ON OPTIMAL UNIFORM APPROXIMATION OF LÉVY PROCESSES ON BANACH SPACES WITH FINITE VARIATION PROCESSES

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Abstract. For a general càdlàg Lévy process X on a separable Banach space V we estimate values of $\inf_{c \geq 0} \left\{ \psi(c) + \inf_{Y \in \mathcal{A}_X(c)} \mathbb{E}\mathrm{TV}(Y, [0, T]) \right\}$, where $\mathcal{A}_X(c)$ is the family of processes on V adapted to the natural filtration of X, a.s. approximating paths of X uniformly with accuracy c, ψ is a penalty function with polynomial growth and $\mathrm{TV}(Y, [0, T])$ denotes the total variation of the process Y on the interval $[0, T]$. Next, we apply obtained estimates in three specific cases: Brownian motion with drift on $\mathbb{R}$, standard Brownian motion on $\mathbb{R}^d$ and a symmetric α-stable process ($\alpha \in (1, 2)$) on $\mathbb{R}$.

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1. Introduction and formulation of the problem

Let X_t, $t \geq 0$, be a càdlàg Lévy process X taking its values in a separable Banach space V (*i.e.* a process with a.s. càdlàg paths and independent and stationary increments). We also assume that X has the strong Markov property with respect to its natural filtration. By $|\cdot|$ we denote the norm in V. For $T > 0$ and two processes $Y, Z : \Omega \times [0, +\infty) \to V$ with càdlàg trajectories we denote the random variable

$$\|Y - Z\|_{\infty, [0, T]} := \sup_{0 \leq t \leq T} |Y_t - Z_t|.$$

Let us note that $\|Y - Z\|_{\infty, [0, T]}$ is indeed a measurable random variable, since, by the fact that Y and Z have càdlàg trajectories, denoting by $\mathbb{Q}$ the set of rational numbers, we get

$$\|Y - Z\|_{\infty, [0, T]} := \sup_{0 \leq t \leq T, t \in \mathbb{Q}} |Y_t - Z_t|.$$

By $\mathcal{A}_X(c)$ we denote the family of V-valued processes Y_t, $t \geq 0$, adapted to the natural filtration of X and such that $\|Y - X\|_{\infty,[0,T]} \leq c$ a.s. Also

$$\text{TV}(Y, [0,T]) := \sup_n \sup_{0 \leq t_0 < t_1 < \cdots < t_n \leq T} \sum_{i=1}^n \left| Y_{t_i} - Y_{t_{i-1}} \right|,$$

that is for $\omega \in \Omega$, $\text{TV}(Y(\omega), [0,T])$ is the total variation of the trajectory $Y(\omega)$ on the interval $[0,T]$.

In this paper we deal with the following optimisation problem. Given are $T > 0$ and a non-decreasing function $\psi : [0, +\infty) \to [0, +\infty)$ calculate (or estimate up to universal constants)

$$V_X(\psi) := \inf_{c>0} \left\{ \psi(c) + \inf_{Y \in \mathcal{A}_X(c)} \mathbb{E}\text{TV}(Y, [0,T]) \right\}. \tag{1.1}$$

Thus we want to find a process with possibly small total variation, adapted to the natural filtration of X and whose trajectories a.s. uniformly approximate trajectories of X with given accuracy c, but the worse accuracy of the approximation is, the bigger penalty $\psi(c)$ we pay. Interestingly, $V_X(\psi)$ may be finite even if the total variation of X is infinite. Lower and upper bounds for $V_X(\psi)$ are given in Theorem 2.6. For finite-dimensional processes finiteness of these bounds is equivalent with $\mathbb{E}|X_1 - X_0| < +\infty$, see for example Theorem 25.3 of [11].

This type of optimization problems appear naturally in several situations. For example, in financial models with small proportional transaction costs where X is the process representing optimal investment strategy on (frictionless) market without transaction costs, while Y is the approximation of X and the total variation of Y is proportional to the transactions costs of the implementation of the strategy Y, see for example [6, 7]. This type of optimization problems have no unified, algorithmic solution since the generator of the total variation functional is not well defined. Moreover, we deal with very general Lévy processes taking their values in general Banach spaces.

Another problems when approximation by functions with possible small total variation is used, is image denoising, see for example [4]. Here however one deals with more complicated domains (usually $\mathbb{R}^d$) and different norms measuring the quality of the approximation (like L^1).

Using well known results of the renewal theory, some ad-hoc reasoning, results obtained for the functional called *truncated variation* and assuming that ψ grows no faster than some polynomial, we will be able to estimate (1.1) up to universal constants, depending on ψ and some simpler quantities describing of the process X (like its Lévy measure and laws of its exit times from the balls centered at X_0). Together with the estimates we will provide the construction of the process Z uniformly approximating X, for which these estimates hold.

From the triangle inequality we immediately get that if $\|X - Y\|_{\infty,[0,T]} \leq c$ then for any $0 \leq s \leq t \leq T$, $|Y_t - Y_s| \geq \max\{|X_t - X_s| - 2c, 0\}$, thus

$$\text{TV}(Y, [0,T]) = \sup_n \sup_{0 \leq t_0 < t_1 < \cdots < t_n \leq T} \sum_{i=1}^n \left| Y_{t_i} - Y_{t_{i-1}} \right|$$

$$\geq \sup_n \sup_{0 \leq t_0 < t_1 < \cdots < t_n \leq T} \sum_{i=1}^n \max\left\{ \left| X_{t_i} - X_{t_{i-1}} \right| - 2c, 0 \right\}$$

$$=: \text{TV}^{2c}(X, [0,T]). \tag{1.2}$$

The quantity in the last line of (1.2) is called the truncated variation of X. In the case when $V = \mathbb{R}$, from the results of Remark 15 in [10] it is possible to prove that for any $c > 0$ there exists a process $X^c \in \mathcal{A}_X(c)$ such that $\|X - X^c\|_{\infty,[0,T]} \leq c$ and

$$\text{TV}^{2c}(X, [0,T]) \leq \text{TV}(X^c, [0,T]) \leq \text{TV}^{2c}(X, [0,T]) + 2c,$$

thus in the case $V = \mathbb{R}$ we have the estimate

$$\inf_{c>0} \left\{ \psi(c) + \mathbb{E}\mathrm{TV}^{2c}(X, [0, T]) \right\}$$

$$\leq \inf_{c>0} \left\{ \psi(c) + \inf_{Y \in \mathcal{A}_X(c)} \mathbb{E}\mathrm{TV}(Y, [0, T]) \right\}$$

$$\leq \inf_{c>0} \left\{ \psi(c) + \mathbb{E}\mathrm{TV}^{2c}(X, [0, T]) + 2c \right\},$$

which means that if ψ grows no faster than some polynomial (and no slower than some increasing linear function) then $\inf_{c>0} \left\{ \psi(c) + \mathbb{E}\mathrm{TV}^{2c}(X, [0, T]) \right\}$ and $V_X(\psi)$ are comparable up to universal constants depending on ψ only.

For a general Banach space-valued Lévy process, using similar construction as in the proof of Theorem 1 from [9], we get that there exists a process $Y^c \in \mathcal{A}_X(c)$ such that $\|X - Y^c\|_{\infty, [0, T]} \leq c$ and

$$\mathbb{E}\mathrm{TV}^{2c}(X, [0, T]) \leq \mathbb{E}\mathrm{TV}(Y^c, [0, T]) \tag{1.3}$$

$$\leq \inf_{\lambda > 1} \lambda \cdot \mathbb{E}\mathrm{TV}^{(\lambda-1)c/\lambda}(X, [0, T]).$$

From this, assuming that there exists a constant K_ψ such that for any $a \geq 0$, $\psi(2a) \leq K_\psi \cdot \psi(a)$, we get

$$\inf_{c>0} \left\{ \psi(c) + \mathbb{E}\mathrm{TV}^{2c}(X, [0, T]) \right\}$$

$$\leq \inf_{c>0} \left\{ \psi(c) + \inf_{Y \in \mathcal{A}_X(c)} \mathbb{E}\mathrm{TV}(Y, [0, T]) \right\}$$

$$\leq \inf_{c>0} \left\{ \psi(c) + \mathbb{E}\mathrm{TV}(Y^c, [0, T]) \right\} \tag{1.4}$$

$$\leq \inf_{c>0} \left\{ \psi(c) + \inf_{\lambda > 1} \lambda \cdot \mathbb{E}\mathrm{TV}^{(\lambda-1)c/\lambda}(X, [0, T]) \right\}$$

$$= \inf_{c>0} \left\{ \psi(4c) + \inf_{\lambda > 1} \lambda \cdot \mathbb{E}\mathrm{TV}^{(\lambda-1)4c/\lambda}(X, [0, T]) \right\}$$

$$\leq \inf_{c>0} \left\{ \psi(4c) + 2 \cdot \mathbb{E}\mathrm{TV}^{4c/2}(X, [0, T]) \right\}$$

$$\leq \max \left(K_\psi^2, 2 \right) \inf_{c>0} \left\{ \psi(c) + \mathbb{E}\mathrm{TV}^{2c}(X, [0, T]) \right\}$$

thus again we see that both quantities: $\inf_{c>0} \left\{ \psi(c) + \mathbb{E}\mathrm{TV}^{2c}(X, [0, T]) \right\}$ and $V_X(\psi)$ are comparable up to universal constants (depending on ψ only). Since X has càdlàg trajectories, the construction of the process Y^c appearing in (1.3) and (1.4) simplifies to the following one. First, we define stopping times $\tau_0^c = 0$ and for $n = 1, 2, \ldots$

$$\tau_n^c = \begin{cases} \inf \left\{ t > \tau_{n-1}^c : \left| X_{\tau_{n-1}^c} - X_t \right| \geq c \right\} & \text{if } \tau_{n-1}^c < +\infty; \\ +\infty & \text{if } \tau_{n-1}^c = +\infty \end{cases} \tag{1.5}$$

and then we define

$$Y_t^c = \sum_{n=0}^{+\infty} X_{\tau_n^c} \mathbf{1}_{\left[\tau_n^c; \tau_{n+1}^c \right)}(t). \tag{1.6}$$

To avoid technical problems with these definitions we apply the convention that $\inf \emptyset = +\infty$, $X_\infty = X_0$ and that $[+\infty; +\infty) = \emptyset$.

Remark 1.1. The construction in the proof of Theorem 1 in [9] rather uses times $\tilde{\tau}_n^c$ defined in the following way

$$\tilde{\tau}_n^c = \begin{cases} \inf \left\{ t > \tilde{\tau}_{n-1}^c : \left| X_{\tilde{\tau}_{n-1}^c} - X_t \right| > c \right\} & \text{if } \tilde{\tau}_{n-1}^c < +\infty; \\ +\infty & \text{if } \tilde{\tau}_{n-1}^c = +\infty, \end{cases}$$

which may be not stopping times, but it is straightforward to verify that for the times defined by (1.5) and Y^c defined by (1.6) the estimates (1.4) hold as well (see the proof of [9], Thm. 1).

In what follows, we will use the presented construction to obtain more straightforward estimates of $V_X(\psi)$ in terms of the characteristics of the process X.

This paper is organised as follows. In the next section we prove useful estimates of $\mathbb{E}\mathrm{TV}(Y^c, [0, T])$, where Y^c is the process defined by equation (1.6), and then prove two universal estimates of $V_X(\psi)$ (Thms. 2.5 and 2.6) expressed in terms of simpler functionals of X. In the last, third section, we apply obtained estimates in three specific cases, namely when: (1) X is a Brownian motion with drift on $\mathbb{R}$, (2) X is a standard Brownian motion on $\mathbb{R}^d$ and (3) X is a symmetric α-stable process ($\alpha \in (1, 2)$) on $\mathbb{R}$.

2. ESTIMATION OF $\mathbb{E}\mathrm{TV}(Y^c, [0, T])$ AND $V_X(\psi)$.

First, using the strong Markov property and the independence of the increments of the process X, we will estimate $\mathbb{E}\mathrm{TV}(Y^c, [0, T])$, where Y^c is the process defined by equation (1.6).

For $t > 0$ let us define few auxiliary quantities

$$\sigma^c(t) := \min \{k : \tau_k^c > t\} = \sum_{n=1}^{+\infty} \mathbf{1}_{\left\{\tau_{n-1}^c \leq t\right\}},$$

$$U^c(t) := \mathbb{E}\sigma^c(t) = \sum_{n=1}^{+\infty} \mathbb{P}\left(\tau_{n-1}^c \leq t\right) = \sum_{n=1}^{+\infty} \mathbb{P}\left(\sigma^c(t) \geq n\right)$$

and

$$f^c(t) := \mathbb{E}\left(\left| X_{\tau_1^c} - X_0 \right| \mathbf{1}_{\left\{\tau_1^c \leq t\right\}}\right)$$

where stopping times τ_n^c, $n = 0, 1, \ldots$, are defined by formula (1.5).

Lemma 2.1. *For the process Y^c defined by equation (1.6) the following inequalities hold:*

$$\frac{1}{2} U^c(T) f^c(T) \leq \mathbb{E}\mathrm{TV}(Y^c, [0, T]) \leq U^c(T) f^c(T).$$

Proof. Let us first notice that

$$\mathrm{TV}(Y^c, [0, T]) = \sum_{n=1}^{+\infty} \left| X_{\tau_n^c} - X_{\tau_{n-1}^c} \right| \mathbf{1}_{\{\tau_n^c \leq T\}}, \tag{2.1}$$

where stopping times τ_n^c, $n = 0, 1, \ldots$, are defined by formula (1.5). From this, using independence of increments of A and the strong Markov property we get an upper bound for $\mathbb{E}\mathrm{TV}(Y^c, [0, T])$, which reads

$$
\begin{aligned}
\mathbb{E}\mathrm{TV}(Y^c, [0, T]) &= \mathbb{E} \sum_{n=1}^{+\infty} \left| X_{\tau_n^c} - X_{\tau_{n-1}^c} \right| \mathbf{1}_{\{\tau_n^c \leq T\}} \\
&\leq \mathbb{E} \sum_{n=1}^{+\infty} \left| X_{\tau_n^c} - X_{\tau_{n-1}^c} \right| \mathbf{1}_{\left\{\tau_n^c - \tau_{n-1}^c \leq T\right\}} \mathbf{1}_{\left\{\tau_{n-1}^c \leq T\right\}} \\
&= \sum_{n=1}^{+\infty} \mathbb{E} \left(\left| X_{\tau_n^c} - X_{\tau_{n-1}^c} \right| \mathbf{1}_{\left\{\tau_n^c - \tau_{n-1}^c \leq T\right\}} \right) \mathbb{E} \mathbf{1}_{\left\{\tau_{n-1}^c \leq T\right\}} \\
&= \sum_{n=1}^{+\infty} \mathbb{E} \left(\left| X_{\tau_1^c} - X_0 \right| \mathbf{1}_{\left\{\tau_1^c \leq T\right\}} \right) \mathbb{E} \mathbf{1}_{\left\{\tau_{n-1}^c \leq T\right\}} \\
&= U^c(T) f^c(T).
\end{aligned}
$$

To bound $\mathbb{E}\mathrm{TV}(Y^c, [0, T])$ from below we write

$$
\begin{aligned}
\mathrm{TV}(Y^c, [0, T]) &= \sum_{n=1}^{\sigma^c(T)-1} \left| X_{\tau_n^c} - X_{\tau_{n-1}^c} \right| \\
&= \sum_{n=1}^{\sigma^c(T)-1} \left| X_{\tau_n^c} - X_{\tau_{n-1}^c} \right| \mathbf{1}_{\left\{\tau_n^c - \tau_{n-1}^c \leq T\right\}} \mathbf{1}_{\left\{\tau_{n-1}^c \leq T\right\}}.
\end{aligned} \tag{2.2}
$$

We will use the notion of *stochastic domination*. We say that a real random variable Q stochastically dominates a real random variable P if for any $x \in \mathbb{R}$, $\mathbb{P}(Q \geq x) \geq \mathbb{P}(R \geq x)$. We denote this by $Q \succeq P$. We have that (even if $\mathbb{P}\left(\tau_{\sigma^c(T)}^c = +\infty\right) > 0$, applying the convention that $X_\infty = X_0$)

$$
\mathrm{TV}(Y^c, [0, T]) \succeq \left| X_{\tau_{\sigma^c(T)}^c} - X_{\tau_{\sigma^c(T)-1}^c} \right| \mathbf{1}_{\left\{\tau_{\sigma^c(T)}^c - \tau_{\sigma^c(T)-1}^c \leq T\right\}} \mathbf{1}_{\left\{\tau_{\sigma^c(T)-1}^c \leq T\right\}}. \tag{2.3}
$$

The domination in (2.3) follows from the strong Markov property of X since, by the stationarity of the increments of X, the variable

$$
\left| X_{\tau_{\sigma^c(T)}^c} - X_{\tau_{\sigma^c(T)-1}^c} \right| \mathbf{1}_{\left\{\tau_{\sigma^c(T)}^c - \tau_{\sigma^c(T)-1}^c \leq T\right\}}
$$

conditioned on the event that $\tau_{\sigma^c(T)-1}^c < +\infty$ has the same law as the variable $\left| X_{\tau_1^c} - X_{\tau_0^c} \right| \mathbf{1}_{\left\{\tau_1^c - \tau_0^c \leq T\right\}}$ which, by (2.1), stochastically dominates $\mathrm{TV}(Y^c, [0, T])$.

Taking expectations of both sides of relations (2.2) and (2.3) and adding them we get

$$
\begin{aligned}
2 \cdot \mathbb{E}\mathrm{TV}(Y^c, [0, T]) &\geq \mathbb{E} \sum_{n=1}^{\sigma^c(T)} \left| X_{\tau_n^c} - X_{\tau_{n-1}^c} \right| \mathbf{1}_{\left\{\tau_n^c - \tau_{n-1}^c \leq T\right\}} \mathbf{1}_{\left\{\tau_{n-1}^c \leq T\right\}} \\
&= \mathbb{E} \sum_{n=1}^{+\infty} \left| X_{\tau_n^c} - X_{\tau_{n-1}^c} \right| \mathbf{1}_{\left\{\tau_n^c - \tau_{n-1}^c \leq T\right\}} \mathbf{1}_{\left\{\tau_{n-1}^c \leq T\right\}} \\
&= U^c(T) f^c(T),
\end{aligned}
$$

where we used the fact that for $n > \sigma^c(T)$, $\mathbf{1}_{\left\{\tau^c_{n-1} \leq T\right\}} \equiv 0$ $\qquad\qquad\square$

Let us denote $\tau^c = \tau^c_1$.

Remark 2.2. The function $U^c(T)$ is an example of a *renewal function*, a well known object in the *renewal theory*. *Elementary renewal theorem* states that

$$\lim_{T \to +\infty} \frac{U^c(T)}{T} = \frac{1}{\mathbb{E}\tau^c},$$

where in the case $\mathbb{E}\tau^c = +\infty$ we set $1/\mathbb{E}\tau^c = 0$.

Remark 2.3. We have the following estimates which are special cases of results obtained by Erickson in [5]:

$$\frac{t}{m^c(t)} \leq U^c(T) \leq \frac{2t}{m^c(t)},$$

where $m^c(t) = \mathbb{E}\left(\tau^c \wedge t\right) = \int_0^t \mathbb{P}\left(\tau^c > s\right) ds$. This gives in the case $\mathbb{E}\tau^c = +\infty$ the proper order of growth of the renewal function.

Sometimes (and this is often the case when one deals with Lévy processes) it is easier to deal with the Laplace transform of τ^c than with the function $U^c(T)$.

Lemma 2.4. *For the process Y^c defined by equation (1.6) the following inequalities hold*

$$\mathbb{E}\mathrm{TV}(Y^c, [0, T]) \leq 2\frac{f^c(T)}{1 - \mathbb{E}2^{-\tau^c/T}} \tag{2.4}$$

and

$$\mathbb{E}\mathrm{TV}(Y^c, [0, T]) \geq \frac{1}{4}\frac{f^c(T)}{1 - \mathbb{E}2^{-\tau^c/T}}. \tag{2.5}$$

Proof. Both estimates follow from Lemma 2.1 and elementary estimates of $U^c(T)$. The estimate from above follows from the estimate

$$U^c(T) = \sum_{n=1}^{+\infty} \mathbb{E}\mathbf{1}_{\left\{\tau^c_{n-1} \leq T\right\}} \leq \sum_{n=1}^{+\infty} \mathbb{E}2^{1-\tau^c_{n-1}/T} \tag{2.6}$$

which is the consequence of the elementary estimate $\mathbf{1}_{\{x \leq T\}} \leq 2^{1-x/T}$ valid for any $x \in \mathbb{R}$. Further, we have

$$\begin{aligned}
\sum_{n=1}^{+\infty} \mathbb{E}2^{1-\tau^c_{n-1}/T} &= 2\sum_{n=1}^{+\infty} \mathbb{E}2^{-\tau^c_{n-1}/T} \\
&= 2\sum_{n=1}^{+\infty} \left(\mathbb{E}2^{-\tau^c/T}\right)^{n-1} \\
&= \frac{2}{1 - \mathbb{E}2^{-\tau^c/T}}.
\end{aligned} \tag{2.7}$$

From (2.6) and (2.7) we get estimate (2.4).

To bound $\mathbb{E}\mathrm{TV}(Y^c, [0, T])$ from below for $t > 0$ we define $\sigma^{c,0}(t) = 0$ and for $k = 1, 2, \ldots$, such that $\tau^c_{\sigma^{c,k-1}(t)} < +\infty$ let $\sigma^{c,k}(t)$ be the smallest integer such that $\tau^c_{\sigma^{c,k}(t)} - \tau^c_{\sigma^{c,k-1}(t)} > t$ (we naturally have $\sigma^{c,1}(t) = \sigma^c(t)$ and also have $\tau^c_{\sigma^{c,k}(t)-1} - \tau^c_{\sigma^{c,k-1}(t)} \leq t$). For $k = 1, 2, \ldots$, such that $\tau^c_{\sigma^{c,k-1}(t)} = +\infty$ we set $\sigma^{c,k}(t) = \sigma^{c,k-1}(t) + 1$. This yields that $\tau^c_{\sigma^{c,k}(t)} \geq k \cdot t$ and for $k = 0, 1, 2, \ldots$ we have

$$2^{-k}U^c(T) \geq \mathbb{E}\left(2^{-\tau^c_{\sigma^{c,k}(T)}/T} \sum_{n=\sigma^{c,k}(T)+1}^{\sigma^{c,k+1}(T)} 1\right). \tag{2.8}$$

Summing estimates (2.8) over $k = 0, 1, 2, \ldots$ we have

$$\begin{aligned}
2U^c(T) = \sum_{k=0}^{+\infty} 2^{-k}U^c(T) &\geq \sum_{k=0}^{+\infty} \mathbb{E}\left(2^{-\tau^c_{\sigma^{c,k}(T)}/T} \sum_{n=\sigma^{c,k}(T)+1}^{\sigma^{c,k+1}(T)} 1\right) \\
&\geq \sum_{k=0}^{+\infty} \mathbb{E}\left(2^{-\tau^c_{\sigma^{c,k}(T)}/T} \sum_{n=\sigma^{c,k}(T)+1}^{\sigma^{c,k+1}(T)} 2^{-\left(\tau^c_{n-1}/T - \tau^c_{\sigma^{c,k}(T)}/T\right)}\right) \\
&= \sum_{k=0}^{+\infty} \mathbb{E} \sum_{n=\sigma^{c,k}(T)+1}^{\sigma^{c,k+1}(T)} 2^{-\tau^c_{n-1}/T} = \sum_{n=1}^{+\infty} \mathbb{E}2^{-\tau^c_{n-1}/T} \\
&= \sum_{n=1}^{+\infty} \left(\mathbb{E}2^{-\tau^c/T}\right)^{n-1} = \frac{1}{1 - \mathbb{E}2^{-\tau^c/T}}. \tag{2.9}
\end{aligned}$$

Lemma 2.1 and (2.9) yield the estimate from below (2.5). $\qquad\square$

Now, using Lemma 2.4 and estimates (1.4) we obtain the following result.

Theorem 2.5. *Let X_t, $t \geq 0$, be a Lévy process on a separable Banach space V with the norm $|\cdot|$ and let $\mathcal{A}_X$ be the class of processes adapted to the natural filtration of X. Let $\psi : [0, +\infty) \to [0, +\infty)$ be a non-decreasing function such that for $a \geq 0$, $\psi(2a) \leq K_\psi \cdot \psi(a)$. For any $T > 0$ the following estimates hold:*

$$\begin{aligned}
V_X(\psi) = \inf_{c>0}\left\{\psi(c) + \inf_{Y \in \mathcal{A}_X(c)} \mathbb{E}\,TV(Y, [0, T])\right\}. \\
\leq \inf_{c>0}\left\{\psi(c) + 2\frac{\mathbb{E}\left(|X_{\tau^c} - X_0|\,\mathbf{1}_{\{\tau^c \leq T\}}\right)}{1 - \mathbb{E}2^{-\tau^c/T}}\right\} \tag{2.10} \\
\leq 2\inf_{c>0}\left\{\psi(c) + \frac{\mathbb{E}\left(|X_{\tau^c} - X_0|\,\mathbf{1}_{\{\tau^c \leq T\}}\right)}{1 - \mathbb{E}2^{-\tau^c/T}}\right\}
\end{aligned}$$

and

$$\begin{aligned}
V_X(\psi) = \inf_{c>0}\left\{\psi(c) + \inf_{Y \in \mathcal{A}_X(c)} \mathbb{E}\,TV(Y, [0, T])\right\} \\
\geq \frac{1}{\max\left(K_\psi^2, 2\right)} \inf_{c>0}\left\{\psi(c) + \frac{1}{4}\frac{\mathbb{E}\left(|X_{\tau^c} - X_0|\,\mathbf{1}_{\{\tau^c \leq T\}}\right)}{1 - \mathbb{E}2^{-\tau^c/T}}\right\} \tag{2.11}
\end{aligned}$$

$$\geq \frac{1}{4\max\left(K_\psi^2, 2\right)} \inf_{c>0} \left\{\psi\left(c\right) + \frac{\mathbb{E}\left(\left|X_{\tau^c} - X_0\right| \mathbf{1}_{\{\tau^c \leq T\}}\right)}{1 - \mathbb{E}2^{-\tau^c/T}}\right\},$$

where $\tau^c = \inf\left\{t > 0 : |X_t - X_0| \geq c\right\}.$

In what follows we will estimate $\mathbb{E}\left|X_{\tau^c} - X_0\right| \mathbf{1}_{\{\tau^c \leq T\}}$ to obtain the following theorem.

Theorem 2.6. *Let X_t, $t \geq 0$, be a Lévy process on a separable Banach space V with the norm $|\cdot|$ and let $\mathcal{A}_X$ be the class of processes adapted to the natural filtration of X. Let $\psi : [0, +\infty) \to [0, +\infty)$ be a non-decreasing function such that for $a \geq 0$, $\psi\left(2a\right) \leq K_\psi \cdot \psi\left(a\right)$. For any $T, > 0$ the following estimates hold:*

$$V_X\left(\psi\right) = \inf_{c>0}\left\{\psi\left(c\right) + \inf_{Y \in \mathcal{A}_X(c)} \mathbb{E}\, TV(Y, [0, T])\right\} \tag{2.12}$$

$$\leq \inf_{c>0}\left\{\psi\left(c\right) + \frac{6c \cdot \mathbb{P}\left(\tau^c \leq T\right)}{1 - \mathbb{E}2^{-\tau^c/T}} + \frac{4}{\ln 2}T \int_{(2c,+\infty]} y \cdot \Pi\left(\mathrm{d}y\right)\right\}$$

and

$$V_X\left(\psi\right) = \inf_{c>0}\left\{\psi\left(c\right) + \inf_{Y \in \mathcal{A}_X(c)} \mathbb{E}\, TV(Y, [0, T])\right\} \tag{2.13}$$

$$\geq \frac{1}{\max\left(K_\psi^2, 2\right)} \inf_{c>0}\left\{\psi\left(c\right) + \frac{1}{8}\frac{c \cdot \mathbb{P}\left(\tau^c \leq T\right)}{1 - \mathbb{E}2^{-\tau^c/T}} + \frac{1}{32\ln 2}T \int_{(2c,+\infty]} y \cdot \Pi\left(\mathrm{d}y\right)\right\}$$

where $\tau^c = \inf\left\{t > 0 : |X_t - X_0| \geq c\right\}$ and Π is the image of the Lévy measure of the process X under the transformation $x \mapsto |x|$.

Proof. Let $\Delta X_{\tau^c} = X_{\tau^c} - X_{\tau^c-}$ denote the jump of the process at the moment τ^c and let us notice that by the triangle inequality and the definition of τ^c, for $\tau^c < +\infty$ we have

$$|X_{\tau^c} - X_0| \geq |\Delta X_{\tau^c}| - |X_{\tau^c-} - X_0| \geq |\Delta X_{\tau^c}| - c.$$

Thus for $\tau^c < +\infty$ it follows that

$$|\Delta X_{\tau^c}| \leq c + |X_{\tau^c} - X_0| \leq 2\left|X_{\tau^c} - X_0\right|$$

and we have

$$|X_{\tau^c} - X_0| \geq \frac{1}{2}\left|\Delta X_{\tau^c}\right|. \tag{2.14}$$

Let now μ be the joint law of $\left(|\Delta X_{\tau^c}|, \tau^c\right)$. For $y \in (c, +\infty)$ and $t \in (0, +\infty)$ one has

$$\mathrm{d}\mu\left(y, t\right) = \mathbb{P}\left(\sup_{0 \leq s < t} |X_s - X_0| < c\right) \mathrm{d}\Pi\left(y\right) \mathrm{d}t,$$

where $\mathrm{d}t$ denotes the Lebesgue measure. This observation follows from the fact that for $y \in (2c, +\infty)$ and $t \in (0, +\infty)$ the event

$$\left\{|\Delta X_{\tau^c}| \in [y, y + \mathrm{d}y), \tau^c \in [t, t + \mathrm{d}t)\right\}$$

is equal the intersection of two independent events

$$\left\{ \sup_{0 \leq s < t} |X_s - X_0| < c \right\} \text{ and } \{|X_{t+\mathrm{d}t} - X_{t-}| \in [y, y + \mathrm{d}y)\}$$

which follows from (2.14) and the Lévy-Ito decomposition (see [1]). Now, using (2.14) we easily estimate

$$
\begin{aligned}
\mathbb{E} |X_{\tau^c} - X_0| \, \mathbf{1}_{\{\tau^c \leq T\}} &\geq \mathbb{E} |X_{\tau^c} - X_0| \, \mathbf{1}_{\{|\Delta X_{\tau^c}| > 2c\}} \mathbf{1}_{\{\tau^c \leq T\}} \\
&\geq \frac{1}{2} \mathbb{E} |\Delta X_{\tau^c}| \, \mathbf{1}_{\{|\Delta X_{\tau^c}| > 2c\}} \mathbf{1}_{\{\tau^c \leq T\}} \\
&= \frac{1}{2} \int_{(2c,+\infty) \times (0,T]} y \cdot \mathrm{d}\mu(y,t) \\
&= \frac{1}{2} \int_{(2c,+\infty) \times (0,T]} y \cdot \mathbb{P} \left(\sup_{0 \leq s < t} |X_s - X_0| < c \right) \mathrm{d}\Pi(y) \, \mathrm{d}t \\
&= \frac{1}{2} \int_{(2c,+\infty)} y \cdot \Pi(\mathrm{d}y) \int_{(0,T]} \mathbb{P} \left(\sup_{0 \leq s < t} |X_s - X_0| < c \right) \mathrm{d}t \\
&= \frac{1}{2} \int_{(2c,+\infty)} y \cdot \Pi(\mathrm{d}y) \int_{(0,T]} \mathbb{P}(\tau^c \geq t) \, \mathrm{d}t.
\end{aligned} \tag{2.15}
$$

We naturally also have

$$\mathbb{E} |X_{\tau^c} - X_0| \, \mathbf{1}_{\{\tau^c \leq T\}} \geq c \cdot \mathbb{P}(\tau^c \leq T). \tag{2.16}$$

From (2.15) and (2.16) we get

$$\mathbb{E} |X_{\tau^c} - X_0| \, \mathbf{1}_{\{\tau^c \leq T\}} \geq \frac{1}{2} c \cdot \mathbb{P}(\tau^c \leq T) + \frac{1}{4} \int_{(2c,+\infty)} y \cdot \Pi(\mathrm{d}y) \int_{(0,T]} \mathbb{P}(\tau^c \geq t) \, \mathrm{d}t. \tag{2.17}$$

On the other hand, by the definition of τ^c, for $\tau^c < +\infty$ we have

$$
\begin{aligned}
|X_{\tau^c} - X_0| &\leq |X_{\tau^c-} - X_0| + |\Delta X_{\tau^c}| \\
&\leq c + |\Delta X_{\tau^c}|
\end{aligned}
$$

from which we get the estimate

$$
\begin{aligned}
\mathbb{E} |X_{\tau^c} - X_0| \, \mathbf{1}_{\{\tau^c \leq T\}} &\leq c \cdot \mathbb{E} \mathbf{1}_{\{\tau^c \leq T\}} + \mathbb{E} |\Delta X_{\tau^c}| \, \mathbf{1}_{\{\tau^c \leq T\}} \\
&= c \cdot \mathbb{P}(\tau^c \leq T) + \mathbb{E} |\Delta X_{\tau^c}| \, \mathbf{1}_{\{|\Delta X_{\tau^c}| \leq 2c\}} \mathbf{1}_{\{\tau^c \leq T\}} \\
&\quad + \mathbb{E} |\Delta X_{\tau^c}| \, \mathbf{1}_{\{|\Delta X_{\tau^c}| > 2c\}} \mathbf{1}_{\{\tau^c \leq T\}} \\
&\leq 3c \cdot \mathbb{P}(\tau^c \leq T) + \int_{(2c,+\infty)} y \cdot \Pi(\mathrm{d}y) \int_{(0,T]} \mathbb{P}(\tau^c \geq t) \, \mathrm{d}t.
\end{aligned} \tag{2.18}
$$

To deal with the integral $\int_{(0,T]} \mathbb{P}(\tau^c \geq t) \, \mathrm{d}t$ let us notice that the following estimates hold:

$$\int_{(0,T]} \mathbb{P}(\tau^c \geq t) \, \mathrm{d}t \leq 2 \int_0^{+\infty} 2^{-t/T} \mathbb{P}(\tau^c \geq t) \, \mathrm{d}t$$

and

$$
\begin{aligned}
\int_0^{+\infty} 2^{-t/T} \mathbb{P}\left(\tau^c \geq t\right) \mathrm{d}t &= \sum_{k=1}^{+\infty} \int_{(k-1)T}^{kT} 2^{-t/T} \mathbb{P}\left(\tau^c \geq t\right) \mathrm{d}t \\
&\leq \sum_{k=1}^{+\infty} \int_{(k-1)T}^{kT} 2^{-(k-1)T/T} \mathbb{P}\left(\tau^c \geq t - (k-1)T\right) \mathrm{d}t \\
&= \sum_{k=1}^{+\infty} 2^{-(k-1)} \int_0^T \mathbb{P}\left(\tau^c \geq t\right) \mathrm{d}t \\
&= 2 \int_{(0,T]} \mathbb{P}\left(\tau^c \geq t\right) \mathrm{d}t.
\end{aligned}
$$

Thus, we have the double-sided estimate

$$
\begin{aligned}
\frac{1}{2} \int_0^{+\infty} 2^{-t/T} \mathbb{P}\left(\tau^c \geq t\right) \mathrm{d}t &\leq \int_{(0,T]} \mathbb{P}\left(\tau^c \geq t\right) \mathrm{d}t \\
&\leq 2 \int_0^{+\infty} 2^{-t/T} \mathbb{P}\left(\tau^c \geq t\right) \mathrm{d}t.
\end{aligned} \tag{2.19}
$$

Finally, let us notice that (by integration by parts)

$$
\begin{aligned}
\int_0^{+\infty} 2^{-t/T} \mathbb{P}\left(\tau^c \geq t\right) \mathrm{d}t &= \frac{T}{\ln 2} - \frac{T}{\ln 2} \int_0^{+\infty} 2^{-t/T} \mathbb{P}\left(\tau^c \in \mathrm{d}t\right) \\
&= \frac{T}{\ln 2} \left(1 - \mathbb{E} 2^{-\tau^c/T}\right). \tag{2.20}
\end{aligned}
$$

Now, from (2.10), (2.18), (2.19) and (2.20) we get (2.12) while from (2.11), (2.17), (2.19) and (2.20) we get (2.13). □

3. EXAMPLES

In this section we will apply the obtained estimates in three special cases. In the first case the process X will be a real-valued Brownian motion with drift, in the second case it will be a standard Brownian motion on $\mathbb{R}^d$, $d = 2, 3, \ldots$, and in the third case it will be a real valued, symmetric α-stable process with $\alpha \in (1, 2)$.

3.1. Estimates of $V_X(\psi)$ in the case when X is a Brownian motion with drift

Let now B be a standard Brownian motion starting from 0 and $X_t = B_t + \mu t$ be a (real-valued) Brownian motion with drift μ. From Theorem 2.5 it follows that in order to estimate $V_X(\psi)$ it is sufficient to estimate (up to universal constants) the quantity $\mathbb{E}\left(|X_{\tau^c}| \mathbf{1}_{\{\tau^c \leq T\}}\right) / \left(1 - \mathbb{E} 2^{-\tau^c/T}\right)$. From the continuity of Brownian paths we immediately get that $|X_{\tau^c}| = c$ and

$$
\mathbb{E}\left(|X_{\tau^c}| \mathbf{1}_{\{\tau^c \leq T\}}\right) = c \cdot \mathbb{P}\left(\tau^c \leq T\right) = c \cdot \mathbb{P}\left(\sup_{0 \leq t \leq T} |B_t + \mu t| \geq c\right).
$$

Now let us consider two cases.

Case 1. $c \leq \sqrt{T} + |\mu| T$. Let $\tilde{B} = \text{sign}(\mu)B$, where $\text{sign}(\mu) = -1$ if $\mu < 0$, $\text{sign}(\mu) = 1$ if $\mu \geq 0$. We get

$$
\mathbb{P}\left(\sup_{0 \leq t \leq T} |B_t + \mu t| \geq c\right) \geq \mathbb{P}\left(|B_T + \mu T| \geq \sqrt{T} + |\mu| T\right)
$$
$$
= \mathbb{P}\left(|\text{sign}(\mu)B_T + \text{sign}(\mu)\mu T| \geq \sqrt{T} + |\mu| T\right)
$$
$$
\geq \mathbb{P}\left(\text{sign}(\mu)B_T + |\mu| T \geq \sqrt{T} + |\mu| T\right)
$$
$$
= \mathbb{P}\left(\tilde{B}_T \geq \sqrt{T}\right) \geq 1 - \Phi(1) > \frac{1}{7},
$$

where $\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-t^2/2} \mathrm{d}t$ is the cumulative probability function of a standard normal variable.

Case 2. $c > \sqrt{T} + |\mu| T$. In this case we get the following lower bound

$$
\mathbb{P}\left(\sup_{0 \leq t \leq T} |B_t + \mu t| \geq c\right) \geq \mathbb{P}(|B_T + \mu T| \geq c)
$$
$$
= \mathbb{P}(|\text{sign}(\mu)B_T + \text{sign}(\mu)\mu T| \geq c)
$$
$$
\geq \mathbb{P}(\text{sign}(\mu)B_T + |\mu| T \geq c)
$$
$$
= \mathbb{P}\left(\tilde{B}_T \geq c - |\mu| T\right) = 1 - \Phi\left(\frac{c - |\mu| T}{\sqrt{T}}\right).
$$

On the other hand, in both cases we have

$$
\mathbb{P}\left(\sup_{0 \leq t \leq T} |B_t + \mu t| \geq c\right) \leq \mathbb{P}\left(\sup_{0 \leq t \leq T} |B_s| + |\mu| T \geq c\right)
$$
$$
\leq 2 \cdot \mathbb{P}\left(\sup_{0 \leq t \leq T} B_t \geq c - |\mu| T\right)
$$
$$
= 4 \cdot \mathbb{P}(B_T \geq c - |\mu| T)
$$
$$
= 4\left(1 - \Phi\left(\frac{c - |\mu| T}{\sqrt{T}}\right)\right).
$$

Thus, in both cases

$$
\mathbb{P}\left(\sup_{0 \leq t \leq T} |B_t + \mu t| \geq c\right) = \kappa_1 \left(1 - \Phi\left(\frac{c - |\mu| T}{\sqrt{T}}\right)\right),
$$

where $\kappa_1 \in \left[\frac{1}{7}, 4\right]$.

Now we turn to look at $1 - \mathbb{E}2^{-\tau^c/T} = \mathbb{E}\left(1 - e^{-\ln 2 \cdot \tau^c/T}\right)$. Let τ be an exponentially distributed random variable, independent from B, with the cumulative probability function $\mathbb{P}(\tau < t) = \left(1 - e^{-\ln 2 \cdot t/T}\right) \mathbf{1}_{\{t>0\}}$. By formula 1.15.2 on p. 270 in [3] we get

$$
\mathbb{E}\left(1 - 2^{-\tau^c/T}\right) = \mathbb{P}(\tau < \tau^c)
$$
$$
= \mathbb{P}\left(\inf_{0 \leq s \leq \tau}(B_s + \mu s) > -c, \sup_{0 \leq s \leq \tau}(B_s + \mu s) < c\right)
$$

$$= 1 - \frac{(e^{-\mu c} + e^{\mu c}) \sinh\left(c\sqrt{\frac{2\ln 2}{T} + \mu^2}\right)}{\sinh\left(2c\sqrt{\frac{2\ln 2}{T} + \mu^2}\right)}$$

$$= 1 - \frac{\cosh(\mu c)}{\cosh\left(c\sqrt{\frac{2\ln 2}{T} + \mu^2}\right)}.$$

Finally, from Theorem 2.5 we get that

$$V_X(\psi) = \kappa_2 \inf_{c>0} \left\{ \psi(c) + c\frac{1 - \Phi\left(\frac{c - |\mu|T}{\sqrt{T}}\right)}{1 - \frac{\cosh(\mu c)}{\cosh\left(c\sqrt{\frac{2\ln 2}{T} + \mu^2}\right)}} \right\},$$

where $\kappa_2 \in \left[\frac{1}{28 \max\left(K_\psi^2, 2\right)}, 8\right]$.

3.2. Estimates of $V_X(\psi)$ in the case when X is a d-dimensional standard Brownian motion ($d \geq 2$)

Let $B^{(1)}, B^{(2)}, \ldots, B^{(d)}$ be d independent ($d \geq 2$), standard, one-dimensional Brownian motions, starting from 0, and let $X = \left(B^{(1)}, B^{(2)}, \ldots, B^{(d)}\right)$.

Again, by Theorem 2.5 it is sufficient to estimate the ratio

$$\frac{\mathbb{E}\left(|X_{\tau^c}| \mathbf{1}_{\{\tau^c \leq T\}}\right)}{1 - \mathbb{E}2^{-\tau^c/T}}$$

and again, by the continuity of X, we get

$$\mathbb{E}\left(|X_{\tau^c} - X_0| \mathbf{1}_{\{\tau^c \leq T\}}\right) = c \cdot \mathbb{P}\left(\tau^c \leq T\right).$$

Moreover, recall that the process R defined by

$$R = \sqrt{\left(B^{(1)}\right)^2 + \left(B^{(2)}\right)^2 + \ldots + \left(B^{(d)}\right)^2}$$

is called d-dimensional Bessel process or a Bessel process of order d or a Bessel process with index $\nu = d/2 - 1$.

Using results of Serafin [12], we will obtain estimates of $V_X(\psi)$ which are universal up to a constant depending on ψ and ν (but not T). Using Corollary 3.4 of [12] and scaling properties of the standard Brownian motion for $t > 0$ we get

$$\mathbb{P}\left(\tau^c \in dt\right) = \kappa(t, \nu) \frac{1}{c^2} \left(1 + \frac{c^2}{t}\right)^{\nu+2} \exp\left(-\frac{c^2}{2t} - j_{\nu,1}^2 \frac{t}{2c^2}\right) dt,$$

where $\kappa(t, \nu) \in [\kappa_3(\nu), \kappa_4(\nu)]$ and $\kappa_4(\nu) > \kappa_3(\nu) > 0$ are constants depending on ν only, and $j_{\nu,1}$ denotes the smallest positive zero of the the Bessel function J_ν of the first kind which is defined as

$$J_\nu(y) = \left(\frac{y}{2}\right)^\nu \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!\Gamma(m + \nu + 1)} \left(\frac{y}{2}\right)^{2m}.$$

This gives the following estimates.

Case 1. $c^2 \leq T$. In this case

$$
\mathbb{P}\left(\tau^c \leq T\right) = \int_0^T \mathbb{P}\left(\tau^c \in \mathrm{d}t\right) \geq \int_{2c^2}^{4c^2} \mathbb{P}\left(\tau^c \in \mathrm{d}t\right)
$$

$$
\geq \kappa_3(\nu)\frac{1}{c^2}\int_{2c^2}^{4c^2}\left(1+\frac{c^2}{t}\right)^{\nu+2}\exp\left(-\frac{c^2}{2t}-j_{\nu,1}^2\frac{t}{2c^2}\right)\mathrm{d}t
$$

$$
\geq \kappa_3(\nu)\frac{1}{c^2}\int_{2c^2}^{4c^2}\exp\left(-\frac{1}{4}-2j_{\nu,1}^2\right)\mathrm{d}t
$$

$$
= \kappa_5(\nu).
$$

For the bound from above we use trivial bound

$$
\mathbb{P}\left(\tau^c \leq T\right) \leq 1.
$$

Case 2. $c^2 > T$. In this case

$$
\mathbb{P}\left(\tau^c \leq T\right) = \int_0^T \mathbb{P}\left(\tau^c \in \mathrm{d}t\right)
$$

$$
\geq \kappa_3(\nu)\frac{1}{c^2}\int_0^T\left(1+\frac{c^2}{t}\right)^{\nu+2}\exp\left(-\frac{c^2}{2t}-j_{\nu,1}^2\frac{t}{2c^2}\right)\mathrm{d}t
$$

$$
\geq \kappa_3(\nu)\frac{1}{c^2}\int_0^T\left(\frac{c^2}{t}\right)^{\nu+2}\exp\left(-\frac{c^2}{2t}-\frac{j_{\nu,1}^2}{2}\right)\mathrm{d}t
$$

$$
\geq \kappa_6(\nu)\frac{1}{c^2}\int_0^T\left(\frac{c^2}{t}\right)^{\nu+2}\exp\left(-\frac{c^2}{2t}\right)\mathrm{d}t
$$

$$
= \kappa_7(\nu)\Gamma\left(\nu+1,\frac{c^2}{2T}\right),
$$

where for $a, y > 0$, $\Gamma\left(a, y\right)$ denotes the incomplete gamma function,

$$
\Gamma\left(a, y\right) = \int_y^{+\infty} x^{a-1}e^{-x}\mathrm{d}x.
$$

Similarly we can obtain a bound from above:

$$
\mathbb{P}\left(\tau^c \leq T\right) = \int_0^T \mathbb{P}\left(\tau^c \in \mathrm{d}t\right)
$$

$$
\leq \kappa_4(\nu)\frac{1}{c^2}\int_0^T\left(1+\frac{c^2}{t}\right)^{\nu+2}\exp\left(-\frac{c^2}{2t}-j_{\nu,1}^2\frac{t}{2c^2}\right)\mathrm{d}t
$$

$$
\leq \kappa_4(\nu)\frac{1}{c^2}\int_0^T\left(\frac{c^2}{t}+\frac{c^2}{t}\right)^{\nu+2}\exp\left(-\frac{c^2}{2t}\right)\mathrm{d}t
$$

$$
\leq \kappa_8(\nu)\frac{1}{c^2}\int_0^T\left(\frac{c^2}{t}\right)^{\nu+2}\exp\left(-\frac{c^2}{2t}\right)\mathrm{d}t
$$

$$= \kappa_9(\nu)\Gamma\left(\nu+1, \frac{c^2}{2T}\right).$$

We notice that in both $(c^2 \le T$ and $c^2 > T)$ cases

$$\mathbb{P}\left(\tau^c \le T\right) = \kappa_{10}\Gamma\left(\nu+1, \frac{1}{2}\max\left(1, \frac{c^2}{T}\right)\right), \tag{3.1}$$

where $\kappa_{10} \in [\kappa_{11}(\nu), \kappa_{12}(\nu)]$ and $\kappa_{12}(\nu) > \kappa_{11}(\nu) > 0$ are constants depending on ν only (for $\kappa_{11}(\nu)$ we may take $\min\{\kappa_7(\nu), \kappa_5(\nu)/\Gamma(\nu+1, 1/2)\}$ while for $\kappa_{12}(\nu)$ we may take $\max\{\kappa_9(\nu), 1/\Gamma(\nu+1, 1/2)\}$).

Next, the term $1 - \mathbb{E}2^{-\tau^c/T} = 1 - \mathbb{E}\exp\left(-(\ln 2/T)\cdot \tau^c\right)$ is given by an explicit formula. For a Bessel process with index ν, starting from x and $\lambda \ge 0$:

$$1 - \mathbb{E}^x\exp\left(-\lambda\cdot\tau^c\right) = 1 - c^\nu\frac{x^{-\nu}I_\nu\left(x\sqrt{2\lambda}\right)}{I_\nu\left(c\sqrt{2\lambda}\right)}$$

(see [3], formula 1.1.2 on p. 373). Here I_ν denotes the modified Bessel function

$$I_\nu(y) = \left(\frac{y}{2}\right)^\nu \sum_{m=0}^{+\infty}\frac{1}{m!\Gamma(m+\nu+1)}\left(\frac{y}{2}\right)^{2m}$$

and in our case (for $x=0$) we get $x^{-\nu}I_\nu\left(x\sqrt{2\lambda}\right) = (\lambda/2)^{\nu/2}/\Gamma(\nu+1)$, hence, substituting $\lambda = \ln 2/T$,

$$1 - \mathbb{E}2^{-\tau^c/T} = 1 - \frac{\left(c^2\ln 2/(2T)\right)^{\nu/2}}{\Gamma(\nu+1)I_\nu\left(c\sqrt{2\ln 2/T}\right)}. \tag{3.2}$$

Equations (3.1) and (3.2) together with Theorem 2.5 allow to estimate $V_X(\psi)$ up to a constants depending on ν and ψ only:

$$V_X(\psi) \le 2\inf_{c>0}\left\{\psi(c) + c\frac{\kappa_{12}(\nu)\Gamma\left(\nu+1, \frac{1}{2}\max\left(1, \frac{c^2}{T}\right)\right)}{1 - \frac{(c^2\ln 2/(2T))^{\nu/2}}{\Gamma(\nu+1)I_\nu\left(c\sqrt{2\ln 2/T}\right)}}\right\}$$

and

$$V_X(\psi) \ge \frac{1}{4\max\left(K_\psi^2, 2\right)}\inf_{c>0}\left\{\psi(c) + c\frac{\kappa_{11}(\nu)\Gamma\left(\nu+1, \frac{1}{2}\max\left(1, \frac{c^2}{T}\right)\right)}{1 - \frac{(c^2\ln 2/(2T))^{\nu/2}}{\Gamma(\nu+1)I_\nu\left(c\sqrt{2\ln 2/T}\right)}}\right\}.$$

Remark 3.1. By formula 1.1.4 on p. 373 in [3], which is due to Kent [8], we get exact formula for $\mathbb{P}\left(\tau^c \le T\right)$:

$$\mathbb{P}\left(\tau^c \le T\right) = 1 - \frac{2^{1-\nu}}{\Gamma(\nu+1)}\sum_{k=1}^{+\infty}\frac{j_{\nu,k}^{\nu-1}}{J_{\nu+1}\left(j_{\nu,k}\right)}e^{-2j_{\nu,k}^2T/c^2}, \tag{3.3}$$

where $j_{\nu,1} < j_{\nu,2} < \dots$ denote consecutive positive zeros of J_ν. Unfortunately, the series in formula (3.3) is convenient when dealing with larger times, and for $T \geq 2\mathbb{E}\tau^c = 2c^2/d$ we naturally have $\mathbb{P}(\tau^c \leq T) \geq 1/2$. Unfortunately, for smaller Ts this sum is oscillating and in that case the cancellations between the terms really matter in the context of asymptotic behaviour.

3.3. Estimates of $V_X(\psi)$ in the case when X is a symmetric, real-valued, strictly α-stable motion ($\alpha \in (1,2)$)

Let now X_t, $t \geq 0$, be a symmetric, real-valued, strictly α-stable motion ($\alpha \in (1,2)$) such that $X_0 = 0$. X has the following scaling property: for $t \geq 0$ and $a > 0$

$$X_{at} \overset{law}{=} a^{1/\alpha} X_t. \tag{3.4}$$

To fix our attention to the process of a given magnitude, we will assume that X_1 has the following characteristic function

$$\mathbb{E}\exp\left(i\xi X_1\right) = \exp\left(\int_{-\infty}^{\infty} e^{i\xi x} - 1 - i\xi x \frac{\mathrm{d}x}{|x|^{\alpha+1}}\right) = e^{-\sigma_\alpha |\xi|^\alpha},$$

where $\xi \in \mathbb{R}$, $\sigma_\alpha = 2\Gamma(-\alpha)\cos\frac{(2-\alpha)\pi}{2}$. Let β_α be such that

$$\mathbb{P}\left(|X_1| \geq \beta_\alpha\right) = \frac{1}{3e^5}. \tag{3.5}$$

To estimate $V_X(\psi)$ we will apply Theorem 2.6.

First we need to estimate $1 - \mathbb{E}2^{-\tau^c/T} = \frac{\ln 2}{T}\int_0^{+\infty} 2^{-t/T}\mathbb{P}(\tau^c \geq t)\,\mathrm{d}t$. Let us define the function $(0, +\infty) \ni c \mapsto u(c) \in (0, +\infty)$ such that

$$\mathbb{P}\left(\left|X_{u(c)}\right| \geq c\right) = \frac{1}{3e^5}.$$

By scaling property of X it is equivalent to

$$\mathbb{P}\left(|X_1| \geq (u(c))^{-1/\alpha} c\right) = \frac{1}{3e^5} = \mathbb{P}\left(|X_1| \geq \beta_\alpha\right).$$

Thus

$$u(c) = \frac{c^\alpha}{(\beta_\alpha)^\alpha}. \tag{3.6}$$

Next, by symmetry and strong Markov property of X, we have the estimate

$$\begin{aligned}
\frac{\mathbb{P}\left(\left|X_{u(c)}\right| \geq c\right)}{\mathbb{P}\left(\tau^c \leq u(c)\right)} &= \mathbb{P}\left(\left|X_{u(c)}\right| \geq c \,\middle|\, \tau^c \leq u(c)\right) \\
&\geq \mathbb{P}\left(\mathrm{sign}\left(X_{\tau^c}\right)\left(X_{u(c)} - X_{\tau^c}\right) \geq 0 \,\middle|\, \tau^c \leq u(c)\right) \\
&\geq \frac{1}{2}
\end{aligned} \tag{3.7}$$

(recall that $|X_{\tau^c}| \geq c$) from which it follows $\mathbb{P}(\tau^c \leq u(c)) \leq 2\mathbb{P}(|X_{u(c)}| \geq c)$ and

$$\mathbb{P}(\tau^c > u(c)) = 1 - \mathbb{P}(\tau^c \leq u(c))$$

$$\geq 1 - 2 \cdot \mathbb{P}(|X_{u(c)}| \geq c) \geq 1 - 2\frac{1}{3e^5} > \frac{1}{2}. \tag{3.8}$$

On the other hand, using the independence of the increments and scaling properties of X, for $k = 1, 2, \ldots$, we estimate

$$\mathbb{P}(\tau^c > 2^\alpha k \cdot u(c)) = \mathbb{P}\left(\sup_{0 \leq s \leq 2^\alpha k \cdot u(c)} |X_s| < c\right)$$

$$\leq \mathbb{P}\left(\sup_{0 \leq s < t \leq 2^\alpha k \cdot u(c)} |X_s - X_t| < 2c\right)$$

$$\leq \mathbb{P}\left(\sup_{2^\alpha(j-1)\cdot u(c) \leq s < t \leq 2^\alpha j \cdot u(c)} |X_s - X_t| < 2c \text{ for } j = 1, 2, \ldots, k\right)$$

$$= \prod_{j=1}^k \mathbb{P}\left(\sup_{2^\alpha(j-1)\cdot u(c) \leq s < t \leq 2^\alpha j \cdot u(c)} |X_s - X_t| < 2c\right)$$

$$= \left(\mathbb{P}\left(\sup_{0 \leq s < t \leq 2^\alpha u(c)} |X_s - X_t| < 2c\right)\right)^k$$

$$\leq \left(\mathbb{P}\left(\sup_{0 \leq s \leq 2^\alpha u(c)} |X_s| < 2c\right)\right)^k = \left(\mathbb{P}\left(\sup_{0 \leq s \leq u(c)} 2|X_s| < 2c\right)\right)^k$$

$$= \left(1 - \mathbb{P}\left(\sup_{0 \leq s \leq u(c)} |X_s| \geq c\right)\right)^k \leq (1 - \mathbb{P}(|X_{u(c)}| \geq c))^k$$

$$= \left(1 - \frac{1}{3e^5}\right)^k. \tag{3.9}$$

From (3.8) we estimate

$$1 - \mathbb{E}2^{-\tau^c/T} = \frac{\ln 2}{T}\int_0^{+\infty} 2^{-t/T}\mathbb{P}(\tau^c \geq t)\,\mathrm{d}t$$

$$\geq \frac{\ln 2}{T}\int_0^{u(c)} 2^{-t/T}\frac{1}{2}\mathrm{d}t$$

$$= \frac{1}{2}\left(1 - 2^{-u(c)/T}\right) \tag{3.10}$$

and from (3.9) we estimate

$$1 - \mathbb{E}2^{-\tau^c/T} = \frac{\ln 2}{T}\int_0^{+\infty} 2^{-t/T}\mathbb{P}(\tau^c \geq t)\,\mathrm{d}t$$

$$= \frac{\ln 2}{T}\sum_{k=1}^{+\infty}\int_{2^\alpha(k-1)\cdot u(c)}^{2^\alpha k \cdot u(c)} 2^{-t/T}\mathbb{P}(\tau^c \geq t)\,\mathrm{d}t$$

$$\leq \frac{\ln 2}{T} \sum_{k=1}^{+\infty} \int_{2^\alpha (k-1) \cdot u(c)}^{2^\alpha k \cdot u(c)} 2^{-t/T} \left(1 - \frac{1}{3e^5}\right)^{k-1} \mathrm{d}t$$

$$= \frac{1}{1 - \left(1 - \frac{1}{3e^5}\right) 2^{-2^\alpha u(c)}} \left(1 - e^{-2^\alpha u(c)/T}\right)$$

$$\leq 3e^5 \cdot 2^\alpha \left(1 - 2^{-u(c)/T}\right) \leq 12e^5 \left(1 - 2^{-u(c)/T}\right). \tag{3.11}$$

The last but one inequality follows from the estimates: $1 - \left(1 - \frac{1}{3e^5}\right) 2^{-2^\alpha u(c)} \geq \frac{1}{3e^5}$ and $1 - 2^{-2^\alpha u(c)/T} \leq 2^\alpha \left(1 - 2^{-u(c)/T}\right)$ which is the consequence of the concavity of the function $x \mapsto 1 - 2^{-x}$:

$$\frac{1}{2^\alpha} \left(1 - 2^{-2^\alpha u(c)/T}\right) + \left(1 - \frac{1}{2^\alpha}\right)\left(1 - 2^{-0}\right) \leq 1 - 2^{-\frac{1}{2^\alpha} 2^\alpha u(c)/T}.$$

Next, we need to estimate $\mathbb{P}\left(\tau^c \leq T\right) = \mathbb{P}\left(\sup_{0 \leq s \leq T} |X_s| \geq c\right)$. Using similar reasoning as in (3.7) we get:

$$\mathbb{P}\left(|X_T| \geq c\right) \leq \mathbb{P}\left(\tau^c \leq T\right) \leq 2\mathbb{P}\left(|X_T| \geq c\right).$$

This and scaling properties of X yield:

$$\mathbb{P}\left(|X_1| \geq \frac{c}{T^{1/\alpha}}\right) \leq \mathbb{P}\left(\tau^c \leq T\right) \leq 2 \cdot \mathbb{P}\left(|X_1| \geq \frac{c}{T^{1/\alpha}}\right). \tag{3.12}$$

Finally, using Theorem 2.6, (3.12), (3.10), (3.6) and the fact that $\Pi\left(\mathrm{d}y\right) = |y|^{-\alpha-1}\,\mathrm{d}y$, we obtain estimate from above:

$$V_X\left(\psi\right) \leq \inf_{c>0} \left\{\psi\left(c\right) + 24 \frac{c \cdot \mathbb{P}\left(|X_1| \geq \frac{c}{T^{1/\alpha}}\right)}{1 - 2^{-c^\alpha/((\beta_\alpha)^\alpha T)}} + \frac{4}{\ln 2} \frac{T}{(\alpha - 1)(2c)^{\alpha-1}}\right\}. \tag{3.13}$$

Similarly, using Theorem 2.6, (3.12), (3.11), (3.6) and the fact that $\Pi\left(\mathrm{d}y\right) = |y|^{-\alpha-1}\,\mathrm{d}y$, we obtain estimate from below:

$$V_X\left(\psi\right) \geq \tag{3.14}$$

$$\frac{1}{\max\left(K_\psi^2, 2\right)} \inf_{c>0} \left\{\psi\left(c\right) + \frac{1}{96e^5} \frac{c \cdot \mathbb{P}\left(|X_1| \geq \frac{c}{T^{1/\alpha}}\right)}{1 - 2^{-c^\alpha/((\beta_\alpha)^\alpha T)}} + \frac{1}{32 \ln 2} \frac{T}{(\alpha - 1)(2c)^{\alpha-1}}\right\}.$$

From (4.32), (4.33) of [2] it follows that $\beta_\alpha \geq \frac{\tilde{d}}{\sqrt{2-\alpha}}$ and $\beta_\alpha \leq \frac{\tilde{D}}{\sqrt{2-\alpha}}$ for some universal constants $0 < \tilde{d} \leq \tilde{D}$. From this and concavity of the function $x \mapsto 1 - 2^{-x}$ we obtain that

$$d\left(1 - 2^{-c^\alpha/\left(\left(\frac{1}{\sqrt{2-\alpha}}\right)^\alpha T\right)}\right) \leq 1 - 2^{-c^\alpha/((\beta_\alpha)^\alpha T)} \leq D\left(1 - 2^{-c^\alpha/\left(\left(\frac{1}{\sqrt{2-\alpha}}\right)^\alpha T\right)}\right) \tag{3.15}$$

for some universal constants $0 < d \leq D$. Moreover, from Theorem 4.12 in [2] it also follows that

$$\mathbb{P}\left(|X_1| \geq y\right) \leq C \min\left\{1, \max\left\{y^{-\alpha}, e^{-(2-\alpha)(y/4)^2}\right\}\right\} \tag{3.16}$$

and

$$\mathbb{P}\left(|X_1| \geq y\right) \geq c \min \left\{1, \max \left\{y^{-\alpha}, e^{-(2-\alpha)\left(\sqrt{8}y\right)^2}\right\}\right\} \tag{3.17}$$

for some universal constants $0 < c \leq C$. Thus, from (3.13)–(3.17) we get that for some positive constant L_ψ, depending on ψ only, we have

$$V_X\left(\psi\right) \leq L_\psi \inf_{c>0} \left\{\psi\left(c\right) + \frac{c \cdot F\left(c \cdot T^{-1/\alpha}\right)}{1 - 2^{-c^\alpha (2-\alpha)^{\alpha/2}/T}} + \frac{T}{(\alpha - 1) c^{\alpha-1}}\right\} \tag{3.18}$$

and

$$V_X\left(\psi\right) \geq \frac{1}{L_\psi} \inf_{c>0} \left\{\psi\left(c\right) + \frac{c \cdot F\left(c \cdot T^{-1/\alpha}\right)}{1 - 2^{-c^\alpha (2-\alpha)^{\alpha/2}/T}} + \frac{T}{(\alpha - 1) c^{\alpha-1}}\right\}, \tag{3.19}$$

where

$$F(y) = \min \left\{1, \max \left\{y^{-\alpha}, e^{-(2-\alpha)y^2}\right\}\right\}.$$

References

[1] D. Applebaum, Lévy processes ans stochastic integrals in Banach spaces. *Probab. Math. Statist.* **27** (2007) 75–88.

[2] W.M. Bednorz, R.M. Łochowski and R. Martynek, On tails of symmetric and totally asymmetric α-stable distributions. *Prob. Math. Statist.* **41** (2021) 321–345.

[3] A.N. Borodin and P. Salminen, *Handbook of Brownian Motion – Facts and Formulae. Second Edition.* Birkhäuser (2002).

[4] T.F. Chan and S. Esedoglu, Aspects of total variation regularized L^1 function approximation. *SIAM J. Appl. Math.* **65** (2005) 1817–837.

[5] K.B. Erickson, The strong law of large numbers when the mean is undefined. *Trans. Amer. Math. Soc.* **185** (1973) 371–381.

[6] J. Kallsen and J. Muhle-Karbe, Option pricing and hedging with small transaction costs. *Math. Finance* **25** (2015) 702–723.

[7] J. Kallsen and J. Muhle-Karbe, The general structure of optimal investment and consumption with small transaction costs. *Math. Finance* **27** (2017) 659–703.

[8] J.T. Kent, Eigenvalue expansions for diffusion hitting times. *Z. Wahrscheinlichkeitstheor. Verw. Geb.* **52** (1980) 309–319.

[9] R.M. Łochowski, A new inequality for the Riemann-Stieltjes integrals driven by irregular signals in Banach spaces. *J. Inequal. Appl.* **2018(20)** (2018) 1–20.

[10] R.M. Łochowski and R. Ghomrasni, The play operator, the truncated variation and the generalisation of the Jordan decomposition. *Math. Methods Appl. Sci.* **38** (2015) 403–419.

[11] K. Sato, Lévy processes and infinitely divisible distributions, volume 68 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press (1999).

[12] G. Serafin, Exit times densities of the Bessel process. *Proc. Am. Math. Soc.* **145** (2017) 3165–3178.