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AN N-POLICY DISCRETE-TIME $GEO/G/1$ QUEUE WITH MODIFIED MULTIPLE SERVER VACATIONS AND BERNOULLI FEEDBACK *

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Abstract. This paper deals with a single-server discrete-time $Geo/G/1$ queueing model with Bernoulli feedback and N-policy where the server leaves for modified multiple vacations once the system becomes empty. Applying the law of probability decomposition, the renewal theory and the probability generating function technique, we explicitly derive the transient queue length distribution as well as the recursive expressions of the steady-state queue length distribution. Especially, some corresponding results under special cases are directly obtained. Furthermore, some numerical results are provided for illustrative purposes. Finally, a cost optimization problem is numerically analyzed under a given cost structure.

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1. Introduction

The research of discrete-time queueing systems have caught numerous scholars' interest during the last few decades. In fact, the applications of discrete-time queues can be found in many real-world situations, such as flexible manufacturing systems, production/inventory systems and the digital computer and communication systems. In particular, in computer and communication systems (*e.g.*, time division multiple access (TDMA)), the inter-arrival times of packets and their forward transmission times are the elementary units of time like bits and bytes. These queueing activities, on a discrete-time basis, can only occur at regularly spaced epochs, which has become a powerful incentive to the investigation of discrete-time queueing theory. In addition, discrete-time queues are more suitable than their continuous-time counterparts for characterizing the behaviors of data communication and computer networks, and can be used to approximate the continuous systems but not vice versa. For details on discussion and applications in the area of discrete-time queues, we refer the reader to the monographs by Hunter [8], Bruneel and Kim [4] and Woodward [17].

In the context of utilizing effectively the server's idle time for different purposes (*e.g.*, carrying out maintenance for service facility or taking a break), Levy and Yechiali [18] first proposed the concept of vacation policy. Since then, many researchers have been attracted to study queueing systems with vacations and considerable

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research results on this topic have been achieved. Up to now, vacation queueing models have been developed to successfully analyze the operations of various stochastic service systems and have been regarded as effective tools in modeling and analyzing complex computer and communication networks. The assumption that when the system becomes empty the server leaves for vacation is not only more realistic, but also provides more flexibility in optimal design and control of a queueing system. A comprehensive review of vacation queues can be found in the surveys of Doshi [5] and Ke *et al.* [12], and the monographs of Takagi [24] and Tian and Zhang [25]. In the recent past, remarkable contributions on discrete-time queueing systems with vacations have been made by many authors (see *e.g.*, Zhang and Tian [32], Tang *et al.* [23], Wang *et al.* [27], Wang [28], Goswami and Mund [10], Laxmi and Jyothsna [20], and Gao and Wang [11]).

In the aforementioned papers, it is generally assumed that at the completion epoch of a vacation the server either is available thereafter (single vacation policy) or takes another vacation (multiple vacations policy). The purpose of the present study is to extend the above work by incorporating a new feature, named standby period. When a customer who has just been served leaves an empty system, the server stays idle for some random length of time (called standby period), instead of leaving for vacation immediately. If there are arrivals occurring during the standby period, the server is immediately available and begins to serve the customers until the system becomes empty again. If no customer arrives at the system at the end of the standby period, the server starts his vacation at once. Further, if no customer is found in the system at the completion epoch of a vacation, the server commences a new standby period. The introduction of standby period generalizes the standard multiple vacations policy and makes it possible for the immediate availability of the server, whereas in the classical vacation queues the server is available only when he completes his vacation.

Over the last few decades, one of the most extensive studies regarding queueing models is the optimal design and control of the behavior of queues. The main objective of investigating controllable queueing systems is to actualize the precise control and help system managers economize the operating cost. Among many control policies, the most popular policy is the N-policy which was first proposed in Yadin and Naor [30]. The N-policy states that the server is turned off whenever the system becomes empty and is turned on when the number of waiting customers reaches the predetermined threshold value N. Queueing systems with N-policy have been well studied by many researchers. Hernández−Díaz and Moreno [7] studied a discrete-time queue with early setup and Bernoulli feedback in which the system operates under N-policy. They discussed a cost optimization problem to manage the system at a minimum cost. By the supplementary variable technique, Moreno [21] considered a discrete-time $Geo/G/1$ queue with a generalized N-policy and setup-closedown times. Lim et al. [13] discussed a $GI/Geo/1$ queueing model with N-policy and derived the stationary queue length distributions at various time epochs. For more discussion and applications concerning N-policy, interested readers may refer to Luo *et al.* [14], Lee and Yang [16], Wei *et al.* [29], Aksoy and Gupta [1], and references therein.

A queueing system with feedback phenomenon is characterized by the feature that when a customer is dissatisfied with the current service, it may retry again and again until it has obtained a satisfactory service before leaving the system. This kind of queues frequently arises in our day-to-day life. For instance, in multiple access telecommunication systems, the data packet with errors at the destination will be sent repeatedly until the data packet is successfully transmitted. The research of queueing system with Bernoulli feedback was initiated by Takacs [26]. Atencia and Moreno [2] addressed a discrete-time $Geo^X/G_H/1$ retrial queue with Bernoulli feedback. Atencia *et al.* [3] provided an extensive analysis for a discrete-time retrial queueing system with starting failures, Bernoulli feedback and general retrial times. More discussion for the related topic concerning discrete-time queues with Bernoulli feedback can be referred to Liu and Gao [19], Gao and Liu [9] and Yang *et al.* [31].

In this paper, we propose to investigate a discrete-time $Geo/G/1$ queueing system with Bernoulli feedback, modified multiple vacations and N-policy. This model is useful to control the system queue length and reduce the customers' waiting time. Moreover, it can also greatly economize the switching costs of the system. To the best of our knowledge, there is no research work on the proposed model. The main contributions of this paper are as follows. It is the first time in the queueing literature to consider a discrete-time queue by incorporating Bernoulli feedback, modified multiple vacations and N-policy simultaneously, which makes the model much closer to many

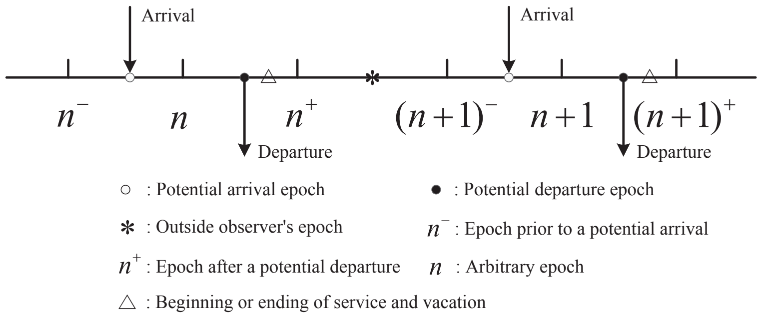

FIGURE 1. Various time epochs in a late arrival system with delayed access (LAS-DA).

practical situations. Also, the analysis technique used here is different from the supplementary variable method and embedded Markov chain technique. Using a new and direct approach, we obtain the transient solution of the queue size distribution and the explicit recursive formulas of the stationary queue length distribution. The explicit recursive formulas can used to calculate accurately the probabilities of queue length. Finally, in order to save the operating cost, we develop a long-run expected cost function per unit time to discuss the cost optimization problem.

The organization of the remaining paper is structured as follows. In Section 2, the considered mathematical model is formulated and some preliminaries are given. Section 3 is devoted to analyzing some queueing characteristics including the transient queue length distribution, the steady-state queue length distribution, *etc.* Section 4 is related to exploring the effect of some key parameters on system performance measures. Moreover, in Section 5, we consider a cost problem for minimizing the system cost. Finally, some conclusions are given in Section 6.

2. Model description and preliminaries

We consider a discrete-time $Geo/G/1$ queueing system with Bernoulli feedback and N-policy where the server leaves for modified multiple vacations as soon as there is no customers in the system. Different from continuous-time queues, all the queueing activities (*e.g.*, arrivals and departures of customers), in discrete time queues, are nonnegative integer-valued random variables. Assume that the time axis is segmented into equal length intervals (called slots) and let the time axis be marked with $0, 1, 2, \ldots, n, \ldots$ All the arrivals and departures only happen at boundary epochs of time slots in discrete-time regime. In view of this fact, one arrival and one departure may take place simultaneously within a slot. So, it is necessary to stipulate the order of arrivals and departures. Generally speaking, there are two types of discrete-time models, namely, the early arrival system (EAS) and the late arrival system (LAS). And the late arrival system (LAS) can further be subdivided into late arrival system with delayed access (LAS-DA) and late arrival system with immediate access (LAS-IA). More discussion regarding these concepts can be referred to Hunter [8]. In this study, we consider the late arrival system with delayed access (LAS-DA), that is, the arrivals take place within (n^-, n), $n = 0, 1, 2, \ldots$, and the departures occur within (n, n^+), $n = 1, 2, \ldots$ Moreover, it is assumed that there is no arrival within $(0^-, 0)$ and no departure within $(0, 0^+)$. To make it clearer, the various time epochs at which queueing events occur are displayed in Figure 1.

The detailed mathematical model is described as follows.

(1) Arrival process: Customers arrive at the system according to a Bernoulli process with rate $p (0 < p < 1)$, *i.e.*, only one customer arrives with probability p and no customer arrives with probability $1 - p$ in every slot, which ensures that the customers' inter-arrival times $\{\tau_k, k \geq 1\}$ are independent and identically distributed

(i.i.d.) with a geometric distribution $\mathrm{P}\{\tau_k = j\} = p(1-p)^{j-1}$, $j = 1, 2, \ldots$ The probability generating function (PGF) of the number of arrivals during a random slot is then given by $A(z) = 1 - p + pz$, $|z| < 1$.

(2) Service process: There is only one service station in the system and the customers are served one by one according to the first-come-first-served (FCFS) discipline. The service times of the customers, denoted by $\{\chi_k, k \geq 1\}$, are i.i.d. discrete-time random variables and arbitrarily distributed with probability mass function (PMF) $\mathrm{P}\{\chi_k = j\} = g_j$, $j \geq 1$, and PGF $G(z) = \sum_{j=1}^{\infty} g_j z^j$, $|z| < 1$. Both the mean service time μ and the second moment $\mathrm{E}[(\chi_k)^2]$ are finite. After a customer is served, he will decide either to leave the system forever with probability $\alpha\,(0 < \alpha \leq 1)$ or join the queue again for another service with complementary probability $1 - \alpha$, which implies that the total service number of a customer, denoted by ξ, follows a geometric distribution with parameter α, *i.e.*, $\mathrm{P}\{\xi = j\} = \alpha(1-\alpha)^{j-1}$, $j \geq 1$.

(3) Modified multiple vacations and N-policy: Whenever all the customers in the system are completely served, the server enters standby state instead of leaving for vacation immediately. The duration of standby period, denoted by Y, is a discrete random variable with PMF $\mathrm{P}\{Y = j\} = y_j$, $j = 0, 1, 2, \ldots$ and PGF $Y(z) = \sum_{j=0}^{\infty} y_j z^j$. If there are arrivals occurring during server standby period, the server is available and begins to serve the customers immediately until the system becomes empty again. And the server renewedly enters standby state. If no customer arrives at the system during standby period, the server starts his/her vacation at once. The vacation time, designated by V, is also a discrete random variable with PMF $\mathrm{P}\{V = k\} = v_k$, $k = 1, 2, \ldots$ and PGF $V(z) = \sum_{j=1}^{\infty} v_j z^j$. During a server's vacation, if the number of customers in the system reaches a predetermined N, the server interrupts his/her vacation and begins to provide service for the customers at once. If there are less than N customers in the queue, he/she stays vacation state until the vacation period ends. On return from the vacation, if he/she finds one or more customers (less than N customers) waiting for service, he/she serves these customers immediately until the system is empty again. Otherwise, if no customer is found in the system at the completion epoch of a vacation, the server commences a new standby period which may be followed by another vacation.

(4) It is assumed that if there are no customers in the system at initial time $n^+ = 0^+$, the server stays idle and waits for the first arrival (this hypothesis is not only more realistic in real-world situations, but also has no effect on the stationary queue length distribution). That is to say, after the first busy period, the system begins to take modified multiple vacations and N-policy. Further, various random variables involved in the system are independent of each other.

For later discussion, we first present some preliminaries as follows.

Definition 2.1. "System idle period" is the time interval that starts at the instant at which the system becomes empty and ends at the instant when the first customer arrives. Obviously, the system idle period is the remaining time of an arrival interval.

Let $\tilde{\tau}_k (k = 1, 2, \ldots)$ be the kth system idle period. Thus, $\tilde{\tau}_k$ are independent mutually and satisfy the same geometric distribution with parameter p (*i.e.*, $\mathrm{P}\{\tilde{\tau}_k = j\} = p(1-p)^{j-1}$, $j \geq 1$) due to the Markovian property of geometric distribution.

Definition 2.2. "Server idle period" is the time interval that commences when the system is completely empty and finishes when the server begins to serve the waiting customers.

Denote by I the server idle period. In our model, the server idle period may contain the standby period and the vacation period.

Definition 2.3. "Total service time" is the time length from the time when the service of a customer begins until the time when the customer is served completely and leaves the system, which consists of the possible multiple services caused by Bernoulli feedback.

Let $\tilde{\chi}$ be the total service time of a customer and $\tilde{g}_j = \mathrm{P}\{\tilde{\chi} = j\}$ be the corresponding PMF. According to the model assumptions, we have $\tilde{\chi} = \sum_{i=1}^{\xi} \chi_i$, where $\chi_i,\ i \geq 1$ and ξ are independent of each other. Then, for $j \geq 1$, we can get

$$\tilde{g}_j = \mathrm{P}\{\tilde{\chi} = j\} = \mathrm{P}\left\{\sum\nolimits_{i=1}^{\xi} \chi_i = j\right\} = \sum_{k=1}^{j} \alpha (1-\alpha)^{k-1} \mathrm{P}\left\{\sum\nolimits_{i=1}^{k} \chi_i = j\right\}. \tag{2.1}$$

From the above equation, the PGF of $\tilde{\chi}$ is given by

$$\tilde{G}(z) = \sum_{j=1}^{\infty} \tilde{g}_j z^j = \frac{\alpha G(z)}{1-(1-\alpha)G(z)},\ |z|<1, \tag{2.2}$$

which indicates that the mean of the total service time $\tilde{\chi}$ is

$$\mathrm{E}[\tilde{\chi}] = \frac{\mathrm{d}\tilde{G}(z)}{\mathrm{d}z}|_{z=1} = \frac{\mu}{\alpha}\cdot \tag{2.3}$$

Remark 2.4. If $\tilde{\chi}$ is regarded as the service time of a customer, the queueing system under consideration is equal to a discrete-time $Geo/G/1$ queue with N-policy and modified multiple server vacations in which the service times of customers are i.i.d. with distribution $\{\tilde{g}_j,\ j \geq 1\}$ and mean $\mathrm{E}[\tilde{\chi}] = \frac{\mu}{\alpha}$.

Definition 2.5. "Server busy period" is the time interval that starts at the time point at which the server begins to serve customers and finishes when the system becomes empty again.

Denote by b the length of the server busy period initiated with only one customer. Then, similar to the analysis in Bruneel and Kim [4], we have Lemma as follows.

Lemma 2.6. *Let $B(z) = \sum_{j=1}^{\infty} P\{b=j\} z^j$ be the PGF of b. For $|z|<1$, $B(z)$ is the root of the equation $B(z) = \tilde{G}((\bar{p}+pB(z))z)$, and the mean value is given by*

$$E[b] = \begin{cases} \dfrac{\rho}{p(1-\rho)}, & \rho < 1, \\ \infty, & \rho \geq 1, \end{cases}$$

where $\bar{p} = 1-p$, $\rho = p\mu/\alpha$ represents the traffic intensity of the model under consideration.

Let $b^{\langle i \rangle}$ be the server busy period initiated with $i\,(i \geq 1)$ customers. Due to the Markovian property of geometric distribution, $b^{\langle i \rangle}$ can be expressed as $b^{\langle i \rangle} = b_1 + b_2 + \ldots + b_i$, where $b_1, b_2, \ldots, b_i$ are independent of each other and follow the same distribution as b. Thus, the PGF of $b^{\langle i \rangle}$ is given by $\sum_{j=i}^{\infty} \mathrm{P}\left\{b^{\langle i \rangle} = j\right\} z^j = B^i(z),\ |z|<1$

Let $L(n^+)$ designate the number of customers in the system at time epoch n^+ and $Q_j(n^+) = \mathrm{P}\{b > n^+, L(n^+) = j\},\ j \geq 1$ denote the transient probability of there being j customers at epoch n^+ in busy period b, where the epoch $n^+ = 0^+$ is the beginning of b. According to the definition of b, we have the boundary condition $Q_1(0^+) = 1,\ Q_j(0^+) = 0,\ j = 2, 3, \ldots$ Similar to the discussion in reference Wei *et al.* [29], we have the following lemma.

Lemma 2.7. *Let $Q_j^+(z) = \sum_{n=0}^{\infty} Q_j(n^+) z^n,\ |z|<1, j \geq 1$ be the PGF of $Q_j(n^+)$, then the recursive formulas of $Q_j^+(z)$ can be expressed as*

$$\begin{aligned} Q_j^+(z) = &\frac{1}{\tilde{G}(\bar{p}z)} \left\{ B(z) \sum_{n=j-1}^{\infty} z^n \sum_{k=n+1}^{\infty} \tilde{g}_k \binom{n}{j-1} p^{j-1} \bar{p}^{n-j+1} + \sum_{i=1}^{j-1} \frac{Q_{j-i}^+(z)}{B^i(z)} \right. \\ &\left. \times \left[B(z) - \tilde{G}(\bar{p}z) - \sum_{m=1}^{i} \sum_{k=m}^{\infty} \binom{k}{m} \tilde{g}_k z^k [pB(z)]^m \bar{p}^{k-m} \right] \right\}, \end{aligned}$$

where $B(z)$ *is defined as in Lemma* 2.6, $\tilde{g}_k$ *is given by* (2.1), $\tilde{G}(\bar{p}z) = \sum_{j=1}^{\infty} \tilde{g}_j (\bar{p}z)^j$, $\binom{k}{m} = \frac{k!}{m!(k-m)!}$, $k \geq m \geq 1$, $\binom{k}{0} = 1$ *and* $\binom{k}{m} = 0$ *if* $k < m$. $\sum_{i=1}^{j} = 0$ *if* $j < i$.

3. Analysis of system queue length distributions

In this section, by the law of probability decomposition and renewal theory, we first derive the PGF of the transient queue length distribution at any epoch n^+. Then, based on the transient results, the explicit formulas for the steady-state queue length distribution are obtained.

3.1. The transient distribution of the queue length at epoch n^+

Let $P_{ij}(n^+) = \mathrm{P}\{L(n^+) = j \,|L(0^+) = i\}$ be the conditional probability that there are j customers at epoch n^+ under arbitrary initial state $L(0^+) = i (i = 0, 1, \ldots)$. The PGF of $P_{ij}(n^+)$ is given by $P_{ij}^+(z) = \sum_{n=0}^{\infty} P_{ij}(n^+) z^n$, $i, j = 0, 1, 2, \ldots$. The expressions of $P_{ij}^+(z)$ with respect to different i and j, will be derived in Theorems 3.1–3.3.

Theorem 3.1. *For* $|z| < 1$*, we have*

$$P_{00}^+(z) = \frac{1}{1 - z\bar{p}} \left[1 + \frac{F(z) B(z) [1 - V(z\bar{p}) Y(z\bar{p})]}{1 - \Delta(z)}\right], \tag{3.1}$$

$$P_{i0}^+(z) = \frac{1}{1 - z\bar{p}} \frac{B^i(z) [1 - V(z\bar{p}) Y(z\bar{p})]}{1 - \Delta(z)}, \ i \geq 1, \tag{3.2}$$

where

$$\begin{aligned} F(z) &= \frac{pz}{1 - \bar{p}z}, \\ \Delta(z) &= V(z\bar{p}) Y(z\bar{p}) + F(z) B(z) [1 - Y(z\bar{p})] \\ &\quad + B^N(z) Y(z\bar{p}) \sum_{u=N}^{\infty} P\{V = u\} \sum_{m=N}^{u} z^m \binom{m-1}{N-1} p^N \bar{p}^{m-N} \\ &\quad + Y(z\bar{p}) \sum_{r=1}^{N-1} \sum_{u=r}^{\infty} P\{V = u\} z^u \binom{u}{r} [pB(z)]^r \bar{p}^{u-r}. \end{aligned}$$

Proof. See Appendix A for details. □

Theorem 3.2. *For* $|z| < 1$*,* $i \geq 1$ *and* $j = 1, 2, \ldots, N-1$*, we have*

$$P_{0j}^+(z) = F(z) \left\{Q_j^+(z) + \frac{F(z) B(z) Q_j^+(z) [1 - Y(z\bar{p})] + Y(z\bar{p}) \sigma_j(z)}{1 - \Delta(z)}\right\}, \tag{3.3}$$

$$P_{ij}^+(z) = \sum_{k=1}^{i} B^{k-1}(z) Q_{j-i+k}^+(z) + \frac{F(z) B^i(z) Q_j^+(z) [1 - Y(z\bar{p})] + B^{i-1}(z) Y(z\bar{p}) \sigma_j(z)}{1 - \Delta(z)}, \tag{3.4}$$

where $\Delta(z)$ is given by Theorem 3.1, $Q_j^+(z)$ is determined by Lemma 2.7, and

$$\begin{aligned}\sigma_j(z) = &\sum_{k=N-j+1}^{N} Q_{j-N+k}^+(z)\, B^k(z) \sum_{u=N}^{\infty} P\{V=u\} \sum_{m=N}^{u} z^m \binom{m-1}{N-1} p^N \bar{p}^{m-N} \\ &+ \sum_{r=1}^{j}\sum_{k=1}^{r} B^k(z)\, Q_{j-r+k}^+(z) \sum_{u=r}^{\infty} P\{V=u\} \binom{u}{r} (zp)^r (z\bar{p})^{u-r} \\ &+ \sum_{r=j+1}^{N-1}\sum_{k=r-j+1}^{r} B^k(z)\, Q_{j-r+k}^+(z) \sum_{u=r}^{\infty} P\{V=u\} \binom{u}{r} (zp)^r (z\bar{p})^{u-r} \\ &+ B(z) \sum_{u=j+1}^{\infty} P\{V=u\} \sum_{k=j}^{u-1} z^k \binom{k}{j} p^j \bar{p}^{k-j}.\end{aligned}$$

Proof. See Appendix B for details. □

Theorem 3.3. *For $|z|<1$, $i \geq 1$ and $j = N, N+1, \ldots$, we have*

$$P_{0j}^+(z) = F(z)\left\{ Q_j^+(z) + \frac{F(z)\, B(z)\, Q_j^+(z)\,[1-Y(z\bar{p})] + Y(z\bar{p})\,\theta_j(z)}{1-\Delta(z)} \right\}, \tag{3.5}$$

$$P_{ij}^+(z) = \sum_{k=1}^{i} B^{k-1}(z)\, Q_{j-i+k}^+(z) + \frac{F(z)\, B^i(z)\, Q_j^+(z)\,[1-Y(z\bar{p})] + B^{i-1}(z)\, Y(z\bar{p})\,\theta_j(z)}{1-\Delta(z)}, \tag{3.6}$$

where $\Delta(z)$ is given by Theorem 3.1, $Q_j^+(z)$ is determined by Lemma 2.7, and

$$\begin{aligned}\theta_j(z) = &\sum_{k=1}^{N} Q_{j-N+k}^+(z)\, B^k(z) \sum_{u=N}^{\infty} P\{V=u\} \sum_{m=N}^{u} z^m \binom{m-1}{N-1} p^N \bar{p}^{m-N} \\ &+ \sum_{r=1}^{N-1}\sum_{k=1}^{r} B^k(z)\, Q_{j-r+k}^+(z) \sum_{u=r}^{\infty} P\{V=u\} \binom{u}{r} (zp)^r (z\bar{p})^{u-r}.\end{aligned}$$

Proof. For $j = N, N+1, \ldots, L(n^+) = j$ indicates that epoch n^+ is only located in the server busy period with j customers in the system. Using the similar analytical technique and derivation process of the proof of Theorem 3.1 (see Appendix A), we can complete the proof of Theorem 3.3, so we omit it here. □

3.2. The recursive formulas for the steady-state distribution of queue length at epoch n^+

On the basis of the transient distribution of the queue length at an arbitrary epoch n^+ derived in Theorems 3.1–3.3, the recursive expressions of the steady-state queue length distribution at epoch n^+ will be investigated in this subsection.

Theorem 3.4. *Let $p_j^+ = \lim_{n\to\infty} P\{L(n^+)=j\}$, $j = 0,1,2,\ldots$ be the steady-state queue length distribution of our queueing system. We have*

(i) *If $\rho = p\mu/\alpha \geq 1$, then $p_j^+ = 0$, $j = 0,1,2,\ldots$*

(ii) *For $\rho = p\mu/\alpha < 1$, the steady-state queue length distribution $\{p_j^+, j \geq 0\}$ at epoch n^+ exists and forms a probability distribution. The recursive expressions are given by*

$$p_0^+ = (1-\rho)\,\frac{1 - V(\bar{p})\,Y(\bar{p})}{1 + Y(\bar{p})\,(\Delta_N - 1)}, \tag{3.7}$$

$$p_j^+ = p\,(1-\rho)\,\frac{[1 - Y(\bar{p})]\,Q_j + Y(\bar{p})\,\sigma_j}{1 + Y(\bar{p})\,(\Delta_N - 1)},\ \ j = 1, 2, \ldots, N-1, \tag{3.8}$$

$$p_j^+ = p\,(1-\rho)\,\frac{[1 - Y(\bar{p})]\,Q_j + Y(\bar{p})\,\theta_j}{1 + Y(\bar{p})\,(\Delta_N - 1)},\ \ j = N, N+1, \ldots, \tag{3.9}$$

where

$$\begin{aligned}
\Delta_N &= \sum_{r=1}^{N}\sum_{u=r}^{\infty} P\{V = u\} \sum\nolimits_{m=r}^{u} \binom{m-1}{r-1} p^r \bar{p}^{m-r},\\
Q_j &= \frac{1}{\tilde{G}(\bar{p})}\left\{\sum_{n=j-1}^{\infty}\sum_{k=n+1}^{\infty} \tilde{g}_k \binom{n}{j-1} p^{j-1}\bar{p}^{n-j+1}\right.\\
&\quad \left. + \sum_{i=1}^{j-1} Q_{j-i}\left[1 - \tilde{G}(\bar{p}) - \sum_{m=1}^{i}\sum_{k=m}^{\infty}\binom{k}{m}\tilde{g}_k p^m \bar{p}^{k-m}\right]\right\},\ j = 1, 2, \ldots,\\
\sigma_j &= \sum_{k=N-j+1}^{N} Q_{j-N+k} \sum_{u=N}^{\infty} P\{V = u\} \sum_{m=N}^{u}\binom{m-1}{N-1} p^N \bar{p}^{m-N}\\
&\quad + \sum_{r=1}^{j}\sum_{k=1}^{r} Q_{j-r+k} \sum_{u=r}^{\infty} P\{V = u\}\binom{u}{r} p^r \bar{p}^{u-r}\\
&\quad + \sum_{r=j+1}^{N-1}\sum_{k=r-j+1}^{r} Q_{j-r+k} \sum_{u=r}^{\infty} P\{V = u\}\binom{u}{r} p^r \bar{p}^{u-r}\\
&\quad + \sum_{u=j+1}^{\infty} P\{V = u\}\sum_{k=j}^{u-1}\binom{k}{j} p^j \bar{p}^{k-j},\ j = 1, 2, \ldots, N-1,\\
\theta_j &= \sum_{k=1}^{N} Q_{j-N+k}\sum_{u=N}^{\infty} P\{V = u\}\sum_{m=N}^{u}\binom{m-1}{N-1} p^N \bar{p}^{m-N}\\
&\quad + \sum_{r=1}^{N-1}\sum_{k=1}^{r} Q_{j-r+k}\sum_{u=r}^{\infty} P\{V = u\}\binom{u}{r} p^r \bar{p}^{u-r},\ j = N, N+1, \ldots
\end{aligned}$$

Proof. In discrete-time environment we have $p_j^+ = \lim_{z \to 1^-} (1-z)\,P_{ij}^+(z)$ (see Jury [6]). Applying Lemma 2.6, Theorems 3.1–3.3 and L'Hospital's rule, the formulas (3.7)–(3.9) of Theorem 3.4 can be derived. It is noted that the L'Hospital's rule used by $\lim_{z \to 1^-} (1-z)\,P_{ij}^+(z)$ can yield E$[b]$ in the denominator, and from Lemma 2.6, we know that E$[b] = \infty$ under $\rho \geq 1$, which leads to $p_j^+ = 0$, $j = 0, 1, 2, \ldots$ Furthermore, by manipulating direct calculations, the formula $\sum_{j=0}^{\infty} p_j^+ = 1$ holds, *i.e.*, $\{p_j^+, j = 0, 1, 2, \ldots\}$ is a probability distribution. □

Theorem 3.5. *Let* $\Pi^{+}(z) = \sum_{j=0}^{\infty} p_j^{+} z^j$, $|z| < 1$ *be the PGF of the stationary queue size distribution* $\{p_j^{+}, j = 0, 1, 2, \ldots\}$ *at epoch* n^{+}. *For* $\rho = p\mu/\alpha < 1$ *and* $|z| < 1$, *we have*

$$\Pi^{+}(z) = \frac{(1-\rho)(1-z)\tilde{G}(zp+\bar{p})}{\tilde{G}(zp+\bar{p}) - z} \times \frac{1 - Y(\bar{p}) + Y(\bar{p}) \sum\limits_{r=0}^{N-1} z^r \sum\limits_{u=r+1}^{\infty} P\{V=u\} \sum\limits_{m=r+1}^{u} \binom{m-1}{r} p^{r+1} \bar{p}^{m-r-1}}{1 + Y(\bar{p})(\Delta_N - 1)}. \tag{3.10}$$

The mean steady-state queue length, denoted by $E[L^{+}]$, *is given by*

$$E[L^{+}] = \rho + \frac{p^2}{2(1-\rho)} E[\tilde{\chi}(\tilde{\chi}-1)] + \frac{Y(\bar{p}) \sum\limits_{r=1}^{N-1} r \sum\limits_{u=r+1}^{\infty} P\{V=u\} \sum\limits_{m=r+1}^{u} \binom{m-1}{r} p^{r+1} \bar{p}^{m-r-1}}{1 + Y(\bar{p})(\Delta_N - 1)}, \tag{3.11}$$

where $\tilde{\chi}$ *denotes the total service time,* $E[\tilde{\chi}(\tilde{\chi}-1)] = \sum\limits_{j=1}^{\infty} j(j-1) P\{\tilde{\chi} = j\}$, Δ_N *is given by Theorem 3.4.*

Proof. Utilizing the formulas of p_j^{+} given in Theorem 3.4 and noticing that

$$\sum_{j=1}^{\infty} Q_j z^j = \frac{z\left[1 - \tilde{G}\left(z\lambda + \bar{\lambda}\right)\right]}{\lambda\left[\tilde{G}\left(z\lambda + \bar{\lambda}\right) - z\right]},$$

the expression (3.10) can be obtained by manipulating some algebraic simplification on $\Pi^{+}(z) = \sum_{j=0}^{\infty} p_j^{+} z^j$. Meanwhile, (3.11) can be derived by $\mathrm{E}[L^{+}] = \frac{\mathrm{d}}{\mathrm{d}z}[\Pi^{+}(z)]|_{z=1}$. □

Corollary 3.6. *In a discrete-time Geo/G/1 with Bernoulli feedback,* N*-policy and modified multiple vacations, the steady-state queue size* L^{+} *consists of two independent random variables:* $L^{+} = L_0^{+} + L_d^{+}$. L_0^{+} *is the steady-state queue length of the Geo/G/1 queue with Bernoulli feedback and the corresponding PGF is* $((1-\rho)(1-z)\tilde{G}(zp+\bar{p}))/(\tilde{G}(zp+\bar{p})-z)$. L_d^{+} *is the additional queue size caused by* N*-policy and modified multiple vacations, and the probability distribution of* L_d^{+} *is given by*

$$P\left\{L_d^{+} = 0\right\} = \frac{1 - V(\bar{p})}{1 + Y(\bar{p})(\Delta_N - 1)}, \tag{3.12}$$

$$P\left\{L_d^{+} = r\right\} = \frac{Y(\bar{p}) \sum\limits_{u=r+1}^{\infty} P\{V=u\} \sum\limits_{m=r+1}^{u} \binom{m-1}{r} p^{r+1} \bar{p}^{m-r-1}}{1 + Y(\bar{p})(\Delta_N - 1)}, \quad r = 1, 2, \ldots, N-1. \tag{3.13}$$

Proof. From (3.10), the steady-state queue length is composed of two independent parts, and the PGF of L_d^{+} is given by

$$\Pi_d^{+}(z) = \frac{1 - Y(\bar{p}) + Y(\bar{p}) \sum\limits_{r=0}^{N-1} z^r \sum\limits_{u=r+1}^{\infty} \mathrm{P}\{V=u\} \sum\limits_{m=r+1}^{u} \binom{m-1}{r} p^{r+1} \bar{p}^{m-r-1}}{1 + Y(\bar{p})(\Delta_N - 1)}. \tag{3.14}$$

Hence the PMF of additional queue size L_d^{+} can be derived by $\mathrm{P}\left\{L_d^{+} = r\right\} = \frac{1}{r!} \cdot \frac{\mathrm{d}^r}{\mathrm{d}z^r}\left[\Pi_d^{+}(z)\right]|_{z=0}$. □

Remark 3.7. (Special cases) In this remark, we consider some special cases of our model by taking specific values for the parameters.

(1) When $\mathrm{P}\{\xi = 1\} = 1(i.e.,\ \alpha = 1)$ and $\mathrm{P}\{Y = 0\} = 1$, our model is equivalent to a discrete-time $Geo/G/1$ queue with N-policy and regular multiple vacations. For $\rho = p\mu < 1$ and $|z| < 1$, we have

$$\Pi^{+}(z) = \frac{(1-\rho)(1-z)G(zp+\bar{p})}{G(zp+\bar{p})-z} \times \frac{\sum_{r=0}^{N-1} z^r \sum_{u=r+1}^{\infty} \mathrm{P}\{V=u\} \sum_{m=r+1}^{u} \binom{m-1}{r} p^{r+1}\bar{p}^{m-r-1}}{\Delta_N}.$$

(2) When $N = 1$ and $\mathrm{P}\{Y = 0\} = 1$, the model investigated in this paper becomes a discrete-time $Geo/G/1$ queue with Bernoulli feedback. For $\rho = p\mu/\alpha < 1$ and $|z| < 1$, we have

$$\Pi^{+}(z) = \frac{(1-\rho)(1-z)\tilde{G}(\bar{p}+zp)}{\tilde{G}(\bar{p}+zp)-z},$$

which is in accordance with the corresponding result in Luo *et al.* [15].

(3) When $\mathrm{P}\{V = \infty\} = 1$, the considered model reduces to a discrete-time $Geo/G/1$ queueing system with Bernoulli feedback and modified N-policy. For $\rho = p\mu/\alpha < 1$ and $|z| < 1$, we obtain

$$\Pi^{+}(z) = \frac{(1-\rho)(1-z)\tilde{G}(\bar{p}+zp)}{\tilde{G}(\bar{p}+zp)-z} \cdot \frac{1-z+Y(\bar{p})\left(z-z^N\right)}{(1-z)\left[1+(N-1)Y(\bar{p})\right]}.$$

(4) When $N \to \infty$, $\alpha = 1$ and $\mathrm{P}\{Y = 0\} = 1$, our model is equal to a discrete-time $Geo/G/1$ queueing system with regular multiple vacations. For $\rho = p\mu < 1$ and $|z| < 1$, we obtain

$$\Pi^{+}(z) = \frac{(1-\rho)(1-z)G(\bar{p}+zp)}{G(\bar{p}+zp)-z} \cdot \frac{1-V(\bar{p}+zp)}{pE[V](1-z)},$$

which matches with the corresponding result in Takagi [22].

4. Numerical results

In this section, some numerical examples are carried out to illustrate the validity and practicality of the expressions derived in previous sections and to qualitatively describe the behavior of the queueing system under investigation. All the notations used in subsequent discussion are the same as those defined in the previous sections. Meanwhile, all the computations are implemented in the software of MATLAB and the data are given in five decimal places. Of course, the values of all the parameters are chosen so as to satisfy the stability condition $\rho = \lambda\mu/\alpha < 1$. For simplicity, we assume that the service time χ, server vacation time V and the length of standby period Y follow the geometric distributions with parameters β, θ and η, respectively.

4.1. Computation of the steady-state queue length

Based on the recursive solutions provided by Theorem 3.4, we can compute the steady-state queue length at time epochs n^{+}. The numerical results are displayed in the form of a table (see Tab. 1). Here the default system parameters for the numerical results shown in Table 1 are set as $p = 0.15$, $\beta = 0.45$, $\alpha = 0.5$, $\eta = 0.3$, $\theta = 0.2$ and $N = 7$.

TABLE 1. The steady-state queue length distribution at epochs n^+.

j	p_j^+	j	p_j^+	j	p_j^+
0	0.26343	11	0.00238	22	0.00001
1	0.25042	12	0.00145	23	0.00001
2	0.17638	13	0.00088	24	0.00000
3	0.11627	14	0.00054	25	0.00000
4	0.07407	15	0.00033	26	0.00000
5	0.04630	16	0.00020	27	0.00000
6	0.02862	17	0.00012	28	0.00000
7	0.01745	18	0.00007	29	0.00000
8	0.01061	19	0.00004	30	0.00000
9	0.00645	20	0.00003	31	0.00000
10	0.00392	21	0.00002	mean	2.03199

4.2. Sensitivity analysis of operating characteristics

We first investigate the sensitivity of the expected additional queue size $\mathrm{E}\left[L_d^+\right]$ to the changes of N and θ. From Theorem 3.5, the expected length of additional queue size can be explicitly expressed as

$$\mathrm{E}\left[L_d^+\right] = \frac{\dfrac{\eta\bar{p}}{\eta+p\bar{\eta}}\left\{\dfrac{p}{\theta}\left(\dfrac{p\bar{\theta}}{\theta}+N\right)\left[1-\left(\dfrac{p\bar{\theta}}{\theta+p\bar{\theta}}\right)^N\right]-\dfrac{Np}{\theta}\right\}}{1+\dfrac{\eta\bar{p}}{\eta+p\bar{\eta}}\left\{\dfrac{p}{\theta}\left[1-\left(\dfrac{p\bar{\theta}}{\theta+p\bar{\theta}}\right)^N\right]-1\right\}}. \tag{4.1}$$

In (4.1), setting $p = 0.3$, $\beta = 0.45$, $\alpha = 0.5$ and $\eta = 0.3$, the effect of N on $\mathrm{E}\left[L_d^+\right]$ for different values of θ is plotted in Figure 2. It is observed from Figure 2 that for a fixed value of θ, $\mathrm{E}\left[L_d^+\right]$ increases initially as N increases but finally it keep unchangeable with further increment of N. This is due to the fact that as the threshold value N gradually increases, it plays a smaller role under the modified multiple vacations and N-policy. When the threshold N is large enough, the mean additional queue length $\mathrm{E}\left[L_d^+\right]$ is completely determined by the server's vacation time V, that is to say, the queueing system behaves as pure multiple vacations queue. Furthermore, it can be also seen from Figure 2 that the $\mathrm{E}\left[L_d^+\right]$ shows a decreasing trend with the value of θ getting larger. The reason is that the length of the vacation time V is being shortened with the growth of θ, and therefore the customers who just come to the system have a greater chance to be served, which matches with our practical scenario.

We now analyze the impact of system parameters on the mean steady-state queue length $\mathrm{E}[L^+]$ and the probability that the system is empty p_0^+.

Figure 3 depicts the impact of α on $\mathrm{E}\left[L^+\right]$ and p_0^+ for different values of θ. We set default parameters for Figure 3 as $p = 0.05$, $\beta = 0.45$, $\eta = 0.3$ and $N = 7$. It is observed from Figure 3a that the average system size $\mathrm{E}\left[L^+\right]$ decreases monotonously for increasing values of α for any θ. This is because as α increases, the probability that the customer who has just been served leaves the system also becomes larger, which leads to the decrease of system queue length. Furthermore, as expected, for a fixed α, $\mathrm{E}\left[L^+\right]$ shows a trend of decrease with the increase of vacation rate θ. In Figure 3b, the reverse trend is displayed for p_0^+. That is, for a fixed θ, the probability that the system is empty p_0^+increases monotonously when α increases. The reason is that the larger the probability α is, the greater the probability that the served customer leaves the system is, which indicates the probability that the system becomes empty is larger. Also, p_0^+ is an increasing function of θ for a fixed α, which is in accordance with our expectation.

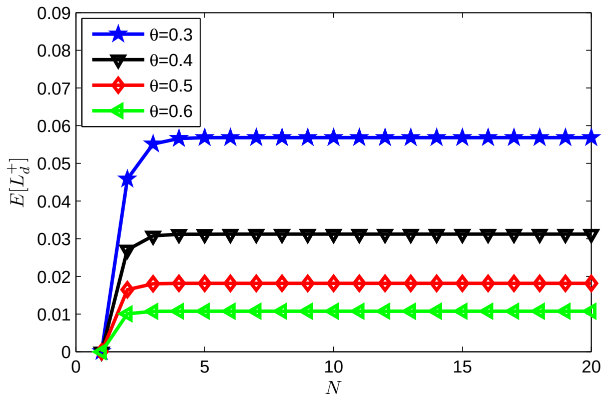

FIGURE 2. The effect of N on $\mathrm{E}\left[L_d^+\right]$ for different values of θ.

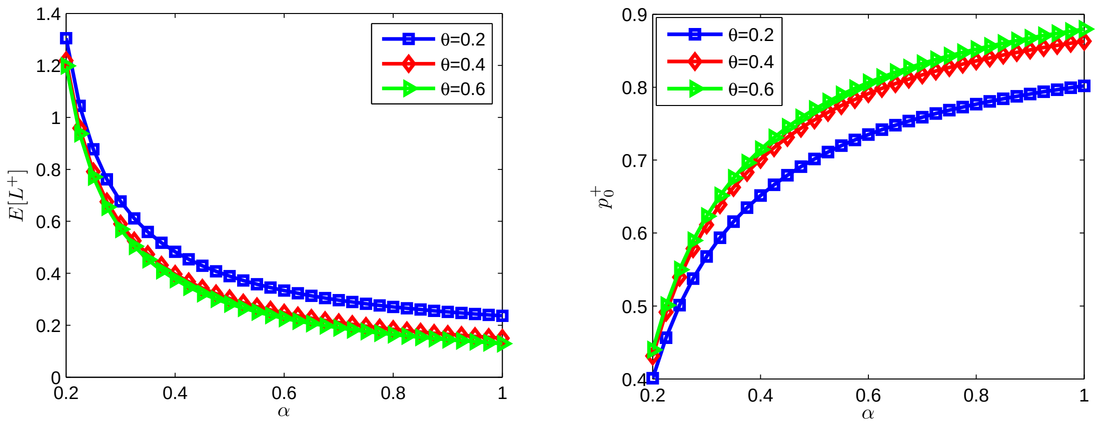

FIGURE 3. The effect of α on (a) $\mathrm{E}\left[L^+\right]$ (b) p_0^+ for different values of θ.

The effect of arrival rate p on $\mathrm{E}\left[L^+\right]$ and p_0^+ for different values of η is plotted in Figure 4. We set default parameters for Figure 4 as $\beta = 0.65$, $\alpha = 0.5$, $\theta = 0.1$ and $N = 5$. It can be seen from Figures 4a and 4b that $\mathrm{E}\left[L^+\right]$ increases with the increment of p while p_0^+ decreases as p increases. This is intuitively true because increasing p implies that the probability that an arrival occurs in a time slot becomes larger, which can obviously lead to the growth of the system queue length and the decrease of the probability that the system being empty. Similarly, for a fixed value of p, as η increases, $\mathrm{E}\left[L^+\right]$ increases but p_0^+ decreases.

The above numerical analysis not only demonstrates the validity of our analytical results, but also can provide significant insight to the concerned system designers or decision makers so as to reduce the congestion problem encountered in practical scenarios, such as manufacturing systems, computer systems and telecommunication network.

5. Cost optimization problem

In practice, the operating cost of system is closely connected with the system benefit and system designers/managers are interested in minimizing system operating cost of per unit time. For the sake of demonstrating the applicability of the results obtained in the previous discussion, a cost optimization analysis is carried out from an economic point of view. We establish an expected operating cost function per unit time for the queueing

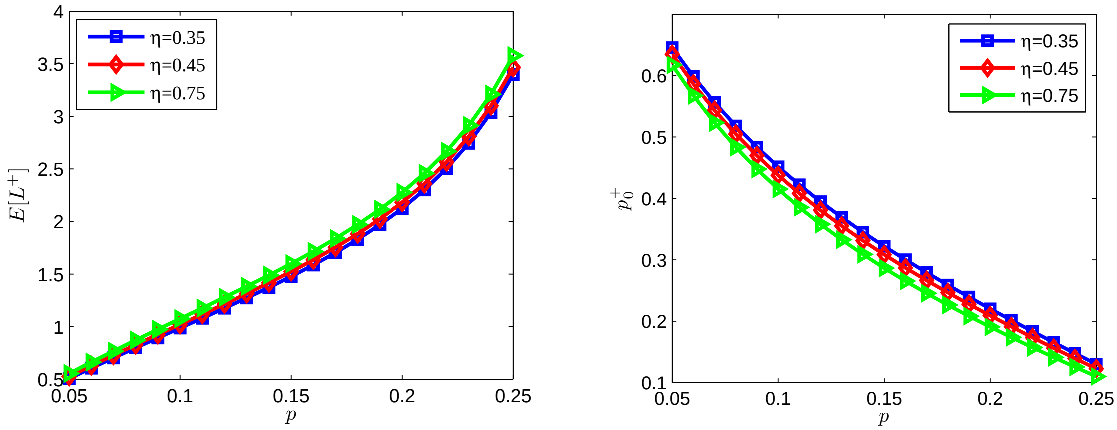

FIGURE 4. The effect of p on (a) $\mathrm{E}\left[L^{+}\right]$ (b) p_0^+ for different values of η.

model investigated in this paper, in which N is decision variable. Then, a numerical example is provided to find the optimum threshold value N, say N^*, to minimize the long-run expected cost per unit time.

To do this, we first discuss the expected length of the busy cycle period C. The busy cycle period consists of a server idle period and an adjacent server busy period. Let B be a server busy period selected randomly, Q_B be the number of customers in the system at the beginning of the server busy period B. It is noticed from the model assumptions that the number of the customers at the beginning of a server busy period may be $1, 2, \ldots, N$. Therefore, we have

$$\mathrm{P}\left\{Q_B=1\right\}=\frac{1-Y(\bar{p})}{1-V(\bar{p})Y(\bar{p})}+\frac{1}{1-V(\bar{p})Y(\bar{p})}\sum_{r=1}^{\infty}v_r r p\bar{p}^{r-1},$$

$$\mathrm{P}\left\{Q_B=j\right\}=\frac{1}{1-V(\bar{p})Y(\bar{p})}\sum_{r=j}^{\infty}v_r\binom{r}{j}p^j\bar{p}^{r-j},\ j=2,3,\ldots,N-1,$$

$$\mathrm{P}\left\{Q_B=N\right\}=\frac{1}{1-V(\bar{p})Y(\bar{p})}\sum_{j=N}^{\infty}\sum_{r=j}^{\infty}v_r\binom{r}{j}p^j\bar{p}^{r-j}.$$

Using the three formulas above, the mean of Q_B is given by

$$\mathrm{E}\left[Q_B\right]=\sum_{k=1}^{N}k\,\mathrm{P}\left\{Q_B=k\right\}=\frac{1+Y\left(\bar{p}\right)\left(\Delta_N-1\right)}{1-V\left(\bar{p}\right)Y\left(\bar{p}\right)},$$

where Δ_N is determined by Theorem 3.4.

The mean of the server busy period B, with the help of Lemma 2.6, is

$$\mathrm{E}\left[B\right]=\mathrm{E}\left[b\right]\cdot\mathrm{E}\left[Q_B\right]=\frac{\rho}{p\left(1-\rho\right)}\cdot\frac{1+Y\left(\bar{p}\right)\left(\Delta_N-1\right)}{1-V\left(\bar{p}\right)Y\left(\bar{p}\right)},\ \rho<1. \tag{5.1}$$

Since the inter-arrival times during server idle period follow independent and identical geometric distribution with parameter p, the expected length of the server idle period is obtained as

$$\mathrm{E}[I]=\frac{\mathrm{E}\left[Q_B\right]}{p}=\frac{1+Y\left(\bar{p}\right)\left(\Delta_N-1\right)}{p(1-V\left(\bar{p}\right)Y\left(\bar{p}\right))}\cdot \tag{5.2}$$

TABLE 2. The average cost per unit time for different values of N.

N	$C(N)$	N	$C(N)$	N	$C(N)$	N	$C(N)$	N	$C(N)$
1	155.00000	8	150.24239	15	151.77267	22	151.81720	29	151.81816
2	142.75000	9	150.83548	16	151.79156	23	151.81762	30	151.81817
3	**142.07469**	10	151.21468	17	151.80269	24	151.81786	31	151.81818
4	144.02233	11	151.45209	18	151.80920	25	151.81800	32	151.81818
5	146.21953	12	151.59832	19	151.81299	26	151.81808	33	151.81818
6	148.02396	13	151.68724	20	151.81519	27	151.81812	34	151.81818
7	149.34066	14	151.74075	21	151.81646	28	151.81815	35	151.81818

From (5.1) and (5.2), the expected length of the busy cycle period, denoted by $\mathrm{E}\,(C)$, can be expressed as

$$\mathrm{E}[C] = \mathrm{E}[B] + \mathrm{E}[I] = \frac{1}{p\,(1-\rho)} \cdot \frac{1 + Y\,(\bar{p})\,(\varDelta_N - 1)}{1 - V\,(\bar{p})\,Y\,(\bar{p})},\ \ \rho < 1. \tag{5.3}$$

We now begin to study the cost optimization problem. Let us define the cost structure as follows.

$C_h \equiv$ cost per unit time for each customer present in the system (this cost originates from the customer's sojourn time that consists of the waiting time and the total service time).

$C_s \equiv$ fixed setup cost per unit time for per busy cycle (this cost is due to the switch-over between server busy period and server idle period).

Utilizing the definitions of each cost element listed above and the corresponding system performance measures obtained previously, the long-run expected cost function per unit time, denoted by $C\,(N)$, is given by

$$\begin{aligned} C\,(N) &= \mathrm{E}\left[L^{+}\right] C_h + \frac{1}{\mathrm{E}\,[C]} C_s \\ &= \frac{p\,(1-\rho)\,[1 - V\,(\bar{p})\,Y\,(\bar{p})]}{1 + Y\,(\bar{p})\,(\varDelta_N - 1)} C_s + \left[\rho + \frac{p^2}{2\,(1-\rho)}\,\mathrm{E}\,[\tilde{\chi}\,(\tilde{\chi} - 1)] \right. \\ &\quad \left. + \frac{Y\,(\bar{p}) \sum\limits_{r=1}^{N-1} r \sum\limits_{u=r+1}^{\infty} \mathrm{P}\,\{V = u\} \sum\limits_{m=r+1}^{u} \binom{m-1}{r} p^{r+1} \bar{p}^{m-r-1}}{1 + Y\,(\bar{p})\,(\varDelta_N - 1)} \right] C_h,\ \ \rho < 1. \end{aligned} \tag{5.4}$$

We can see from (5.4) that the cost function is extremely complex and non-linear, which poses a hard task to achieve the analytic results for the optimum value of N. Therefore, we will search the optimum value N^* for cost function $C\,(N)$ through numerical experience.

Example 5.1. In this example, it is assumed that the service time χ, server vacation time V and the duration of standby period Y are governed by the geometric distributions with parameters $\beta = 0.75$, $\theta = 0.2$ and $\eta = 0.4$, respectively. The default values of other system parameters and cost elements are taken as $p = 0.3$, $\alpha = 0.8$, $C_h = 50$, $C_s = 800$. Since N is a discrete variable, the optimal value N^* can be found by utilizing direct substitution of successive values of N into the profit function (5.4) until the minimum value is attained. Substituting these parameters into (5.4) and developing MATLAB program, the numerical results of the long-run expected cost per unit time under different threshold value N are displayed in the Table 2 and Figure 5.

It is observed from Table 2 and Figure 5 that the long-run expected operating cost $C(3) = 142.07469$ is the minimum value, *i.e.*, the optimum threshold value of N is $N^* = 3$.

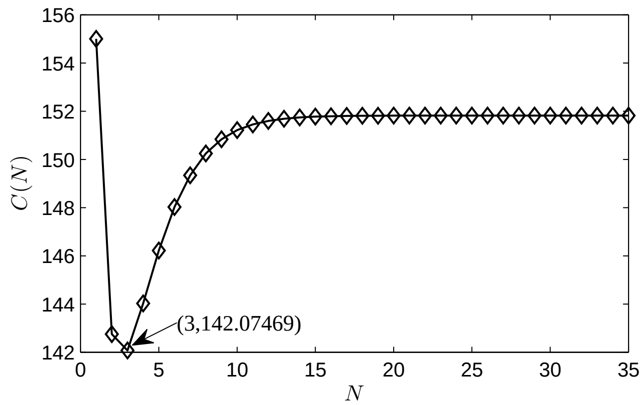

FIGURE 5. The plot of $C(N)$ against N.

6. Conclusions

The present investigation analyzed a discrete-time $Geo/G/1$ queueing model with Bernoulli feedback, modified multiple vacations and N-policy. Employing the law of probability decomposition, renewal theory, and probability generating function method, the analytical expressions for the transient and the steady-state queue length distribution were derived. It is remarkable that the explicit recursion formulas for the steady-state queue length distribution (given by Thm. 3.4) are computationally tractable to handle the congestion problems in real-life situations. Furthermore, some numerical examples were provided to study the effect of some key parameters on the operating characteristics of the system. Finally, we established a cost structure to investigate a cost optimization problem.

The analysis results of this paper can provide a potentially practical application for practitioners in telecommunication systems, queueing networks, flexible manufacturing systems, inventory problems and so forth. For further research, one can extend this model by incorporating more complex scenarios like discrete-time Markovian arrival process (DMAP) of customers, unreliable server and multi-optional services.

Appendix A. Proof of Theorem 3.1

Proof. Let $S_k = \sum_{i=1}^{k}(V_i + Y_i)$, $l_k = \sum_{i=1}^{k}\tau_i$, $k = 1, 2, \ldots$ and $S_0 = l_0 = 0$, where V_i and Y_i denote the ith vacation period and standby period, respectively. It is noted that $P_{00}(n^+)$ indicates that there are no customers in the queue at epoch n^+ under initial state $L(0^+) = 0$, *i.e.*, epoch n^+ is located in the system idle period. Based on the previous model assumptions, the beginning and ending epochs of the server busy period are renewal points. Using renewal theory and the law of total probability decomposition, we obtain

$$
\begin{aligned}
P_{00}\left(n^+\right) = {} & \mathrm{P}\left\{0 \le n^+ < \tilde{\tau}_1\right\} + \mathrm{P}\left\{\tilde{\tau}_1 + b_1 \le n^+ < \tilde{\tau}_1 + b_1 + \tilde{\tau}_2\right\} \\
& + \sum_{k=1}^{\infty} \mathrm{P}\left\{\tilde{\tau}_1 + b_1 + \tilde{\tau}_2 \le n^+, S_{k-1} < \tilde{\tau}_2 \le S_{k-1} + Y_k, L\left(n^+\right) = 0\right\} \\
& + \sum_{k=1}^{\infty} \mathrm{P}\left\{\tilde{\tau}_1 + b_1 + \tilde{\tau}_2 \le n^+, S_{k-1} + Y_k < \tilde{\tau}_2 \le S_k, L\left(n^+\right) = 0\right\},
\end{aligned} \tag{A.1}
$$

where $\tilde{\tau}_k$ and b_k, $k \ge 1$ represent the kth system idle period and server busy period, respectively.

The first term of (A.1) means that epoch n^+ is located in the first system idle period, which is equal to

$$\mathrm{P}\left\{0 \le n^+ < \tilde{\tau}_1\right\} = \sum_{t=n+1}^{\infty} \mathrm{P}\left\{\tilde{\tau}_1 = t\right\} = \sum_{t=n+1}^{\infty} p\bar{p}^{t-1} = \bar{p}^n. \tag{A.2}$$

The second term of (A.1) is the probability that epoch n^+ is located in the second system idle period, so we have

$$\begin{aligned}\mathrm{P}\left\{\tilde{\tau}_1 + b_1 \le n^+ < \tilde{\tau}_1 + b_1 + \tilde{\tau}_2\right\} &= \sum_{t=2}^{n} \mathrm{P}\left\{\tilde{\tau}_1 + b_1 = t\right\} \sum_{j=n-t+1}^{\infty} \mathrm{P}\left\{\tilde{\tau}_2 = j\right\} \\ &= \sum_{t=2}^{n} \mathrm{P}\left\{\tilde{\tau}_1 + b_1 = t\right\} \bar{p}^{n-t}. \end{aligned}\tag{A.3}$$

The third term of (A.1) indicates that the time epoch n^+ is located after the second system idle period and the first arrival occurs during the kth server standby period. Under such scenario, the server is immediately available and begins to serve the customers until the system becomes empty. Thus, the third term of (A.1) is equivalent to

$$\begin{aligned}&\sum_{k=1}^{\infty} \mathrm{P}\left\{\tilde{\tau}_1 + b_1 + \tilde{\tau}_2 \le n^+, S_{k-1} < \tilde{\tau}_2 \le S_{k-1} + Y_k, L\left(n^+\right) = 0\right\} \\ &= \sum_{k=1}^{\infty} \sum_{t=2}^{n} \mathrm{P}\left\{\tilde{\tau}_1 + b_1 = t\right\} \sum_{m=1}^{n-t} \mathrm{P}\left\{\tilde{\tau}_2 = m\right\} P_{10}\left((n-t-m)^+\right) \\ &\quad \times \mathrm{P}\left\{S_{k-1} < m \le S_{k-1} + Y_k\right\} \\ &= \sum_{k=1}^{\infty} \sum_{t=2}^{n} \mathrm{P}\left\{\tilde{\tau}_1 + b_1 = t\right\} \sum_{m=1}^{n-t} \mathrm{P}\left\{\tilde{\tau}_2 = m\right\} P_{10}\left((n-t-m)^+\right) \\ &\quad \times \left(\sum_{w=2k-2}^{m} \mathrm{P}\left\{S_{k-1} = w\right\} - \sum_{w=2k-1}^{m} \mathrm{P}\left\{S_{k-1} + Y_k = w\right\}\right). \end{aligned}\tag{A.4}$$

In the fourth term of (A.1), the restrictions "$\tilde{\tau}_1 + b_1 + \tilde{\tau}_2 \le n^+$" and "$S_{k-1} + Y_k < \tilde{\tau}_2 \le S_k$" implies that the epoch n^+ is located after the second system idle period and the first arrival occurs in the kth server vacation period. So some other potential customers may arrive in the system during this server vacation period. From our model description, the number of arrivals during this server vacation period is either N or r $(1 \le r \le N-1)$. Hence, the fourth term of equation (A.1) can be further expressed as

$$\begin{aligned}&\sum_{k=1}^{\infty} \mathrm{P}\left\{\tilde{\tau}_1 + b_1 + \tilde{\tau}_2 \le n^+, S_{k-1} + Y_k < \tilde{\tau}_2 \le S_k, L\left(n^+\right) = 0\right\} \\ &= \sum_{k=1}^{\infty} \mathrm{P}\left\{\tilde{\tau}_1 + b_1 + \tilde{\tau}_2 + l_{N-1} \le n^+, S_{k-1} + Y_k < \tilde{\tau}_2, \tilde{\tau}_2 + l_{N-1} \le S_k, L\left(n^+\right) = 0\right\} \\ &\quad + \sum_{k=1}^{\infty} \sum_{r=1}^{N-1} \mathrm{P}\left\{\tilde{\tau}_1 + b_1 + S_k \le n^+, S_{k-1} + Y_k < \tilde{\tau}_2 \le S_k, \tilde{\tau}_2 + l_{r-1} \le S_k < \tilde{\tau}_2 + l_r, L\left(n^+\right) = 0\right\}, \end{aligned}\tag{A.5}$$

where the first term and the second term in (A.5) indicate N customers and r $(1 \le r \le N-1)$ customers arrive at the system during server vacation period, respectively. Since the ending point of server vacation is the beginning epoch of a new server busy period, which is a renewal point, the fourth term of equation (A.1) is finally given by

$$\begin{aligned}
&\sum_{k=1}^{\infty}\mathrm{P}\left\{\tilde{\tau}_1+b_1+\tilde{\tau}_2\le n^+, S_{k-1}+Y_k<\tilde{\tau}_2\le S_k, L\left(n^+\right)=0\right\}\\
&=\sum_{k=1}^{\infty}\sum_{t=2}^{n}\mathrm{P}\left\{\tilde{\tau}_1+b_1=t\right\}\sum_{w=2k-1}^{n-t}\mathrm{P}\left\{S_{k-1}+Y_k=w\right\}\bar{p}^{w}\sum_{h=N}^{n-t-w}\mathrm{P}\left\{\tilde{\tau}_2+l_{N-1}=h\right\}\\
&\quad\times\mathrm{P}\left\{V_k\ge h\right\}P_{N0}\left(\left(n-t-w-h\right)^+\right)\\
&\quad+\sum_{k=1}^{\infty}\sum_{r=1}^{N-1}\sum_{t=2}^{n}\mathrm{P}\left\{\tilde{\tau}_1+b_1=t\right\}\sum_{w=2k-1}^{n-t}\mathrm{P}\left\{S_{k-1}+Y_k=w\right\}\sum_{u=1}^{n-t-w}\mathrm{P}\left\{V_k=u\right\}\\
&\quad\times\binom{u}{r}p^r\bar{p}^{u+w-r}P_{r0}\left(\left(n-t-w-u\right)^+\right).
\end{aligned}\tag{A.6}$$

Substituting equations (A.2)–(A.4) and (A.6) into (A.1), multiplying (A.1) by z^n and summing over n, it finally leads to

$$\begin{aligned}
P_{00}^{+}(z)&=\frac{1}{1-z\bar{p}}+\frac{F(z)\,B(z)}{1-z\bar{p}}+\frac{F^2(z)\,B(z)\left[1-Y(z\bar{p})\right]}{1-V(z\bar{p})\,y(z\bar{p})}P_{10}^{+}(z)\\
&\quad+\frac{F(z)\,B(z)\,Y(z\bar{p})}{1-V(z\bar{p})\,Y(z\bar{p})}\left\{\sum_{u=N}^{\infty}\mathrm{P}\left\{V=u\right\}\sum_{m=N}^{u}z^m\,\mathrm{P}\left\{\tilde{\tau}_2+l_{N-1}=m\right\}\right.\\
&\quad\left.\times P_{N0}^{+}(z)+\sum_{r=1}^{N-1}\sum_{u=r}^{\infty}\mathrm{P}\left\{V=u\right\}\binom{u}{r}(zp)^r(z\bar{p})^{u-r}P_{r0}^{+}(z)\right\}.
\end{aligned}\tag{A.7}$$

For $i\ge1$, similar to the analysis of $P_{00}\left(n^+\right)$, we can get

$$\begin{aligned}
P_{i0}\left(n^+\right)&=\mathrm{P}\left\{b^{\langle i\rangle}\le n^+<b^{\langle i\rangle}+\tilde{\tau}_1\right\}+\mathrm{P}\left\{b^{\langle i\rangle}+\tilde{\tau}_1\le n^+, L\left(n^+\right)=0\right\}\\
&=\sum_{t=i}^{n}\mathrm{P}\left\{b^{\langle i\rangle}=t\right\}\bar{p}^{n-t}+\sum_{k=1}^{\infty}\sum_{t=i}^{n}\mathrm{P}\left\{b^{\langle i\rangle}=t\right\}\sum_{m=1}^{n-t}\mathrm{P}\left\{\tilde{\tau}_1=m\right\}\\
&\quad\times P_{10}\left(\left(n-t-m\right)^+\right)\left(\sum_{w=2k-2}^{m}\mathrm{P}\left\{S_{k-1}=w\right\}-\sum_{w=2k-1}^{m}\mathrm{P}\left\{S_{k-1}+Y_k=w\right\}\right)\\
&\quad+\sum_{k=1}^{\infty}\sum_{t=i}^{n}\mathrm{P}\left\{b^{\langle i\rangle}=t\right\}\sum_{w=2k-1}^{n-t}\mathrm{P}\left\{S_{k-1}+Y_k=w\right\}\bar{p}^{w}\sum_{h=N}^{n-t-w}\mathrm{P}\left\{\tilde{\tau}_1+l_{N-1}=h\right\}\\
&\quad\times\mathrm{P}\left\{V_k\ge h\right\}P_{N0}\left(\left(n-t-w-h\right)^+\right)\\
&\quad+\sum_{k=1}^{\infty}\sum_{r=1}^{N-1}\sum_{t=i}^{n}\mathrm{P}\left\{b^{\langle i\rangle}=t\right\}\sum_{w=2k-1}^{n-t}\mathrm{P}\left\{S_{k-1}+Y_k=w\right\}\sum_{u=1}^{n-t-w}\mathrm{P}\left\{V_k=u\right\}\\
&\quad\times\binom{u}{r}p^r\bar{p}^{u+w-r}P_{r0}\left(\left(n-t-w-u\right)^+\right).
\end{aligned}\tag{A.8}$$

Multiplying (A.8) by z^n and summing over n, it gives

$$P_{i0}^{+}(z) = \frac{B^i(z)}{1-z\bar{p}} + \frac{F(z)B^i(z)\left[1-Y(z\bar{p})\right]}{1-V(z\bar{p})Y(z\bar{p})}P_{10}^{+}(z)$$

$$+\frac{B^i(z)Y(z\bar{p})}{1-V(z\bar{p})Y(z\bar{p})}\left\{\sum_{u=N}^{\infty}\mathrm{P}\{V=u\}\sum_{m=N}^{u}z^m\,\mathrm{P}\{\tilde{\tau}_1+l_{N-1}=m\}\right.$$

$$\left.\times P_{N0}^{+}(z)+\sum_{r=1}^{N-1}\sum_{u=r}^{\infty}\mathrm{P}\{V=u\}\binom{u}{r}(zp)^r(z\bar{p})^{u-r}P_{r0}^{+}(z)\right\}. \tag{A.9}$$

Solving equations (A.7) and (A.9) leads to the expressions of $P_{00}^{+}(z)$ and $P_{i0}^{+}(z)$ provided in Theorem 3.1. □

Appendix B. Proof of Theorem 3.2

Proof. For $j=1,2,\ldots,N-1$, there are j customers in the system at epoch n^+ if and only if time epoch n^+ is located in the server busy period or server vacation period with j customers. Using similar probabilistic argument as in the proof of Theorem 3.1, it makes

$$\begin{aligned}
P_{0j}\left(n^+\right) =& \sum_{t=1}^{n}\mathrm{P}\{\tilde{\tau}_1=t\}\,Q_j\left((n-t)^+\right)\\
&+\sum_{k=1}^{\infty}\sum_{t=2}^{n}\mathrm{P}\{\tilde{\tau}_1+b_1=t\}\sum_{m=1}^{n-t}\mathrm{P}\{\tilde{\tau}_2=m\}P_{1j}\left((n-t-m)^+\right)\\
&\times\left(\sum_{w=2k-2}^{m}\mathrm{P}\{S_{k-1}=w\}-\sum_{w=2k-1}^{m}\mathrm{P}\{S_{k-1}+Y_k=w\}\right)\\
&+\sum_{k=1}^{\infty}\sum_{t=2}^{n}\mathrm{P}\{\tilde{\tau}_1+b_1=t\}\sum_{w=2k-1}^{n-t}\mathrm{P}\{S_{k-1}+Y_k=w\}\\
&\times\mathrm{P}\{V_k>n-t-w\}\binom{n-t-w}{j}p^j\bar{p}^{n-t-j}\\
&+\sum_{k=1}^{\infty}\sum_{t=2}^{n}\mathrm{P}\{\tilde{\tau}_1+b_1=t\}\sum_{w=2k-1}^{n-t}\mathrm{P}\{S_{k-1}+Y_k=w\}\bar{p}^w\\
&\times\sum_{h=N}^{n-t-w}\mathrm{P}\{\tilde{\tau}_2+l_{N-1}=h\}\,\mathrm{P}\{V_k\geq h\}\,P_{Nj}\left((n-t-w-h)^+\right)\\
&+\sum_{k=1}^{\infty}\sum_{r=1}^{N-1}\sum_{t=2}^{n}\mathrm{P}\{\tilde{\tau}_1+b_1=t\}\sum_{w=2k-1}^{n-t}\mathrm{P}\{S_{k-1}+Y_k=w\}\\
&\times\sum_{u=1}^{n-t-w}\mathrm{P}\{V_k=u\}\binom{u}{r}p^r\bar{p}^{u+w-r}P_{rj}\left((n-t-w-u)^+\right),
\end{aligned} \tag{B.1}$$

where $Q_j((n-t)^+) = \mathrm{P}\{b>(n-t)^+, L((n-t)^+)=j\}$.

Analogously, for $i \geq 1$, we obtain

$$\begin{aligned} P_{ij}\left(n^{+}\right) =& \sum_{k=1}^{i} \sum_{m=k-1}^{n} \mathrm{P}\left\{b_1 + b_2 + \ldots + b_{k-1} = m\right\} Q_{j-i+k}\left((n-m)^{+}\right) \\ & + \sum_{k=1}^{\infty} \sum_{t=i}^{n} \mathrm{P}\left\{b^{\langle i \rangle} = t\right\} \sum_{m=1}^{n-t} \mathrm{P}\left\{\tilde{\tau}_2 = m\right\} P_{1j}\left((n-t-m)^{+}\right) \\ & \times \left(\sum_{w=2k-2}^{m} \mathrm{P}\left\{S_{k-1} = w\right\} - \sum_{w=2k-1}^{m} \mathrm{P}\left\{S_{k-1} + Y_k = w\right\} \right) \\ & + \sum_{k=1}^{\infty} \sum_{t=i}^{n} \mathrm{P}\left\{b^{\langle i \rangle} = t\right\} \sum_{w=2k-1}^{n-t} \mathrm{P}\left\{S_{k-1} + Y_k = w\right\} \\ & \times \mathrm{P}\left\{V_k > n-t-w\right\} \binom{n-t-w}{j} p^j \bar{p}^{n-t-j} \\ & + \sum_{k=1}^{\infty} \sum_{t=i}^{n} \mathrm{P}\left\{b^{\langle i \rangle} = t\right\} \sum_{w=2k-1}^{n-t} \mathrm{P}\left\{S_{k-1} + Y_k = w\right\} \bar{p}^{w} \\ & \times \sum_{h=N}^{n-t-w} \mathrm{P}\left\{\tilde{\tau}_2 + l_{N-1} = h\right\} \mathrm{P}\left\{V_k \geq h\right\} P_{Nj}\left((n-t-w-h)^{+}\right) \\ & + \sum_{k=1}^{\infty} \sum_{r=1}^{N-1} \sum_{t=i}^{n} \mathrm{P}\left\{b^{\langle i \rangle} = t\right\} \sum_{w=2k-1}^{n-t} \mathrm{P}\left\{S_{k-1} + Y_k = w\right\} \\ & \times \sum_{u=1}^{n-t-w} \mathrm{P}\left\{V_k = u\right\} \binom{u}{r} p^r \bar{p}^{u+w-r} P_{rj}\left((n-t-w-u)^{+}\right). \end{aligned} \tag{B.2}$$

Multiplying (B.1) and (B.2) by z^n and adding over n from 0 to ∞, it gives

$$\begin{aligned} P_{0j}^{+}(z) =& F(z) Q_j^{+}(z) + \frac{F^2(z) B(z)\left[1 - Y(z\bar{p})\right]}{1 - V(z\bar{p}) Y(z\bar{p})} P_{1j}^{+}(z) \\ & + \frac{F(z) B(z) Y(z\bar{p})}{1 - V(z\bar{p}) Y(z\bar{p})} \sum_{u=j+1}^{\infty} \mathrm{P}\left\{V = u\right\} \sum_{k=j}^{u-1} z^k \binom{k}{j} p^j \bar{p}^{k-j} \\ & + \frac{F(z) B(z) Y(z\bar{p})}{1 - V(z\bar{p}) Y(z\bar{p})} \left\{ \sum_{u=N}^{\infty} \mathrm{P}\left\{V = u\right\} \sum_{m=N}^{u} z^m \, \mathrm{P}\left\{\tilde{\tau}_2 + l_{N-1} = m\right\} \right. \\ & \left. \times P_{Nj}^{+}(z) + \sum_{r=1}^{N-1} \sum_{u=r}^{\infty} \mathrm{P}\left\{V = u\right\} \binom{u}{r} (zp)^r (z\bar{p})^{u-r} P_{rj}^{+}(z) \right\}, \end{aligned} \tag{B.3}$$

$$\begin{aligned} P_{ij}^{+}(z) =& \sum_{k=1}^{i} B^{k-1}(z) Q_{j-i+k}^{+}(z) + \frac{F(z) B^i(z)\left[1 - Y(z\bar{p})\right]}{1 - V(z\bar{p}) Y(z\bar{p})} P_{1j}^{+}(z) \\ & + \frac{B^i(z) Y(z\bar{p})}{1 - V(z\bar{p}) Y(z\bar{p})} \sum_{u=j+1}^{\infty} \mathrm{P}\left\{V = u\right\} \sum_{k=j}^{u-1} z^k \binom{k}{j} p^j \bar{p}^{k-j} \end{aligned}$$

$$+\frac{B^{i}(z)\,Y(z\bar{p})}{1-V(z\bar{p})\,Y(z\bar{p})}\left\{\sum_{u=N}^{\infty}\mathrm{P}\{V=u\}\sum_{m=N}^{u}z^{m}\,\mathrm{P}\{\tilde{\tau}_2+l_{N-1}=m\}\right.$$

$$\left.\times P_{Nj}^{+}(z)+\sum_{r=1}^{N-1}\sum_{u=r}^{\infty}\mathrm{P}\{V=u\}\binom{u}{r}(zp)^{r}(z\bar{p})^{u-r}P_{rj}^{+}(z)\right\}. \tag{B.4}$$

Solving equations (B.3) and (B.4) for $P_{0j}^{+}(z)$ and $P_{ij}^{+}(z)$, we can obtain the desired formulas (3.3) and (3.4). □

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