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CLIQUE-CONNECTING FOREST AND STABLE SET POLYTOPES

Denis Cornaz[1]

Abstract. Let $G = (V, E)$ be a simple undirected graph. A forest $F \subseteq E$ of G is said to be *clique-connecting* if each tree of F spans a clique of G. This paper adresses the clique-connecting forest polytope. First we give a formulation and a polynomial time separation algorithm. Then we show that the nontrivial nondegenerate facets of the stable set polytope are facets of the clique-connecting polytope. Finally we introduce a family of rank inequalities which are facets, and which generalize the clique inequalities.

Keywords. Graph, polytope, separation, facet.

Mathematics Subject Classification. 05C15, 90C09.

1. Introduction

Let $G = (V, E)$ be a simple undirected graph (without loop or multiple edge). A *stable set of* G is a subset of pairwise nonadjacent vertices, the *stable set polytope of* G is the convex-hull of the characteristics vectors (in $\{0, 1\}^V$) of the stable sets of G. This polytope has been extensively studied in litterature (see *e.g.* [11]).

A subset of edges $F \subseteq E$ is called a *forest of* G if the number k of the connected components of the partial subgraph (V, F) of G satisfies $|F| + k = |V|$. (Note that some components of (V, F) may be isolated vertices.) In other words, F is a forest of G if and only if $|F(U)| \leq |U| - 1$ for every nonempty subset of vertices $U \subseteq V$ (where $F(U)$ denotes the subset of the edges of F with both vertices in U). Given a weight vector $c \in \mathbb{Z}^E$ associated to the edges of G, several well-known greedy

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[1] LIMOS, Complexe scientifique des Cézeaux, 63177 Aubiere Cedex, France; cornaz@isima.fr

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algorithms (*e.g.* Kruskal, Prim, see [11]) find a forest F maximizing its weight $c(F) = \sum_{e \in F} c_e$ in polynomial time. The *forest polytope of* G is the convex-hull of the characteristic vectors (in $\{0,1\}^E$) of the forests of G. Edmonds [5] showed that for any graph G, its forest polytope is described by the following system of linear inequalities:

$$0 \leq x_e \leq 1 \quad \text{for } e \in E, \tag{1}$$

$$x(E(U)) \leq |U| - 1 \quad \text{for nonempty } U \subseteq V. \tag{2}$$

Optimizing over (1)–(2) is, given a vector $c \in \mathbb{Q}^E$, to determine max cx over $x \in \mathbb{Q}^E$ satisfying (1)–(2). Separating over (1)–(2) is, given a vector $\bar{x} \in \mathbb{Q}^E$, to decide if $\bar{x}$ satisfies (1)–(2) or to find a constraint violated by $\bar{x}$. The greedy algorithms together with the result by Edmonds show that we can optimize over (1)–(2) in polynomial time (see *e.g.* [11]). A consequence of the optimization-separation theorem by [6] is that separating over (1)–(2) can be done in polynomial time. Independently, Picard and Queyranne [9] and Padberg and Wolsey [8] gave a polynomial combinatorial algorithm separating (1)–(2).

A *matching of* G is a subset of pairwise disjoint edges. After giving a combinatorial algorithm finding a matching optimizing any linear function, Edmonds [4] showed that the matching polytope is described by the following system of linear inequalities:

$$x_e \geq 0 \quad \text{for } e \in E, \tag{3}$$

$$x(\delta(v)) \leq 1 \quad \text{for } v \in V, \tag{4}$$

$$2x(E(U)) \leq |U| - 1 \quad \text{for odd cardinality } U \subseteq V. \tag{5}$$

(Where, as usual, $\delta(v)$ is the set of the edges of G incident with the vertex v.)

A *clique of* G is a subset of vertices any two of which are adjacent. Determining the minimum number of cliques in a partition of V into cliques of G is NP-hard, see *e.g.* [11] (it is equivalent to the graph coloring problem). A forest $F \subseteq E$ of $G = (V, E)$ is said to be *clique-connecting* if each tree of F spans a clique of G, that is, if each connected component of the partial subgraph (V, F) of G induces a complete subgraph of G. Clique-connecting stars are considered in [1] for the representative formulation of graph coloring. It is also used, with additional restrictions, in [2] to show a one-to-one correspondence beetween the colorings of G and the stable sets of $\tilde{G}$, where $\tilde{G}$ is a partial subgraph of the line graph of the complementary $\overline{G}$ of G.

This paper adresses the *clique-connecting forest polytope of* G, that is the convex-hull of the incidence vectors of the clique-connecting forests of G. A first motivation for studying that polytope is that it is a "coloring polytope", in the sense that optimizing $1^T x$ over it is equivalent to determining the chromatic number [2]. A second motivation for studying the clique-connecting forest polytope is that it lays beetween two well described polytopes. Indeed, it is obvious that the forest polytope of G contains the clique-connecting forest polytope of G. Futhermore, since any matching of G is a clique-connecting forest of G, the clique-connecting forest polytope contains the matching polytope.

In this paper, we give a formulation for the clique-connecting forest polytope. That is, we define a polyhedron the integer vectors of which are precisely the characteristic vectors of the clique-connecting forests. The number of linear inequalities defining the polyhedron may be exponential with respect to the size of G, but we give a polynomial time combinatorial algorithm separating over them. Consequently, by [6], there is a polynomial time algorithm optimizing over the polyhedron.

Then we study the facial structure of the polytope. We show that every nontrivial nondegenerate facet of the stable set polytope corresponds to a facet of the clique-connecting forest polytope. Futhermore, we give a set of rank facets of the clique-connecting forest polytope which cannot be deduced from facets of the stable set polytope. These facets, called the *K-complete set inequalities*, are associated with each (not necessarily maximal) clique K and they generalize the clique facets.

2. Formulation and separation

Let $G = (V, E)$ be a graph. For each edge subset $F \subseteq E$ the characteristic vector of F is the vector $x \in \{0,1\}^E$ such that for each $e \in E$, then $x_e = 1$ if $e \in F$, and $x_e = 0$ if $e \notin F$. We claim that an integer vector $x \in \mathbb{Z}^E$ is the characteristic vector of a clique-connecting forest of G if and only if x satisfies:

$$0 \le x_e \le 1 \quad \text{for each } e \in E, \tag{6}$$

$$x(E(U)) \le |U| - \begin{cases} 1 \text{ if } U \text{ is a clique of } G \\ 2 \text{ otherwise} \end{cases} \quad \text{for nonempty } U \subseteq V. \tag{7}$$

To see sufficiency, first observe that if x satisfies (6)–(7) then it satisfies (1)–(2). Hence, since x is integer, it is the characteristic vector of a forest F of G. Now if U is the vertex set of a connected component of F, then $x(E(U)) = |U| - 1$. Since $x(E(U)) \le |U| - 2$ for each subset $U \subseteq V$ which is **not** a clique of G, hence U is a clique of G. It follows that F is clique-connecting. To see necessity, let x be the characteristic vector of a clique-connecting forest F of G. Clearly, x is integer and satisfies (1)–(2). Given a subset $U \subseteq V$ which is not a clique of G, then no tree of F spans U. It follows that $|F(U)| \le |U| - 2$, and hence x satisfies (6)–(7).

Now let us consider the polyhedron defined by the vectors of $\mathbb{R}^E$ satisfying (6)–(7). We claim that one can optimize over that polyhedron in a polynomial time. By [6] we only need to solve in polynomial time the separation problem which can be stated as follows:

Separation problem. Given $\bar{x} \in \mathbb{Q}_+^E$, decide if $\bar{x}$ satisfies (6)–(7), and if not, find an inequality violated by $\bar{x}$.

One can adapt the proof in [8,9] for the separation problem associated with (1)–(2) in order to prove that separating over (6)–(7) can be done in a polynomial time.

The proof of [8,9] is based on Theorem 2.1 below. For the sake of completeness we give a proof (similar to that by Schrijver in [11], Vol. B, p. 880). For any graph $G=(V,E)$ and $U \subseteq V$, let $\delta_G(U)$ be the cut composed by the edges of G incident with one vertex in U and the other vertex in $V \setminus U$ outside.

Theorem 2.1 (Rhys [10])**.** *Given a graph $G=(V,E)$, two vectors $x \in \mathbb{Q}_+^E$ and $y \in \mathbb{Q}^V$, and a subset $S \subseteq V$, we can find in a strongly polynomial time a set U with $S \subseteq U \subseteq V$ minimizing $x(\delta_G(U)) + y(U)$.*

Proof. We transform the graph G into a graph H by the following operations: First we extend the graph by two new vertices s and t, by new edges $\{s,v\}$ for each $v \in V$ with $y_v > 0$, and by new edges $\{v,t\}$ for each $v \in V$ with $y_v < 0$, so we obtain a new graph with edge set $\tilde{E}$; Then we replace the vertices in $S \cup \{s\}$ by the vertex s, that is, we contract $S \cup \{s\}$ (we can assume that the loops are deleted). For each $e \in E$ define the capacity c_e of e by $c_e := x_e$. For each $\{s,v\} \in \tilde{E} \setminus E$ define the capacity c_e of e by $c_e := y_v$. For each $\{v,t\} \in \tilde{E} \setminus E$ define the capacity c_e of e by $c_e := |y_v|$. Then one has for any $U \subseteq V \setminus S$:

$$\begin{aligned} c(\delta_H(U \cup \{t\})) &= x(\delta_G(U)) + \sum_{v \in U: y_v > 0} y_v + \sum_{v \in V \setminus U: y_v < 0} |y_v| \\ &= x(\delta_G(U)) + y(U) + \sum_{v \in V: y_v < 0} |y_v|. \end{aligned}$$

Since $\sum_{v \in V: y_v < 0} |y_v|$ is a constant, it follows that minimizing $x(\delta_G(U)) + y(U)$ reduces to finding a minimum-capacity cut separating s and t in H; which can be done in a strongly polynomial time with a max-flow algorithm (see *e.g.* [11]). □

For the sake of completeness we give a proof (similar to that by Schrijver in [11], Vol. B, p. 881) of the corollary below.

Corollary 2.2 [8,9]**.** *Given a graph $G=(V,E)$ and a vector $x \in \mathbb{Q}_+^E$, we can decide if x satisfies (1)–(2) (that is, if x belongs to the forest polytope of G), and if not, find the most violated inequality among (1)–(2), in strongly polynomial time.*

Proof. We can assume that x satisfies (1). Define $y_v := 2 - x(\delta(\{v\}))$ for $v \in V$. Then $2(x(E(U)) - |U|) = -x(\delta(U)) - y(U)$. So any set $U \subseteq V$, such that U contains a given vertex u, minimizing $x(\delta(U)) + y(U)$, maximizes $x(E(U)) - |U|$. By Theorem 2.1 (with $S = \{u\}$), we can find such a U in polynomial time. Hence, by finding $|V|$ such sets, one for each $u \in V$, we can assume that U is a nonempty subset of V maximizing $x(E(U)) - |U|$. If $x(E(U)) - |U| \le -1$, then x satisfies (2), and otherwise U gives a most violated inequality. □

Now we can state Corollary 2.3 below, which implies that optimizing over (6)–(7) is polynomial.

Corollary 2.3. *The separation problem for (6)–(7) can be solved in polynomial time.*

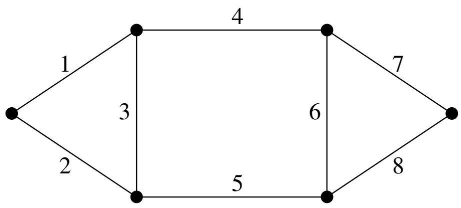

FIGURE 1. A graph $G = (V, E)$ with $E = \{1, 2, 3, 4, 5, 6, 7\}$.

Proof. By Corollary 2.2, we can assume that x satisfies (1)–(2). Hence we only need to separate

$$x(E(U)) \leq |U| - 2 \quad \text{for each subset } U \subseteq V \text{ which is not a clique of } G. \tag{8}$$

Let $\mathcal{S}$ be the set of the stable sets of G with cardinality 2. Note that the cardinality of $\mathcal{S}$ is polynomial. As in Corollary 2.2, we define $y_v := 2 - x(\delta(\{v\}))$ for $v \in V$. So, for any $S \in \mathcal{S}$, any set U with $S \subseteq U \subseteq V$ minimizing $x(\delta(U)) + y(U)$ maximizes $x(E(U)) - |U|$. By Theorem 2.1, we can find such a U in polynomial time. By finding $|\mathcal{S}|$ such sets, one for each $S \in \mathcal{S}$, we can assume that U is a nonempty subset of V maximizing $x(E(U)) - |U|$. If $x(E(U)) - |U| \leq -2$, then x satisfies (8), and otherwise U gives a violated inequality. □

3. Facets

In this section we focus on the vectors of the polyhedron (6)–(7) which are not in the clique-connecting polytope of G. Our aim is to find new *valid* inequalities, that is, cutting some of these vectors out of the polyhedron but keeping the vectors of the polytope in the polyhedron.

First we remark that if each component of G is a complete graph, then the clique-connecting forests of G are the forests of G, so (6)–(7) is equal to (1)–(2) and it decribes the clique-connecting forest polytope of G. Futhermore, we remark that if G has no triangle, then the clique-connecting forests of G are the matchings of G and so the polytope is described by (3)–(5). Finally, we note that the clique-connecting polytope is full-dimensional, since it contains the matching polytope.

Now let us consider for instance the graph of Figure 1.

Assume that the edges 4 and 5 have a weight $c_4 = c_5 = 3$, and that the other edges have a weight 2. It is easily seen that the maximum of cx over x in the clique-connecting forest polytope is 8 (the maximum is obtained by a forest with 4 edges with weight 2). However the maximum of cx over x satisfying (6)–(7) is strictly greater since the vector $\bar{x}$ with $\bar{x}_e = 1/2$ for each $e \in E$ satisfies (6)–(7) and it gives $c\bar{x} = 9$. Now let us give a description of the clique-connecting forest polytope of the graph of Figure 1. It has 18 facets each of which is defined by one of the 18 linear inequalities below:

$$\begin{array}{rrrrrrrrl}
x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7, & x_8 & \geq 0\\
x_1 & & & +x_4 & & & & & \leq 1\\
& x_2 & & & +x_5 & & & & \leq 1\\
& & x_3 & +x_4 & & & & & \leq 1\\
& & x_3 & & +x_5 & & & & \leq 1\\
& & & x_4 & & +x_6 & & & \leq 1\\
& & & x_4 & & & +x_7 & & \leq 1\\
& & & & x_5 & +x_6 & & & \leq 1\\
& & & & x_5 & & & +x_8 & \leq 1\\
x_1 & +x_2 & +x_3 & +x_4 & +x_5 & & & & \leq 2\\
& & & x_4 & +x_5 & +x_6 & +x_7 & +x_8 & \leq 2.
\end{array}$$

(To see this first note that the 18 inequalities are valid and that they imply (6)–(7). So, one must check that each vector satisfying at least 8 linearly independant inequalities with equality among the 18 is integer.) Of course the vector $\bar{x}$ with components $1/2$ does not belong to the polytope. Indeed, it violates the last two constraints with right-hand-side 2. (The other constraints are of type (6)–(7).) One can already remark that the inequalities of type $x_e \geq 0$, namely the nonnegativity constraints, always define facets. Indeed, these are already facets of the forest polytope and of the matching polytope. To make it even clearer, the vector $x = 0$ together with the vectors x_f with all components 0 except $\bar{x}_f = 1$ for $f \in E \setminus \{e\}$ form $|E|$ affinely independant vectors in the clique-connecting forest polytope such that $x_e = 0$.

In a general context, a valid inequality with 0-1 coefficients and integer right-and-side (up to multiplying by a scalar) is called a *rank inequality*. For instances, (2) for the forest polytope, and (4)–(5) for the matching polytope. A rank inequality with right-and-side 1 is called a *clique inequality*. A clique inequality is said to be *maximal* if it is not dominated by another clique inequality. Very often, *e.g.* the stable set polytope [7], the maximal clique inequalities define facets. In contrast, a rank inequality generally does not define a facet even if it is dominated by no other rank inequality. For instance for the clique-connecting forest polytope of the graph of Figure 1: $\sum_{i=1}^{i=8} x_i \leq 4$. Futhermore, of course, the rank inequalities does not describe the polytope in general, *e.g.* the wheel inequalities are non-rank facets of the stable set polytope [3].

In the following we enlight a strong link beetween the clique-connecting forest polytope and the stable set polytope.

For any vertex u of G, let $N(u)$ be the set of the vertices in V adjacent with v. For any $F \subseteq \delta(u)$, we let $N_F(u)$ denotes the set of the vertices in $N(u)$ incident with an edge in F. For any $U \subseteq V$, let G_U be the subgraph of G induced by the vertices in U. As usual, $\overline{G}$ is the complementary graph of G. It is not hard to see that this lemma holds:

Lemma 3.1. *For any vertex u of G, $F \subseteq \delta(u)$ is a clique-connecting forest of G if and only if $N_F(u)$ is a clique of $G_{N(u)}$, that is, a stable set of $\overline{G}_{N(u)}$.*

Note now that, given a vertex u, there is a natural one-to-one correspondence beetween the vectors in $\mathbb{R}^{\delta(u)}$ and the vectors in $\mathbb{R}^{N(u)}$. It follows from the above lemma that a vector $x \in \mathbb{R}^E$ with $x_e = 0$ for each $e \in E \setminus \delta(u)$ is in the clique-connecting forest polytope of G if and only if its corresponding vector in $\mathbb{R}^{N(u)}$ is in the stable set polytope of $\overline{G}_{N(u)}$. A facet of a polytope is called *trivial* if it has at most one nonzero coefficient. For instance the trivial facets of the stable set polytope are those of the form $x_v \geq 0$ or, if v is isolated, $x_v \leq 1$. Suppose that G is composed of tree vertices u, v, w and two edges uv, vw. Then the facets of the clique-connecting forest polytope of G corresponds to the stable set polytope of $\overline{G}_{N(v)}$ but not to the the facets of the stable set polytope of $\overline{G}_{N(u)}$. Indeed, $x_v \leq 1$ is a trivial facet of this polytope but $x_{uv} \leq 1$ is not a facet of the clique-connecting forest polytope. Suppose now that G is the graph of Figure 1. One can observe that each of the eight facets with two nonzero coefficients of the clique-connecting forest polytope is one of the two nontrivial facets of the stable set polytope of $\overline{G}_{N(u)}$, where u is one of the four degree 3 vertices.

Let us define one classe of facets:

Definition 3.2. Let $G = (V, E)$ be a graph and let $\sum_{u \in V} a_u x_u \leq \alpha$ be an inequality which defines a nontrivial facet of the stable set polytope of G. If there exists two nonadjacent vertices v, w of G such that every stable set S with $\sum_{u \in S} a_u = \alpha$ contains either v or w, then the facet is called *degenerate*.

Note that classical nontrivial facets, namely the clique inequalities, the odd-cycle inequalities and the wheel inequalities, are nondegenerate. Actually the following problem is open:

Open problem 1. Is there a graph the stable set polytope of which has degenerate facets?

Theorem 3.4 below shows that for each vertex v, each nontrivial and nondegenerate facet of the stable set polytope of $\overline{G}_{N(v)}$ is a facet of the clique-connecting forest polytope. In order to prove the theorem we need the following lemma.

Lemma 3.3. *Let $G = (V, E)$ be a graph and let $\sum_{v \in V} a_v x_v \leq \alpha$ be an inequality which defines a nontrivial and nondegenerate facet of the stable set polytope of G. Then*

(i) *For any vertex v of G, there exists a stable set of G whose incidence vector is in the facet, which does not contains v.*
(ii) *For any nonadjacent vertices v and w of G, there exists a stable set of G which contains neither v nor w, and the incidence vector of which is in the facet.*

Proof. (i) If every stable set whose incidence vector is in the facet contains the vertex v, then the facet is included in the hyperplan $x_v = 1$. Since the stable set polytope is full-dimensional, it follows that $\sum_{v \in V} a_v x_v \leq \alpha$ is equivalent to the trivial valid inequality $x_v \leq 1$, leading to a contradiction.
(ii) Trivial since the facet is nondegenerate. □

Now we can prove Theorem 3.4.

Theorem 3.4. *Let u be a vertex of G, and let us denote each edge uv in $\delta(u)$ by e_v. If an inequality $\sum_{v\in N(u)} a_v x_v \le \alpha$ defines a nontrivial and nondegenerate facet of the stable set polytope of $\overline{G}_{N(v)}$, then the corresponding inequality*

$$\sum_{e_v\in\delta(u)} a_v x_{e_v} \le \alpha \tag{9}$$

defines a facet of the clique-connecting forest polytope of G.

Proof. By Lemma 3.1, (9) is valid for the clique-connecting forest polytope. We only need to show that there are $|E|$ affinely independant vectors x in the clique-connecting forest polytope satisfying (9) with equality. There are already $d := |\delta(u)|$ such vectors, since $\sum_{v\in N(u)} a_v x_v \le \alpha$ defines a facet of the stable set polytope of $\overline{G}_{N(u)}$, which is full-dimensional. We can take the d vectors in $\{0,1\}^{\delta(u)}$ and we let $F_1,\dots,F_d \subseteq \delta(u)$ be the corresponding clique-connecting forests. Now let $e \in E \setminus \delta(u)$. We claim that there is some $i \in \{1,\dots,d\}$ such that $F_i \cup \{e\}$ is a clique-connecting forest. If the claim holds, it makes $|E|$ affinely independant vectors satisfying (9) with equality and the proof is done. If e is incident with no vertex in $N(u)$ the claim is clearly true. If e is incident with one vertex v in $N(u)$, by Lemma 3.3(i), we can assume that there is a n F_i which does not contains e_v. Hence $F_i \cup \{e\}$ is a clique-connecting forest. We can suppose now that e is incident with two vertices v and w in $N(u)$. By Lemma 3.3(ii), there is an F_i which contains neither e_v nor e_w. Hence $F_i \cup \{e\}$ is a clique-connecting forest. Hence the claim holds, and then the theorem is true also. □

Our aim for the rest of the paper is to present a family of rank inequalities defining facets of the clique-connecting forest polytope and which are not facet of the stable set polytope of $\overline{G}_{N(u)}$ for some vertex u. The *rank* of a subset of edges E' of G, denoted by $r(E')$, is equal to the maximum cardinality of a clique-connecting forest F of G such that $F \subseteq E'$. So, a rank inequality for the clique connecting-forest polytope of G is an inequality $x(E') \le r(E')$ for some $E' \subseteq E$. Let us identify briefly the most natural ones. First, it is not hard to see that a subset of edges E' of G has rank 1 if and only if it is an induced star of G (that is, there is a stable set $\{v_1,\dots,v_{|E'|}\}$ of G and a vertex u such that $E' = \{uv_i : i = 1,\dots,|E'|\}$). Second, for every $v \in V$, the rank of $\delta(u)$ is equal to the maximum cardinality of a clique of $G_{N(u)}$. Finally, for every $U \subseteq V$, then $r(E(U)) = |U| - \overline{\chi}(G_U)$, where $\overline{\chi}(G_U)$ is the minimum number of cliques in a clique partition of G_U. In the following we present a family of rank inequalities associated with edge subsets which are not necessarily of the form $\delta(u)$ for some $u \in V$ or of the form $E(U)$ for some $U \subseteq V$. The form of these subsets is described in the definition below:

Definition 3.5. Let $K \subseteq V$ be a clique of G with at least two vertices and let $E(K)$ be the set of the edges of K. A subset $Q \subseteq E$ containing $E(K)$ is said to be a *K-complete set of* G if it is an inclusionwise maximal subset such that:

(i) Every edge in Q is incident with a node in K.

(ii) If there exist two edges in $Q \setminus E(K)$ incident to a same vertex in K, then the other extremities of the two edges are nonadjacent in G.

(iii) For every edge in Q with one vertex $v \notin K$, there exists at least one vertex $u \in K$ which is incident to no edge in $Q \setminus E(K)$ and such that u and v are nonadjacent in G.

The *complete set inequalities* are

$$x(Q) \leq |K| - 1 \quad \text{for every clique } K \text{ with } |K| > 1 \text{ and every } K\text{-complete set } Q. \tag{10}$$

Notice that the K-clique-complete sets with $|K| = 2$ are precisely the maximal induced stars. So, the complete set inequalities generalize the clique inequalities.

Example 3.6. For the clique-connecting forest polytope of the graph of Figure 1, the complete set inequalities are precisely the 10 facets with non-zero right-hand-side.

Proposition 3.7 below states the main property of K-complete sets, which asserts that the complete set inequalities are valid.

Proposition 3.7. *Let K be a clique with at least two vertices and let $Q \subseteq E$ be a subset with $E(K) \subseteq Q$. Then $r(Q) = |K| - 1$ if and only if Q satisfies (i)–(iii) of Definition 3.5.*

Proof. Necessity. Suppose $r(Q) = |K| - 1$. Assume that there is an edge $e \in Q \setminus E(K)$ disjoint from K. Taking a tree T spanning K, we have a clique-connecting forest $T \cup \{e\} \subseteq Q$. So $r(Q) > |K| - 1$, which is impossible; hence Q satisfies (i). Suppose that there are two edges $e, f \in Q \setminus E(K)$ incident to a same vertex w of K whose extremities are adjacent in G. Taking a tree T' spanning $K \setminus \{w\}$, we have a clique-connecting forest $T' \cup \{e, f\} \subseteq Q$. Again impossible, hence Q satisfies (ii). Now let $e \in Q \setminus E(K)$ be incident to $v \notin K$. Let K' be the set of the vertices of K which are not adjacent with v. If every vertex of K' is incident with an edge in $Q \setminus E(K)$, then taking e, taking a tree spanning $K \setminus K'$ and taking one edge in $Q \setminus E(K)$ for each vertex in K', one finds a clique-connecting forest with $|K|$ edges. This is a contradiction, hence Q satisfies (iii).

Sufficiency. Suppose that Q satisfies the three properties of Definition 3.5. Let F be a clique-connecting forest of G such that $F \subseteq Q$ with $|F|$ maximum, so $r(Q) = |F|$. Since K is a clique, then $|F| \geq |K| - 1$. To show that $|F| \leq |K| - 1$, we assume that F has been chosen with $|F \cap E(K)|$ maximum, and we only have to prove that $F \setminus E(K)$ is empty. Suppose on the contrary that there is an edge $vw \in F \setminus E(K)$. By (i) we can assume that $w \in K$ (and $v \notin K$). By (iii) there is a vertex $u \in K$ which is not adjacent with v and which is incident with no edge in $Q \setminus E(K)$. Since u and v are not adjacent, no path in F links u and w, and in particular $uw \in E(K) \setminus F$. Since $|F \cap E(K)|$ is maximal, then $(F \setminus \{vw\}) \cup \{uw\}$ is not a clique-connecting forest. It follows, since u is incident with no edge in $F \setminus E(K)$, that there is an edge $wv' \in F \setminus E(K)$ (where $v' \notin K$ is not adjacent with u). But then v and v' are adjacent; this contradicts (ii). □

Corollary 3.8. *Let K be a clique with at least two vertices. Then Q is an inclusionwise maximal subset containing $E(K)$ with rank $|K|-1$ if and only if it is a K-complete set.*

A consequence of the corollary is that no complete set inequalities is dominated by another rank inequality. Now we can state the main result of the section.

Theorem 3.9. *Let G be a graph. Each complete set inequality (10) defines a facet of the clique-connecting forest polytope of G*

Proof. Let K be a clique of G with at least two vertices and Q a K-complete set. Since by Proposition 3.7, $r(Q) = |K| - 1$, then inequality (10) is valid. To see that (10) determines a facet, let $\sum_{e \in E} a_e x_e = \beta$ be satisfied by all x in the clique-connecting forest polytope of G with $x(Q) = |K| - 1$. We only need to prove that $\sum_{e \in E} a_e x_e = \beta$ is some multiple of $x(Q) = |K| - 1$. So $a(F) := \sum_{e \in F} a_e = \beta$ for each clique-connecting forest F with $|F \cap Q| = |K| - 1$. If $|K| = 2$, there is a clique-connecting forest $\{e\}$ where e is the unique edge in $E(K)$, then $a_e = \beta$. Otherwise let e_1, e_2 be two distinct edges in $E(K)$. Since K is a clique, we can assume that there exists two trees spanning K, say T_1 and T_2, such that $T_1 \setminus T_2 = e_1$ and $T_2 \setminus T_1 = e_2$. Then $a(T_1) - a(T_2) = a_{e_1} - a_{e_2} = \beta - \beta = 0$. It follows that $a_e = \beta/(|K| - 1)$ for every $e \in E(K)$. (If $|K| = 2$ this still holds.) Now let $e \in Q \setminus E(K)$. Since every edge in Q is incident to at least one node in K, we can suppose that $e = vw$ with $w \in K$. Let T be a spanning tree of $K \setminus \{w\}$. Since $F = T \cup \{e\}$ is a clique-connecting forest such that $|F \cap Q| = |K| - 1$, $a(F) = a(T) + a_e = \beta$. Since $a(T) = (|K|-2) \times \beta/(|K|-1)$, then $a_e = \beta/(|K|-1)$. Also, $a_f = 0$ for each $f \in E \setminus Q$. Indeed, since by Corollary 3.8, Q is maximal, then $r(Q \cup \{f\}) = r(Q) + 1$, and then there is a clique-connecting forest F such that $F \setminus Q = \{f\}$ and $|F \cap Q| = |K| - 1$, and hence $a(F) = \beta = a(F \setminus \{f\})$. So $a_f = 0$. In conclusion, $\sum_{e \in E} a_e x_e = \beta$ is some multiple of $x(Q) = |K| - 1$, and hence (10) define facets. □

As it is implied by the proposition below, unsurprisingly, one cannot separate the complete set inequalities.

Proposition 3.10. *It is NP-complete to separating the clique inequalities.*

Proof. Recall that $x(Q) \leq 1$ is a clique inequality if and only if Q is a maximal induced star of G. Hence if we could separate the clique inequalities in polynomial time, we could decide if G contains an induced star with a weight >1 with respect to a weight function x. In particular we could find an induced star with maximum cardinality in polynomial time. Suppose that we want to find a stable set with maximum cardinality in a graph H. It suffices to finding an induced star with maximum cardinality in the graph obtained from H by adding a (universal) vertex adjacent to all the other vertices of H. Since finding a maximum cardinality stable set is NP-hard, it follows that separating the clique inequalities is NP-hard. □

4. Conclusion

We have studied a coloring polytope, namely the clique-connecting forest polytope, and we have showed its link with the stable set polytope. In light of the 1-to-1 correspondence in [2], this is not too surprising to have a stronk link between these two polytopes. An open problem raises naturally concerning the clique-connecting polytope: Does this polytope have other rank facets than the ones mentionned in the present paper? Also, concerning the stable set polytope, we raised one open problem: Can it have degenerate facet? Where a "degenerate facet" is a nontrivial one with a pair of nonadjacent vertices intersecting every stable set saturating the facet.

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