SOME RESULTS ABOUT COMPONENT FACTORS IN GRAPHS *

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Abstract. For a set $\mathcal{H}$ of connected graphs, a spanning subgraph H of a graph G is called an $\mathcal{H}$-factor of G if every component of H is isomorphic to a member of $\mathcal{H}$. An $\mathcal{H}$-factor is also referred as a component factor. If each component of H is a star (resp. path), H is called a star (resp. path) factor. By a $P_{\geq k}$-factor (k positive integer) we mean a path factor in which each component path has at least k vertices (*i.e.* it has length at least $k - 1$). A graph G is called a $P_{\geq k}$-factor covered graph, if for each edge e of G, there is a $P_{\geq k}$-factor covering e. In this paper, we prove that (i) a graph G has a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}\}$-factor if and only if $\mathrm{bind}(G) \geq \frac{1}{k}$, where $k \geq 2$ is an integer; (ii) a connected graph G is a $P_{\geq 2}$-factor covered graph if $\mathrm{bind}(G) > \frac{2}{3}$; (iii) a connected graph G is a $P_{\geq 3}$-factor covered graph if $\mathrm{bind}(G) \geq \frac{3}{2}$. Furthermore, it is shown that the results in this paper are best possible in some sense.

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1. Introduction

The graphs considered in this paper are finite undirected graphs without loops or multiple edges. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For $x \in V(G)$, we denote by $N_G(x)$ the set of vertices adjacent to x in G. Let S be a subset of $V(G)$. We write $N_G(S) = \cup_{x \in S} N_G(x)$. We use $G[S]$ to denote the subgraph of G induced by S, and $G - S = G[V(G) \setminus S]$. We say that S is independent if $N_G(S) \cap S = \emptyset$. Let $I(G)$ be the set of isolated vertices of G and $i(G)$ the number of isolated vertices of G. The binding number $\mathrm{bind}(G)$ of G is defined by

$$\mathrm{bind}(G) = \min\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\}.$$

We use K_n and $K_{n,m}$ to denote the complete graph and the complete bipartite graph, respectively.

For a set $\mathcal{H}$ of connected graphs, a spanning subgraph H of a graph G is called an $\mathcal{H}$-factor of G if every component of H is isomorphic to a member of $\mathcal{H}$. An $\mathcal{H}$-factor is also referred as a component factor. A graph G is called an $\mathcal{H}$-factor covered graph, if for each edge e of G, there is a $\mathcal{H}$-factor including e. By a $P_{\geq k}$-factor (k positive integer) we mean a path factor in which each component path has at least k vertices (*i.e.* it has length

at least $k - 1$). A graph G is called a $P_{\geq k}$-factor covered graph, if for each edge e of G, there is a $P_{\geq k}$-factor covering e.

Kano and Saito [8] showed a sufficient condition for a graph to have a $\{K_{1,l} : m \leq l \leq 2m\}$-factor. Kano, Lu and Yu [7] obtained a sufficient condition for a graph to have a $\{K_{1,2}, K_{1,3}, K_5\}$-factor. Akiyama, Avis and Era [2] gave a necessary and sufficient condition for a graph with a $P_{\geq 2}$-factor. Kaneko [6] obtained a necessary and sufficient condition for a graph to have a $P_{\geq 3}$-factor. Li and Zhang [11] showed that the square of any connected graph on at least 6 vertices has a $\{P_3, P_4\}$-factor. Bazgan *et al.* [5] gave a toughness condition for a graph with a $P_{\geq 3}$-factor. The following results on component factors are known.

Amahashi and Kano [3] gave two necessary and sufficient conditions for the existence of a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}\}$-factor in a graph.

Theorem 1.1. *(Amahashi and Kano [3]). Let k be an integer with $k \geq 2$. Then a graph G has a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}\}$-factor if and only if $i(G - S) \leq k|S|$ for every subset S of $V(G)$, or $|N_G(X)| \geq \frac{|X|}{k}$ for every independent set X of G.*

Zhang and Zhou [21] obtained respective necessary and sufficient conditions defining a $P_{\geq 2}$-factor covered graph and a $P_{\geq 3}$-factor covered graph.

Theorem 1.2. *(Zhang and Zhou [21]). Let G be a connected graph. Then G is a $P_{\geq 2}$-factor covered graph if and only if $i(G - S) \leq 2|S| - \varepsilon_1(S)$ for any subset S of $V(G)$, where $\varepsilon_1(S)$ is defined by*

$$\varepsilon_1(S) = \begin{cases} 2, \textit{ if } S \textit{ is not an independent set}; \\ 1, \textit{ if } S \textit{ is a nonempty independent set and } G - S \textit{ has a nontrivial component}; \\ 0, \textit{ otherwise}. \end{cases}$$

A graph R is said to be factor-critical if $R - x$ admits a 1-factor (K_2-factor) for each $x \in V(R)$. A graph H is said to be a sun if $H = K_1$, $H = K_2$ or H is the corona of a factor-critical graph R with order at least three, *i.e.*, H is obtained from R by adding a new vertex $w = w(v)$ together with a new edge vw for each $v \in V(R)$. A sun with order at least 6 is said to be a big sun. We denote by $\text{sun}(G)$ the number of sun components of G.

Theorem 1.3. *(Zhang and Zhou [21]). Let G be a connected graph. Then G is a $P_{\geq 3}$-factor covered graph if and only if $\text{sun}(G - S) \leq 2|S| - \varepsilon_2(S)$ for any subset S of $V(G)$, where $\varepsilon_2(S)$ is defined by*

$$\varepsilon_2(S) = \begin{cases} 2, \textit{ if } S \textit{ is not an independent set}; \\ 1, \textit{ if } S \textit{ is a nonempty independent set and } G - S \textit{ has a non} - \textit{sun component}; \\ 0, \textit{ otherwise}. \end{cases}$$

The binding number of a graph was first introduced by Woodall [13]. There is a significant number of papers dealing with connections between binding number and factors in graphs. For instance, Katerinis and Woodall [10] showed the binding number conditions for graphs to have k-factors; Kano and Tokushige [9] posed the binding number condition for a graph to have an f-factor; Zhou [17] presented a binding number condition for the existence of $[a, b]$-factor with prescribed properties; Yu and Liu [14] got the binding number conditions for graphs having fractional factors; Zhou, Bian and Sun [20] gave the binding number condition for graphs to be all fractional (a, b, k)-critical graphs; Zhou [16] posed a binding number condition for a graph to be a fractional ID-k-factor-critical graph. A recent book discussing both binding numbers and component factors (among many other related subjects) is [15]. Some surveys or collections of results on the subject are [1, 4, 12, 18, 19]. In this paper we investigate the relationship between the binding number and component factors in graphs, and obtain some results on $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}\}$-factors of graphs and $P_{\geq k}$-factor covered graphs. The main results will be shown in Sections 2, 3 and 4, respectively.

2. Binding number and component factors in graphs

Theorem 2.1. *Let k be an integer with $k \geq 2$. Then a graph G has a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}\}$-factor if and only if $\operatorname{bind}(G) \geq \frac{1}{k}$.*

Proof. Sufficiency: Suppose that $\operatorname{bind}(G) \geq \frac{1}{k}$, but G has no $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}\}$-factor. Then by Theorem 1.1, $|N_G(X)| < \frac{|X|}{k}$ for some independent set X of G and, in particular, $\operatorname{bind}(G) < \frac{1}{k}$, which contradicts $\operatorname{bind}(G) \geq \frac{1}{k}$. Hence, G has a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}\}$-factor.

Necessity: Suppose that G has a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}\}$-factor, but $\operatorname{bind}(G) < \frac{1}{k}$. Then there exists a subset Y of $V(G)$ with $Y \neq \emptyset$, such that $\frac{|N_G(Y)|}{|Y|} = \operatorname{bind}(G) < \frac{1}{k}$. Thus, we obtain

$$|N_G(Y)| < \frac{1}{k}|Y|. \tag{2.1}$$

According to (2.1), $G[Y]$ has at least one isolated vertex. Set $S = N_G(Y) \setminus (N_G(Y) \cap Y)$. Obviously, $S \neq \emptyset$. In terms of (2.1) and the definition of S, we have

$$\begin{aligned} i(G - S) &\geq |Y \setminus (N_G(Y) \cap Y)| = |Y| - |N_G(Y) \cap Y| \\ &> k|N_G(Y)| - |N_G(Y) \cap Y| \geq k(|N_G(Y)| - |N_G(Y) \cap Y|) \\ &= k|N_G(Y) \setminus (N_G(Y) \cap Y)| = k|S|, \end{aligned}$$

that is,

$$i(G - S) > k|S|. \tag{2.2}$$

On the other hand, since G has a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}\}$-factor, by Theorem 1.1 we get

$$i(G - S) \leq k|S|.$$

Which contradicts (2.2). Hence, $\operatorname{bind}(G) \geq \frac{1}{k}$. This completes the proof of Theorem 2.1. $\square$

Note that a $P_{\geq 2}$-factor is equivalent to a $\{K_{1,1}, K_{1,2}\}$-factor. Hence, we obtain immediately the following corollary.

Corollary 2.2. *A graph G has a $P_{\geq 2}$-factor if and only if $\operatorname{bind}(G) \geq \frac{1}{2}$.*

3. Binding number and $P_{\geq 2}$-factor covered graphs

Theorem 3.1. *Let G be a connected graph. Then G is a $P_{\geq 2}$-factor covered graph if $\operatorname{bind}(G) > \frac{2}{3}$.*

Proof. Suppose that $\operatorname{bind}(G) > \frac{2}{3}$, but G is not a $P_{\geq 2}$-factor covered graph. Then by Theorem 1.2, there exists a subset S of $V(G)$ such that

$$i(G - S) > 2|S| - \varepsilon_1(S) \tag{3.1}$$

Now, we prove the following claims.

Claim 1. $|S| \geq 2$.

Proof. Assume that $S = \emptyset$. Then $\varepsilon_1(S) = 0$. Combining this with (3.1), we have

$$i(G) > 0. \tag{3.2}$$

On the other hand, G is a connected graph, so $i(G) = 0$. Which contradicts (3.2).

If $|S| = 1$ (we write $S = \{u\}$), then $\varepsilon_1(S) \leq 1$. Using (3.1), we obtain

$$i(G - S) > 2|S| - \varepsilon_1(S) \geq 2 - 1 = 1.$$

Since $i(G - S)$ is an integer, we have

$$i(G - S) \geq 2.$$

Note that G is a connected graph. Thus, there exist two vertices x_1 and x_2 of $V(G)$ such that $x_i u \in E(G)$ and $x_i v \notin E(G)$, $i = 1, 2$, where $v \in V(G) \setminus S$. In terms of the definition of $\text{bind}(G)$, we obtain

$$\text{bind}(G) \leq \frac{|N_G(\{x_1, x_2\})|}{|\{x_1, x_2\}|} = \frac{1}{2},$$

which contradicts that $\text{bind}(G) > \frac{2}{3}$. This completes the proof of Claim 1.   □

Claim 2. $i(G - S) \geq 3$.

Proof. From (3.1), Claim 1 and $\varepsilon_1(S) \leq 2$, we obtain

$$i(G - S) > 2|S| - \varepsilon_1(S) \geq 4 - 2 = 2.$$

Since $i(G - S)$ is an integer, we have

$$i(G - S) \geq 3.$$

The proof of Claim 2 is complete.   □

According to Claim 2, we have $I(G - S) \neq \emptyset$ and $N_G(I(G - S)) \neq V(G)$. Then by the definition of $\text{bind}(G)$ and the condition of the theorem, we obtain

$$\frac{2}{3} < \text{bind}(G) \leq \frac{|N_G(I(G - S))|}{|I(G - S)|} \leq \frac{|S|}{i(G - S)},$$

which implies

$$i(G - S) < \frac{3}{2}|S|. \tag{3.3}$$

On the other hand, since $\varepsilon_1(S) \leq 2$ and $i(G - S)$ is an integer, (3.1) yields, we have

$$i(G - S) \geq 2|S| - 1.$$

Combining this with (3.3), we get

$$2|S| - 1 \leq i(G - S) < \frac{3}{2}|S|,$$

that is,

$$|S| < 2,$$

which contradicts Claim 1. This completes the proof of Theorem 5.   □

Remark 3.2. We show that the condition $\text{bind}(G) > \frac{2}{3}$ in Theorem 3.1 is sharp, which is shown in the following example:

We construct a graph $G_m = K_2 \vee (3K_1 \cup K_m)$, where $m \geq 2$ is an integer. Obviously, $\text{bind}(G_m) = \frac{2}{3}$. Set $S = V(K_2)$. Then $i(G_m - S) = 3$ and $\varepsilon_1(S) = 2$. Thus, we have

$$i(G_m - S) > 2|S| - \varepsilon_1(S).$$

According to Theorem 1.2, G_m is not a $P_{\geq 2}$-factor covered graph.

4. Binding number and $P_{\geq 3}$-factor covered graphs

Woodall [13] prove that the graphs with $\mathrm{bind}(G) \geq \frac{3}{2}$ have Hamiltonian circuits. Moreover, we verify that the graphs with $\mathrm{bind}(G) \geq \frac{3}{2}$ are $P_{\geq 3}$-factor covered graphs, which is the following theorem.

Theorem 4.1. *Let G be a connected graph. Then G is a $P_{\geq 3}$-factor covered graph if* $\mathrm{bind}(G) \geq \frac{3}{2}$.

Proof. Suppose that $\mathrm{bind}(G) \geq \frac{3}{2}$, but it is not a $P_{\geq 3}$-factor covered graph. Then by Theorem 3.1, there exists a subset S of $V(G)$ satisfying

$$\mathrm{sun}(G - S) > 2|S| - \varepsilon_2(S). \tag{4.1}$$

$\square$

Claim 3. $S \neq \emptyset$.

Proof. If $S = \emptyset$, then by (4.1) we have

$$\mathrm{sun}(G) > 0. \tag{4.2}$$

Note that G is a connected graph. It follows from (4.2) that

$$\mathrm{sun}(G) = 1.$$

Thus, it is easy to see that $G \neq K_1$ and $G \neq K_2$. In the following, we consider that G is a big sun. We use R to denote the factor-critical subgraph of G, and write $Q = V(R)$. Obviously, $i(G - Q) = |Q| = \frac{|V(G)|}{2}$. Combining this with the definition of $\mathrm{bind}(G)$, we obtain

$$\mathrm{bind}(G) \leq \frac{|N_G(I(G - Q))|}{|I(G - Q)|} = \frac{|Q|}{|Q|} = 1,$$

which contradicts $\mathrm{bind}(G) \geq \frac{3}{2}$. This completes the proof of Claim 3. $\square$

Assume that there exist a isolated vertices, b K_2's and c big sun components $H_1, H_2, \ldots, H_c$, where $|H_i| \geq 6$, in $G - S$. Clearly, we have

$$\mathrm{sun}(G - S) = a + b + c. \tag{4.3}$$

According to (4.1), (4.3), Claim 3 and $\varepsilon_2(S) \leq 2$, we obtain

$$a + b + c = \mathrm{sun}(G - S) > 2|S| - \varepsilon_2(S) \geq 0,$$

that is,

$$a + b + c \geq 1. \tag{4.4}$$

Claim 4. $a + c \geq 1$.

Proof. Assume that $a + c = 0$. Then by (4.4), we have

$$b \geq 1,$$

and so,

$$\mathrm{sun}(G - S) = b. \tag{4.5}$$

In terms of the definition of $\mathrm{bind}(G)$, we obtain

$$\mathrm{bind}(G) \leq \frac{|N_G(V(bK_2) \setminus x)|}{|V(bK_2) \setminus x|} \leq \frac{|S| + 2b - 1}{2b - 1} \tag{4.6}$$

for any $x \in V(bK_2)$. If $b = 1$, then by (4.1), (4.5) and $\varepsilon_2(S) \leq 2$ we have

$$1 = b = \mathrm{sun}(G - S) > 2|S| - \varepsilon_2(S) \geq 2|S| - 2,$$

which implies

$$|S| < \frac{3}{2}.$$

In terms of Claim 3 and the integrity of $|S|$, we obtain

$$|S| = 1.$$

Combining this with the definition of $\varepsilon_2(S)$, we have

$$\varepsilon_2(S) \leq 1.$$

Then by using (4.1), we obtain

$$1 = b = \mathrm{sun}(G - S) > 2|S| - \varepsilon_2(S) = 2|S| - 1 = 1,$$

which is a contradiction.

If $b = 2$, then by (4.1), (4.5) and $\varepsilon_2(S) \leq 2$ we obtain

$$2 = b = \mathrm{sun}(G - S) > 2|S| - \varepsilon_2(S) \geq 2|S| - 2,$$

which implies

$$|S| < 2.$$

According to Claim 3 and the integrity of $|S|$, we have

$$|S| = 1.$$

Combining this with (4.6) and $\mathrm{bind}(G) \geq \frac{3}{2}$, we obtain

$$\frac{3}{2} \leq \mathrm{bind}(G) \leq \frac{|S| + 2b - 1}{2b - 1} = \frac{2b}{2b - 1}$$

that is,

$$b \leq \frac{3}{2},$$

which contradicts $b = 2$.

If $b \geq 3$, then by (4.1), (4.5), (4.6), $\varepsilon_2(S) \leq 2$ and $\mathrm{bind}(G) \geq \frac{3}{2}$ we have

$$\frac{3}{2} \leq \mathrm{bind}(G) \leq \frac{|S| + 2b - 1}{2b - 1} < \frac{\frac{b+2}{2} + 2b - 1}{2b - 1} = \frac{5b}{4b - 2},$$

which implies

$$b < 3,$$

which contradicts $b \geq 3$. The proof of Claim 4 is complete. □

We choose one vertex from each K_2 component of $G - S$, and denote by X the set of such vertices. For each H_i, we use R_i to denote the factor-critical subgraph of H_i, and write $Y_i = V(R_i)$. Then $|X| = b$ and $i(H_i - Y_i) = |Y_i| = \frac{|H_i|}{2}$. We write $Y = \bigcup_{i=1}^{c} Y_i$. Then we have

$$i(G - (S \cup X \cup Y)) = a + b + \sum_{i=1}^{c} \frac{|H_i|}{2}. \tag{4.7}$$

In terms of (4.4), (4.7) and $|H_i| \geq 6$ $(1 \leq i \leq c)$, we have

$$I(G - (S \cup X \cup Y)) \neq \emptyset \cdot \tag{4.8}$$

According to (4.3), (4.7), (4.8), the definition of $\text{bind}(G)$ and the condition of the theorem, we obtain

$$\frac{3}{2} \leq \text{bind}(G) \leq \frac{|N_G(I(G - (S \cup X \cup Y)))|}{|I(G - (S \cup X \cup Y))|} \leq \frac{|S| + |X| + |Y|}{i(G - (S \cup X \cup Y))} = \frac{|S| + b + \sum_{i=1}^{c} |Y_i|}{a + b + \sum_{i=1}^{c} \frac{|H_i|}{2}}$$

$$= \frac{|S| + b + \sum_{i=1}^{c} \frac{|H_i|}{2}}{a + b + \sum_{i=1}^{c} \frac{|H_i|}{2}} = \frac{|S| + \text{sun}(G - S) - a - c + \sum_{i=1}^{c} \frac{|H_i|}{2}}{\text{sun}(G - S) - c + \sum_{i=1}^{c} \frac{|H_i|}{2}},$$

which implies

$$\text{sun}(G - S) \leq 2|S| - \sum_{i=1}^{c} \frac{|H_i|}{2} + c - 2a. \tag{4.9}$$

It follows from (4.9) and $|H_i| \geq 6$ that

$$\text{sun}(G - S) \leq 2|S| - \sum_{i=1}^{c} 3 + c - 2a = 2|S| - 2(a + c). \tag{4.10}$$

It follows from (4.10), Claim 4 and $\varepsilon_2(S) \leq 2$ that

$$\text{sun}(G - S) \leq 2|S| - 2(a + c) \leq 2|S| - 2 \leq 2|S| - \varepsilon_2(S),$$

which contradicts (4.1). This completes the proof of Theorem 4.1.

Remark 4.2. Let us show that $\text{bind}(G) \geq \frac{3}{2} = \frac{15}{10}$ in Theorem 4.1 cannot be replaced by $\text{bind}(G) \geq \frac{14}{10} = \frac{7}{5}$. We show this by constructing a graph $G = K_n \vee ((2n - 1)K_2)$, where $n \geq 2$ is a positive integer. In terms of the definition of $\text{bind}(G)$, we have

$$\frac{3}{2} > \text{bind}(G) = \frac{|N_G(V((2n-1)K_2) \setminus x)|}{|V((2n-1)K_2) \setminus x|} = \frac{n + 2(2n-1) - 1}{2(2n-1) - 1} = \frac{5n - 3}{4n - 3} > \frac{5}{4},$$

where $x \in V((2n-1)K_2)$. Let $S = V(K_n)$. Then by the definition of $\varepsilon_2(S)$, we obtain $\varepsilon_2(S) = 2$. Thus, we have

$$\text{sun}(G - S) = 2n - 1 > 2n - \varepsilon_2(S) = 2|S| - \varepsilon_2(S).$$

According to Theorem 1.3, G is not a $P_{\geq 3}$-factor covered graph. The binding number $\text{bind}(G) = \frac{7}{5}$ when $n = 2$.

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