), RL (reverse logistics), and EPQ (economic production quantity). After reviewing the literature on product deterioration, reverse supply chain, and sustainability, the authors propose a system of two supply chains for deteriorating products that work parallel to each other and have equal cycle time, highlighting the need for sustainability and resource conservation. The structure of the paper is as follows: Section 2 presents comprehensive problem definition. Section 3 contains mathematical model formulation, notation, assumptions, model development and solution methodology. A special case is presented in Section 4. Section 5 exhibits numerical experiments and sensitivity analysis. Section 6 provides pertinent managerial insights of the model. Conclusions and future directions are given in Section 7.

2. Problem definition

2.1. Problem statement

This study considers one cycle of a two-supply chain system that work in parallel and have equal period lengths T, for shelf life products, as illustrated in Figure 1. The first supply chain is treated as a forward system and the second as a reverse supply chain. As explained in the figure, the jth cycle involves the jth period of supply chain 1 and the *(j-1)*th period of supply chain 2. EOU items of supply chain 1 from its previous period are used as raw material for the product of supply chain 2 in its current period. The system operates for *(n+1)* number of cycles, and this study discusses a single cycle.

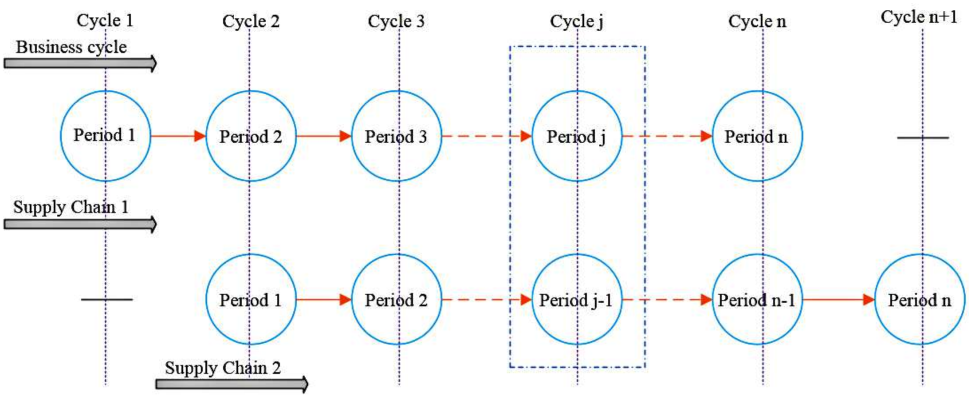


Figure 1. System flow.

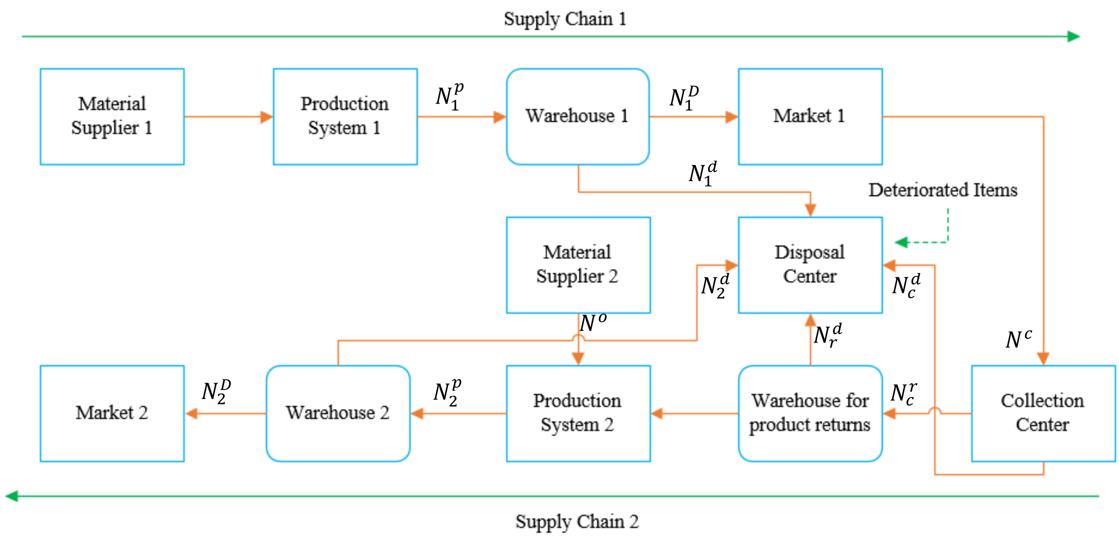


Figure 2. Process flow diagram.

Process flow from material suppliers to customer markets is described in Figure 2 During the jth cycle, material is purchased, and product 1 is produced at its production facility. Finished product items are stored as inventory to fulfill customer demand during the planning horizon. A presumed fraction of EOU product items from a previous period of supply chain 1 are received at the collection center and sorted into two types, the items which are reworkable (*RWA*), and others which are non-reworkable (*NRWA*). *RWA* items are sent to returned inventory, and *NRWA* items are forwarded to a disposal center. Returned inventory provides raw material for product 2. If raw material requirements of product 2 cannot be fulfilled from returned inventory, material is purchased from other suppliers at some fixed cost. Product 2 is produced at its production facility and is stored as inventory to satisfy customer demand during the planning horizon. These inventory items deteriorate at a specific rate and deteriorated items are forwarded to a disposal center. Rate of deterioration depends on the maximum lifetime of the product and varies directly with the time in inventory. System costs include setup, material, production, inventory holding, disposal, return collection, and sorting cost, while hidden cost consists of the loss due to wastage of deteriorated items. Importance of production-inventory planning is high in such a system where rate of deterioration is a function of time. This study formulates a non-linear mathematical programming model that calculates the optimal solution for the production and inventory schedules while minimizing the total cost. Another model is formulated that is a special case of the first model and involves deterioration in the first supply chain only while the products in the second supply chain do not deteriorate.

2.2. Notation

The abbreviations and notation for the mathematical model of proposed system is defined as follows:

2.2.1. Index

i $\quad i = 1, 2$, index i is used for product 1 and 2

2.2.2. Variables

T planning horizon / cycle time (time units)
t_i production time of product 1 and 2 (time units)

2.2.3. Parameters

C_i^s setup cost per setup of product i, $i = 1, 2$ (\$/setup)
C_r^s setup cost per return setup (\$/setup)
C_i^{mt} material purchase cost per unit of product i, $i = 1, 2$ (\$/unit)
C_i^p production cost per unit of product i, $i = 1, 2$ (\$/unit)
C^c collection cost per unit (\$/unit)
h_i inventory holding cost per unit per unit time of product i, $i = 1, 2$ (\$/unit/unit time)
h_r inventory holding cost per unit per unit time of *RWA* returned product (\$/unit/unit time)
C^d disposal cost per unit of deteriorated items (\$/unit)
TC total cost per unit time (\$/unit time)
D_i demand per unit time of product i, $i = 1, 2$ (units/unit time)
N_i^D total demand per cycle of product i, $i = 1, 2$ (units/cycle)
P_i production rate of product i, $i = 1, 2$ (units/unit time)
R rate of product return (units/unit time)
θ_i rate of deterioration of product i, $i = 1, 2$
L_i maximum lifetime of product i, $i = 1, 2$ (time units)
I_i^a on-hand inventory of product i at any time t, $0 \leqslant t \leqslant t_i$ (units)
I_i^b on-hand inventory of product i at any time t, $t_i \leqslant t \leqslant T$ (units)
I_i total inventory per cycle of product i, $i = 1, 2$ (units/cycle)
I_r^a on-hand inventory of returned product at any time t, $0 \leqslant t \leqslant t_2$ (units)
I_r^b on-hand inventory of returned product at time t, $t_2 \leqslant t \leqslant T$ (units)

I_r total inventory per cycle of returned product (units/cycle)
N_i^p quantity produced per cycle of product i, $i = 1, 2$ (units/cycle)
N^c quantity of collected items at collection center per cycle (units/cycle)
N_c^r *RWA* items received at collection center per cycle (units/cycle)
N_c^d *NRWA* items received at collection center per cycle (units/cycle)
N^o raw material required to produce product 2 (units/cycle)
N_i^d deteriorated items per cycle of product i, $i = 1, 2$ (units/cycle)
N_r^d deteriorated items per cycle of returned products (units/cycle)
β fraction of returned products at collection center, $0 \leqslant \beta \leqslant 1$
μ fraction of returned *RWA* items at collection center, $0 \leqslant \mu \leqslant 1$
k_i proportionality constant within production p_i and demand D_i, $i = 1, 2$

2.3. Assumptions

(1) A single-cycle of a forward and reverse supply chain system is considered, which produces two different products. The proposed cycle involves the jth period of supply chain 1 and the *(j-1)*th period of supply chain 2, which work parallel to each other and have equal period lengths.
(2) Products deteriorate with time, and the rate of deterioration depends on the maximum lifetime of the product. Rate of deterioration is calculated as $\theta = \frac{1}{1+L-t}$, where L is the maximum lifetime of the product [32]. When $t \to 0$, deterioration rate is the minimum. Similarly when $t \to L$, the deterioration rate is 1, which means that all of the products are deteriorated at its maximum lifetime.
(3) A certain fraction of the returned product items μR is *RWA*. Remaining quantity $(1-\mu)\,R$ is *NRWA* and is disposed.
(4) One item of *RWA* returned product produces α number of items of product 2.
(5) If raw material requirements of product 2 are not fulfilled with the *RWA* returned product, then required material is purchased from the supplier.
(6) Demand and production rate are constant and known over the planning horizon. During production cycle, production rate is higher than demand, *i.e.*, $k_i > 1$.
(7) The deteriorated items are not repairable and are disposed.
(8) Inspection cost is negligible.

3. Model formulation

3.1. Model development

The proposed model assumes that product 1 and 2 deteriorate with time as soon as they are received in inventory. Inventory of the products is maintained to fulfill the demand of customer within planning horizon. Excessive amount of inventory incurs more holding cost and may cause the products to expire, while less amount of inventory can create shortages. Therefore, it is important to estimate an optimum level of inventory. In the proposed system demand of both the products remains constant during one cycle. The rate of demand remains D_1 and D_2 during complete cycle for the product 1 and 2, respectively. Total number of items demanded of product 1 and 2 per cycle are calculated in following equations:

$$N_1^D = \int_0^T D_1 \mathrm{d}t = D_1 T,$$

$$N_2^D = \int_0^T D_2 \mathrm{d}t = D_2 T.$$

Products are produced at their respective production facilities and the rate of production is a function of the demand, which is expressed below.

$$P_1 = k_1 D_1$$

$$P_2 = k_2 D_2$$

In their respective inventories, both the products deteriorate and rate of deterioration at any time t is defined by the following equations.

$$\theta_1 = \frac{1}{1 + L_1 - t}$$

$$\theta_2 = \frac{1}{1 + L_2 - t}$$

From market of product 1, the used products are returned to the collection center. The rate of return depends on the demand of product 1. It is assumed that a certain fraction of the demand of product 1 will be returned, which is defined in following equation:

$$R = \beta D_1$$

Quantity produced during production cycle fulfills customer demand and compensates the quantity that deteriorates during complete cycle. For both supply chains, during the interval $0 \to t_i$, $i = 1, 2$, at any time t, P_i number of product items are produced and added to the product inventory while D_i number of items are demanded and $\theta_i I_i$ number of items are deteriorated and removed from the inventory. Level of product inventory increases during the production cycle due to the rate of production being higher than the accumulative demand and deterioration rate. During interval $t_i \to T$, there is no production and inventory stock is depleted by the number of items demanded and number of items deteriorated. Conversely, level of returned inventory decreases

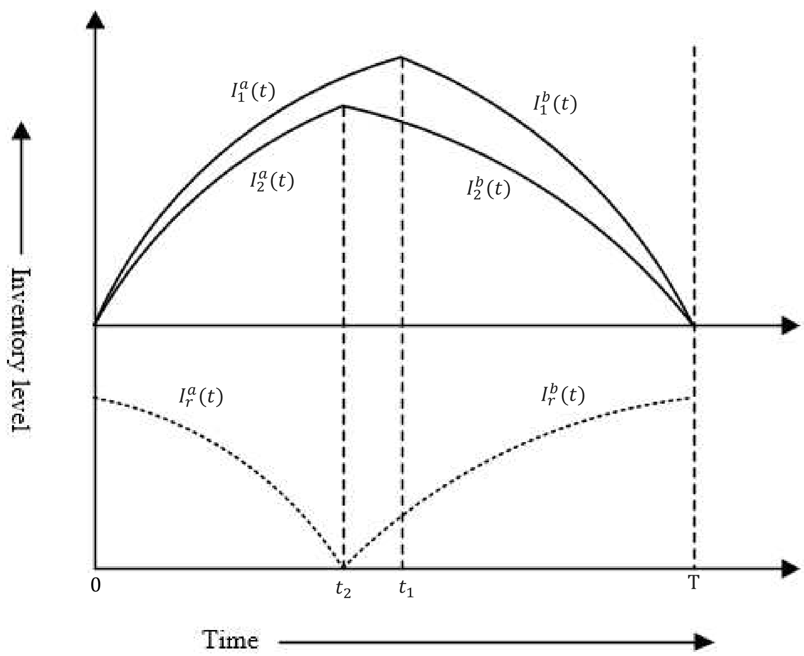


FIGURE 3. Inventory behavior during one cycle.

for the interval $0 \to t_2$ and is replenished during the interval $t_2 \to T$. During the interval $0 \to t_2$, μR number of RWA returned items are added while P_2 and $\theta_1 I_r$ number of items are removed from the returned inventory. During the interval $t_2 \to T$, there is no production of product 2, so only $\theta_1 I_r$ are removed while adding μR number of items to the returned inventory. The behavior of inventory level during the production-inventory cycle is depicted in Figure 3.

Three inventory stocks are calculated and are used to determine the total inventory and number of items deteriorated per cycle. The governing differential equations of current inventory are expressed as in following equations.

Inventory of product 1 is replenished during production cycle and is depleted during the inventory cycle. Rate of change of inventory level of product 1 is as given below:

$$\frac{\mathrm{d}I_1^a(t)}{\mathrm{d}t} = P_1 - D_1 - \theta_1 I_1^a,\ 0 \leqslant t \leqslant t_1, \tag{3.1}$$

$$\frac{\mathrm{d}I_1^b(t)}{\mathrm{d}t} = -D_1 - \theta_1 I_1^b,\ t_1 \leqslant t \leqslant T. \tag{3.2}$$

Similarly the rate of change of inventory level of product 2 is as given below:

$$\frac{\mathrm{d}I_2^a(t)}{\mathrm{d}t} = P_2 - D_2 - \theta_2 I_2^a,\ 0 \leqslant t \leqslant t_2, \tag{3.3}$$

$$\frac{\mathrm{d}I_2^b(t)}{\mathrm{d}t} = -D_2 - \theta_2 I_2^b,\ t_2 \leqslant t \leqslant T. \tag{3.4}$$

Inventory of return products is depleted during the interval as the product 2 is produced and raw material is taken from returned inventory. Returned inventory is replenished during the interval due to return products at collection center, which are added to returned inventory.

$$\frac{\mathrm{d}I_r^a(t)}{\mathrm{d}t} = \mu R - P_2 - \theta_1 I_r^a,\ 0 \leqslant t \leqslant t_2 \tag{3.5}$$

$$\frac{\mathrm{d}I_r^b(t)}{\mathrm{d}t} = \mu R - \theta_1 I_r^b,\ t_2 \leqslant t \leqslant T \tag{3.6}$$

The above differential equations are solved using the inventory conditions as given below and on-hand inventory is calculated at different stocks at any time t.

Level of inventory of product 1 is zero when $t = 0$ or $t = T$. These inventory conditions are given as below:

$$I_1^a(t) = 0 \qquad \text{at} \qquad t = 0,$$

$$I_1^b(t) = 0 \qquad \text{at} \qquad t = T.$$

Considering the above inventory conditions, solution of (3.1) and (3.2) is as given below:

$$I_1^a(t) = D_1(k_1 - 1)(1 + L_1 - t)\left(\ln\frac{1 + L_1}{1 + L_1 - t}\right), \qquad 0 \leqslant t \leqslant t_1 \tag{3.7}$$

$$I_1^b(t) = D_1(1 + L_1 - t)\left(\ln\frac{1 + L_1 - t}{1 + L_1 - T}\right), \qquad t_1 \leqslant t \leqslant T \tag{3.8}$$

Similarly level of inventory of product 1 is zero when or . These inventory conditions are expressed as below:

$$I_2^a(t) = 0 \qquad \text{at} \qquad t = 0,$$

$$I_2^b(t) = 0 \qquad \text{at} \qquad t = T.$$

Considering the above inventory conditions, solution of (3.3) and (3.4) is as given below:

$$I_2^a(t) = D_2(k_2-1)(1+L_2-t)\left(\ln\frac{1+L_2}{1+L_2-t}\right), \qquad 0 \leqslant t \leqslant t_2 \tag{3.9}$$

$$I_2^b(t) = D_2(1+L_2-t)\left(\ln\frac{1+L_2-t}{1+L_2-T}\right), \qquad t_2 \leqslant t \leqslant T \tag{3.10}$$

The level of inventory of return products is zero when $t = t_2$. The inventory conditions are expressed as below:

$$I_r^a(t) = 0 \qquad \text{at} \qquad t = t_2,$$

$$I_r^b(t) = 0 \qquad \text{at} \qquad t = t_2.$$

Considering these inventory conditions, solution of (3.5) and (3.6) is as given below:

$$I_r^a(t) = (k_2D_2 - \mu\beta D_1)(1+L_1-t)\left(\ln\frac{1+L_1-t}{1+L_1-t_2}\right), \qquad 0 \leqslant t \leqslant t_2 \tag{3.11}$$

$$I_r^b(t) = \mu\beta D_1(1+L_1-t)\left(\ln\frac{1+L_1-t_2}{1+L_1-t}\right), \qquad t_2 \leqslant t \leqslant T \tag{3.12}$$

Using (3.7) and (3.8), total inventory of product 1 carried per cycle is calculated and demonstrated below:

$$\begin{aligned} I_1 &= \int_0^{t_1} I_1^a(t)\,\mathrm{d}t + \int_{t_1}^{T} I_1^b(t)\,\mathrm{d}t \\ &= \frac{1}{4}D_1\left\{\begin{array}{l} (k_1-1)\left((1+L_1)^2-(1+L_1-t_1)^2\left(1+2\ln\left(\frac{1+L_1}{1+L_1-t_1}\right)\right)\right)+T(T-2L_1-2) \\ +2(1+L_1)^2\ln(1+L_1-t_1) \\ -t_1(2+2L_1-t_1)\left(2\ln\left(\frac{1+L_1-t_1}{1+L_2-T}\right)-1\right)-2(1+L_1)^2\ln(1+L_1-T) \end{array}\right\} \end{aligned} \tag{3.13}$$

Similarly, using (3.9) and (3.10), total inventory of product 2 carried per cycle is calculated and demonstrated below:

$$\begin{aligned} I_2 &= \int_0^{t_2} I_2^a(t)\,\mathrm{d}t + \int_{t_2}^{T} I_2^b(t)\,\mathrm{d}t \\ &= \frac{1}{4}D_2\left\{\begin{array}{l} (k_2-1)\left((1+L_2)^2-(1+L_2-t_2)^2\left(1+2\ln\left(\frac{1+L_2}{1+L_2-t_2}\right)\right)\right)+T(T-2L_2-2) \\ +2(1+L_2)^2\ln(1+L_2-t_2) \\ -t_2(2+2L_2-t_2)\left(2\ln\left(\frac{1+L_2-t_2}{1+L_2-T}\right)-1\right)-2(1+L_2)^2\ln(1+L_2-T) \end{array}\right\} \end{aligned} \tag{3.14}$$

RWA returned items of product 1 are stored as returned inventory which are used as raw material for product 2. Total returned inventory carried per cycle is calculated using (3.11) and (3.12), which is expressed in below equation:

$$\begin{aligned} I_r &= \int_0^{t_2} I_r^a(t)\,\mathrm{d}t + \int_{t_2}^{T} I_r^b(t)\,\mathrm{d}t \\ &= \frac{1}{4}\left\{\begin{array}{l} 2(k_2D_2-\mu\beta D_1)(1+L_1)^2\ln(1+L_1) - \mu\beta D_1(1+L_1-T)^2\left(1+2\ln\left(1+\frac{T-t_2}{1+L_1-T}\right)\right) \\ \mu\beta D_1(1+L_1-t_2)^2+(k_2D_2-\mu\beta D_1)\left(t_2^2-2t(1+L_1)-2(1+L_1)^2\ln(1+L_1-t_2)\right) \end{array}\right\} \end{aligned} \tag{3.15}$$

The system's total cost is incurred by the setup cost, material purchase cost, production cost, collection and sorting cost, inventory carrying cost, and disposal cost.

Setup cost.

This cost comprises of the expenses for production and inventory facilities, apparatus for production, and transportation of materials. Investment in setup cost is a strategic decision. The setup cost is planned per setup of production and inventory, which is calculated for the proposed system as is expressed as in the following equation.

$$\text{Setup cost} = C_1^s + C_2^s + C_r^s$$

Inventory holding cost.

Inventory of any product is maintained to fulfill the customer demand and improve service level. Storage of the produced items is an important task for any kind of products. The storage conditions vary according to the nature of the produced products. Some products need very precise temperature and humidity conditions while others need only a storage facility. The cost invested to store the produced products before selling to the customers is called inventory holding cost. For the proposed system, inventory holding cost is calculated per cycle.

$$\text{Inventory holding cost} = h_1 I_1 + h_2 I_2 + h_r I_r$$

where the values of I_1, I_2, and I_r are defined in (3.13), (3.14), and (3.15), respectively.

Cost of material.

Material is arranged according to produced quantity of the products. Cost of material per cycle depends on the number of items produced during the cycle and per unit cost of material. Using the number of items produced per cycle and the cost of material per unit item, total cost of material per cycle is calculated in the following equation:

$$\text{Material cost} = C_1^{mt} N_1^p + C_2^{mt} N^o$$

where $N_1^p$is the number of items produced of product 1, as calculated below.

$$N_1^p = \int_0^{t_1} P_1 \mathrm{d}t = \int_0^{t_1} k_1 D_1 \mathrm{d}t = k_1 D_1 t_1$$

Similarly, the number of items produced of product 2 per cycle are determined below.

$$N_2^p = \int_0^{t_2} P_2 \mathrm{d}t = \int_0^{t_2} k_2 D_2 \mathrm{d}t = k_2 D_2 t_2$$

It is assumed that one item of RWA returned product 1 produces α items of product 2. If the demand of product 2 is more than the quantity that can be produced from the RWA returned items, then the remaining required material is purchased in order to satisfy the customer demand. The raw material quantity (N^o) to be purchased in order to produce product 2 is calculated by the conditional expression as given below.

$$N^o = \left\{ \begin{array}{lll} 0 & \text{If} & N_c^r \geqslant \alpha N_2^D \\ \alpha N_2^D - N_c^r & \text{Otherwise} & \end{array} \right\}$$

Cost of production.

Production of a single unit of a product involves several operations and each operation involves investment in energy, machinery, labor, and overheads. These costs are added at each level to compute unit production cost. Production cost is calculated per cycle by using unit production cost and the number of items produced per cycle, which is expressed in the following equation.

$$\text{Production cost} = C_1^p N_1^p + C_2^p N_2^p$$

Collection cost.

At collection centers, returned products are received from the customers for some incentive price and are sorted into *RWA* and *NRWA* products. The cost incurred to acquire and segregate these products is termed as collection cost. In order to calculate the collection cost per cycle, the number of items collected per cycle are computed below.

$$N^c = \int_0^T R\mathrm{d}t = \int_0^T \beta D_1 \mathrm{d}t = \beta D_1 T$$

Collection cost for one cycle is given below.

$$\text{Return collection and sorting cost} = C^c N^c$$

Disposal cost.

Segregation of the collected items into *RWA* and *NRWA* items is important to decide on the products, which can be used to produce some other products. The following equations calculate the numbers of *RWA* and *NRWA* items per cycle.

$$N_c^r = \mu N^c = \mu\beta D_1 T$$

$$N_c^d = (1-\mu)\, N^c = (1-\mu)\, \beta D_1 T$$

Some of the items from inventory stocks are deteriorated and are removed from other good products. These deteriorated items are forwarded to the disposal center. To calculate the disposal cost, it is required to compute the number of items deteriorated in their respective inventories per cycle. The number of items deteriorated in the inventories of products 1 and 2 and the returned inventory per cycle are expressed in the following equations.

$$\begin{aligned} N_1^d &= \int_0^{t_1} \theta_1 I_1^a(t)\,\mathrm{d}t + \int_{t_1}^{T} \theta_1 I_1^b(t)\,\mathrm{d}t \\ &= D_1 \left\{ \begin{array}{l} (k_1-1)\left((1+L_1-t_1)\ln\left(1+\frac{t_1}{1+L_1-t_1}\right) - (1+L_1-T)\ln\left(1+\frac{T}{1+L_1-T}\right) + T - t_1\right) \\ (1+L_1)\ln(1+L_1) - (1+L_1)\ln(1+L_1-t_1) + t_1\ln\left(1+\frac{T-t_1}{1+L_1-T}\right) - t_1 \end{array} \right\} \end{aligned}$$

$$\begin{aligned} N_2^d &= \int_0^{t_2} \theta_2 I_2^a(t)\,\mathrm{d}t + \int_{t_2}^{T} \theta_2 I_2^b(t)\,\mathrm{d}t \\ &= D_2 \left\{ \begin{array}{l} (k_2-1)\left((1+L_2-t_2)\ln\left(1+\frac{t_2}{1+L_2-t_2}\right) - (1+L_2-T)\ln\left(1+\frac{T}{1+L_2-T}\right) + T - t_2\right) \\ (1+L_2)\ln(1+L_2) - (1+L_2)\ln(1+L_2-t_2) + t_2\ln\left(1+\frac{T-t_2}{1+L_2-T}\right) - t_2 \end{array} \right\} \end{aligned}$$

$$\begin{aligned} N_r^d &= \int_0^{t_2} \theta_1 I_r^a(t)\,\mathrm{d}t + \int_{t_2}^{T} \theta_1 I_r^b(t)\,\mathrm{d}t \\ &= \left\{ \begin{array}{l} \mu\beta D_1\left(T - t_2 - (1+L_1-T)\ln\left(1+\frac{T-t_2}{1+L_1-T}\right)\right) \\ +(k_2D_2 - \mu\beta D_1 - \mu\beta D_1)((1+L_1)\ln(1+L_1) - (t_2 + (1+L_1)\ln(1+L_1-t_2))) \end{array} \right\} \end{aligned}$$

The deteriorated items are sent to the specified disposal centers for proper disposal for a specific cost, in order to fulfil legislative requirements. Total cost of disposal per cycle is calculated by using unit disposal cost and total number of deteriorated items per cycle.

$$\text{Disposal cost} = C^d N^d,$$

where $N^d = N_1^d + N_2^d + N_r^d + N_c^d$

Total cost.

The total cost per unit time of the given system is calculated in following equation:

$$TC(t_1,t_2,T) = \frac{1}{T}\left\{\begin{array}{l} C_1^s + C_2^s + C_r^s + C_1^{mt}N_1^p + C_2^{mt}N^o + C^cN^c \\ +C_1^pN_1^p + C_2^pN_2^p + h_1I_1 + h_2I_2 + h_rI_r + C^d\left(N_1^d + N_2^d + N_r^d + N_c^d\right) \end{array}\right\}$$

$$= \frac{1}{T}\left[\begin{array}{l}
C_1^s + C_2^s + C_r^s + \left(C_1^{mt} + C_1^p\right)k_1D_1t_1 + C_2^{mt}N^o + C_2^pk_2D_2t_2 + C^c\beta D_1T \\
+\frac{1}{4}h_1D_1\left\{\begin{array}{l} (k_1-1)\left((1+L_1)^2 - (1+L_1-t_1)^2\left(1+2\ln\left(\frac{1+L_1}{1+L_1-t_1}\right)\right)\right) \\ +2(1+L_1)^2\ln(1+L_1-t_1) - t_1(2+2L_1-t_1)\left(2\ln\left(\frac{1+L_1-t_1}{1+L_1-T}\right)-1\right) \\ -2(1+L_1)^2\ln(1+L_1-T) + T(T-2L_1-2) \end{array}\right\} \\
+\frac{1}{4}h_2D_2\left\{\begin{array}{l} (k_2-1)\left((1+L_2)^2 - (1+L_2-t_2)^2\left(1+2\ln\left(\frac{1+L_2}{1+L_2-t_2}\right)\right)\right) \\ +2(1+L_2)^2\ln(1+L_2-t_2) - t_2(2+2L_2-t_2)\left(2\ln\left(\frac{1+L_2-t_2}{1+L_2-T}\right)-1\right) \\ -2(1+L_2)^2\ln(1+L_2-T) + T(T-2L_2-2) \end{array}\right\} \\
+\frac{1}{4}h_r\left\{\begin{array}{l} 2(k_2D_2-\mu\beta D_1)(1+L_1)^2\ln(1+L_1) - \mu\beta D_1(1+L_1-T)^2\left(1+2\ln\left(1+\frac{T-t_2}{1+L_1-T}\right)\right) \\ \mu\beta D_1(1+L_1-t_2)^2 + (k_2D_2-\mu\beta D_1)\left(t_2^2 - 2t(1+L_1) - 2(1+L_1)^2\ln(1+L_1-t_2)\right) \end{array}\right\} \\
+C^d\left\{\begin{array}{l}
D_1\left\{\begin{array}{l} (k_1-1)\left((1+L_1-t_1)\ln\left(1+\frac{t_1}{1+L_1-t_1}\right) - (1+L_1-T)\right. \\ \left.\times\ln\left(1+\frac{T}{1+L_1-T}\right) + T - t_1\right) \\ (1+L_1)\ln(1+L_1) - (1+L_1)\ln(1+L_1-t_1) + t_1\ln\left(1+\frac{T-t_1}{1+L_1-T}\right) - t_1 \end{array}\right\} \\
+D_2\left\{\begin{array}{l} (k_2-1)\left((1+L_2-t_2)\ln\left(1+\frac{t_2}{1+L_2-t_2}\right) - (1+L_2-T)\right. \\ \left.\times\ln\left(1+\frac{T}{1+L_2-T}\right) + T - t_2\right) \\ (1+L_2)\ln(1+L_2) - (1+L_2)\ln(1+L_2-t_2) + t_2\ln\left(1+\frac{T-t_2}{1+L_2-T}\right) - t_2 \end{array}\right\} \\
+\left\{\begin{array}{l} \mu\beta D_1\left(T - t_2 - (1+L_1-T)\ln\left(1+\frac{T-t_2}{1+L_1-T}\right)\right) \\ +(k_2D_2-\mu\beta D_1)\left((1+L_1)\ln(1+L_1) - (t_2 + (1+L_1)\ln(1+L_1-t_2))\right) \end{array}\right\} \\
+(1-\mu)\beta D_1T
\end{array}\right\}
\end{array}\right] \tag{3.16}$$

The aim of this study is to obtain the optimal values of decision variables that minimize total cost per unit time, which is expressed as below.

Minimize $TC(t_1,t_2,T)$

subject to the following constraints:

$$N_1^p - N_1^d \geqslant N_1^D \tag{3.17}$$

$$N_2^p - N_2^d \geqslant N_2^D \tag{3.18}$$

$$N_c^r \geqslant N_r^d + N_2^p \tag{3.19}$$

$$t_1, t_2, T \geqslant 0 \tag{3.20}$$

Objective function (3.16) is the total cost per unit time. Constraints (3.17) and (3.18) ensure that the demand of product 1 and that of product 2 is satisfied. Constraint (3.19) is the flow constraint at returned inventory that ensures that the number of items removed from the returned inventory is less than or equal to the number of items added to the returned inventory per cycle. Constraint (3.20) shows non-negativity of the decision variables.

3.2. Solution methodology

As the objective function is non-linear with inequality constraints, thus Kuhn−Tucker method is an appropriate solution technique .[50]. As the values of decision variables cannot be obtained directly and a closed-form solution is not possible due to high degree of non-linearity, therefore a solution algorithm is designed to calculate the optimal solution. Solution methodology is explained below.

3.2.1. Kuhn−Tucker method

Lagrange function of the objective is given below.

$$L\left(t_1, t_2, T, \lambda_1, \lambda_2, \lambda_3\right) = TC + \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3$$

TC is defined in (3.1), λ_u, $u = 1, 2, 3$ are Lagrange multipliers, and X_1, X_2, X_3 are calculated using constraints (17)–(19) as given below.

$$X_1 = N_1^D + N_1^d - N_1^p \leqslant 0$$
$$X_2 = N_2^D + N_2^d - N_2^p \leqslant 0$$
$$X_3 = N_r^d + N_2^p - N_c^r \leqslant 0$$

Kuhn−Tucker necessary conditions for the optimal point giving the minimum cost value, considering the defined constraints, are given below.

$$\frac{\partial L\left(t_1, t_2, T, \lambda_1, \lambda_2, \lambda_3\right)}{\partial t_1} \geqslant 0, \qquad t_1\left\{\frac{\partial L\left(t_1, t_2, T, \lambda_1, \lambda_2, \lambda_3\right)}{\partial t_1}\right\} = 0$$
$$\frac{\partial L\left(t_1, t_2, T, \lambda_1, \lambda_2, \lambda_3\right)}{\partial t_2} \geqslant 0, \qquad t_2\left\{\frac{\partial L\left(t_1, t_2, T, \lambda_1, \lambda_2, \lambda_3\right)}{\partial t_2}\right\} = 0$$
$$\frac{\partial L\left(t_1, t_2, T, \lambda_1, \lambda_2, \lambda_3\right)}{\partial T} \geqslant 0, \qquad T\left\{\frac{\partial L\left(t_1, t_2, T, \lambda_1, \lambda_2, \lambda_3\right)}{\partial T}\right\} = 0$$
$$\lambda_u X_u = 0,\ u = 1, 2, 3.$$

Considering the fact that t_1, t_2 and T cannot be zero, the above conditions are satisfied when the first derivative of the Lagrange function with respect to each decision variable is equal to zero. These calculations are given below.

Derivative of the Lagrange function with respect to t_1

$$\begin{aligned}
&\frac{\partial L(t_1,t_2,T,\lambda_1,\lambda_2,\lambda_3)}{\partial t_1} = 0\\
&D_1\left\{\begin{array}{l}\frac{k_1\left(C_1^{mt}+C_1^p\right)}{T} + \frac{h_1(1+L_1-T)}{T}\left((k_1-1)\ln\frac{1+L_1}{1+L_1-t_1} - \ln\frac{1+L_1-t_1}{1+L_1-T}\right)\\ +\frac{C^d}{T}\left((1-k_1)\ln\frac{1+L_1}{1+L_1-t_1} + \ln\left(1+\frac{T-t_1}{1+L_1-T}\right)\right)\end{array}\right\}\\
&+\lambda_1 D_1\left\{(1-k_1)\ln\frac{1+L_1}{1+L_1-t_1} + \ln\left(1+\frac{T-t_1}{1+L_1-T}\right) - k_1\right\} = 0
\end{aligned} \tag{3.21}$$

Derivative of the Lagrange function with respect to t_2

$$\begin{aligned}
&\frac{\partial L(t_1,t_2,T,\lambda_1,\lambda_2,\lambda_3)}{\partial t_2} = 0\\
&\frac{1}{T}\left\{\begin{array}{l}C_2^p k_2 D_2 + h_2 D_2\left(1+L_2-t_2\right)\left((k_2-1)\ln\frac{1+L_2}{1+L_2-t_2} - \ln\frac{1+L_2-t_2}{1+L_2-T}\right)\\ +\frac{h_r}{4(1+L_1-t_2)}\left(2T\mu\beta D_1\left(T-2L_1-2\right) + 2k_2 D_2 t_2\left(2+2L-t_2\right)\right)\\ +C^d\left\{D_2\left((1-k_2)\ln\frac{1+L_2}{1+L_2-t_2} - k_2 + \ln\left(1+\frac{T-t_2}{1+L_2-T}\right)\right) + \frac{k_2 D_2(1+L_1)-\mu\beta D_1 T}{1+L_1-t_2}\right\}\end{array}\right\}\\
&+\lambda_2 D_2\left\{(1-k_2)\ln\frac{1+L_2}{1+L_2-t_2} - k_2 + \ln\left(1+\frac{T-t_2}{1+L_2-T}\right)\right\} + \lambda_3\frac{k_2 D_2(1+L_1)-\mu\beta D_1 T}{1+L_1-t_2} = 0
\end{aligned} \tag{3.22}$$

Derivative of the Lagrange function with respect to T

$$
\begin{aligned}
&\frac{\partial L\left(t_1,t_2,T,\lambda_1,\lambda_2,\lambda_3\right)}{\partial T}=0\\
&\frac{1}{T^*}\left[\begin{array}{l}
\beta C^c D_1+C_2^{mt}\left(D_2-\mu\beta D_1\right)+D_1h_1\left(T^*-t_1\right)\left(1+\ln\left(1+L_1-T^*\right)\right)\\
+C^d\left\{\begin{array}{l}
\beta D_1\left(1-\mu\right)+D_1\left(1+\ln\left(1+L_1-T^*\right)\right)\\
\ln\dfrac{1+L_1-t_1}{1+L_1-T^*}+D_2\left(1+\ln\left(1+L_2-T^*\right)\right)\ln\dfrac{1+L_2-t_2}{1+L_2-T^*}\\
+\mu\beta D_1\left(1-\ln\left(1+L_1-T^*\right)+\dfrac{T^*-\left(1+L_1\right)+\left(1+L_2-t_2\right)\ln\left(1+L_2-t_2\right)}{\left(1+L_1-T^*\right)}\right)
\end{array}\right\}\\
+D_2h_2\left(T^*-t_2\right)\left(1+\ln\left(1+L_2-T^*\right)\right)+\dfrac{\mu\beta D_1h_r}{2}\left\{\left(1+L_1-T^*\right)\left(1-2\ln\left(1+L_1-T^*\right)\right)\right.\\
\left.-\left(1+L_1-T^*\right)+2\left(1+L_1-t_2\right)\ln\left(1+L_1-t_2\right)\right\}
\end{array}\right]\\
&+\frac{1}{T^{*2}}\left[\begin{array}{l}
C_1^s+C_2^s+C_r^s+\beta C^cD_1T^*+C_2^{mt}\left(D_2T^*-\mu\beta D_1T^*\right)+\left(C_1^{mt}+C_1^p\right)k_1D_1t_1+C_2^pk_2D_2t_2\\
+\dfrac{D_1h_1}{4}\left\{\begin{array}{l}
\left(1+L_1-T^*\right)^2\left(1-2\ln\left(1+L_1-T^*\right)\right)-4\left(T^*-t_1\right)\left(1+L_1-T^*\right)\\
\times\ln\left(1+L_1-T^*\right)-\left(1+L_1-t_1\right)^2\left(1-2\ln\left(1+L_1-t_1\right)\right)\\
\left(k_1-1\right)\left\{\left(1+L_1\right)^2\left(1-2\ln\left(1+L_1\right)\right)-\left(1+L_1-t_1\right)^2\right.\\
\left.\left(1-2\ln\left(1+L_1-t_1\right)\right)+4t_1\left(1+L_1\right)\ln\left(1+L_1\right)\right\}
\end{array}\right\}\\
+\dfrac{D_2h_2}{4}\left\{\begin{array}{l}
\left(1+L_2-T^*\right)^2\left(1-2\ln\left(1+L_2-T^*\right)\right)-4\left(T^*-t_2\right)\left(1+L_2-T^*\right)\\
\times\ln\left(1+L_2-T^*\right)-\left(1+L_2-t_2\right)^2\left(1-2\ln\left(1+L_2-t_2\right)\right)\\
\left(k_2-1\right)\left\{\left(1+L_2\right)^2\left(1-2\ln\left(1+L_2\right)\right)-\left(1+L_2-t_2\right)^2\left(1-2\ln\left(1+L_2-t_2\right)\right)\right.\\
\left.+4t_2\left(1+L_2\right)\ln\left(1+L_2\right)\right\}
\end{array}\right\}\\
+\dfrac{h_r}{4}\left\{\begin{array}{l}
\mu\beta D_1\left(4\left(T^*-t_2\right)\left(1+L_2-t_2\right)\ln\left(1+L_2-t_2\right)-\left(1+L_1-T^*\right)^2\left(1-2\ln\left(1+L_1-T^*\right)\right)\right.\\
\left.+\left(1+L_1-t_2\right)^2\left(1-2\ln\left(1+L_1-t_2\right)\right)\right)\\
+\left(k_2D_2-\mu\beta D_1\right)\left(2\left(1+L_1\right)^2\left(\ln\left(1+L_1\right)-\ln\left(1+L_1-t_2\right)\right)\right.\\
\left.-2t_2\left(1+L_1\right)+\left(1+2\ln\left(1+L_1-t_2\right)\right)t_2^2\right)
\end{array}\right\}\\
+C^d\left\{\begin{array}{l}
\left(1-\mu\right)\beta D_1T^*+D_1\left(\begin{array}{l}
T^*\ln\left(1+L_1-T^*\right)-\left(1+L_1-T^*\right)\ln\left(1+L_1-T^*\right)\\
\times\ln\dfrac{1+L_1-t_1}{1+L_1-T^*}+\left(1+L_1\right)\times\ln\dfrac{1+L_1-t_1}{1+L_1-T^*}+t_1-T^*\\
-t_1\ln\left(1+L_1-t_1\right)+\left(k_1-1\right)\left(t_1-t_1\ln\left(1+L_1-t_1\right)\right.\\
-\left(1+L_1\right)\left(1-\ln\left(1+L_1\right)\right)\\
\left.\ln\dfrac{1+L_1}{1+L_1-t_1}\right)
\end{array}\right)\\
+\left(k_2D_2-\mu\beta D_1\right)\left(t_2\ln\left(1+L_1-t_2\right)+\left(1+L_1-\left(1+L_1-t_2\right)\ln\left(1+L_1-t_2\right)\right)\right)\\
\left.\times\ln\dfrac{1+L_1}{1+L_1-t_2}-t_2\right)\\
+\mu\beta D_1\left(T^*-T^*\ln\left(1+L_1-T^*\right)-\left(1+L_1-\left(1+L_1-t_2\right)\right.\right.\\
\left.\times\ln\left(1+L_1-t_2\right)\right)\ln\dfrac{1+L_1-t_2}{1+L_1-T^*}+t_2\ln\left(1+L_1-t_2\right)-t_2\Big)\\
+D_2\left(\begin{array}{l}
T^*\ln\left(1+L_2-T^*\right)-\left(1+L_2-T^*\right)\ln\left(1+L_2-T^*\right)\ln\dfrac{1+L_2-t_2}{1+L_2-T^*}+\left(1+L_2\right)\\
\times\ln\dfrac{1+L_2-t_2}{1+L_2-T^*}+t_2-T^*\\
+t_2\ln\left(1+L_2-t_2\right)+\left(k_2-1\right)\left(t_2-t_2\ln\left(1+L_2-t_2\right)-\left(1+L_2\right)\left(1-\ln\left(1+L_2\right)\right)\right.\\
\left.\times\ln\dfrac{1+L_2}{1+L_2-t_2}\right)
\end{array}\right)
\end{array}\right\}
\end{array}\right]\\
&+\left\{D_1+D_1\left(1+\ln\left(1+L_1-T^*\right)\right)\ln\frac{1+L_1-t_1}{1+L_1-T^*}\right\}\lambda_1+\left\{D_2+D_2\left(1+\ln\left(1+L_2-T^*\right)\right)\ln\frac{1+L_2-t_2}{1+L_2-T^*}\right\}\lambda_2\\
&+\left\{\mu\beta D_1\left(1-\ln\left(1+L_1-T^*\right)+\frac{T^*-\left(1+L_1\right)+\left(1+L_1-t_2\right)\ln\left(1+L_1-t_2\right)}{\left(1+L_1-T^*\right)}\right)-\mu\beta D_1\right\}\lambda_3=0
\end{aligned}
\tag{3.23}
$$

$$\lambda_1 X_1 = 0$$
$$\lambda_1\left[T^* D_1 - k_1 D_1 t_1^* + D_1\left\{\begin{array}{l} T^*\ln(1+L_1-T^*) - T^* - (1+L_1-T^*)\ln(1+L_1-T^*)\ln\dfrac{1+L_1-t_1^*}{1+L_1-T^*} \\ +(1+L_1)\ln\dfrac{1+L_1-t_1^*}{1+L_1-T^*} + t_1^* - t_1^*\ln(1+L_1-t_1^*) \\ +(k_1-1)\left(t_1^* - t_1^*\ln(1+L_1-t_1^*) - (1+L_1)(1-\ln(1+L_1))\ln\dfrac{1+L_1}{1+L_1-t_1^*}\right)\end{array}\right\}\right] = 0 \tag{3.24}$$

$$\lambda_2 X_2 = 0$$
$$\lambda_2\left[T^* D_2 - k_2 D_2 t_2^* + D_2\left\{\begin{array}{l} T^*\ln(1+L_2-T^*) - T^* - (1+L_2-T^*)\ln(1+L_2-T^*)\ln\dfrac{1+L_2-t_2^*}{1+L_2-T^*} \\ +(1+L_2)\ln\dfrac{1+L_2-t_2^*}{1+L_2-T^*} + t_2^* - t_2^*\ln(1+L_2-t_2^*) \\ +(k_2-1)\left(t_2^* - t_2^*\ln(1+L_2-t_2^*) - (1+L_2)(1-\ln(1+L_2))\ln\dfrac{1+L_2}{1+L_2-t_2^*}\right)\end{array}\right\}\right] = 0 \tag{3.25}$$

$$\lambda_3 X_3 = 0$$
$$\lambda_3\left[\begin{array}{l} k_2 D_2 t_2^* - \mu\beta D_1 T^* + (k_2 D_2 - \mu\beta D_1)\left\{(1+L_1-t_2^*)(1+L_1-\ln(1+L_1-t_2^*))\right. \\ \left.\ln\left(\dfrac{1+L_1}{1+L_1-t_2^*}\right) - t_2^* + t_2^*\ln(1+L_1-t_2^*)\right\} \\ +\mu\beta D_1\left\{T^* - T^*\ln(1+L_1-T^*) - (1+L_1-t_2^*)(1+L_1-\ln(1+L_1-t_2^*))\right. \\ \left.\ln\left(\dfrac{1+L_1-t_2^*}{1+L_1-T^*}\right) - t_2^* + t_2^*\ln(1+L_1-t_2^*)\right\}\end{array}\right] = 0 \tag{3.26}$$

The values of $\lambda_1, \lambda_2,$ and λ_3 are calculated using (21)–(23) as is given below.

$$\lambda_1 = \frac{\xi_1}{\xi_2} \tag{3.27}$$

$$\lambda_2 = \frac{\xi_3 - \xi_5\xi_6}{\xi_5\xi_7 - \xi_4} \tag{3.28}$$

$$\lambda_3 = \frac{\xi_3(\xi_4 - \xi_5\xi_7) + \xi_4(\xi_5\xi_6 - \xi_3)}{\xi_5(\xi_5\xi_7 - \xi_4)} \tag{3.29}$$

where the value of ξ_x, $x = 1, ..., 8$ is defined in the Appendix

3.2.2. WB Solution algorithm

Step 1. Input initial values of λ_1, λ_2, λ_3, and all system parameters.
Step 2. Calculate values of t_1^*, t_2^*, T^* using (21-23), which satisfy (24-26).
Step 3 Using the values of t_1^*, t_2^*, T^*, calculated in Step 2, compute the value of total cost (TC) ... defined in (3.16).
Step 4. From (27)−(29), using values of t_1^*, t_2^*, T^*, calculated in Step 2, compute values of λ_1, λ_2, ... and λ_3.
Step 5. Using values ofλ_1, λ_2, and λ_3, calculated in Step 4, repeat Step 1, 2, and 3 for n times ... until the value of TC stops decreasing.

Step 6. Stop further iterations. The values of $t_1^*, t_2^*,$ and T^*calculated in the final iteration are ... optimum, and (TC) is the minimum cost.

4. Special case

In this case the product of supply chain 2 does not deteriorate, while the product of supply chain 1 does deteriorates. The following inventory governing equations for product 2 change.

$$\frac{\mathrm{d}I_2^a(t)}{\mathrm{d}t} = P_2 - D_2,\ 0 \leqslant t \leqslant t_2 \tag{4.1}$$

$$\frac{\mathrm{d}I_2^b(t)}{\mathrm{d}t} = -D_2,\ t_2 \leqslant t \leqslant T \tag{4.2}$$

Solving these differential equations, inventory value at any time t is given as below.

$$I_2^a(t) = (k_2 - 1) D_2 t,\ 0 \leqslant t \leqslant t_2 \tag{4.3}$$

$$I_2^b(t) = -D_2 t,\ t_2 \leqslant t \leqslant T \tag{4.4}$$

Hence, the total inventory carried per cycle for product 2 is as follows:

$$I_2 = \int_0^{t_2} I_2^a(t)\,\mathrm{d}t + \int_{t_2}^{T} I_2^b(t)\,\mathrm{d}t = \frac{D_2}{2}\left(k_2 t_2^2 - T^2\right) \tag{4.5}$$

As the product 2 does not deteriorate, thus, $N_2^d = 0$

The total cost per unit time of the given two-supply chain model when product 2 does not deteriorate, is given by

$$TC(t_1, t_2, T) = \frac{1}{T}\left[\begin{array}{l}
C_1^s + C_2^s + C_r^s + \left(C_1^{mt} + C_1^p\right) k_1 D_1 t_1 + +C_2^{mt} N^o + C_2^p k_2 D_2 t_2 + C^c \beta D_1 T + \frac{h_2 D_2}{2}\left(k_2 t_2^2 - T^2\right) \\
+\frac{1}{4} h_1 D_1 \left\{\begin{array}{l}
(1 + L_1 - T)^2 (1 - 2\ln(1 + L_1 - T)) - 4(T - t_1)(1 + L_1 - T)\ln(1 + L_1 - T) \\
-(1 + L_1 - t_1)^2 (1 - 2\ln(1 + L_1 - t_1)) \\
+(k_1 - 1)\left\{(1 + L_1)^2 (1 - 2\ln(1 + L_1)) - (1 + L_1 - t_1)^2 (1 - 2\ln(1 + L_1 - t_1))\right. \\
\left. +4t_1 (1 + L_1)\ln(1 + L_1)\right\}
\end{array}\right\} \\
+\frac{1}{4} h_r \left\{\begin{array}{l}
\mu\beta D_1 \left\{\begin{array}{l}
-(1 + L_1 - T)^2 (1 - 2\ln(1 + L_1 - T)) + 4(T - t_2)(1 + L_1 - t_2)\ln(1 + L_1 - t_2) \\
+(1 + L_1 - t_2)^2 (1 - 2\ln(1 + L_1 - t_2))
\end{array}\right\} \\
+(k_2 D_2 - \mu\beta D_1)\left\{2(1 + L_1)^2 (\ln(1 + L_1)\right. \\
\left. -\ln(1 + L_1 - t_2)) - 2t_2(1 + L_1) + (1 + 2\ln(1 + L_1 - t_2)) t_2^2\right\}
\end{array}\right\} \\
+C^d \left\{\begin{array}{l}
D_1 \left\{\begin{array}{l}
T\ln(1 + L_1 - T) - t_1 \ln(1 + L_1 - t_1) \\
-(1 + L_1 - T)\ln(1 + L_1 - T)\ln\left(\frac{1 + L_1 - t_1}{1 + L_1 - T}\right) + (1 + L_1)\ln\left(\frac{1 + L_1 - t_1}{1 + L_1 - T}\right) + t_1 - T \\
+(k_1 - 1)\left\{(1 + L_1)(\ln(1 + L_1) - 1)\ln\left(\frac{1 + L_1}{1 + L_1 - t_1}\right) - t_1 \ln(1 + L_1 - t_1) + t_1\right\}
\end{array}\right\} \\
+\left\{\begin{array}{l}
(k_2 D_2 - \mu\beta D_1)\left\{\{(1 + L_1) - (1 + L_1 - t_2)\ln(1 + L_1 - t_2)\}\right. \\
\left. \times \ln\left(\frac{1 + L_1}{1 + L_1 - t_2}\right) + t_2 \ln(1 + L_1 - t_2) - t_2\right\} \\
+\mu\beta D_1 \left\{\begin{array}{l}
\{(1 + L_1 - t_2)\ln(1 + L_1 - t_2) - (1 + L_1)\}\ln\left(\frac{1 + L_1 - t_2}{1 + L_1 - T}\right) - T\ln(1 + L_1 - T) \\
+t_2 \ln(1 + L_1 - t_2) - t_2 + T
\end{array}\right\}
\end{array}\right\} \\
+(1 - \mu)\beta D_1 T
\end{array}\right\}
\end{array}\right] \tag{4.6}$$

This model contains the same constraints and is solved by the same method as the main model.

5. Numerical experiments

The proposed models are validated through numerical experiments, and related results are reported in this section. Two examples are considered for the proposed models (3.16) and (4.6). Mathematica 9 is used to find the optimal solution. Input parameters and results are summarized in the following sections.

5.1. Input parameters

Most of the input parameters are taken from Alamri [1], and product lifetime is considered from Sarkar [32]. Those parameters not related to these references are assumed by the authors. Input parameters are summarized in Table 2.

Assuming$\alpha = 1$, input values are taken such that $N_c^r \geqslant N_2^D$ and $N^o = 0$.

5.2. Results and discussion

Optimal results are obtained and summarized for both examples as given below.

Example 5.1. Optimal solution for the model in which both the products deteriorate is summarized in Table 3.

This optimal solution satisfies the model constraints. Optimal values of the decision variables are found on the boundary of the feasible region defined by the model constraints. One can observe in Table 3 that the optimal time of production 1 is 0.37 month, and that of production 2 is 0.32 month. The optimal cycle time is 1.12 months and the minimum cost per month is \$17354.5. The solution is illustrated graphically in Figures 4a−4e, illustrating the convexity of the model for TC at optimal values of t_1, t_2 and T. Although total cost can further be reduced below the optimal value, but at those values of decision variables, model constraints are violated and the solution becomes invalid. Behavior of Figures 4a−4e is explained as following:

Figure 4a shows the relation between production time of product 1 t_1 and the total costTC. According to the obtained optimal solutions, total cost is minimum at $t_1^* = 0.37$ provided the model constraints are not violated. By decreasing the value of t_1 below the optimal point the total cost can further be reduced, but at those points model constrains are not satisfied. The region of graph above the optimal value of t_1 is within the feasible region, but the cost increases due to increased rate of deterioration.

The relation between production time of product 2 t_2 and total cost TC is illustrated in Figure 4b. The graph shows that total cost is minimum at $t_2^* = 0.32$, which fulfils the defined constraints. As both the products are deteriorating, therefore behavior of the graph of product 2 is similar to that of product 1. Beyond the optimal value of t_2, the total cost increases due to the results of deterioration.

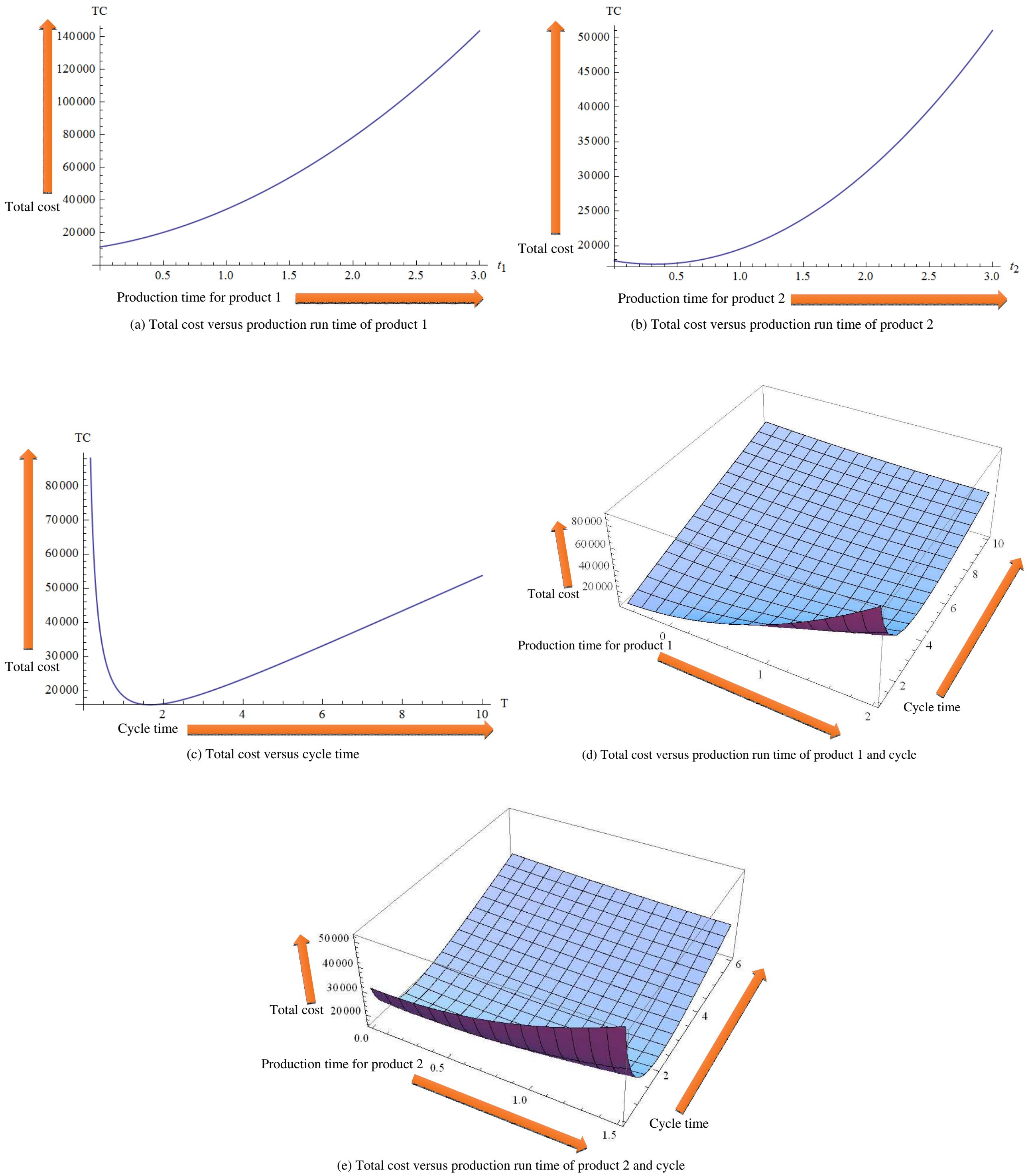
Figure 4c shows variation in total cost TC with respect to the cycle time T. One can observe that the total cost becomes higher when length of planning horizon is decrease below or increase beyond the optimal length of planning horizon $T^* = 1.12$. Therefore, it is important to precisely plan the inventory schedule for deteriorating

Table 2. Input parameters for Example 1 and 2

$C_1^s = \$2400/\text{cycle}$	$h_1 = \$1.6/\text{unit/month}$	$C_1^{mt} = \$5/\text{unit}$	$C^c = \$1/\text{unit}$
$C_2^s = \$1600/\text{cycle}$	$h_2 = \$1.2/\text{unit/month}$	$C_2^{mt} = \$2/\text{unit}$	$L_1 = 36\,\text{months}$
$C_r^s = \$1200/\text{cycle}$	$D_1 = 1000\,\text{units / month}$	$C_1^p = \$2/\text{unit}$	$L_2 = 50\,\text{months}$
$h_r = \$1/\text{unit/month}$	$D_2 = 250\,\text{units / month}$	$C_2^p = \$1.2/\text{unit}$	$C^d = \$0.2/\text{unit}$
$k_1 = 3.6$	$k_2 = 4$	$\mu = 0.8$	$\beta = 0.7$

Table 3. Optimal solutions of Example 1.

$TC^* = \$17354.5/\text{month}$	$t_1^* = 0.37\,\text{month}$	$t_2^* = 0.32\,\text{month}$	$T^* = 1.12\,\text{months}$



(a) Total cost versus production run time of product 1

(b) Total cost versus production run time of product 2

(c) Total cost versus cycle time

(d) Total cost versus production run time of product 1 and cycle

(e) Total cost versus production run time of product 2 and cycle

FIGURE 4. Total cost versus production run time of product 1.

products because with every passing day, more number of items deteriorate, which increases the cost and makes the system less profitable.

Figure 4d depicts the joint effect of variation in the values of production and cycle time of the product 1 on the value of total cost. The trough of graph indicates the minimum value of the cost with respect to both the variables of product 1. The curve is more inclined towards small periods of production and inventory schedules, which is an obvious result of time-varying deterioration. Therefore, this is an important conclusion for deteriorating products to plan frequent inventory replenishments such that fewer items deteriorate.

Value of total cost varies with respect to the values of production and cycle time of product 2 and effect of both these variables on the total cost is illustrated in Figure 4e. Corresponding points of production and cycle time at the lowest value of cost indicate the optimal solution for the problem under consideration. As explained earlier, shorter production and cycle time is preferable for deteriorating products. But these schedules cannot be shortened infinitesimally small because it will increase the cost drastically, as is obvious from the curve illustrated above.

Example 5.2. Optimal solution for the special case, when product 2 does not deteriorate and only product 1 deteriorates is summarized in Table 4.

Table 4 shows that optimal production time of product 1 is 0.40 month and that of product 2 is 0.34 month. The optimal cycle time is 1.22 months and the minimum total cost per month is \$16582.4. Comparing these results with the results of Example1, the total cost per month is reduced. The optimal solution indicates that when product 2 does not deteriorate, the loss due to deterioration is reduced, and hence, the total cost is reduced. Convexity of the model is illustrated through graphical representation in Figures 5a–5e, which proves the model validity.

Total cost of the system varies with production time of the product 1. Figure 5a shows the convexity of the total cost with respect to the production time of product 1. The cost is minimum when production time of product 1 is 0.40 months.

Figure 5b illustrates the effect of variation in production time for product 2 on the total cost, which is minimum at $t_2^* = 0.34$. The graph shows that the total cost function is convex with respect to the production time for product 2.

It is obvious from Figure 5c that total cost is the minimum within defined constraints when $T^* = 1.22$. As compared to the model 1, length of planning horizon has increased for this model because product 2 does not deteriorate and it can be stored as inventory for longer periods of time.

Both the production and cycle time of product 1 affect the total cost of the system, which is illustrated in Figure 5d. As is shown in the figure that the trend of variation in total cost with respect to these variables is not linear. Therefore, the minimum cost is found on the lowest point of the curve, which represents the optimal values of the production and cycle time of product 1. Moving to either side of the optimal point increases the cost and makes the system less profitable.

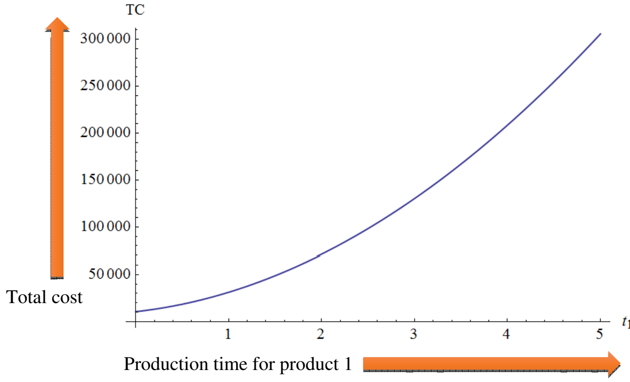
Figure 5e shows the combine effect of variation in the values of production and cycle time of the product 2 on the value of total cost. The trough of the graph indicates the minimum value of the cost with respect to both the variables of product 2.

5.3. Sensitivity analysis

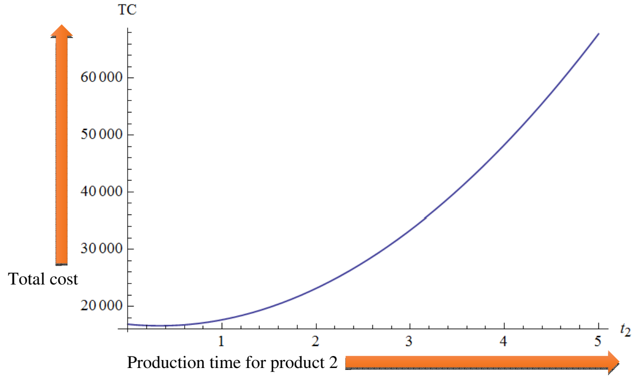
Sensitivity analysis is performed for the model in which both products deteriorate. Table 5 and Figures 6a– 6d show the effect of variation by certain percentages (–50%, –25%, + 25%, + 50%) of the specific costs on the optimal values of decision variables and that of the objective function.

TABLE 4. Optimal solutions of Example 2.

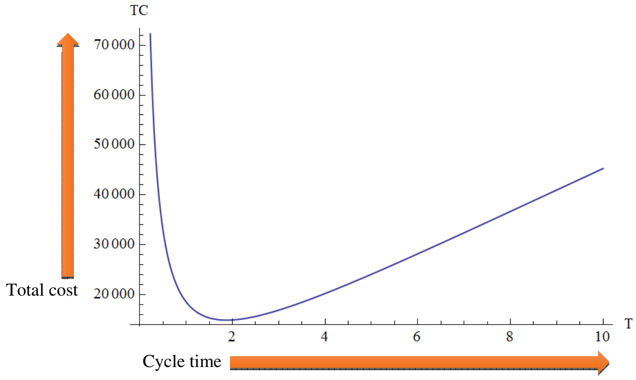
$TC^* = \$16582.4/\text{month}$	$t_1^* = 0.40\,\text{month}$	$t_2^* = 0.34\,\text{month}$	$T^* = 1.22\,\text{months}$



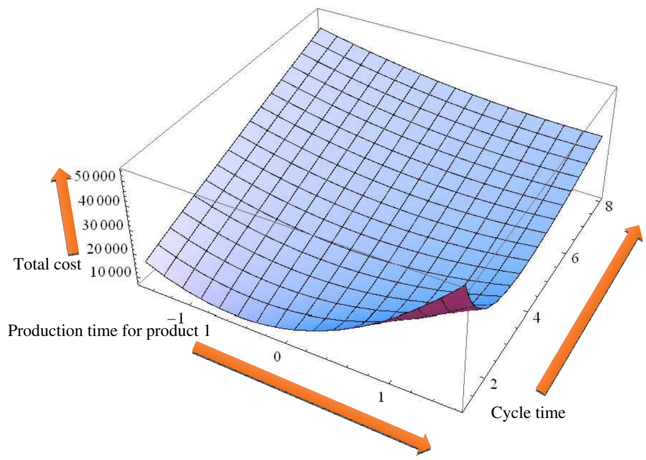
(a) Total cost versus production run time of product 1



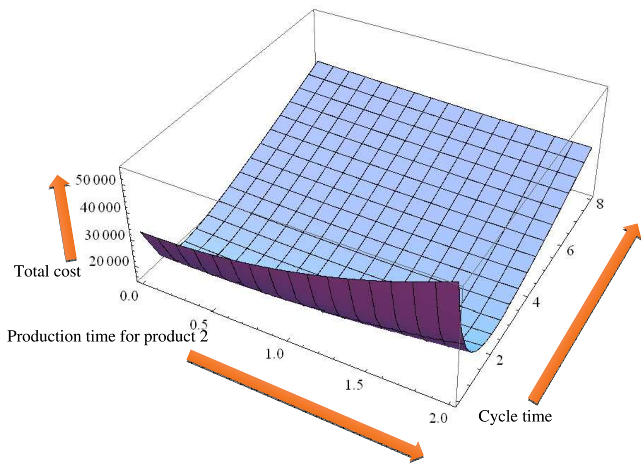
(b) Total cost versus production run time of product 2



(c) Total cost versus cycle time

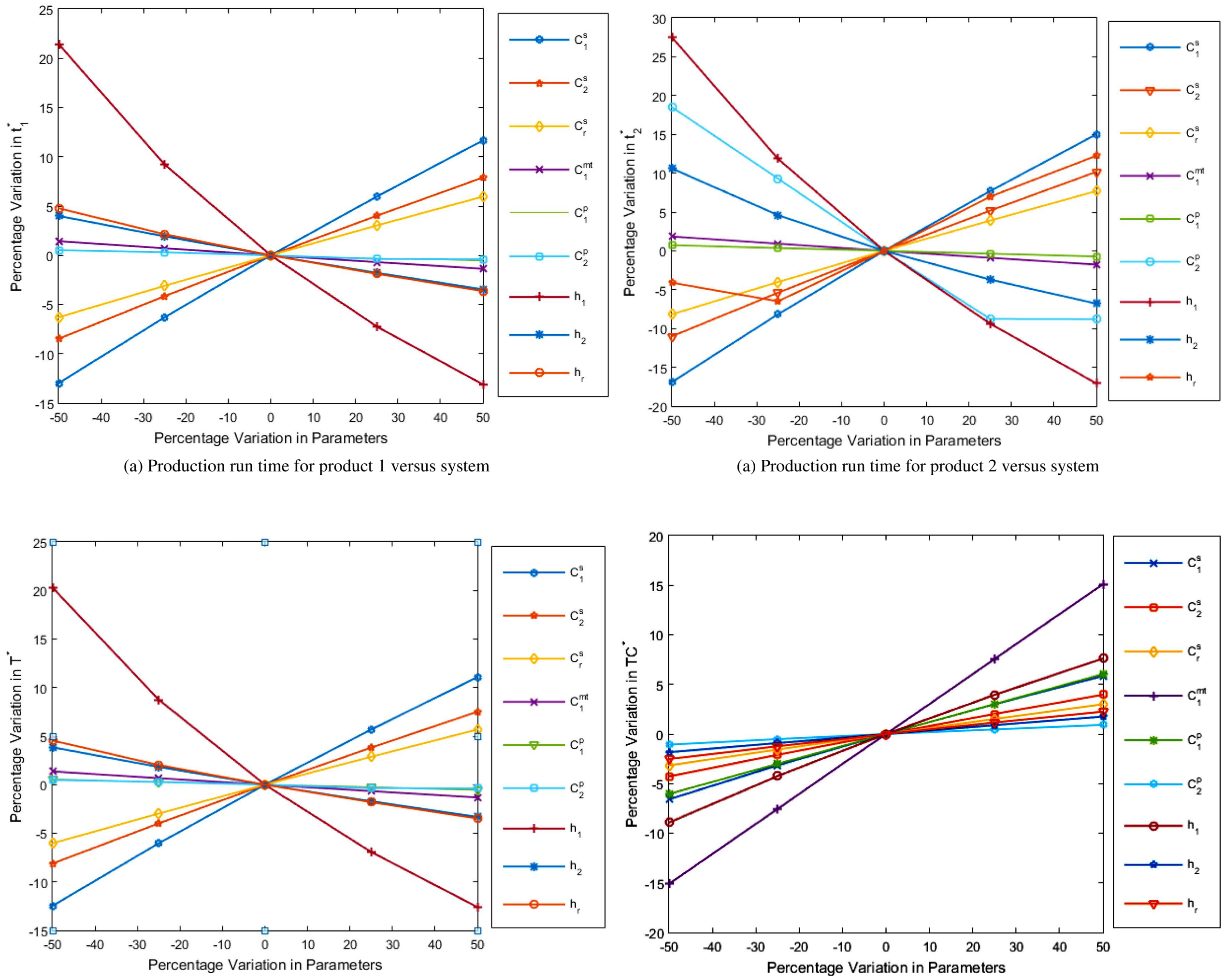


(d) Total cost versus production run time of product 1 and cycle



(e) Total cost versus production run time of product 2 and cycle

FIGURE 5.



(a) Production run time for product 1 versus system

(a) Production run time for product 2 versus system

(c) Cycle time versus system parameters

(d) Total cost versus system parameters

Figure 6.

Analysis of the above results is given below.

- Different types of cost parameters affect the decision variables in different way. Values of t_1, t_2 and T vary directly with variation in setup costs and inversely with variation in material, production, and inventory holding costs.
- For variation in the value of a specific cost parameter, decision variables show similar trends of variation, but the extent of variation is different for different cost parameters.
- The values of t_1, t_2 and T are most sensitive to variation in setup costs and inventory holding costs, while the effect of variation in other cost parameters is moderate. Remarkable variation is observed in the value of t_2 when cost of production of product 2 is changed.
- Optimal value of the total cost varies directly with variation in the value of individual cost parameters. In order to reduce the total cost, reduction in the individual cost parameters will also change the production

TABLE 5. Sensitivity analysis of key parameter of Example 1.

Parameters		t_1^*	t_2^*	T^*	TC^*
	% Change	% Change	% Change	% Change	% Change
C_1^s	– 50	– 12.98	– 16.90	– 12.47	– 6.58
	– 25	– 6.29	– 8.17	– 6.03	– 3.18
	+ 25	+ 5.96	+ 7.70	+ 5.68	+ 3.00
	+ 50	+ 11.65	+ 15.01	+ 11.08	+ 5.85
C_2^s	– 50	– 8.47	– 11.01	– 8.12	– 4.29
	– 25	– 4.15	– 5.39	– 3.98	– 2.10
	+ 25	+ 4.01	+ 5.18	+ 3.82	+ 2.02
	+ 50	+ 7.89	+ 10.18	+ 7.51	+ 3.96
C_r^s	– 50	– 6.29	– 8.17	– 6.03	– 3.18
	– 25	– 3.10	– 4.02	– 2.97	– 1.57
	+ 25	+ 3.02	+ 3.90	+ 2.88	+ 1.52
	+ 50	+ 5.96	+ 7.70	+ 5.68	+ 3.00
C_1^{mt}	– 50	+ 1.43	+ 1.85	+ 1.37	– 15.11
	– 25	+ 0.71	+ 0.92	+ 0.68	– 7.55
	+ 25	– 0.69	– 0.90	– 0.66	+ 7.55
	+ 50	– 1.37	– 1.78	– 1.31	+ 15.10
C_1^p	– 50	+ 0.57	+ 0.73	+ 0.54	– 6.04
	– 25	+ 0.28	+ 0.36	+ 0.27	– 3.02
	+ 25	– 0.28	– 0.36	– 0.27	+ 3.02
	+ 50	– 0.56	– 0.72	– 0.53	+ 6.04
C_2^p	– 50	+ 0.52	+ 18.50	+ 0.49	– 1.07
	– 25	+ 0.30	+ 9.31	+ 0.29	– 0.51
	+ 25	– 0.36	– 8.77	– 0.35	0.47
	+ 50	– 0.40	– 8.80	– 0.38	0.92
h_1	– 50	+ 21.43	+ 27.49	+ 20.29	– 8.92
	– 25	+ 9.20	+ 11.86	+ 8.75	– 4.25
	+ 25	– 7.23	– 9.39	– 6.93	+ 3.94
	+ 50	– 13.11	– 17.07	– 12.60	+ 7.62
h_2	– 50	+ 3.99	+ 10.60	+ 3.85	– 1.84
	– 25	+ 1.91	+ 4.60	+ 1.83	– 0.91
	+ 25	– 1.78	– 3.71	– 1.71	+ 0.89
	+ 50	– 3.46	– 6.80	– 3.31	+ 1.76
h_r	– 50	+ 4.77	– 4.10	+ 4.55	– 2.53
	– 25	+ 2.13	– 6.50	+ 2.03	– 1.24
	+ 25	– 1.88	+ 6.98	– 1.80	+ 1.16
	+ 50	– 3.64	+ 12.25	– 3.48	+ 2.26

and cycle time. Managers have to consider schedule flexibility, *i.e.*, whether the schedule can be planned for longer or shorter time.

- Total cost per unit time is most sensitive to variations in C_1^s, C_1^{mt}, C_1^p & h_1, moderately sensitive to C_2^s & C_r^s and least sensitive to the other cost parameters.

Figure 6a shows the effect of variation in several cost parameters on the optimal value of production time of product 1. One can observe from the figure that production time of product 1 is most sensitive to the inventory holding cost of product 1 and least sensitive to the production costs for both the products. The optimal value of production time increase with decrease in the value of material cost, production cost, and inventory holding cost and vice versa. In contrast, the optimal value of production time increase with increase in the value of setup cost and vice versa.

Variation in the values of several cost parameters varies the optimal value of production time of product 2. This effect is illustrated in Figure 6b, which shows that the production time of product 2 is most sensitive to the inventory holding cost of product 1 and is least sensitive to the production cost of product 2. Variation in setup costs and returned products' inventory holding cost affects the optimal value of production time of product 2 positively, while this effect is negative in case of material cost, production cost and inventory holding cost of product 1 and product 2.

Trend of variation in the optimal value of cycle time with variation in the values of several cost parameters is the same as for the production time of product 1. This effect is illustrated graphically in Figure 6c, which shows that the cycle time is most sensitive to the inventory holding cost of product 1 and is least sensitive to the production costs for both the products. The optimal value of cycle time increases with decrease and the values of material cost, production cost, and inventory holding cost and vice versa. In contrast, the optimal value of cycle time increase with increase in the value of setup cost and vice versa.

Figure 6d illustrates the variation in the optimal value of total cost of the proposed system with respect to several cost parameters. Total cost is most sensitive to the variation in material cost of product 1 and is least sensitive to the inventory holding cost of product 2. All the cost parameters affect the optimal value of the total cost positively. Thus, the total cost can be reduced by reducing any of the individual costs. From the results of sensitivity analysis, it is concluded that variation in any of the cost parameters will vary the optimal value of the total cost as well as the production and cycle time of both the products. Therefore, it is important to consider the flexibility of schedules and capacity of the production systems while reducing these costs

6. Managerial insights

This model of research is appropriate for those deteriorating products, which, after their expiration, can be recycled into other useful products, which is a step towards sustainable system. Some pertinent managerial insights regarding the suggested framework of research are provided as below.

Insight 1.
The results of numerical calculations suggest that the length of production and inventory cycle should be kept short for deteriorating products. Therefore, there should be frequent inventory replenishments, such that fewer number of items would deteriorate. With longer periods of inventory replenishment, more number of items would deteriorate that will not only waste the materials but will also create environmental pollution.

Insight 2.
Cost reduction is one of the major objectives of a supply chain system and managers try to reduce several expenses to increase the yield at lower costs. Sensitivity analysis provides the managers with important insights about reducing total cost of the system. Reducing the individual cost parameters not only reduces the total cost but it also affects the production and inventory cycle time. Therefore, managers should consider the flexibility of their planning and the capacity of their production systems while reducing such cost parameters.

Insight 3.
Incorporating a secondary supply chain in order to recycle the waste food products into useful product generates good amount of revenue, which not only conserves the natural resources but also protects environment. Managers can use the proposed model to adopt such policies of product recycling and enhance productivity.

7. Conclusions

This study discussed a forward and reverse supply chain system for deteriorating products and highlighted the necessity of sustainable systems and resource conservation. EOU and EOL products from one supply chain, instead of being wasted, were used as raw material to produce different products in another supply chain, playing a significant role in resource conservation and environment protection. Kuhn−Tucker method solved the suggested model with inequality constraints, and the proposed algorithm, while minimizing total cost, calculated the optimum schedule of production and inventory. These are critical variables for deteriorating products due to the fact that rate of deterioration is time-varying. This research proposed the precise and short schedule planning of production and inventory cycles for deteriorating products. A practical view of the study was presented through numerical examples and optimal solutions were illustrated graphically to validate the proposed models. Sensitivity analysis provided important managerial insights that would help managers to decide on cost reduction by varying different system parameters considering the schedule flexibility and production capacity. The authors suggest that the model can be extended by considering the products with time-varying demand, as proposed by as proposed by Chakrabarty *et al.* [8]. It can also be extended by introducing preservation technology for deteriorating items, as suggested by Dye [12]. The deterioration can cause shortages of the products, therefore this model can be extended by considering shortages and backlogging, as proposed by Goel *et al.* [16].

Appendix A.

$$\begin{aligned}
\xi_1 =& \frac{k_1 D_1 \left(C_1^{mt} + C_1^{p}\right)}{T^*} + \frac{D_1 \left(C^d + h_1 \left(1 + L_1 - t_1^*\right)\right)}{\left(1 + L_1 - t_1^*\right) T^*} \\
&\times \begin{pmatrix} L_1 \left(\ln\left(1 + L_1 - T^*\right) - \ln\left(1 + L_1\right) - \ln\left(1 + L_1\right) - \left(T^* - 1\right) \ln\left(1 + L_1 - T^*\right)\right) \\ +k_1 \left(\left(1 + L_1\right)\left(\ln\left(1 + L_1\right) - \ln\left(1 + L_1 - T^*\right)\right) t_1^* \ln\left(1 + L_1 - t_1^*\right)\right) \end{pmatrix}
\end{aligned}$$

$$\xi_2 = \frac{D_1}{1 + L_1 - t_1^*} \left\{ \begin{array}{l} \ln\left(1 + L_1\right) + \left(T^* - 1\right) \ln\left(1 + L_1 - T^*\right) + L_1 \left(\ln\left(1 + L_1\right) - \ln\left(1 + L_1 - T^*\right)\right) \\ +k_1 \left(\left(1 + L_1\right)\left(1 - \ln\left(1 + L_1\right) + \ln\left(1 + L_1 - T^*\right)\right) - t_1^* \left(1 + n\left(1 + L_1 - t_1^*\right)\right)\right) \end{array} \right\}$$

$$\xi_3 = \frac{1}{T^*} \left[\begin{array}{l} -\mu\beta D_1 \left(1 + \ln\left(1 + L_1 - t_2^*\right)\right) \left(C^d \ln \dfrac{1 + L_1}{1 + L_1 - T^*} + h_r T^*\right) \\
+\frac{D_2}{1 + L_2 - t_2^*} \left\{ \begin{array}{l} C^d \begin{pmatrix} \left(1 - T^*\right) \ln\left(1 + L_2 - T^*\right) - \ln\left(1 + L_2\right) + L_2 \left(\ln\left(1 + L_2 - T^*\right) - \ln\left(1 + L_2\right)\right) \\ -t_2^* \left\{ \left(1 + \ln\left(1 + L_1 - t_2^*\right)\right) \ln \dfrac{1 + L_1}{1 + L_1 - t_2^*} - \ln\left(1 + L_1 - t_2^*\right) \right\} \\ +k_2 \left\{ \left(1 + L_2\right) \left(\ln\left(1 + L_2\right) - \ln\left(1 + L_1 - t_2^*\right) + \left(1 + \ln\left(1 + L_1 - t_2^*\right)\right) \right. \right. \\ \left. \left. \ln \dfrac{1 + L_1}{1 + L_1 - t_2^*}\right) \right\} \end{pmatrix} \\
+\left(1 + L_2 - t_2^*\right) \left\{ \begin{array}{l} k_2 \left(C_2^{p} + h_r t_2^* \left(1 + \ln\left(1 + L_1 - t_2^*\right)\right)\right) \\ +h_2 k_2 \left(\left(1 + L_2\right) \left(\ln\left(1 + L_2\right) - \ln\left(1 + L_2 - t_2^*\right)\right) + t_2^* \right. \\ \left. \ln\left(1 + L_1 - t_2^*\right)\right) \\ -h_2 \ln\left(1 + L_2\right) - h_2 \left(T^* - 1\right) \ln\left(1 + L_2 - T^*\right) \\ +h_2 L_2 \left(\ln\left(1 + L_2 - T^*\right) - \ln\left(1 + L_2\right)\right) \end{array} \right\} \end{array} \right\} \end{array} \right]$$

$$\xi_4 = \frac{D_2}{1 + L_2 - t_2^*} \left\{ \begin{array}{l} L_2 \left(\ln\left(1 + L_2 - T^*\right) - \ln\left(1 + L_2\right)\right) - \left(T^* - 1\right) \ln\left(1 + L_2 - T^*\right) - \ln\left(1 + L_2\right) \\ +k_2 \left(\left(1 + L_2\right) \left(\ln\left(1 + L_2\right) - \ln\left(1 + L_2 - t_2^*\right) - 1\right) + t_2^* \left(1 + \ln\left(1 + L_2 - t_2^*\right)\right)\right) \end{array} \right\}$$

$$\xi_5 = k_2 D_2 - \left(1 + \ln\left(1 + L_1 - t_2^*\right)\right) \left(\mu\beta D_1 \ln \frac{1 + L_1}{1 + L_1 - T^*} - k_2 D_2 \ln \frac{1 + L_1}{1 + L_1 - t_2^*}\right)$$

$$\xi_6 = \frac{1}{T^*}\left[\begin{array}{l} \beta C^c D_1 + C_2^{mt}\left(D_2 - \mu\beta D_1\right) + D_1 h_1\left(T^* - t_1^*\right)\left(1 + \ln\left(1 + L_1 - T^*\right)\right) \\ +C^d \left\{\begin{array}{l} \beta D_1\left(1-\mu\right) + D_1\left(1 + \ln\left(1 + L_1 - T^*\right)\right)\ln\frac{1+L_1-t_1^*}{1+L_1-T^*} \\ +D_2\left(1 + \ln\left(1 + L_2 - T^*\right)\right)\ln\frac{1+L_2-t_2^*}{1+L_2-T^*} \\ +\mu\beta D_1\left(1 - \ln\left(1 + L_1 - T^*\right) + \frac{T^* - (1+L_1) + (1+L_2-t_2^*)\ln(1+L_2-t_2^*)}{(1+L_1-T^*)}\right) \end{array}\right\} \\ +D_2 h_2\left(T^* - t_2^*\right)\left(1 + \ln\left(1 + L_2 - T^*\right)\right) + \frac{\mu\beta D_1 h_r}{2}\left\{\left(1 + L_1 - T^*\right)\left(1 - 2\ln\left(1 + L_1 - T^*\right)\right)\right. \\ \left. -\left(1 + L_1 - T^*\right) + 2\left(1 + L_1 - t_2^*\right)\ln\left(1 + L_1 - t_2^*\right)\right\} \end{array}\right]$$

$$+\frac{1}{T^{*2}}\left[\begin{array}{l} C_1^s + C_2^s + C_r^s + \beta C^c D_1 T^* + C_2^{mt}\left(D_2 T^* - \mu\beta D_1 T^*\right) + \left(C_1^{mt} + C_1^p\right)k_1 D_1 t_1^* + C_2^p k_2 D_2 t_2^* \\ +\frac{D_1 h_1}{4}\left\{\begin{array}{l} \left(1 + L_1 - T^*\right)^2\left(1 - 2\ln\left(1 + L_1 - T^*\right)\right) - 4\left(T^* - t_1^*\right)\left(1 + L_1 - T^*\right) \\ \times\ln\left(1 + L_1 - T^*\right) - \left(1 + L_1 - t_1^*\right)^2\left(1 - 2\ln\left(1 + L_1 - t_1^*\right)\right) \\ \left(k_1 - 1\right)\left\{\left(1 + L_1\right)^2\left(1 - 2\ln\left(1 + L_1\right)\right) - \left(1 + L_1 - t_1^*\right)^2\left(1 - 2\ln\left(1 + L_1 - t_1^*\right)\right)\right. \\ \left. +4t_1^*\left(1 + L_1\right)\ln\left(1 + L_1\right)\right\} \end{array}\right\} \\ +\frac{D_2 h_2}{4}\left\{\begin{array}{l} \left(1 + L_2 - T^*\right)^2\left(1 - 2\ln\left(1 + L_2 - T^*\right)\right) - 4\left(T^* - t_2^*\right)\left(1 + L_2 - T^*\right) \\ \times\ln\left(1 + L_2 - T^*\right) - \left(1 + L_2 - t_2^*\right)^2\left(1 - 2\ln\left(1 + L_2 - t_2^*\right)\right) \\ \left(k_2 - 1\right)\left\{\left(1 + L_2\right)^2\left(1 - 2\ln\left(1 + L_2\right)\right) - \left(1 + L_2 - t_2^*\right)^2\left(1 - 2\ln\left(1 + L_2 - t_2^*\right)\right)\right. \\ \left. +4t_2\left(1 + L_2\right)\ln\left(1 + L_2\right)\right\} \end{array}\right\} \\ +\frac{h_r}{4}\left\{\begin{array}{l} \mu\beta D_1\left(4\left(T^* - t_2^*\right)\left(1 + L_2 - t_2^*\right)\ln\left(1 + L_2 - t_2^*\right) - \left(1 + L_1 - T^*\right)^2\left(1 - 2\ln\left(1 + L_1 - T^*\right)\right)\right. \\ \left. +\left(1 + L_1 - t_2^*\right)^2\left(1 - 2\ln\left(1 + L_1 - t_2^*\right)\right)\right) \\ +\left(k_2 D_2 - \mu\beta D_1\right)\left(2\left(1 + L_1\right)^2\left(\ln\left(1 + L_1\right) - \ln\left(1 + L_1 - t_2^*\right)\right) - 2t_2^*\left(1 + L_1\right)\right. \\ \left. +\left(1 + 2\ln\left(1 + L_1 - t_2^*\right)\right)t_2^{*2}\right) \end{array}\right\} \\ +C^d\left\{\begin{array}{l} \left(1-\mu\right)\beta D_1 T^* + D_1\left(\begin{array}{l} T^*\ln\left(1 + L_1 - T^*\right) - \left(1 + L_1 - T^*\right)\ln\left(1 + L_1 - T^*\right) \\ \times\ln\frac{1+L_1-t_1^*}{1+L_1-T^*} + \left(1 + L_1\right)\ln\frac{1+L_1-t_1^*}{1+L_1-T^*} + t_1^* - T^* \\ -t_1^*\ln\left(1 + L_1 - t_1^*\right) + \left(k_1 - 1\right)\left(t_1^* - t_1^*\right. \\ \left.\ln\left(1 + L_1 - t_1^*\right) - \left(1 + L_1\right)\left(1 - \ln\left(1 + L_1\right)\right)\ln\frac{1+L_1}{1+L_1-t_1^*}\right) \end{array}\right) \\ +\left(k_2 D_2 - \mu\beta D_1\right)\left(t_2^*\ln\left(1 + L_1 - t_2^*\right)\right. \\ \left. +\left(1 + L_1 - \left(1 + L_1 - t_2^*\right)\ln\left(1 + L_1 - t_2^*\right)\right)\ln\frac{1+L_1}{1+L_1-t_2^*} - t_2^*\right) \\ +\mu\beta D_1\left(T^* - T^*\ln\left(1 + L_1 - T^*\right) - \left(1 + L_1 - \left(1 + L_1 - t_2^*\right)\ln\left(1 + L_1 - t_2^*\right)\right)\right. \\ \left. \times\ln\frac{1+L_1-t_2^*}{1+L_1-T^*} + t_2^*\ln\left(1 + L_1 - t_2^*\right) - t_2^*\right) \\ +D_2\left(\begin{array}{l} T^*\ln\left(1 + L_2 - T^*\right) - \left(1 + L_2 - T^*\right)\ln\left(1 + L_2 - T^*\right)\ln\frac{1+L_2-t_2^*}{1+L_2-T^*} + \left(1 + L_2\right) \\ \times\ln\frac{1+L_2-t_2^*}{1+L_2-T^*} + t_2^* - T^* \\ +t_2^*\ln\left(1 + L_2 - t_2^*\right) + \left(k_2 - 1\right)\left(t_2^* - t_2^*\right. \\ \left. \times\ln\left(1 + L_2 - t_2^*\right) - \left(1 + L_2\right)\left(1 - \ln\left(1 + L_2\right)\right)\ln\frac{1+L_2}{1+L_2-t_2^*}\right) \end{array}\right) \end{array}\right\} \end{array}\right]$$

$$\xi_7 = \left\{D_1 + D_1\left(1 + \ln\left(1 + L_1 - T^*\right)\right)\ln\frac{1 + L_1 - t_1^*}{1 + L_1 - T^*}\right\}\frac{\xi_1}{\xi_2} + \left\{D_2 + D_2\left(1 + \ln\left(1 + L_2 - T^*\right)\right)\ln\frac{1 + L_2 - t_2^*}{1 + L_2 - T^*}\right\}$$

$$\xi_8 = \mu\beta D_1\left(1 - \ln\left(1 + L_1 - T^*\right) + \frac{T^* - \left(1 + L_1\right) + \left(1 + L_1 - t_2^*\right)\ln\left(1 + L_1 - t_2^*\right)}{\left(1 + L_1 - T^*\right)}\right) - \mu\beta D_1$$

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