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PROBABILISTIC REPRESENTATION OF INTEGRATION BY PARTS FORMULAE FOR SOME STOCHASTIC VOLATILITY MODELS WITH UNBOUNDED DRIFT

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Abstract. In this paper, we establish a probabilistic representation as well as some integration by parts formulae for the marginal law at a given time maturity of some stochastic volatility model with unbounded drift. Relying on a perturbation technique for Markov semigroups, our formulae are based on a simple Markov chain evolving on a random time grid for which we develop a tailor-made Malliavin calculus. Among other applications, an unbiased Monte Carlo path simulation method stems from our formulas so that it can be used in order to numerically compute with optimal complexity option prices as well as their sensitivities with respect to the initial values or *Greeks* in finance, namely the *Delta* and *Vega*, for a large class of non-smooth European payoff. Numerical results are proposed to illustrate the efficiency of the method.

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1. Introduction

In this work, we consider a two dimensional stochastic volatility model given by the solution of the following stochastic differential equation (SDE for short) with dynamics

$$\begin{cases} S_t &= s_0 + \int_0^t rS_s \,\mathrm{d}s + \int_0^t \sigma_S(s, Y_s) S_s \,\mathrm{d}W_s, \\ Y_t &= y_0 + \int_0^t b_Y(s, Y_s)\,\mathrm{d}s + \int_0^t \sigma_Y(s, Y_s)\,\mathrm{d}B_s, \\ \mathrm{d}\langle B, W\rangle_t &= \rho\,\mathrm{d}t \end{cases} \tag{1.1}$$

where the coefficients b_Y, σ_S, $\sigma_Y : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are smooth functions, $r \in \mathbb{R}$, W and B are one-dimensional standard Brownian motions with correlation factor $\rho \in (-1, 1)$ both being defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The aim of this article is to prove a probabilistic representation formula for two integration by parts (IBP) formulae for the marginal law of the process (S, Y) at a given time maturity T. To be more specific, for a given starting point $(s_0, y_0) \in (0, \infty) \times \mathbb{R}$ and a given finite time horizon $T > 0$, we establish two Bismut-Elworthy-Li

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(BEL) type formulae for the two following quantities

$$\partial_{s_0}\mathbb{E}\left[h(S_T,Y_T)\right] \quad \text{and} \quad \partial_{y_0}\mathbb{E}\left[h(S_T,Y_T)\right] \tag{1.2}$$

where h is a real-valued possibly non-smooth payoff function defined on $[0,\infty)\times\mathbb{R}$.

Such IBP formulae have attracted a lot of interest during the last decades both from a theoretical and a practical point of views as they can be further analyzed to derive properties related to the transition density of the underlying process or to develop Monte Carlo simulation algorithm among other practical applications, see *e.g.* Nualart [17], Malliavin and Thalmaier [16] and the references therein. For instance, they are of major interest for computing sensitivities, also referred as to *Greeks* in finance, of arbitrage price of financial derivatives which is the keystone for hedging purpose, *i.e.* for protecting the value of a portfolio against some possible changes in sources of risk. The two quantities appearing in (1.2) correspond respectively to the Delta and Vega of the European option with payoff $h(S_T,Y_T)$. For a more detailed discussion on this topic, we refer the interested reader to Fournié *et al.* [7, 8] for IBP formulae related to European, Asian options and conditional expectations, Gobet *et al.* [4, 11] for IBP formulae related to some barrier or lookback options. A natural and direct approach to compute the two quantities appearing in (1.2) would be the use of the standard Malliavin calculus machinery as developed in the series of papers by Kusuoka and Stroock [13–15] which can cope with unbounded drift and diffusion coefficients and is compatible with the current regularity and uniform ellipticity assumptions. Another approach would be to use localized IBP formulae as developed *e.g.* in Theorem 2.1 [5, 10] which allow to work with only locally smooth coefficients. Let us importantly point out that, from a numerical point of view, the aforementioned IBP formulae will inevitably involve a time discretization procedure of the underlying process and Malliavin weights, thus introducing two sources of error given by a bias and a statistical error, as it is already the case for the computation of the price $\mathbb{E}[h(S_T,Y_T)]$.

Relying on a perturbation argument for the Markov semigroup generated by the couple (X,Y), we first establish a probabilistic representation formula for the marginal law (S_T,Y_T) for a fixed prescribed maturity $T>0$ based on a simple Markov chain evolving along a random time grid given by the jump times of an independent renewal process. Such type of probabilistic representation formula was first derived in Bally and Kohatsu-Higa [3] for the marginal law of a multi-dimensional diffusion process and of some Lévy driven SDEs with bounded drift, diffusion and jump coefficients. Still in the case of bounded coefficients, it was then further investigated in Labordère *et al.* [12], Agarwal and Gobet [1], Doumbia *et al.* [6] for multi-dimensional diffusion processes and in Frikha *et al.* [9] for one-dimensional killed processes. The major advantage of the aforementioned probabilistic formulae lies in the fact that an unbiased Monte Carlo simulation method directly stems from it. Thus, it may be used to numerically compute an option price with optimal complexity since its computation will be only affected by the statistical error. However, let us emphasize that in general the variance of the Monte Carlo estimator tends to be large or even infinite. In order to circumvent this issue, an importance sampling scheme based on the law of the jump times of the underlying renewal process has been proposed in Anderson and Kohatsu-Higa [2], see also [6], in the multi-dimensional diffusion framework and in [9] for one-dimensional killed processes.

The main novelty of our approach in comparison to the aforementioned works is that we allow the drift coefficient b_Y to be possibly unbounded (with at most linear growth) as it is the case in most stochastic volatility models (Stein-Stein, Heston, etc.). We will assume however that the diffusion coefficients $a_S=\sigma_S^2$ and $a_Y=\sigma_Y^2$ are bounded and uniformly elliptic so that our results do not directly apply to the aforementioned well-known stochastic volatility models. We importantly mention that the boundedness condition on the drift coefficient has appeared persistently in the previous contributions on unbiased Monte Carlo methods and is actually essential since basically it allows to remove the drift in the choice of the approximation process in order to derive the probabilistic representation formula. Importantly, a direct application of the methodology developed in [3, 9, 12] does not work when the drift is unbounded. The key ingredient that we here develop in order to remove this restriction consists in choosing adequatly the approximation process around which the original perturbation argument of the Markov semigroup (X,Y) is done by taking into account the transport of the initial condition by the deterministic ordinary differential equation (ODE) having unbounded coefficient[1]. The approximation process, or equivalently the underlying Markov chain on which the probabilistic representation is based, is then obtained from the original dynamics (1.1) by *freezing the coefficients* b_Y*,* σ_S *and* σ_Y *along the flow of this ODE*. We stress that the previous choice is here crucial since it provides the adequate approximation process on which some good controls on the weights involved in the probabilistic representation formulae can be established. Roughly speaking, it allows to cancel the time singularity generated by the Malliavin IBP operators appearing in the weights. To the best of our knowledge, this feature appears to be new in this context.

Having this probabilistic representation formula at hand together with the tailor-made Malliavin calculus machinery for this well-chosen underlying Markov chain, in the spirit of the BEL formula established in [9] for killed diffusion processes with bounded drift coefficient, we rely on a propagation of the spatial derivatives forward in time then perform local IBP formulae on each time interval of the random time grid and eventually merge them in a suitable manner in order to establish the two BEL formulae for the two quantities (1.2). Following the ideas developed in [2], we achieve finite variance for the Monte Carlo estimators obtained from the probabilistic representation formulas of the couple (S_T, Y_T) and of both IBP formulae by selecting adequately the law of the jump times of the renewal process. We finally provide some numerical tests illustrating our previous analysis. Let us eventually mention that for sake of simplicity in the present paper we have decided to consider only one-dimensional processes S and Y but that some multi-dimensional generalizations of the above formulae could be achieved without major issue at the price of additional technicalities which we believe would be prejudicial to the understanding of the main idea. Finally, let us emphasize once again that the main advantage of our approach compared to the standard Malliavin calculus machinery (as developed in the aforementioned references) is that our unbiased Monte Carlo algorithm is only affected by a statistical error and thus achieves optimal complexity at least from a theoretical point of view. Another advantage of our unbiased Monte Carlo algorithm compared to the standard paradigm of bias-variance trade-off Monte Carlo algorithms is that it can be used automatically to achieve a prescribed error while biased methods usually require an a priori assessment of the bias.

The article is organized as follows. In Section 2, we introduce our assumptions on the coefficients, present the approximation process that will be the main building block for our perturbation argument as well as the Markov chain that will play a central role in our probabilistic representation for the marginal law of the process (X, Y) and for our IBP formulae. In addition, we construct the taillor-made Malliavin calculus machinery related to the underlying Markov chain upon which both IBP formulae are made. In Section 3, relying on the Markov chain introduced in Section 2, we establish in Theorem 3.1 the probabilistic representation formula for the coupled (S_T, Y_T). In Section 4, we establish the BEL formulae for the two quantities appearing in (1.2). The main result of this section is Theorem 4.2. As a proof of concept, some numerical results are presented in Section 5. Clearly, we believe that one needs to study numerical issues in more details and these are left for later studies. The proofs of Theorem 3.1 and of some other technical but important results are postponed to the appendix of Section A.

Notations: For a fixed time T and positive integer n, we will use the following notation for time and space variables $\mathbf{s}_n = (s_1, \cdots, s_n)$, $\mathbf{x}_n = (x_1, \ldots, x_n)$, the differentials $\mathrm{d}\mathbf{s}_n = \mathrm{d}s_1 \cdots \mathrm{d}s_n$, $\mathrm{d}\mathbf{x}_n = \mathrm{d}x_1 \cdots \mathrm{d}x_n$ and also introduce the simplex $\Delta_n(T) := \{\mathbf{s}_n \in [0,T]^n : 0 \leq s_1 < \cdots s_n \leq T\}$.

In order to deal with time-degeneracy estimates, we will often use the following space-time inequality:

$$\forall p \geq 0, q > 0, \forall x \in \mathbb{R}, \quad |x|^p e^{-q|x|^2} \leq (p/(2qe))^{p/2}. \tag{1.3}$$

For two positive real numbers α and β, we define the Mittag-Leffler function $z \mapsto E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} z^k/\Gamma(\alpha k + \beta)$. For a positive integer d, we denote by $\mathcal{C}_p^{\infty}(\mathbb{R}^d)$ the space of real-valued functions which are infinitely differentiable on $\mathbb{R}^d$ with derivatives of any order having polynomial growth.

[1] This dynamical system is obtained by removing the noise, that is, by setting $\sigma_Y \equiv 0$, from the dynamics of Y in (1.1).

2. Preliminaries: assumptions, definition of the underlying Markov chain and related Malliavin calculus

2.1. Assumptions

Throughout the article, we work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which is assumed to be rich enough to support all random variables that we will consider in what follows. Let n be a positive integer. We will work under the following assumptions on the coefficients:

($\mathbf{AR}_n$) The coefficients σ_S and σ_Y are continuous and bounded on $\mathbb{R}_+ \times \mathbb{R}$. Moreover, for any $t \geq 0$, $\sigma_S(t,.)$ and $\sigma_Y(t,.)$ belong to $\mathcal{C}_b^n(\mathbb{R})$. For any $T > 0$, there exists $C > 0$ such that $\sup_{t\in[0,T]} |b(t,x)| \leq C(1+|x|)$. For any $t \geq 0$, the drift coefficient $b_Y(t,)$ belongs to $\mathcal{C}^{n-1}(\mathbb{R})$ and admits derivatives of any order greater than or equal to one which are uniformly bounded with respect to its entries. In particular, the drift coefficient b_Y may be unbounded but is Lipschitz continuous in space uniformly in time on compact sets of $\mathbb{R}_+$. We thus define

$$[b_Y]_T = \sup_{t\in[0,T],\, x\neq y} \frac{|b(t,x) - b(t,y)|}{|x-y|}. \tag{2.1}$$

(ND) There exists $\kappa \geq 1$ such that for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\kappa^{-1} \leq a_S(t,x) \leq \kappa, \quad \kappa^{-1} \leq a_Y(t,x) \leq \kappa, \tag{2.2}$$

where $a_S = \sigma_S^2$ and $a_Y = \sigma_Y^2$. Therefore, without loss of generality, we will assume that both σ_S and σ_Y are positive function.

We will say that **($\mathbf{AR}_\infty$)** is satisfied if **($\mathbf{AR}_n$)** holds for all positive integer n. In what follows, we will typically consider that **($\mathbf{AR}_2$)** and **(ND)** are in force. We will denote by $(S_t^{s,s'}, Y_t^{s,y})_{t\in[s,T]}$ the unique solution to (1.1) starting from $(s',y) \in \mathbb{R}_+ \times \mathbb{R}$ at time $s \geq 0$. When $s=0$, we write $(S_t^{s'}, Y_t^{y})$ or simply (S_t, Y_t). Apply Itô's lemma to $X_t^{s,x} = \ln(S_t^{s,s'})$ where $x = \ln(s')$. We get

$$\begin{cases} X_t^{s,x} &= x + \int_s^t \left(r - \frac{1}{2} a_S(u, Y_u^{s,y})\right) \mathrm{d}u + \int_s^t \sigma_S(u, Y_u^{s,y})\, \mathrm{d}W_u, \\ Y_t^{s,y} &= y + \int_s^t b_Y(u, Y_u^{s,y})\, \mathrm{d}u + \int_s^t \sigma_Y(u, Y_u^{s,y})\, \mathrm{d}B_u, \\ \mathrm{d}\langle B, W\rangle_t &= \rho\, \mathrm{d}t. \end{cases} \tag{2.3}$$

Without loss of generality, we will thus work with the Markov semigroup associated to the process $(X_t^{s,x}, Y_t^{s,y})_{t\in[s,T]}$, namely $P_{s,t}h(x,y) = \mathbb{E}[h(X_t^{s,x}, Y_t^{s,y})]$.

2.2. Choice of the approximation process

As already mentioned in the introduction, our strategy here is based on a probabilistic representation of the marginal law, in the spirit of the unbiased simulation method introduced for multi-dimensional diffusion processes by Bally and Kohatsu-Higa [3], see also Labordère *et al.* [12], Doumbia *et al.* [6], and investigated from a numerical perspective by Andersson and Kohatsu-Higa [2]. We also mention the recent contribution of one of the author with Kohatsu-Higa and Li [9] for IBP formulae for the marginal law of one-dimensional killed diffusion processes.

However, at this stage, it is important to point out that our choice of approximation process significantly differs from the aforementioned references. Indeed, in the previous contributions, the drift is assumed to be bounded and basically plays no role so that one usually removes it in the dynamics of the approximation process. In order to handle the unbounded drift term b_Y appearing in the dynamics of the volatility process, one has to take into account the transport of the initial condition by the ODE obtained by removing the noise in the dynamics of Y. To be more specific, for given *freezing parameters* $(\tau, \xi) \in [0,T] \times \mathbb{R}$, we introduce the flow

$$m_{t,\tau}(\xi) = \xi + \int_\tau^t b_Y(s, m_{s,\tau}(\xi))\,\mathrm{d}s,\ t \geq \tau,$$

and $m_{t,\tau}(\xi) = \xi$ if $t \leq \tau$. We will simplify the notation when $\tau = 0$ and write $m_t(\xi)$ for $m_{t,0}(\xi)$. When there is no ambiguity, we will often omit the dependence with respect to the initial point ξ and we only write $m_{t,\tau}$ for $m_{t,\tau}(\xi)$. We now introduce the approximation process $(\bar{X}^{s,x,(\tau,\xi)}, \bar{Y}^{s,x,(\tau,\xi)})$ defined by

$$\begin{cases} \bar{X}_t^{s,x,(\tau,\xi)} & = x + \int_s^t (r - \frac{1}{2} a_S(u, m_{u,\tau}(\xi)))\,\mathrm{d}u + \int_s^t \sigma_S(u, m_{u,\tau}(\xi))\,\mathrm{d}W_u, \\ \bar{Y}_t^{s,y,(\tau,\xi)} & = m_{t,s}^{(\tau,\xi)}(y) + \int_s^t \sigma_Y(u, m_{u,\tau}(\xi))\,\mathrm{d}B_u, \\ \mathrm{d}\langle B, W\rangle_t & = \rho\,\mathrm{d}t, \end{cases} \tag{2.4}$$

where we denoted

$$m_{t,s}^{(\tau,\xi)}(y) = y + \int_s^t b_Y(u, m_{u,\tau}(\xi))\,\mathrm{d}u. \tag{2.5}$$

Observe that from the very definition (2.5) we have the important property

$$m_{t,s}^{(\tau,\xi)}(y)|_{(\tau,\xi)=(s,y)} = m_{t,s}(y).$$

Observe that the couple $(\bar{X}_t^{s,x,(\tau,\xi)}, \bar{Y}_t^{s,y,(\tau,\xi)})_{t\in[s,T]}$ is a Gaussian process. We will make intensive use of the explicit form of the Markov semigroup $(\bar{P}_{s,t}^{(\tau,\xi)})_{t\in[s,T]}$ defined for any bounded measurable map $h : \mathbb{R}^2 \to \mathbb{R}$ by $\bar{P}_{s,t}^{(\tau,\xi)} h(x,y) = \mathbb{E}[h(\bar{X}_t^{s,x,(\tau,\xi)}, \bar{Y}_t^{s,y,(\tau,\xi)})]$. When $(\tau,\xi) = (s,y)$, we will simplify the notations and write $(\bar{X}_t^{s,x}, \bar{Y}_t^{s,y})$ for $(\bar{X}_t^{s,x,(s,y)}, \bar{Y}_t^{s,y,(s,y)})$ and $\bar{P}_{s,t} h(x,y)$ for $\bar{P}_{s,t}^{(s,y)} h(x,y)$.

Lemma 2.1. *Let $(x,y) \in \mathbb{R}^2$, $\rho \in (-1,1)$ and $t \in (s,\infty)$. Then, for any bounded and measurable map $h : \mathbb{R}^2 \to \mathbb{R}$, it holds*

$$\bar{P}_{s,t}^{(\tau,\xi)} h(x,y) = \int_{\mathbb{R}^2} h(x',y')\,\bar{p}^{(\tau,\xi)}(s,t,x,y,x',y')\,\mathrm{d}x'\mathrm{d}y', \tag{2.6}$$

with

$$\begin{aligned} \bar{p}^{(\tau,\xi)}(s,t,x,y,x',y') = {} & \frac{1}{2\pi \sigma_{S,t,s}\sigma_{Y,t,s}\sqrt{1-\rho_{t,s}^2}} \exp\Big(-\frac{1}{2}\frac{(x'-x-(r(t-s)-\frac{1}{2}a_{S,t,s}))^2}{a_{S,t,s}(1-\rho_{t,s}^2)} - \frac{1}{2}\frac{(y'-m_{t,s}^{(\tau,\xi)}(y))^2}{a_{Y,t,s}(1-\rho_{t,s}^2)}\Big) \\ & \times \exp\Big(\frac{\rho_{s,t}}{(1-\rho_{t,s}^2)}\frac{(x'-x-(r(t-s)-\frac{1}{2}a_{S,t,s}))(y'-m_{t,s}^{(\tau,\xi)}(y))}{\sigma_{S,t,s}\sigma_{Y,t,s}}\Big), \end{aligned}$$

where we introduced the notations

$$\begin{aligned} a_{S,t,s} = a_{S,t,s}(\tau,\xi) := \sigma_{S,t,s}^2 := \int_s^t a_S(u, m_{u,\tau}(\xi))\,\mathrm{d}u, \\ a_{Y,t,s} = a_{Y,t,s}(\tau,\xi) := \sigma_{Y,t,s}^2 := \int_s^t a_Y(u, m_{u,\tau}(\xi))\,\mathrm{d}u, \end{aligned}$$

$$\sigma_{S,Y,t,s} = \sigma_{S,Y,s,t}(\tau,\xi) := \int_s^t (\sigma_S\sigma_Y)(u, m_{u,\tau}(\xi))\,\mathrm{d}u,$$
$$\rho_{t,s} := \rho\sigma_{S,Y,t,s}/(\sigma_{S,t,s}\sigma_{Y,t,s}).$$

Moreover, there exists some positive constant $C := C(T,\rho,a,r,\kappa)$ *such that for any* $t \in (0,T]$ *and any* $(\tau,\xi) \in \mathbb{R}_+ \times \mathbb{R}$

$$\bar{p}^{(\tau,\xi)}(s,t,x,y,x',y') \le C\bar{q}_{4\kappa}^{(\tau,\xi)}(s,t,x,y,x',y'), \tag{2.7}$$

where, for a positive parameter c*, we introduced the density function*

$$(x',y') \mapsto \bar{q}_c^{(\tau,\xi)}(s,t,x,y,x',y') := \frac{1}{2\pi c(t-s)}\exp\Big(-\frac{(x'-x)^2}{2c(t-s)} - \frac{(y'-m_{t,s}^{(\tau,\xi)}(y))^2}{2c(t-s)}\Big). \tag{2.8}$$

When (τ,ξ) *is chosen to be equal to the initial condition* (s,y)*, we will again simplify the notations and write* $\bar{p}(s,t,x,y,x',y')$ *for* $\bar{p}^{(s,y)}(s,t,x,y,x',y')$ *and similarly* $\bar{q}_c(s,t,x,y,x',y')$ *for* $\bar{q}_c^{(s,y)}(s,t,x,y,x',y')$*.*

Proof. We write

$$\begin{aligned}
&(\bar{X}_t^{s,x,(\tau,\xi)}, \bar{Y}_t^{s,y,(\tau,\xi)}) \\
&= \Big(x + r(t-s) - \frac{1}{2}a_{S,t,s} + \int_s^t \sigma_S(u, m_{u,\tau}(\xi))\,\mathrm{d}W_u, m_{t,s}^{(\tau,\xi)}(y) + \int_s^t \sigma_Y(u, m_{u,\tau}(\xi))\Big(\rho \mathrm{d}W_u + \sqrt{1-\rho^2}\mathrm{d}\widetilde{W}_u\Big)\Big),
\end{aligned}$$

where $\widetilde{W}$ is a one-dimensional standard Brownian motion independent of W. We thus deduce that $(\bar{X}_t^{s,x,(\tau,\xi)}, \bar{Y}_t^{s,y,(\tau,\xi)}) \sim \mathcal{N}(\mu(t,s,x,y), \Sigma_{t,s})$ with

$$\mu(t,s,x,y) = \Big(x + r(t-s) - \frac{1}{2}a_{S,t,s}, m_{t,s}^{(\tau,\xi)}(y)\Big) \quad \text{and} \quad \Sigma_{t,s} = \begin{pmatrix} a_{S,t,s} & \rho_{s,t}\sigma_{S,Y,t,s} \\ \rho_{t,s}\sigma_{S,Y,t,s} & a_{Y,t,s}\end{pmatrix}.$$

The expression of the transition density then readily follows. Now, from **(ND)**, it is readily seen that $\kappa^{-1}(t-s) \le a_{S,t,s}, a_{Y,t,s} \le \kappa(t-s)$ so that using the inequalities $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$, $(a-b)^2 \ge \frac{1}{2}a^2 - b^2$ and $\rho_{t,s}^2 \le \rho^2 \le 1$, it holds

$$\begin{aligned}
\bar{p}^{(\tau,\xi)}(s,t,x,y,x',y') &= \frac{1}{2\pi\sigma_{S,t,s}\sigma_{Y,t,s}\sqrt{1-\rho_{t,s}^2}}\exp\Big(-\frac{1}{2}\frac{(x'-x-(r(t-s)-\frac{1}{2}a_{S,t,s}))^2}{a_{S,t,s}(1-\rho_{t,s}^2)} - \frac{1}{2}\frac{(y'-m_{t,s}^{(\tau,\xi)}(y))^2}{a_{Y,t,s}(1-\rho_{t,s}^2)}\Big) \\
&\quad \times \exp\Big(\frac{\rho_{t,s}}{1-\rho_{t,s}^2}\frac{(x'-x-(r(t-s)-\frac{1}{2}a_{S,t,s}))(y'-m_{t,s}^{(\tau,\xi)}(y))}{\sigma_{S,t,s}\sigma_{Y,t,s}}\Big) \\
&\le C\frac{1}{2\pi\kappa(t-s)}\exp\Big(-\frac{1}{2}\frac{(x'-x-(r(t-s)-\frac{1}{2}a_{S,t,s}))^2}{a_{S,t,s}(1-\rho_{t,s}^2)}(1-|\rho_{t,s}|) \\
&\quad -\frac{1}{2}\frac{(y'-m_{t,s}^{(\tau,\xi)}(y))^2}{a_{Y,t,s}(1-\rho_{t,s}^2)}(1-|\rho_{t,s}|)\Big) \\
&\le C\frac{1}{2\pi(4\kappa)(t-s)}\exp\Big(-(4\kappa)^{-1}\frac{(x'-x)^2}{2(t-s)} - (4\kappa)^{-1}\frac{(y'-m_{t,s}^{(\tau,\xi)}(y))^2}{2(t-s)}\Big) \\
&=: C\bar{q}_{4\kappa}^{(\tau,\xi)}(s,t,x,y,x',y'),
\end{aligned}$$

for some positive constants $C := C(T, \lambda, \rho, r, \kappa)$.

$\square$

2.3. Markov chain on random time grid

The first tool that we will employ is a renewal process N that we now introduce.

Definition 2.2. Let $\tau := (\tau_n)_{n\geq 0}$ be a sequence of random variables such that $(\tau_n - \tau_{n-1})_{n\geq 1}$, with the convention $\tau_0 = 0$, are i.i.d. with positive density function f and cumulant distribution function $t \mapsto F(t) = \int_{-\infty}^t f(s)\,\mathrm{d}s$ and τ is independent of $(W_s, B_s)_{0\leq s\leq T}$. Then, the renewal process $N := (N_t)_{t\geq 0}$ with jump times τ is defined by $N_t := \sum_{n\geq 1} \mathbf{1}_{\{\tau_n \leq t\}}$.

It is readily seen that, for any $t > 0$, $\{N_t = n\} = \{\tau_n \leq t < \tau_{n+1}\}$ and by an induction argument that we omit, one may prove that the joint distribution of $(\tau_1, \ldots, \tau_n)$ is given by

$$\mathbb{P}(\tau_1 \in ds_1, \ldots, \tau_n \in ds_n) = \prod_{j=0}^{n-1} f(s_{j+1} - s_j) \mathbf{1}_{\{0<s_1<\cdots<s_n\}},$$

which in turn implies

$$\begin{aligned}\mathbb{E}[\mathbf{1}_{\{N_t=n\}} \Phi(\tau_1, \ldots, \tau_n)] &= \mathbb{E}[\mathbf{1}_{\{\tau_n\leq t<\tau_{n+1}\}} \Phi(\tau_1, \ldots, \tau_n)] \\ &= \int_t^\infty \int_{\Delta_n(t)} \Phi(s_1, \ldots, s_n) \prod_{j=0}^{n} f(s_{j+1} - s_j)\,\mathrm{d}\mathbf{s}_{n+1},\end{aligned}$$

with the convention $s_0 = 0$. Hence, by Fubini's theorem, it holds

$$\mathbb{E}[\mathbf{1}_{\{N_t=n\}} \Phi(\tau_1, \ldots, \tau_n)] = \int_{\Delta_n(t)} \Phi(s_1, \ldots, s_n)(1 - F(t - s_n)) \prod_{j=0}^{n-1} f(s_{j+1} - s_j)\,\mathrm{d}\mathbf{s}_n, \tag{2.9}$$

for any measurable map $\Phi : \Delta_n(t) \to \mathbb{R}$ satisfying $\mathbb{E}[\mathbf{1}_{\{N_t=n\}}|\Phi(\tau_1, \ldots, \tau_n)|] < \infty$.

Usual choices that we will consider are the followings.

Example 2.3.
1. If the density function f is given by $f(t) = \lambda e^{-\lambda t}\mathbf{1}_{[0,\infty)}(t)$ for some positive parameter λ, then N is a Poisson process with intensity λ.
2. If the density function f is given by $f(t) = \frac{1-\alpha}{\bar{\tau}^{1-\alpha}} \frac{1}{t^\alpha} \mathbf{1}_{[0,\bar{\tau}]}(t)$ for some parameters $(\alpha, \bar{\tau}) \in (0, 1) \times (T, \infty)$, then N is a renewal process with $[0, \bar{\tau}]$-valued $Beta(1 - \alpha, 1)$ jump times.
3. More generally, if the density function f is given by $f(t) = \frac{\bar{\tau}^{1-\alpha-\beta}}{B(\alpha,\beta)} \frac{1}{t^{1-\alpha}(\bar{\tau}-t)^{1-\beta}} \mathbf{1}_{[0,\bar{\tau}]}(t)$ for some parameters $(\alpha, \beta, \bar{\tau}) \in (0, 1)^2 \times (T, \infty)$, then N is a renewal process with $[0, \bar{\tau}]$-valued $Beta(\alpha, \beta)$ jump times.

Given a sequence $Z = (Z_n^1, Z_n^2)_{n\geq 1}$ of i.i.d. random vector with law $\mathcal{N}(0, I_2)$ which is independent of (W, B) and a renewal process N independent of Z with jump times $(\tau_i)_{i\geq 0}$, we set $\zeta_i = \tau_i \wedge T$, with the convention $\zeta_0 = 0$, and we consider the two-dimensional Markov chain $(\bar{X}, \bar{Y})$ with $(\bar{X}_0, \bar{Y}_0) = (x_0, y_0)$ at time 0 (evolving on the random time grid $(\zeta_i)_{i\geq 0}$) and with dynamics for any $0 \leq i \leq N_T$

$$\begin{cases} \bar{X}_{i+1} &= \bar{X}_i + \left(r(\zeta_{i+1} - \zeta_i) - \frac{1}{2}a_{S,i}\right) + \sigma_{S,i} Z_{i+1}^1, \\ \bar{Y}_{i+1} &= m_i + \sigma_{Y,i}\left(\rho_i Z_{i+1}^1 + \sqrt{1 - \rho_i^2} Z_{i+1}^2\right), \end{cases} \tag{2.10}$$

where we introduced the notations

$$
\begin{aligned}
a_{S,i} &:= \sigma^2_{S,i} := a_{S,\zeta_{i+1},\zeta_i}(\bar{Y}_i) = \int_{\zeta_i}^{\zeta_{i+1}} a_S(u, m_{u,\zeta_i}(\bar{Y}_i))\,\mathrm{d}u,\\
a_{Y,i} &:= \sigma^2_{Y,i} := a_{Y,\zeta_{i+1},\zeta_i}(\bar{Y}_i) = \int_{\zeta_i}^{\zeta_{i+1}} a_Y(u, m_{u,\zeta_i}(\bar{Y}_i))\,\mathrm{d}u,\\
\sigma_{S,Y,i} &:= \int_{\zeta_i}^{\zeta_{i+1}} (\sigma_S\sigma_Y)(m_{u,\zeta_i}(\bar{Y}_i))\,\mathrm{d}u,\\
\rho_i &:= \rho_{\zeta_{i+1},\zeta_i}(\bar{Y}_i) = \rho\frac{\sigma_{S,Y,i}}{\sigma_{S,i}\sigma_{Y,i}},\\
m_i &:= m_{\zeta_{i+1},\zeta_i}(\bar{Y}_i).
\end{aligned}
$$

We will denote by $\sigma'_{S,i}$ the first derivative of $y \mapsto \sigma_{S,i}(y)$ evaluated at $\bar{Y}_i$ and proceed similarly for the quantities $\sigma'_{Y,i}$, $\sigma'_{S,Y,i}$, ρ'_i and m'_i. We define the filtration $\mathcal{G} = (\mathcal{G}_i)_{i\geq 0}$ where $\mathcal{G}_i = \sigma(Z^1_j, Z^2_j,\ 1 \leq j \leq i)$, for $i \geq 1$ and $\mathcal{G}_0$ stands for the trivial σ-field. We assume that the filtration $\mathcal{G}$ satisfies the usual conditions. For an integer n, we will use the notations $\zeta^n = (\zeta_0, \ldots, \zeta_n)$ and $\tau^n = (\tau_0, \ldots, \tau_n)$.

2.4. Tailor-made Malliavin calculus for the Markov chain $(\bar{X}, \bar{Y})$

In this section we introduce a tailor-made Malliavin calculus for the underlying Markov chain $(\bar{X}, \bar{Y})$ defined by (2.10) which will be employed in order to establish our IBP formulae. Instead of using an infinite dimensional calculus as it is usually done in the literature, see *e.g.* Nualart [17], the approach developed below is based on a finite dimensional calculus for which the dimension is given by the number of jumps of the underlying renewal process involved in the Markov chain $(\bar{X}, \bar{Y})$. In order to simplify the discussion, we will consider that the coefficients a_S, a_Y and b_Y are smooth w.r.t the space variable so that $(\mathbf{AR}_\infty)$ is satisfied.

Definition 2.4. Let $n \in \mathbb{N}$. For any $i \in \{0, \ldots, n\}$, we define the set $\mathbb{S}_{i,n}(\bar{X}, \bar{Y})$, as the space of random variables H such that

- $H = h(\bar{X}_i, \bar{Y}_i, \bar{X}_{i+1}, \bar{Y}_{i+1}, \zeta^{n+1})$, on the set $\{N_T = n\}$, where we recall $\zeta^{n+1} := (\zeta_0, \ldots, \zeta_{n+1}) = (0, \zeta_1, \ldots, \zeta_n, T)$.
- For any $\mathbf{s}_{n+1} \in \Delta_{n+1}(T)$, the map $h(.,.,.,., \mathbf{s}_{n+1}) \in \mathcal{C}^\infty_p(\mathbb{R}^4)$.

For a r.v. $H \in \mathbb{S}_{i,n}(\bar{X}, \bar{Y})$, we will often abuse the notations and write

$$H \equiv H(\bar{X}_i, \bar{Y}_i, \bar{X}_{i+1}, \bar{Y}_{i+1}, \zeta^{n+1}),$$

that is the same symbol H may denote the r.v. or the function in the set $\mathbb{S}_{i,n}(\bar{X}, \bar{Y})$. One can easily define the flow derivatives for $H \in \mathbb{S}_{i,n}(\bar{X}, \bar{Y})$ as follows

$$
\begin{aligned}
&\partial_{\bar{X}_{i+1}} H = \partial_3 h(\bar{X}_i, \bar{Y}_i, \bar{X}_{i+1}, \bar{Y}_{i+1}, \zeta^{n+1}),\\
&\partial_{\bar{Y}_{i+1}} H = \partial_4 h(\bar{X}_i, \bar{Y}_i, \bar{X}_{i+1}, \bar{Y}_{i+1}, \zeta^{n+1}),\\
&\partial_{\bar{X}_i} H = \partial_1 h(\bar{X}_i, \bar{Y}_i, \bar{X}_{i+1}, \bar{Y}_{i+1}, \zeta^{n+1}) + \partial_3 h(\bar{X}_i, \bar{Y}_i, \bar{X}_{i+1}, \bar{Y}_{i+1}, \zeta^{n+1})\partial_{\bar{X}_i}\bar{X}_{i+1},\\
&\partial_{\bar{Y}_i} H = \partial_2 h(\bar{X}_i, \bar{Y}_i, \bar{X}_{i+1}, \bar{Y}_{i+1}, \zeta^{n+1}) + \partial_3 h(\bar{X}_i, \bar{Y}_i, \bar{X}_{i+1}, \bar{Y}_{i+1}, \zeta^{n+1})\partial_{\bar{Y}_i}\bar{X}_{i+1} + \partial_4 h(\bar{X}_i, \bar{Y}_i, \bar{X}_{i+1}, \bar{Y}_{i+1}, \zeta^{n+1})\partial_{\bar{Y}_i}\bar{Y}_{i+1},
\end{aligned}
$$

and from the dynamics (2.10)

$$\partial_{\bar{X}_i}\bar{X}_{i+1} = 1,$$

$$\partial_{\bar{Y}_i}\bar{Y}_{i+1} = m'_i + \sigma'_{Y,i}\Big(\rho_i Z^1_{i+1} + \sqrt{1-\rho_i^2}Z^2_{i+1}\Big) + \sigma_{Y,i}\frac{\rho'_i}{\sqrt{1-\rho_i^2}}\left(\sqrt{1-\rho_i^2}Z^1_{i+1} - \rho_i Z^2_{i+1}\right), \tag{2.11}$$

$$\partial_{\bar{Y}_i}\bar{X}_{i+1} = -\frac{1}{2}a'_{S,i} + \sigma'_{S,i}Z^1_{i+1} = -\frac{1}{2}a'_{S,i} + \frac{\sigma'_{S,i}}{\sigma_{S,i}}\Big(\bar{X}_{i+1} - \bar{X}_i - \Big(r(\zeta_{i+1}-\zeta_i) - \frac{1}{2}a_{S,i}\Big)\Big). \tag{2.12}$$

We now define the integral and derivative operators for $H \in \mathbb{S}_{i,n}(\bar{X},\bar{Y})$, as

$$\mathcal{I}^{(1)}_{i+1}(H) = H\Big[\frac{Z^1_{i+1}}{\sigma_{S,i}(1-\rho_i^2)} - \frac{\rho_i}{1-\rho_i^2}\frac{\rho_i Z^1_{i+1} + \sqrt{1-\rho_i^2}Z^2_{i+1}}{\sigma_{S,i}}\Big] - \mathcal{D}^{(1)}_{i+1}H, \tag{2.13}$$

$$\mathcal{I}^{(2)}_{i+1}(H) = H\Big[\frac{\rho_i Z^1_{i+1} + \sqrt{1-\rho_i^2}Z^2_{i+1}}{\sigma_{Y,i}(1-\rho_i^2)} - \frac{\rho_i}{1-\rho_i^2}\frac{Z^1_{i+1}}{\sigma_{Y,i}}\Big] - \mathcal{D}^{(2)}_{i+1}H, \tag{2.14}$$

$$\mathcal{D}^{(1)}_{i+1}H = \partial_{\bar{X}_{i+1}}H, \tag{2.15}$$

$$\mathcal{D}^{(2)}_{i+1}H = \partial_{\bar{Y}_{i+1}}H. \tag{2.16}$$

Note that due to the above definitions and assumptions $\mathbf{(AR)}_\infty$ and $\mathbf{(ND)}$, it is readily checked that $\mathcal{I}^{(1)}_{i+1}(H)$, $\mathcal{I}^{(2)}_{i+1}(H)$, $\mathcal{D}^{(1)}_{i+1}H$ and $\mathcal{D}^{(2)}_{i+1}H$ are elements of $\mathbb{S}_{i,n}(\bar{X},\bar{Y})$ so that we can define iterations of the above operators. Namely, by induction, for a multi-index $\alpha = (\alpha_1,\ldots,\alpha_p)$ of length p with $\alpha_i \in \{1,2\}$ and $\alpha_{p+1} \in \{1,2\}$, we define

$$\mathcal{I}^{(\alpha,\alpha_{p+1})}_{i+1}(H) = \mathcal{I}^{(\alpha_{p+1})}_{i+1}(\mathcal{I}^{(\alpha)}_{i+1}(H)), \quad \mathcal{D}^{(\alpha,\alpha_{p+1})}_{i+1}H = \mathcal{D}^{(\alpha_{p+1})}_{i+1}(\mathcal{D}^{(\alpha)}_{i+1}H),$$

with the intuitive notation $(\alpha,\alpha_{p+1}) = (\alpha_1,\ldots,\alpha_{p+1})$.

Throughout the article, we will use the following notation for a certain type of conditional expectation that will be frequently employed. For any $X \in L^1(\mathbb{P})$ and any $i \in \{0,\ldots,n\}$,

$$\mathbb{E}_{i,n}[X] = \mathbb{E}[X|\mathcal{G}_i, \tau^{n+1}, N_T = n]$$

where we recall that we employ the notation $\tau^{n+1} = (\tau_0,\ldots,\tau_{n+1})$. Having the above definitions and notations at hand, the following duality formula is satisfied: for any non-empty multi-index α of length p, with $\alpha_i \in \{1,2\}$, for any $i \in \{1,\ldots,p\}$, p being a positive integer, it holds

$$\mathbb{E}_{i,n}\Big[\mathcal{D}^{(\alpha)}_{i+1}f(\bar{X}_{i+1},\bar{Y}_{i+1})H\Big] = \mathbb{E}_{i,n}\Big[f(\bar{X}_{i+1},\bar{Y}_{i+1})\mathcal{I}^{(\alpha)}_{i+1}(H)\Big]. \tag{2.17}$$

In order to obtain explicit norm estimates for random variables in $\mathbb{S}_{i,n}(\bar{X},\bar{Y})$, it is useful to define for $H \in \mathbb{S}_{i,n}(\bar{X},\bar{Y})$, $i \in \{0,\ldots,n\}$ and $p \geq 1$

$$\|H\|^p_{p,i,n} = \mathbb{E}_{i,n}[|H|^p].$$

We will also employ a chain rule formula for the integral operators defined above.

Lemma 2.5. *Let $H \equiv H(\bar{X}_i,\bar{Y}_i,\bar{X}_{i+1},\bar{Y}_{i+1},\zeta^{n+1}) \in \mathbb{S}_{i,n}(\bar{X},\bar{Y})$, for some $i \in \{0,\ldots,n\}$. The following chain rule formulae hold for any $(\alpha_1,\alpha_2) \in \{1,2\}^2$*

$$\partial_{\bar{X}_i}\mathcal{I}^{(\alpha_1)}_{i+1}(H) = \mathcal{I}^{(\alpha_1)}_{i+1}(\partial_{\bar{X}_i}H), \quad \partial_{\bar{X}_i}\mathcal{I}^{(\alpha_1,\alpha_2)}_{i+1}(H) = \mathcal{I}^{(\alpha_1,\alpha_2)}_{i+1}(\partial_{\bar{X}_i}H). \tag{2.18}$$

Moreover, one has

$$\partial_{\bar{Y}_i}\mathcal{I}^{(1)}_{i+1}(H) = \mathcal{I}^{(1)}_{i+1}(\partial_{\bar{Y}_i}H) - \frac{\sigma'_{S,i}}{\sigma_{S,i}}\mathcal{I}^{(1)}_{i+1}(H) - \frac{\rho'_i}{1-\rho_i^2}\frac{\sigma_{Y,i}}{\sigma_{S,i}}\mathcal{I}^{(2)}_{i+1}(H), \tag{2.19}$$

$$\partial_{\bar{Y}_i}\mathcal{I}^{(2)}_{i+1}(H) = \mathcal{I}^{(2)}_{i+1}(\partial_{\bar{Y}_i}H) - \left(\frac{\sigma'_{Y,i}}{\sigma_{Y,i}} - \frac{\rho'_i\rho_i}{1-\rho_i^2}\right)\mathcal{I}^{(2)}_{i+1}(H), \tag{2.20}$$

$$\partial_{\bar{Y}_i}\mathcal{I}^{(1,1)}_{i+1}(H) = \mathcal{I}^{(1,1)}_{i+1}(\partial_{\bar{Y}_i}H) - 2\frac{\sigma'_{S,i}}{\sigma_{S,i}}\mathcal{I}^{(1,1)}_{i+1}(H) - \frac{\rho'_i}{1-\rho_i^2}\frac{\sigma_{Y,i}}{\sigma_{S,i}}\left(\mathcal{I}^{(1,2)}_{i+1}(H) + \mathcal{I}^{(2,1)}_{i+1}(H)\right), \tag{2.21}$$

$$\partial_{\bar{Y}_i}\mathcal{I}^{(2,2)}_{i+1}(H) = \mathcal{I}^{(2,2)}_{i+1}(\partial_{\bar{Y}_i}H) - 2\left(\frac{\sigma'_{Y,i}}{\sigma_{Y,i}} - \frac{\rho'_i\rho_i}{1-\rho_i^2}\right)\mathcal{I}^{(2,2)}_{i+1}(H), \tag{2.22}$$

$$\partial_{\bar{Y}_i}\mathcal{I}^{(1,2)}_{i+1}(H) = \mathcal{I}^{(1,2)}_{i+1}(\partial_{\bar{Y}_i}H) - \left(\frac{\sigma'_{S,i}}{\sigma_{S,i}} + \frac{\sigma'_{Y,i}}{\sigma_{Y,i}} - \frac{\rho'_i\rho_i}{1-\rho_i^2}\right)\mathcal{I}^{(1,2)}_{i+1}(H) - \frac{\rho'_i}{1-\rho_i^2}\frac{\sigma_{Y,i}}{\sigma_{S,i}}\mathcal{I}^{(2,2)}_{i+1}(H). \tag{2.23}$$

Proof. Observe that from the very definitions (2.13) and (2.14), one directly gets

$$\partial_{\bar{X}_i}\mathcal{I}^{(1)}_{i+1}(1) = \partial_{\bar{X}_i}\mathcal{I}^{(2)}_{i+1}(1) = 0,$$

while, also by direct computation, we obtain

$$\begin{aligned}
\partial_{\bar{Y}_i}\mathcal{I}^{(1)}_{i+1}(1) &= -\frac{\sigma'_{S,i}}{\sigma_{S,i}}\mathcal{I}^{(1)}_{i+1}(1) - \frac{\rho'_i}{1-\rho_i^2}\frac{\sigma_{Y,i}}{\sigma_{S,i}}\mathcal{I}^{(2)}_{i+1}(1),\\
\partial_{\bar{Y}_i}\mathcal{I}^{(2)}_{i+1}(1) &= -\left(\frac{\sigma'_{Y,i}}{\sigma_{Y,i}} - \frac{\rho'_i\rho_i}{1-\rho_i^2}\right)\mathcal{I}^{(2)}_{i+1}(1).
\end{aligned}$$

We thus deduce

$$\begin{aligned}
\partial_{\bar{X}_i}\mathcal{I}^{(\alpha_1)}_{i+1}(H) &= \partial_{\bar{X}_i}H\mathcal{I}^{(\alpha_1)}_{i+1}(1) + H\partial_{\bar{X}_i}\mathcal{I}^{(\alpha_1)}_{i+1}(1) - \partial_{\bar{X}_i}\mathcal{D}^{(\alpha_1)}_{i+1}H\\
&= \partial_{\bar{X}_i}H\mathcal{I}^{(\alpha_1)}_{i+1}(1) - \mathcal{D}^{(\alpha_1)}_{i+1}(\partial_{\bar{X}_i}H)\\
&= \mathcal{I}^{(\alpha_1)}_{i+1}(\partial_{\bar{X}_i}H),
\end{aligned}$$

where we used the fact $\mathcal{D}^{(\alpha_1)}_{i+1}\partial_{\bar{X}_i}H = \partial_{\bar{X}_i}\mathcal{D}^{(\alpha_1)}_{i+1}H$ which easily follows by direct computation. As a consequence, it is readily seen that

$$\partial_{\bar{X}_i}\mathcal{I}^{(\alpha_1,\alpha_2)}_{i+1}(H) = \partial_{\bar{X}_i}\mathcal{I}^{(\alpha_2)}_{i+1}(\mathcal{I}^{(\alpha_1)}_{i+1}(H)) = \mathcal{I}^{(\alpha_2)}_{i+1}(\partial_{\bar{X}_i}\mathcal{I}^{(\alpha_1)}_{i+1}(H)) = \mathcal{I}^{(\alpha_2)}_{i+1}(\mathcal{I}^{(\alpha_1)}_{i+1}(\partial_{\bar{X}_i}H)) = \mathcal{I}^{(\alpha_1,\alpha_2)}_{i+1}(\partial_{\bar{X}_i}H).$$

This concludes the proof of (2.18). The chain rule formulae (2.19), (2.20), (2.21), (2.22) and (2.23) follow from similar arguments. Let us prove (2.19) and (2.20). The proofs of (2.21), (2.22) and (2.23) are omitted. Observe first that in general $\mathcal{D}^{(\alpha_1)}_{i+1}\partial_{\bar{Y}_i}H \neq \partial_{\bar{Y}_i}\mathcal{D}^{(\alpha_1)}_{i+1}H$. Indeed, by standard computations, it holds

$$\begin{aligned}
\partial_{\bar{Y}_i}\mathcal{D}^{(1)}_{i+1}H &= \partial_{\bar{Y}_i}\partial_{\bar{X}_{i+1}}h(\bar{X}_i,\bar{Y}_i,\bar{X}_{i+1},\bar{Y}_{i+1},\zeta^{n+1})\\
&= \partial^2_{2,3}h(\bar{X}_i,\bar{Y}_i,\bar{X}_{i+1},\bar{Y}_{i+1},\zeta^{n+1}) + \partial^2_{3}h(\bar{X}_i,\bar{Y}_i,\bar{X}_{i+1},\bar{Y}_{i+1},\zeta^{n+1})\partial_{\bar{Y}_i}\bar{X}_{i+1} + \partial^2_{4,3}h(\bar{X}_i,\bar{Y}_i,\bar{X}_{i+1},\bar{Y}_{i+1},\zeta^{n+1})\partial_{\bar{Y}_i}\bar{Y}_{i+1},\\
\mathcal{D}^{(1)}_{i+1}\partial_{\bar{Y}_i}H &= \partial_{\bar{X}_{i+1}}\partial_{\bar{Y}_i}h(\bar{X}_i,\bar{Y}_i,\bar{X}_{i+1},\bar{Y}_{i+1},\zeta^{n+1})\\
&= \partial_{\bar{X}_{i+1}}(\partial_2 h(\bar{X}_i,\bar{Y}_i,\bar{X}_{i+1},\bar{Y}_{i+1},\zeta^{n+1}) + \partial_3 h(\bar{X}_i,\bar{Y}_i,\bar{X}_{i+1},\bar{Y}_{i+1},\zeta^{n+1})\partial_{\bar{Y}_i}\bar{X}_{i+1} + \partial_4 h(\bar{X}_i,\bar{Y}_i,\bar{X}_{i+1},\bar{Y}_{i+1},\zeta^{n+1})\partial_{\bar{Y}_i}\bar{Y}_{i+1})\\
&= \partial^2_{3,2}h(\bar{X}_i,\bar{Y}_i,\bar{X}_{i+1},\bar{Y}_{i+1},\zeta^{n+1}) + \partial^2_{3}h(\bar{X}_i,\bar{Y}_i,\bar{X}_{i+1},\bar{Y}_{i+1},\zeta^{n+1})\partial_{\bar{Y}_i}\bar{X}_{i+1} + \partial_3 h(\bar{X}_i,\bar{Y}_i,\bar{X}_{i+1},\bar{Y}_{i+1},\zeta^{n+1})\partial_{\bar{X}_{i+1}}\partial_{\bar{Y}_i}\bar{X}_{i+1}
\end{aligned}$$

$$
\begin{aligned}
&+\partial^2_{3,4}h(\bar X_i,\bar Y_i,\bar X_{i+1},\bar Y_{i+1},\zeta^{n+1})\partial_{\bar Y_i}\bar Y_{i+1}+\partial_4 h(\bar X_i,\bar Y_i,\bar X_{i+1},\bar Y_{i+1},\zeta^{n+1})\partial_{\bar X_{i+1}}\partial_{\bar Y_i}\bar Y_{i+1}\\
&=\partial_{\bar Y_i}\mathcal{D}^{(1)}_{i+1}H+\partial_3 h(\bar X_i,\bar Y_i,\bar X_{i+1},\bar Y_{i+1},\zeta^{n+1})\partial_{\bar X_{i+1}}\partial_{\bar Y_i}\bar X_{i+1}+\partial_4 h(\bar X_i,\bar Y_i,\bar X_{i+1},\bar Y_{i+1},\zeta^{n+1})\partial_{\bar X_{i+1}}\partial_{\bar Y_i}\bar Y_{i+1}\\
&=\partial_{\bar Y_i}\mathcal{D}^{(1)}_{i+1}H+\mathcal{D}^{(1)}_{i+1}H\partial_{\bar X_{i+1}}\partial_{\bar Y_i}\bar X_{i+1}+\mathcal{D}^{(2)}_{i+1}H\partial_{\bar X_{i+1}}\partial_{\bar Y_i}\bar Y_{i+1}\\
&=\partial_{\bar Y_i}\mathcal{D}^{(1)}_{i+1}H+\frac{\sigma'_{S,i}}{\sigma_{S,i}}\mathcal{D}^{(1)}_{i+1}H+\frac{\rho'_i}{1-\rho_i^2}\frac{\sigma_{Y,i}}{\sigma_{S,i}}\mathcal{D}^{(2)}_{i+1}H,
\end{aligned}
$$

where we used the two identities $\partial_{\bar X_{i+1}}\partial_{\bar Y_i}\bar X_{i+1}=\frac{\sigma'_{S,i}}{\sigma_{S,i}}$ and $\partial_{\bar X_{i+1}}\partial_{\bar Y_i}\bar Y_{i+1}=\frac{\rho'_i}{1-\rho_i^2}\frac{\sigma_{Y,i}}{\sigma_{S,i}}$ which readily stems from (2.11), (2.12) and the dynamics (2.10).

From (2.13) and the previous identity, we thus obtain

$$
\begin{aligned}
\partial_{\bar Y_i}\mathcal{I}^{(1)}_{i+1}(H)&=\partial_{\bar Y_i}\mathcal{I}^{(1)}_{i+1}(1)H+\mathcal{I}^{(1)}_{i+1}(1)\partial_{\bar Y_i}H-\partial_{\bar Y_i}\mathcal{D}^{(1)}_{i+1}H\\
&=-\frac{\sigma'_{S,i}}{\sigma_{S,i}}\mathcal{I}^{(1)}_{i+1}(1)H-\frac{\rho'_i}{1-\rho_i^2}\frac{\sigma_{Y,i}}{\sigma_{S,i}}\mathcal{I}^{(2)}_{i+1}(1)H+\mathcal{I}^{(1)}_{i+1}(1)\partial_{\bar Y_i}H-\mathcal{D}^{(1)}_{i+1}\partial_{\bar Y_i}H+\frac{\sigma'_{S,i}}{\sigma_{S,i}}\mathcal{D}^{(1)}_{i+1}H\\
&\quad+\frac{\rho'_i}{1-\rho_i^2}\frac{\sigma_{Y,i}}{\sigma_{S,i}}\mathcal{D}^{(2)}_{i+1}H\\
&=-\frac{\sigma'_{S,i}}{\sigma_{S,i}}\left(\mathcal{I}^{(1)}_{i+1}(1)H-\mathcal{D}^{(1)}_{i+1}H\right)+\mathcal{I}^{(1)}_{i+1}(1)\partial_{\bar Y_i}H-\mathcal{D}^{(1)}_{i+1}\partial_{\bar Y_i}H-\frac{\rho'_i}{1-\rho_i^2}\frac{\sigma_{Y,i}}{\sigma_{S,i}}\left(\mathcal{I}^{(2)}_{i+1}(1)H-\mathcal{D}^{(2)}_{i+1}H\right)\\
&=\mathcal{I}^{(1)}_{i+1}(\partial_{\bar Y_i}H)-\frac{\sigma'_{S,i}}{\sigma_{S,i}}\mathcal{I}^{(1)}_{i+1}(H)-\frac{\rho'_i}{1-\rho_i^2}\frac{\sigma_{Y,i}}{\sigma_{S,i}}\mathcal{I}^{(2)}_{i+1}(H).
\end{aligned}
$$

Similarly, after some algebraic manipulations using (2.10) and (2.11), we get $\partial_{\bar Y_{i+1}}\partial_{\bar Y_i}\bar Y_{i+1}=\frac{\sigma'_{Y,i}}{\sigma_{Y,i}}-\frac{\rho'_i\rho_i}{1-\rho_i^2}$ so that

$$
\mathcal{D}^{(2)}_{i+1}\partial_{\bar Y_i}H=\partial_{\bar Y_i}\mathcal{D}^{(2)}_{i+1}H+\mathcal{D}^{(2)}_{i+1}H\partial_{\bar Y_{i+1}}\partial_{\bar Y_i}\bar Y_{i+1}=\partial_{\bar Y_i}\mathcal{D}^{(2)}_{i+1}H+\left(\frac{\sigma'_{Y,i}}{\sigma_{Y,i}}-\frac{\rho'_i\rho_i}{1-\rho_i^2}\right)\mathcal{D}^{(2)}_{i+1}H.
$$

Omitting some technical details, the previous identity implies

$$
\begin{aligned}
\partial_{\bar Y_i}\mathcal{I}^{(2)}_{i+1}(H)&=\mathcal{I}^{(2)}_{i+1}(\partial_{\bar Y_i}H)-\left(\frac{\sigma'_{Y,i}}{\sigma_{Y,i}}-\frac{\rho'_i\rho_i}{1-\rho_i^2}\right)\mathcal{I}^{(2)}_{i+1}(1)H+\left(\frac{\sigma'_{Y,i}}{\sigma_{Y,i}}-\frac{\rho'_i\rho_i}{1-\rho_i^2}\right)\mathcal{D}^{(2)}_{i+1}H\\
&=\mathcal{I}^{(2)}_{i+1}(\partial_{\bar Y_i}H)-\left(\frac{\sigma'_{Y,i}}{\sigma_{Y,i}}-\frac{\rho'_i\rho_i}{1-\rho_i^2}\right)\mathcal{I}^{(2)}_{i+1}(H).
\end{aligned}
$$

The identities (2.21), (2.22) and (2.23) eventually follows from (2.19) and (2.20) using some simple algebraic computations. □

We conclude this section by introducing the following space of random variables which satisfy some time regularity estimates.

Definition 2.6. Let $\ell\in\mathbb{Z}$ and $n\in\mathbb{N}$. For any $i\in\{0,\dots,n\}$, we define the space $\mathbb{M}_{i,n}(\bar X,\bar Y,\ell/2)$ as the set of finite random variables $H\in\mathbb{S}_{i,n}(\bar X,\bar Y)$ satisfying the following time regularity estimate: for any $p\ge 1$, any $c>0$ and any $c'>c$, there exists some positive constant $C:=C(T,c,c')$, $T\mapsto C(T,c,c')$ being non-decreasing such that for any $(x_i,y_i,x_{i+1},y_{i+1},\mathbf{s}_{n+1})\in\mathbb{R}^4\times\Delta_{n+1}(T)$,

$$
\begin{aligned}
&|H(x_i,y_i,x_{i+1},y_{i+1},\mathbf{s}_{n+1})|^p\bar q_c(s_i,s_{i+1},x_i,y_i,x_{i+1},y_{i+1})\\
&\qquad\le C(s_{i+1}-s_i)^{\frac{p\ell}{2}}\bar q_{c'}(s_i,s_{i+1},x_i,y_i,x_{i+1},y_{i+1}),
\end{aligned}\tag{2.24}
$$

where we recall that the density function $\mathbb{R}^2 \ni (x_{i+1}, y_{i+1}) \mapsto \bar{q}_c(s_i, s_{i+1}, x_i, y_i, x_{i+1}, y_{i+1})$ is defined in Lemma 2.1.

We again remark that since the space $\mathbb{M}_{i,n}(\bar{X}, \bar{Y}, \ell/2)$ is a subset of $\mathbb{S}_{i,n}(\bar{X}, \bar{Y})$, when we say that a random variable $\mathbb{M}_{i,n}(\bar{X}, \bar{Y}, \ell/2)$ this statement is always understood on the set $\{N_T = n\}$.

Before proceeding, let us provide a simple example of some random variables that belong to the aforementioned space. From (2.13) and the dynamics (2.10) of the Markov chain $(\bar{X}, \bar{Y})$, it holds

$$\begin{aligned}\mathcal{I}^{(1)}_{i+1}(1) &= \frac{\bar{X}_{i+1} - \bar{X}_i - (r(\zeta_{i+1} - \zeta_i) - \frac{1}{2} a_{S,i})}{a_{S,i}(1-\rho_i^2)} - \frac{\rho_i}{1-\rho_i^2} \frac{\bar{Y}_{i+1} - m_i}{\sigma_{S,i}\sigma_{Y,i}},\\ \mathcal{I}^{(1,1)}_{i+1}(1) &= (\mathcal{I}^{(1)}_{i+1}(1))^2 - \mathcal{D}^{(1)}_{i+1}(\mathcal{I}^{(1)}_{i+1}(1)) = (\mathcal{I}^{(1)}_{i+1}(1))^2 - \frac{1}{a_{S,i}(1-\rho_i^2)},\end{aligned}$$

so that, $\mathcal{I}^{(1)}_{i+1}(1)$ and $\mathcal{I}^{(1,1)}_{i+1}(1)$ belong to $\mathbb{S}_{i,n}(\bar{X}, \bar{Y})$. Moreover, under **(ND)**, for any $p \geq 1$, it holds

$$\left|\mathcal{I}^{(1)}_{i+1}(1)(x_i, y_i, x_{i+1}, y_{i+1}, \mathbf{s}_{n+1})\right|^p \leq C\Big(1 + \frac{|x_{i+1} - x_i|^p}{(s_{i+1} - s_i)^p} + \frac{|y_{i+1} - m_i(y_i)|^p}{(s_{i+1} - s_i)^p}\Big),$$

and similarly,

$$\left|\mathcal{I}^{(1,1)}_{i+1}(1)(x_i, y_i, x_{i+1}, y_{i+1}, \mathbf{s}_{n+1})\right|^p \leq C\Big(1 + \frac{1}{(s_{i+1} - s_i)^p} + \frac{|x_{i+1} - x_i|^{2p}}{(s_{i+1} - s_i)^{2p}} + \frac{|y_{i+1} - m_i(y_i)|^{2p}}{(s_{i+1} - s_i)^{2p}}\Big).$$

Hence, from the space-time inequality (1.3), for any $c > 0$ and any $c' > c$, it holds

$$\left|\mathcal{I}^{(1)}_{i+1}(1)(x_i, y_i, x_{i+1}, y_{i+1}, \mathbf{s}_{n+1})\right|^p \bar{q}_c(s_i, s_{i+1}, x_i, y_i, x_{i+1}, y_{i+1}) \leq C(s_{i+1} - s_i)^{-\frac{p}{2}} \bar{q}_{c'}(s_i, s_{i+1}, x_i, y_i, x_{i+1}, y_{i+1}), \tag{2.25}$$

and

$$\left|\mathcal{I}^{(1,1)}_{i+1}(1)(x_i, y_i, x_{i+1}, y_{i+1}, \mathbf{s}_{n+1})\right|^p \bar{q}_c(s_i, s_{i+1}, x_i, y_i, x_{i+1}, y_{i+1}) \leq C(s_{i+1} - s_i)^{-p} \bar{q}_{c'}(s_i, s_{i+1}, x_i, y_i, x_{i+1}, y_{i+1}), \tag{2.26}$$

for some positive constant $C := C(T, c, c')$, $T \mapsto C(T, c, c')$ being non-decreasing. We thus conclude that $\mathcal{I}^{(1)}_{i+1}(1) \in \mathbb{M}_{i,n}(\bar{X}, \bar{Y}, -1/2)$ and $\mathcal{I}^{(1,1)}_{i+1}(1) \in \mathbb{M}_{i,n}(\bar{X}, \bar{Y}, -1)$ for any $i \in \{0, \dots, n\}$.

A straightforward generalization of the above example is the following property that will be frequently used in the sequel. We omit its proof.

Lemma 2.7. *Fix $n \in \mathbb{N}$ and $i \in \{0, \dots, n\}$.*

- *Let $\ell_1, \ell_2 \in \mathbb{Z}$ and $H_1 \in \mathbb{M}_{i,n}(\bar{X}, \bar{Y}, \ell_1/2), H_2 \in \mathbb{M}_{i,n}(\bar{X}, \bar{Y}, \ell_2/2)$. Then, one has $H_1 H_2 \in \mathbb{M}_{i,n}(\bar{X}, \bar{Y}, (\ell_1 + \ell_2)/2)$.*
- *Let $\ell \in \mathbb{Z}$ and $H \in \mathbb{M}_{i,n}(\bar{X}, \bar{Y}, \ell/2)$ such that $\mathcal{D}^{(\alpha_1)}_{i+1} H \in \mathbb{M}_{i,n}(\bar{X}, \bar{Y}, \ell'/2)$ for some $\alpha_1 \in \{1, 2\}$ and $\ell' \in \mathbb{Z}$.*
  - *It holds that $\mathcal{I}^{(\alpha_1)}_{i+1}(H) \in \mathbb{M}_{i,n}(\bar{X}, \bar{Y}, ((\ell - 1) \wedge \ell')/2)$ and $(\zeta_{i+1} - \zeta_i)\mathcal{I}^{(\alpha_1)}_{i+1}(H_1) \in \mathbb{M}_{i,n}(\bar{X}, \bar{Y}, ((\ell + 1) \wedge (\ell' + 2))/2)$.*
  - *Assume additionally that $\mathcal{D}^{(\alpha_1, \alpha_2)}_{i+1} H \in \mathbb{M}_{i,n}(\bar{X}, \bar{Y}, \ell''/2)$ for some $\ell'' \in \mathbb{Z}$ and $\alpha_2 \in \{1, 2\}$. Then it holds that $\mathcal{I}^{(\alpha_1, \alpha_2)}_{i+1}(H) \in \mathbb{M}_{i,n}(\bar{X}, \bar{Y}, ((\ell - 2) \wedge (\ell' - 1) \wedge \ell'')/2)$.*

Finally, we importantly emphasize that if $H \in \mathbb{M}_{i,n}(\bar{X}, \bar{Y}, \ell/2)$ for some $n \in \mathbb{N}$, $i \in \{0, \dots, n\}$ and $\ell \in \mathbb{Z}$, then, its conditional $L^p(\mathbb{P})$-moment is finite and also satisfies a time regularity estimate. More precisely, for any

$p \geq 1$, it holds

$$\|H\|_{p,i,n} \leq C(\zeta_{i+1} - \zeta_i)^{\ell/2}, \tag{2.27}$$

for some positive constant $C := C(T)$, $T \mapsto C(T)$ being non-decreasing. Indeed, using the fact that the sequence Z is independent of N as well as the upper-estimate (2.7) of Lemma 2.1 and finally (2.24), one directly gets

$$\begin{aligned}
\|H\|_{p,i,n}^p &= \mathbb{E}\Big[|H(\bar{X}_i, \bar{Y}_i, \bar{X}_{i+1}, \bar{Y}_{i+1}, \zeta^{n+1})|^p \Big| \bar{X}_i, \bar{Y}_i, \tau^{n+1}, N_T = n\Big] \\
&= \int_{\mathbb{R}^2} |H(\bar{X}_i, \bar{Y}_i, x_{i+1}, y_{i+1}, \zeta^{n+1})|^p \bar{p}(\zeta_i, \zeta_{i+1}, \bar{X}_i, \bar{Y}_i, x_{i+1}, y_{i+1})\, \mathrm{d}x_{i+1}\mathrm{d}y_{i+1} \\
&\leq C \int_{\mathbb{R}^2} |H(\bar{X}_i, \bar{Y}_i, x_{i+1}, y_{i+1}, \zeta^{n+1})|^p \bar{q}_{4\kappa}(\zeta_i, \zeta_{i+1}, \bar{X}_i, \bar{Y}_i, x_{i+1}, y_{i+1})\, \mathrm{d}x_{i+1}\mathrm{d}y_{i+1} \\
&\leq C(\zeta_{i+1} - \zeta_i)^{p\ell/2},
\end{aligned}$$

so that (2.27) directly follows. The previous conditional $L^p(\mathbb{P})$-moment estimate will be used at several places in the sequel.

3. Probabilistic representation for the couple (S_T, Y_T)

In this section, we establish a probabilistic representation for the marginal law (S_T, Y_T), or equivalently, for the law of (X_T, Y_T) which is based on the Markov chain $(\bar{X}, \bar{Y})$ introduced in the previous section. For $\gamma > 0$, we denote by $\mathcal{B}_\gamma(\mathbb{R}^2)$ the set of Borel measurable map $h : \mathbb{R}^2 \to \mathbb{R}$ satisfying the following exponential growth assumption at infinity, namely, for some positive constant C, for any $(x, y) \in \mathbb{R}^2$,

$$|h(x, y)| \leq C \exp(\gamma(|x|^2 + |y|^2)). \tag{3.1}$$

Theorem 3.1. *Let $T > 0$. Under assumptions* $(\mathbf{AR}_2)$ *and* **(ND)**, *the law of the couple (X_T, Y_T) given by the unique solution to the SDE (2.3) at time T starting from $(x_0 = \ln(s_0), y_0)$ at time 0 satisfies the following probabilistic representation: there exists a positive constant $c = c(T, b_Y, \kappa) := (8(C')^2\kappa T)^{-1}$, where κ and C' are defined respectively in (2.2) and (B.2) such that for any $0 \leq \gamma < c$ and any $h \in \mathcal{B}_\gamma(\mathbb{R}^2)$, it holds*

$$\mathbb{E}[h(X_T, Y_T)] = \mathbb{E}\Big[h(\bar{X}_{N_T+1}, \bar{Y}_{N_T+1}) \prod_{i=1}^{N_T+1} \theta_i\Big], \tag{3.2}$$

where the random variables $\theta_i \in \mathbb{S}_{i-1,n}(\bar{X}, \bar{Y})$ are defined by

$$\theta_i = (f(\zeta_i - \zeta_{i-1}))^{-1}\Big[\mathcal{I}_i^{(1,1)}(c_S^i) - \mathcal{I}_i^{(1)}(c_S^i) + \mathcal{I}_i^{(2,2)}(c_Y^i) + \mathcal{I}_i^{(2)}(b_Y^i) + \mathcal{I}_i^{(1,2)}(c_{Y,S}^i)\Big],\ 1 \leq i \leq N_T, \tag{3.3}$$

$$\theta_{N_T+1} = (1 - F(T - \zeta_{N_T}))^{-1}, \tag{3.4}$$

with

$$\begin{aligned}
c_S^i &:= \frac{1}{2}\Big(a_S(\zeta_i, \bar{Y}_i) - a_S(\zeta_i, m_{i-1})\Big), \\
c_Y^i &:= \frac{1}{2}\Big(a_Y(\zeta_i, \bar{Y}_i) - a_Y(\zeta_i, m_{i-1})\Big), \\
b_Y^i &:= b_Y(\zeta_i, \bar{Y}_i) - b_Y(\zeta_i, m_{i-1}), \\
c_{Y,S}^i &:= \rho((\sigma_S\sigma_Y)(\zeta_i, \bar{Y}_i) - (\sigma_S\sigma_Y)(\zeta_i, m_{i-1})).
\end{aligned}$$

In particular, the random variable appearing inside the expectation in the right-hand side of (3.2) *is in* $L^1(\mathbb{P})$.

Assume furthermore that N *is a renewal process with* $Beta(\alpha, 1)$ *jump times. For any* $p \geq 1$ *satisfying* $p(\frac{1}{2} - \alpha) \leq 1 - \alpha$, *for any* γ *such that* $0 \leq p\gamma < c$ *and any* $h \in \mathcal{B}_\gamma(\mathbb{R}^2)$, *the random variable appearing inside the expectation in the right-hand side of* (3.2) *admits a finite* $L^p(\mathbb{P})$*-moment. In particular, if* $\alpha = 1/2$ *then for any* $p \geq 1$, *for any* $h \in \mathcal{B}_\gamma(\mathbb{R}^2)$ *with* $0 \leq p\gamma < c$, *the* $L^p(\mathbb{P})$*-moment is finite.*

The proof of Theorem 3.1 is postponed to Appendix A.1.

Remark 3.2. We importantly point out that in order to device an unbiased Monte Carlo estimator from the probabilistic representation formula (3.2), one still needs to compute integrals of the form $\int_{\zeta_i}^{\zeta_{i+1}} h(u, m_{u,\zeta_i}(\bar{Y}_i))\, du$ for h given by a_S, a_Y or $\sigma_S \sigma_Y$. In general, the flow $(s, x) \mapsto m_{s,t}(x)$ generated by the unique solution to the ODE $\dot{m}_t = b_Y(t, m_t)$ is not explicit so that one has to resort to some numerical discretization scheme for ODEs. In such situation, numerous schemes with relatively small computational cost (compared to the simulation of the corresponding SDE) can be used. This will inevitably introduce a bias in our Monte Carlo estimator which could be easily quantified. Let us however mention that an unbiased simulation method of the aforementioned integrals could be introduced based on a probabilistic interpretation of the time integral as soon as the flow of the ODE is explicit.

Remark 3.3. We also point out that our approach could be extended without additional difficulties to the case of a multidimensional volatility process with unbounded drift. However, we do not elaborate on this aspect since we believe that this would add an additional layer of technicality that could detract from the understanding of the key idea which is to freeze the coefficients of the dynamics along the flow of the deterministic system obtained by removing the noise in the original SDE satisfied by the stochastic volatility model.

Remark 3.4. In order to apply the above probabilistic representation to the marginal law (S_T, Y_T) and payoff function f, one first writes $f(S_T, Y_T) = f(\exp(X_T), Y_T) := h(X_T, Y_T)$. Hence, if f is a measurable function with polynomial growth at infinity then $h(.,.) := f(\exp(.), .) \in \mathcal{B}_\gamma(\mathbb{R}^2)$ for any positive γ so that any measurable function f with polynomial growth at infinity belong to the admissible class of payoff function for which our probabilistic representation on (S_T, Y_T) holds.

4. Integration by parts formulae

In this section, we establish two IBP formulae for the law of the couple (S_T, Y_T). More precisely, we are interested in providing a Bismut-Elworthy-Li formula for the two quantities

$$\partial_{s_0} \mathbb{E}[h(S_T, Y_T)], \quad \partial_{y_0} \mathbb{E}[h(S_T, Y_T)].$$

Throughout this section, we will assume that $(\mathbf{AR}_3)$ and **(ND)** are in force.

Our strategy is divided into two steps as follows:

Step 1: The first step was performed with the probabilistic representation established in Theorem 3.1 for the couple (X_T, Y_T) involving the two-dimensional Markov chain $(\bar{X}, \bar{Y})$ evolving on a time grid governed by the jump times of the renewal process N. Introducing $h(x, y) = f(e^x, y)$ and assuming that f is of polynomial growth at infinity, it is sufficient to consider the two quantities

$$\partial_{s_0} \mathbb{E}\Big[h(\bar{X}_{N_T+1}, \bar{Y}_{N_T+1}) \prod_{i=1}^{N_T+1} \theta_i\Big], \quad \partial_{y_0} \mathbb{E}\Big[h(\bar{X}_{N_T+1}, \bar{Y}_{N_T+1}) \prod_{i=1}^{N_T+1} \theta_i\Big]$$

recalling that $x_0 = \ln(s_0)$.

Step 2: At this stage, one might be tempted to perform a standard IBP formula as presented in Nualart [17] on the whole time interval $[0, T]$. However, such a strategy is likely to fail. The main reason is that the Skorokhod integral of the product of weights $\prod_{i=1}^{N_T+1} \theta_i$ will inevitably involve the Malliavin derivative of θ_i which will in turn raise some integrability issues of the resulting Malliavin weight. The key idea that we use in order to circumvent this issue consists in performing local IBP formulae on each of the random intervals $[\zeta_i, \zeta_{i+1}]$, $i = 0, \ldots, N_T$, that is, by using the noise of the Markov chain on this specific time interval and then by combining all these local IBP formulae in a suitable way.

To implement successfully our strategy, two main ingredients are needed. Our first ingredient consists in transferring the partial derivatives ∂_{s_0} and ∂_{y_0} on the expectation forward in time from the first time interval $[0, \zeta_1]$ to the interval on which we perform the local IBP formula, say $[\zeta_i, \zeta_{i+1}]$. Our second ingredient consists in combining these various local IBP formulae in an adequate manner. Roughly speaking, we will consider a weighted sum of each IBP formula, the weight being precisely the length of the corresponding time interval.

4.1. The transfer of derivative formula

Lemma 4.1. *Let $h \in \mathcal{C}^1_p(\mathbb{R}^2)$ and $n \in \mathbb{N}$. The maps $\mathbb{R}^2 \ni (x, y) \mapsto \mathbb{E}_{i,n}\Big[h(\bar{X}_{i+1}, \bar{Y}_{i+1})\theta_{i+1}|(\bar{X}_i, \bar{Y}_i) = (x, y)\Big]$, $i \in \{0, \ldots, n\}$, belong to $\mathcal{C}^1_p(\mathbb{R}^2)$ a.s. Moreover, the following transfer of derivative formulae hold*

$$\partial_{s_0}\mathbb{E}_{0,n}\Big[h(\bar{X}_1, \bar{Y}_1)\theta_1\Big] = \mathbb{E}_{0,n}\Big[\partial_{\bar{X}_1} h(\bar{X}_1, \bar{Y}_1)\frac{\theta_1}{s_0}\Big], \tag{4.1}$$

while for $1 \leq i \leq n$,

$$\partial_{\bar{X}_i}\mathbb{E}_{i,n}\Big[h(\bar{X}_{i+1}, \bar{Y}_{i+1})\theta_{i+1}\Big] = \mathbb{E}_{i,n}\Big[\partial_{\bar{X}_{i+1}} h(\bar{X}_{i+1}, \bar{Y}_{i+1})\theta_{i+1}\Big]. \tag{4.2}$$

Similarly, the following transfer of derivative formulae hold: for any $0 \leq i \leq n-1$

$$\begin{aligned}\partial_{\bar{Y}_i}\mathbb{E}_{i,n}\Big[h(\bar{X}_{i+1}, \bar{Y}_{i+1})\theta_{i+1}\Big] &= \mathbb{E}_{i,n}\Big[\partial_{\bar{Y}_{i+1}} h(\bar{X}_{i+1}, \bar{Y}_{i+1})\overrightarrow{\theta}^{e,Y}_{i+1}\Big] + \mathbb{E}_{i,n}\Big[\partial_{\bar{X}_{i+1}} h(\bar{X}_{i+1}, \bar{Y}_{i+1})\overrightarrow{\theta}^{e,X}_{i+1}\Big] \\ &\quad + \mathbb{E}_{i,n}\Big[h(\bar{X}_{i+1}, \bar{Y}_{i+1})\overrightarrow{\theta}^{c}_{i+1}\Big],\end{aligned} \tag{4.3}$$

with

$$\begin{aligned}
\overrightarrow{\theta}^{e,Y}_{i+1} &= (f(\zeta_{i+1} - \zeta_i))^{-1}\Big[\mathcal{I}^{(1,1)}_{i+1}(d^{i+1}_S) + \mathcal{I}^{(2,2)}_{i+1}(d^{i+1}_Y) + \mathcal{I}^{(1)}_{i+1}(e^{Y,i+1}_S) + \mathcal{I}^{(2)}_{i+1}(e^{Y,i+1}_Y) + \mathcal{I}^{(1,2)}_{i+1}(d^{i+1}_{Y,S})\Big], \\
\overrightarrow{\theta}^{e,X}_{i+1} &= (f(\zeta_{i+1} - \zeta_i))^{-1}\mathcal{I}^{(1)}_{i+1}(e^{X,i+1}_S), \\
\overrightarrow{\theta}^{c}_{i+1} &= \mathcal{I}^{(1)}_{i+1}\Big(\partial_{\bar{Y}_i}\bar{X}_{i+1}\theta_{i+1} - \overrightarrow{\theta}^{e,X}_{i+1}\Big) + \partial_{\bar{Y}_i}\theta_{i+1} \\
&\quad + \mathcal{I}^{(2)}_{i+1}\Big(m'_i\theta_{i+1} - \overrightarrow{\theta}^{e,Y}_{i+1} + \Big(\sigma'_{Y,i}\Big(\rho_i Z^1_{i+1} + \sqrt{1-\rho_i^2}Z^2_{i+1}\Big) + \sigma_{Y,i}\frac{\rho'_i}{\sqrt{1-\rho_i^2}}\Big(\sqrt{1-\rho_i^2}Z^1_{i+1} - \rho_i Z^2_{i+1}\Big)\Big)\theta_{i+1}\Big), \\
d^{i+1}_S &= m'_i c^{i+1}_S, \\
d^{i+1}_Y &= m'_i c^{i+1}_Y, \\
d^{i+1}_{Y,S} &= m'_i c^{i+1}_{Y,S}, \\
e^{Y,i+1}_S &= -m'_i c^{i+1}_S + \partial_{\bar{Y}_i} c^{i+1}_{Y,S}, \\
e^{Y,i+1}_Y &= m'_i b^{i+1}_Y + \partial_{\bar{Y}_i} c^{i+1}_Y, \\
e^{X,i+1}_S &= \partial_{\bar{Y}_i} c^{i+1}_S.
\end{aligned}$$

For $i = n$, one also has

$$\partial_{\bar{Y}_n} \mathbb{E}_{n,n}\Big[h(\bar{X}_{n+1}, \bar{Y}_{n+1})\theta_{n+1}\Big] = \mathbb{E}_{n,n}\Big[\partial_{\bar{Y}_{n+1}} h(\bar{X}_{n+1}, \bar{Y}_{n+1}) \overrightarrow{\theta}^{e,Y}_{n+1}\Big] + \mathbb{E}_{n,n}\Big[\partial_{\bar{X}_{n+1}} h(\bar{X}_{n+1}, \bar{Y}_{n+1}) \overrightarrow{\theta}^{e,X}_{n+1}\Big] + \mathbb{E}_{n,n}\Big[h(\bar{X}_{n+1}, \bar{Y}_{n+1}) \overrightarrow{\theta}^{c}_{n+1}\Big], \tag{4.4}$$

with

$$\overrightarrow{\theta}^{e,Y}_{n+1} = (1 - F(T - \zeta_n))^{-1}\Big(m'_n + \sigma'_{Y,n}\Big(\rho_n Z^1_{n+1} + \sqrt{1-\rho_n^2} Z^2_{n+1}\Big) + \sigma_{Y,n} \frac{\rho'_n}{\sqrt{1-\rho_n^2}}\Big(\sqrt{1-\rho_n^2} Z^1_{n+1} - \rho_n Z^2_{n+1}\Big)\Big),$$

$$\overrightarrow{\theta}^{e,X}_{n+1} = (1 - F(T - \zeta_n))^{-1}\Big(-\frac{1}{2} a'_{S,n} + \sigma'_{S,n} Z^1_{n+1}\Big),$$

and we set $\overrightarrow{\theta}^{c}_{n+1} = 0$ for notational convenience.

Finally, the weight sequences $(\overrightarrow{\theta}^{e,Y}_i)_{1 \le i \le n+1}$, $(\overrightarrow{\theta}^{e,X}_i)_{1 \le i \le n+1}$ and $(\overrightarrow{\theta}^{c}_i)_{1 \le i \le n+1}$ defined above satisfy for $i \in \{1, \ldots, n\}$

$$f(\zeta_i - \zeta_{i-1}) \overrightarrow{\theta}^{e,Y}_i,\ f(\zeta_i - \zeta_{i-1}) \overrightarrow{\theta}^{c}_i \in \mathbb{M}_{i-1,n}(\bar{X}, \bar{Y}, -1/2), \quad f(\zeta_i - \zeta_{i-1}) \overrightarrow{\theta}^{e,X}_i \in \mathbb{M}_{i-1,n}(\bar{X}, \bar{Y}, 0),$$

and $(1 - F(T - \zeta_n)) \overrightarrow{\theta}^{e,Y}_{n+1} \in \mathbb{M}_{n,n}(\bar{X}, \bar{Y}, 0)$, $(1 - F(T - \zeta_n)) \overrightarrow{\theta}^{e,X}_{n+1}, \in \mathbb{M}_{n,n}(\bar{X}, \bar{Y}, 1/2)$.

The proof of Lemma 4.1 is postponed to Appendix A.2. The transfer of derivative procedure starts on the first time interval $[0, \zeta_1]$ according to formulae (4.1) and (4.3) (for $i = 0$). It expresses the fact that the flow derivatives ∂_{s_0} and ∂_{y_0} of the conditional expectations on the left-hand side of the equations are transferred to derivative operators $\partial_{\bar{X}_1}$ and $\partial_{\bar{Y}_1}$ on the test function h appearing on the right-hand side. Remark that the first order derivatives of h have been written ubiquitously as $\partial_{\bar{X}_{i+1}} h(\bar{X}_{i+1}, \bar{Y}_{i+1})$ and $\partial_{\bar{Y}_{i+1}} h(\bar{X}_{i+1}, \bar{Y}_{i+1})$.

Then, by the Markov property satisfied by the process $(\bar{X}, \bar{Y})$, the function h appearing inside the (conditional) expectations on the right-hand side of (4.1) and (4.3) (for $i = 0$) will be given by the conditional expectation appearing on the left-hand side of the same equations but for $i = 1$. The transfer of derivative formulae for the following time intervals are obtained by induction using (4.2) and (4.3) up to the last time interval. Doing so, we obtain various transfer of derivative formulae by transferring successively the derivative operators through all intervals forward in time.

4.2. The integration by parts formulae

We first define the weights that will be used in our IBP formulae. For an integer n, on the set $\{N_T = n\}$, for any $k \in \{1, \ldots, n+1\}$ and any $j \in \{1, \ldots, k\}$, we define

$$\begin{aligned}
\overrightarrow{\theta}^{\mathcal{I}^{(1),n+1}_k} &:= \prod_{i=k+1}^{n+1} \theta_i \times \mathcal{I}^{(1)}_k(\theta_k) \times \prod_{i=1}^{k-1} \theta_i, \\
\overrightarrow{\theta}^{C^{n+1}_j} &:= \prod_{i=j+1}^{n+1} \theta_i \times \overrightarrow{\theta}^{c}_j \times \prod_{i=1}^{j-1} \overrightarrow{\theta}^{e,Y}_i, \\
\overrightarrow{\theta}^{\mathcal{I}^{(2),n+1}_k} &:= \prod_{i=k+1}^{n+1} \theta_i \times \mathcal{I}^{(2)}_k(\overrightarrow{\theta}^{e,Y}_k) \times \prod_{i=1}^{k-1} \overrightarrow{\theta}^{e,Y}_i, \\
\overrightarrow{\theta}^{\mathcal{I}^{(1),n+1}_k}_j &:= \prod_{i=k+1}^{n+1} \theta_i \times \mathcal{I}^{(1)}_k(\theta_k) \times \prod_{i=j+1}^{k-1} \theta_i \times \overrightarrow{\theta}^{e,X}_j \times \prod_{i=1}^{j-1} \overrightarrow{\theta}^{e,Y}_i, \quad j = 1, \ldots, k-1,
\end{aligned}$$

$$\overrightarrow{\theta}_k^{\mathcal{I}_k^{(1),n+1}} := \prod_{i=k+1}^{n+1} \theta_i \times \mathcal{I}_k^{(1)}(\overrightarrow{\theta}_k^{e,X}) \times \prod_{i=1}^{k-1} \overrightarrow{\theta}_i^{e,Y},$$

with the convention $\prod_\emptyset \cdots = 1$. Having the above definitions at hand, we are now able to state our IBP formulae.

Theorem 4.2. *Let $T > 0$. Under assumptions* **($\mathbf{AR}_3$)** *and* **(ND)**, *the law of the couple (X_T, Y_T), given by the unique solution to the SDE (2.3) at time T starting from $(x_0 = \ln(s_0), y_0)$ at time 0, satisfies the following Bismut-Elworthy-Li IBP formulae: there exists some positive constant $c := c(T, b_Y, \kappa)$ such that for any $0 \leq \gamma < c$, any $h \in \mathcal{B}_\gamma(\mathbb{R}^2)$ and any $(s_0, y_0) \in (0,\infty) \times \mathbb{R}$, it holds*

$$s_0 T \partial_{s_0} \mathbb{E}\Big[h(X_T, Y_T)\Big] = \mathbb{E}\Big[h(\bar{X}_{N_T+1}, \bar{Y}_{N_T+1}) \sum_{k=1}^{N_T+1} (\zeta_k - \zeta_{k-1}) \overrightarrow{\theta}^{\mathcal{I}_k^{(1),N_T+1}}\Big], \tag{4.5}$$

and

$$T\partial_{y_0}\mathbb{E}\Big[h(X_T, Y_T)\Big] = \mathbb{E}\left[h(\bar{X}_{N_T+1}, \bar{Y}_{N_T+1}) \sum_{k=1}^{N_T+1} (\zeta_k - \zeta_{k-1}) \Big(\overrightarrow{\theta}^{\mathcal{I}_k^{(2),N_T+1}} + \sum_{j=1}^{k} \Big(\overrightarrow{\theta}^{C_j^{N_T+1}} + \overrightarrow{\theta}_j^{\mathcal{I}_k^{(1),N_T+1}}\Big)\Big)\right]. \tag{4.6}$$

Moreover, if N is a renewal process with $Beta(\alpha, 1)$ jump times, then, for any $p \geq 1$ satisfying $p(\frac{1}{2} - \alpha) \leq 1 - \alpha$, for any γ such that $0 \leq p\gamma < c^{-1}$ and any $h \in \mathcal{B}_\gamma(\mathbb{R}^2)$, the random variables appearing inside the expectation in the right-hand side of (4.5) and (4.6) admit a finite $L^p(\mathbb{P})$-moment. In particular, if $\alpha = 1/2$ then for any $p \geq 1$, for any $h \in \mathcal{B}_\gamma(\mathbb{R}^2)$ with $0 \leq p\gamma < c$, the $L^p(\mathbb{P})$-moment is finite.

Proof. We only prove the IBP formula (4.6). The proof of (4.5) follows by completely analogous (and actually more simple) arguments and is thus omitted.

Step 1: proof of the IBP formula (4.6) for $h \in \mathcal{C}_b^1(\mathbb{R}^2)$.
Let $h \in \mathcal{C}_b^1(\mathbb{R}^2)$. From Theorem 3.1 and Fubini's theorem, we write

$$\mathbb{E}[h(X_T, Y_T)] = \sum_{n \geq 0} \mathbb{E}\Big[\mathbb{E}\Big[h(\bar{X}_{n+1}, \bar{Y}_{n+1}) \prod_{i=1}^{n+1} \theta_i | \tau^{n+1}\Big] \mathbf{1}_{\{N_T = n\}}\Big], \tag{4.7}$$

where we used the fact that $\{N_T = n\} = \{\tau_{n+1} > T\} \cap \{\tau_n \leq T\}$. In most of the arguments below, we will work on the set $\{N_T = n\}$. In order to perform our induction argument forward in time through the Markov chain structure, we define for $k \in \{0, \ldots, n\}$ the functions

$$H_k(\bar{X}_k, \bar{Y}_k) := \mathbb{E}_{k,n}\Big[h(\bar{X}_{n+1}, \bar{Y}_{n+1}) \prod_{i=k+1}^{n+1} \theta_i\Big] = \mathbb{E}\Big[h(\bar{X}_{n+1}, \bar{Y}_{n+1}) \prod_{i=k+1}^{n+1} \theta_i | \bar{X}_k, \bar{Y}_k, \tau^{n+1}, N_T = n\Big].$$

We also let $H_{n+1}(\bar{X}_{n+1}, \bar{Y}_{n+1}) := h(\bar{X}_{n+1}, \bar{Y}_{n+1})$. Note that we omit the dependence with respect to the sequence τ^{n+1} in the definition of the (random) maps $(H_k)_{0 \leq k \leq n+1}$. From the above definition and using **(ND)**, **($\mathbf{AR}_3$)**, it follows that the map H_k belongs to $\mathcal{C}_p^1(\mathbb{R}^2)$ *a.s.* for any $0 \leq k \leq n+1$. Moreover, from the tower property of conditional expectation the following relation is satisfied for any $k \in \{0, \ldots, n\}$

$$H_k(\bar{X}_k, \bar{Y}_k) = \mathbb{E}_{k,n}[H_{k+1}(\bar{X}_{k+1}, \bar{Y}_{k+1}) \theta_{k+1}]. \tag{4.8}$$

Now, iterating the transfer of derivative formula (4.3) in Lemma 4.1, for any $k \in \{1, \ldots, n\}$, we obtain[2]

$$\begin{aligned}
\partial_{y_0} H_0(\bar{X}_0, \bar{Y}_0) &= \partial_{y_0} \mathbb{E}_{0,n}[H_1(\bar{X}_1, \bar{Y}_1)\theta_1] \\
&= \mathbb{E}_{0,n}[\partial_{\bar{Y}_1} H_1(\bar{X}_1, \bar{Y}_1) \overrightarrow{\theta}^{e,Y}_1] + \mathbb{E}_{0,n}[\partial_{\bar{X}_1} H_1(\bar{X}_1, \bar{Y}_1) \overrightarrow{\theta}^{e,X}_1] + \mathbb{E}_{0,n}[H_1(\bar{X}_1, \bar{Y}_1) \overrightarrow{\theta}^{c}_1] \\
&= \cdots \\
&= \mathbb{E}_{0,n}[\mathcal{D}^{(2)}_k H_k(\bar{X}_k, \bar{Y}_k) \prod_{i=1}^{k} \overrightarrow{\theta}^{e,Y}_i] + \sum_{j=1}^{k} \mathbb{E}_{0,n}[H_j(\bar{X}_j, \bar{Y}_j) \overrightarrow{\theta}^{c}_j \prod_{i=1}^{j-1} \overrightarrow{\theta}^{e,Y}_i] \\
&\quad + \sum_{j=1}^{k} \mathbb{E}_{0,n}[\mathcal{D}^{(1)}_j H_j(\bar{X}_j, \bar{Y}_j) \overrightarrow{\theta}^{e,X}_j \prod_{i=1}^{j-1} \overrightarrow{\theta}^{e,Y}_i].
\end{aligned} \tag{4.9}$$

Hence, by the Lebesgue differentiation theorem, we deduce

$$\begin{aligned}
\partial_{y_0} \mathbb{E}\Big[h(\bar{X}_{n+1}, \bar{Y}_{n+1}) \prod_{i=1}^{n+1} \theta_i \Big| \tau^{n+1}\Big] &= \partial_{y_0} \mathbb{E}\Big[H_0(\bar{X}_0, \bar{Y}_0) \Big| \tau^{n+1}\Big] \\
&= \mathbb{E}\Big[\partial_{y_0} H_0(\bar{X}_0, \bar{Y}_0) \Big| \tau^{n+1}\Big] \\
&= \mathbb{E}\Big[\mathcal{D}^{(2)}_k H_k(\bar{X}_k, \bar{Y}_k) \prod_{i=1}^{k} \overrightarrow{\theta}^{e,Y}_i \Big| \tau^{n+1}\Big] + \sum_{j=1}^{k} \mathbb{E}\Big[H_j(\bar{X}_j, \bar{Y}_j) \overrightarrow{\theta}^{c}_j \prod_{i=1}^{j-1} \overrightarrow{\theta}^{e,Y}_i \Big| \tau^{n+1}\Big] \\
&\quad + \sum_{j=1}^{k} \mathbb{E}\Big[\mathcal{D}^{(1)}_j H_j(\bar{X}_j, \bar{Y}_j) \overrightarrow{\theta}^{e,X}_j \prod_{i=1}^{j-1} \overrightarrow{\theta}^{e,Y}_i \Big| \tau^{n+1}\Big].
\end{aligned} \tag{4.10}$$

To further simplify the first term appearing on the right-hand side of (4.10), we use the tower property of conditional expectation (w.r.t $\mathbb{E}_{k-1,n}[.]$) and the integration by parts formula (2.17). For any $k \in \{1, \ldots, n\}$, we obtain

$$\mathbb{E}\Big[\mathcal{D}^{(2)}_k H_k(\bar{X}_k, \bar{Y}_k) \overrightarrow{\theta}^{e,Y}_k \Big| \mathcal{G}_{k-1}, \tau^{n+1}\Big] = \mathbb{E}\Big[H_k(\bar{X}_k, \bar{Y}_k) \mathcal{I}^{(2)}_k(\overrightarrow{\theta}^{e,Y}_k) \Big| \mathcal{G}_{k-1}, \tau^{n+1}\Big]. \tag{4.11}$$

We also simplify the third term appearing on the right-hand side of (4.10), by using the transfer of derivatives formula (4.2) up to the time interval $[\zeta_{k-1}, \zeta_k]$. For any $j \in \{1, \ldots, k\}$, it holds

$$\mathbb{E}\Big[\mathcal{D}^{(1)}_j H_j(\bar{X}_j, \bar{Y}_j) \overrightarrow{\theta}^{e,X}_j \prod_{i=1}^{j-1} \overrightarrow{\theta}^{e,Y}_i \Big| \tau^{n+1}\Big] = \mathbb{E}\Big[\mathcal{D}^{(1)}_k H_k(\bar{X}_k, \bar{Y}_k) \prod_{i=j+1}^{k} \theta_i \overrightarrow{\theta}^{e,X}_j \prod_{i=1}^{j-1} \overrightarrow{\theta}^{e,Y}_i \Big| \tau^{n+1}\Big],$$

so that, if $j \in \{1, \ldots, k-1\}$, taking conditional expectation (using again $\mathbb{E}_{k-1,n}[.]$) and then performing an IBP formula on the last time interval $[\zeta_{k-1}, \zeta_k]$ yield

$$\mathbb{E}\Big[\mathcal{D}^{(1)}_k H_k(\bar{X}_k, \bar{Y}_k) \prod_{i=j+1}^{k} \theta_i \overrightarrow{\theta}^{e,X}_j \prod_{i=1}^{j-1} \overrightarrow{\theta}^{e,Y}_i \Big| \tau^{n+1}\Big] = \mathbb{E}\Big[H_k(\bar{X}_k, \bar{Y}_k) \mathcal{I}^{(1)}_k(\theta_k) \prod_{i=j+1}^{k-1} \theta_i \overrightarrow{\theta}^{e,X}_j \prod_{i=1}^{j-1} \overrightarrow{\theta}^{e,Y}_i \Big| \tau^{n+1}\Big], \tag{4.12}$$

[2]As before, we use the convention $\sum_{\emptyset} \cdots = 0$, $\prod_{\emptyset} \cdots = 1$.

while if $j = k$, we obtain

$$\mathbb{E}\Big[\mathcal{D}^{(1)}_k H_k(\bar{X}_k, \bar{Y}_k) \prod_{i=j+1}^{k} \theta_i \overrightarrow{\theta}^{e,X}_j \prod_{i=1}^{j-1} \overrightarrow{\theta}^{e,Y}_i \Big| \tau^{n+1}\Big] = \mathbb{E}\Big[H_k(\bar{X}_k, \bar{Y}_k) \mathcal{I}^{(1)}_k(\overrightarrow{\theta}^{e,X}_k) \prod_{i=1}^{k-1} \overrightarrow{\theta}^{e,Y}_i \Big| \tau^{n+1}\Big]. \tag{4.13}$$

Coming back to (4.10), gathering (4.11), (4.12), (4.13) and using the definition of the maps $(H_k)_{0\leq k\leq n+1}$, we thus deduce

$$\begin{aligned}
\partial_{y_0}\mathbb{E}\Big[h(\bar{X}_{n+1}, \bar{Y}_{n+1}) \prod_{i=1}^{n+1} \theta_i \Big| \tau^{n+1}\Big] &= \mathbb{E}\Big[H_k(\bar{X}_k, \bar{Y}_k) \mathcal{I}^{(2)}_k(\overrightarrow{\theta}^{e,Y}_k) \times \prod_{i=1}^{k-1} \overrightarrow{\theta}^{e,Y}_i \Big| \tau^{n+1}\Big] + \sum_{j=1}^{k} \mathbb{E}\Big[H_j(\bar{X}_j, \bar{Y}_j) \overrightarrow{\theta}^{c}_j \\
&\quad \times \prod_{i=1}^{j-1} \overrightarrow{\theta}^{e,Y}_i \Big| \tau^{n+1}\Big] \\
&\quad + \sum_{j=1}^{k-1} \mathbb{E}\Big[H_k(\bar{X}_k, \bar{Y}_k) \mathcal{I}^{(1)}_k(\theta_k) \times \prod_{i=j+1}^{k-1} \theta_i \times \overrightarrow{\theta}^{e,X}_j \prod_{i=1}^{j-1} \overrightarrow{\theta}^{e,Y}_i \Big| \tau^{n+1}\Big] \\
&\quad + \mathbb{E}\Big[H_k(\bar{X}_k, \bar{Y}_k) \mathcal{I}^{(1)}_k(\overrightarrow{\theta}^{e,X}_k) \prod_{i=1}^{k-1} \overrightarrow{\theta}^{e,Y}_i \Big| \tau^{n+1}\Big] \\
&= \mathbb{E}\Big[h(\bar{X}_{n+1}, \bar{Y}_{n+1}) \prod_{i=k+1}^{n+1} \theta_i \times \mathcal{I}^{(2)}_k(\overrightarrow{\theta}^{e,Y}_k) \times \prod_{i=1}^{k-1} \overrightarrow{\theta}^{e,Y}_i \Big| \tau^{n+1}\Big] \\
&\quad + \sum_{j=1}^{k} \mathbb{E}\Big[h(\bar{X}_{n+1}, \bar{Y}_{n+1}) \prod_{i=j+1}^{n+1} \theta_i \times \overrightarrow{\theta}^{c}_j \times \prod_{i=1}^{j-1} \overrightarrow{\theta}^{e,Y}_i \Big| \tau^{n+1}\Big] \\
&\quad + \sum_{j=1}^{k-1} \mathbb{E}\Big[h(\bar{X}_{n+1}, \bar{Y}_{n+1}) \prod_{i=k+1}^{n+1} \theta_i \times \mathcal{I}^{(1)}_k(\theta_k) \times \prod_{i=j+1}^{k-1} \theta_i \times \overrightarrow{\theta}^{e,X}_j \\
&\quad \times \prod_{i=1}^{j-1} \overrightarrow{\theta}^{e,Y}_i \Big| \tau^{n+1}\Big] \\
&\quad + \mathbb{E}\Big[h(\bar{X}_{n+1}, \bar{Y}_{n+1}) \prod_{i=k+1}^{n+1} \theta_i \times \mathcal{I}^{(1)}_k(\overrightarrow{\theta}^{e,X}_k) \prod_{i=1}^{k-1} \overrightarrow{\theta}^{e,Y}_i \Big| \tau^{n+1}\Big].
\end{aligned} \tag{4.14}$$

In the case $k = n+1$, using the transfer of derivative formulae (4.4), (4.2) of Lemma 4.1 on the last time interval and then performing the IBP formula (2.17), we obtain the representation

$$\begin{aligned}
\partial_{y_0}\mathbb{E}\Big[h(\bar{X}_{n+1}, \bar{Y}_{n+1}) \prod_{i=1}^{n+1} \theta_i \Big| \tau^{n+1}\Big] &= \mathbb{E}\Big[\mathcal{D}^{(2)}_{n+1} h(\bar{X}_{n+1}, \bar{Y}_{n+1}) \prod_{i=1}^{n+1} \overrightarrow{\theta}^{e,Y}_i \Big| \tau^{n+1}\Big] + \sum_{j=1}^{n+1} \mathbb{E}\Big[H_j(\bar{X}_j, \bar{Y}_j) \overrightarrow{\theta}^{c}_j \prod_{i=1}^{j-1} \overrightarrow{\theta}^{e,Y}_i \Big| \tau^{n+1}\Big] \\
&\quad + \sum_{j=1}^{n+1} \mathbb{E}\Big[\mathcal{D}^{(1)}_j H_j(\bar{X}_j, \bar{Y}_j) \overrightarrow{\theta}^{e,X}_j \prod_{i=1}^{j-1} \overrightarrow{\theta}^{e,Y}_i \Big| \tau^{n+1}\Big] \\
&= \mathbb{E}\Big[h(\bar{X}_{n+1}, \bar{Y}_{n+1}) \mathcal{I}^{(2)}_{n+1}(\overrightarrow{\theta}^{e,Y}_{n+1}) \prod_{i=1}^{n} \overrightarrow{\theta}^{e,Y}_i \Big| \tau^{n+1}\Big]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n+1}\mathbb{E}\Big[H_j(\bar{X}_j,\bar{Y}_j)\overrightarrow{\theta}_j^{c}\prod_{i=1}^{j-1}\overrightarrow{\theta}_i^{e,Y}\Big|\tau^{n+1}\Big]\\
&+\sum_{j=1}^{n+1}\mathbb{E}\Big[\mathcal{D}^{(1)}_{n+1}h(\bar{X}_{n+1},\bar{Y}_{n+1})\prod_{i=j+1}^{n+1}\theta_i\times\overrightarrow{\theta}_j^{e,X}\times\prod_{i=1}^{j-1}\overrightarrow{\theta}_i^{e,Y}\Big|\tau^{n+1}\Big]\\
=&\,\mathbb{E}\Big[h(\bar{X}_{n+1},\bar{Y}_{n+1})\mathcal{I}^{(2)}_{n+1}(\overrightarrow{\theta}_{n+1}^{e,Y})\times\prod_{i=1}^{n}\overrightarrow{\theta}_i^{e,Y}\Big|\tau^{n+1}\Big]\\
&+\sum_{j=1}^{n+1}\mathbb{E}\Big[h(\bar{X}_{n+1},\bar{Y}_{n+1})\prod_{i=j+1}^{n+1}\theta_i\times\overrightarrow{\theta}_j^{c}\times\prod_{i=1}^{j-1}\overrightarrow{\theta}_i^{e,Y}\Big|\tau^{n+1}\Big]\\
&+\sum_{j=1}^{n}\mathbb{E}\Big[h(\bar{X}_{n+1},\bar{Y}_{n+1})\mathcal{I}^{(1)}_{n+1}(\theta_{n+1})\times\prod_{i=j+1}^{n}\theta_i\times\overrightarrow{\theta}_j^{e,X}\times\prod_{i=1}^{j-1}\overrightarrow{\theta}_i^{e,Y}\Big|\tau^{n+1}\Big]\\
&+\mathbb{E}\Big[h(\bar{X}_{n+1},\bar{Y}_{n+1})\mathcal{I}^{(1)}_{n+1}(\overrightarrow{\theta}_{n+1}^{e,X})\times\prod_{i=1}^{n}\overrightarrow{\theta}_i^{e,Y}\Big|\tau^{n+1}\Big],
\end{aligned}\tag{4.15}
$$

where, for the last term appearing in the right-hand side of the above identities, we employed the transfer of derivative formula (4.2) up to the last time interval and then performed an IBP formula.

Now, the key point in order to establish the IBP formula (4.6) is to combine in a suitable way the identities (4.14) and (4.15). For each $k\in\{0,\dots,n\}$, we multiply the above formulae by the length of the interval on which the local IBP formula is performed, namely we multiply by $\zeta_k-\zeta_{k-1}$ both sides of (4.14), $k=1,\dots,n-1$, and we multiply by $T-\zeta_n$ both sides of (4.15). We then sum them over all k. Recalling that $\sum_{k=1}^{n+1}\zeta_k-\zeta_{k-1}=T-\zeta_0=T$, we deduce

$$
\begin{aligned}
&T\partial_{y_0}\mathbb{E}\Big[h(\bar{X}_{n+1},\bar{Y}_{n+1})\prod_{i=1}^{n+1}\theta_i\Big|\tau^{n+1}\Big]\\
=&\sum_{k=1}^{n+1}(\zeta_k-\zeta_{k-1})\mathbb{E}\Big[h(\bar{X}_{n+1},\bar{Y}_{n+1})\prod_{i=k+1}^{n+1}\theta_i\times\mathcal{I}^{(2)}_k(\overrightarrow{\theta}_k^{e,Y})\times\prod_{i=1}^{k-1}\overrightarrow{\theta}_i^{e,Y}\Big|\tau^{n+1}\Big]\\
&+\sum_{k=1}^{n+1}(\zeta_k-\zeta_{k-1})\sum_{j=1}^{k}\mathbb{E}\Big[h(\bar{X}_{n+1},\bar{Y}_{n+1})\prod_{i=j+1}^{n+1}\theta_i\times\overrightarrow{\theta}_j^{c}\times\prod_{i=1}^{j-1}\overrightarrow{\theta}_i^{e,Y}\Big|\tau^{n+1}\Big]\\
&+\sum_{k=1}^{n+1}(\zeta_k-\zeta_{k-1})\Big(\sum_{j=1}^{k-1}\mathbb{E}\Big[h(\bar{X}_{n+1},\bar{Y}_{n+1})\prod_{i=k+1}^{n+1}\theta_i\times\mathcal{I}^{(1)}_k(\theta_k)\times\prod_{i=j+1}^{k-1}\theta_i\times\overrightarrow{\theta}_j^{e,X}\times\prod_{i=1}^{j-1}\overrightarrow{\theta}_i^{e,Y}\Big|\tau^{n+1}\Big]\\
&+\mathbb{E}\Big[h(\bar{X}_{n+1},\bar{Y}_{n+1})\prod_{i=k+1}^{n+1}\theta_i\times\mathcal{I}^{(1)}_k(\overrightarrow{\theta}_k^{e,X})\times\prod_{i=1}^{n}\overrightarrow{\theta}_i^{e,Y}\Big|\tau^{n+1}\Big]\Big)\\
&=\mathbb{E}\Big[h(\bar{X}_{n+1},\bar{Y}_{n+1})\sum_{k=1}^{n+1}(\zeta_k-\zeta_{k-1})\Big(\overrightarrow{\theta}^{\mathcal{I}_k^{(2),n+1}}+\sum_{j=1}^{k}\Big(\overrightarrow{\theta}^{C_j^{n+1}}+\overrightarrow{\theta}_j^{\mathcal{I}_k^{(1),n+1}}\Big)\Big)\Big|\tau^{n+1}\Big].
\end{aligned}
$$

We now provide a sharp upper-estimate for the above quantity. From Lemma B.2 and Lemma 4.1, it follows that $f(\zeta_i-\zeta_{i-1})\theta_i$, $f(\zeta_i-\zeta_{i-1})\overrightarrow{\theta}_i^{e,Y}$, $f(\zeta_i-\zeta_{i-1})\overrightarrow{\theta}_i^{c}\in\mathbb{M}_{i-1,n}(\bar{X},\bar{Y},-1/2)$ and $f(\zeta_i-\zeta_{i-1})\overrightarrow{\theta}_i^{e,X}\in\mathbb{M}_{i-1,n}(\bar{X},\bar{Y},0)$ for any $i\in\{1,\dots,n\}$. Moreover, from the very definition of the weights θ_i, $\overrightarrow{\theta}_i^{e,X}$ and $\overrightarrow{\theta}_i^{e,Y}$, after some simple but cumbersome computations that we omit (we also refer the reader to Appendix C which contains some expansion formulae), one has $f(\zeta_i-\zeta_{i-1})\mathcal{D}_i^{(1)}(\theta_i)$, $f(\zeta_i-\zeta_{i-1})\mathcal{D}_i^{(2)}(\overrightarrow{\theta}_i^{e,Y})\in\mathbb{M}_{i-1,n}(\bar{X},\bar{Y},-1)$ and $f(\zeta_i-\zeta_{i-1})\mathcal{D}_i^{(1)}(\overrightarrow{\theta}_i^{e,X})\in\mathbb{M}_{i-1,n}(\bar{X},\bar{Y},-1/2)$ so that from Lemma 2.7 we conclude $f(\zeta_i-\zeta_{i-1})(\zeta_i-\zeta_{i-1})\mathcal{I}_i^{(1)}(\theta_i)\in\mathbb{M}_{i-1,n}(\bar{X},\bar{Y},0)$, $f(\zeta_i-\zeta_{i-1})(\zeta_i-\zeta_{i-1})\mathcal{I}_i^{(2)}(\overrightarrow{\theta}_i^{e,Y})\in\mathbb{M}_{i-1,n}(\bar{X},\bar{Y},0)$ and $f(\zeta_i-\zeta_{i-1})(\zeta_i-\zeta_{i-1})\mathcal{I}_i^{(1)}(\overrightarrow{\theta}_i^{e,X})\in\mathbb{M}_{i-1,n}(\bar{X},\bar{Y},1/2)$. Hence, from the boundedness of h, the tower property of conditional expectation and (2.27), it holds

$$\begin{aligned}&\Big|(\zeta_k-\zeta_{k-1})\mathbb{E}\Big[h(\bar{X}_{n+1},\bar{Y}_{n+1})\prod_{i=k+1}^{n+1}\theta_i\times\mathcal{I}_k^{(2)}(\overrightarrow{\theta}_k^{e,Y})\times\prod_{i=1}^{k-1}\overrightarrow{\theta}_i^{e,Y}\Big|\tau^{n+1}\Big]\Big|\\&\quad\leq C^{n+1}(1-F(T-\zeta_n))^{-1}\prod_{i=k+1}^{n}(f(\zeta_i-\zeta_{i-1}))^{-1}(\zeta_i-\zeta_{i-1})^{-\frac12}(f(\zeta_k-\zeta_{k-1}))^{-1}\prod_{i=1}^{k-1}(f(\zeta_i-\zeta_{i-1}))^{-1}(\zeta_i-\zeta_{i-1})^{-\frac12},\end{aligned}$$

so that using the identity (2.9)

$$\begin{aligned}&\sum_{n\geq0}\mathbb{E}\Big[\sum_{k=1}^{n+1}\Big|(\zeta_k-\zeta_{k-1})\mathbb{E}\Big[h(\bar{X}_{n+1},\bar{Y}_{n+1})\prod_{i=k+1}^{n+1}\theta_i\times\mathcal{I}_k^{(2)}(\overrightarrow{\theta}_k^{e,Y})\times\prod_{i=1}^{k-1}\overrightarrow{\theta}_i^{e,Y}\Big|\tau^{n+1}\Big]\Big|\mathbf{1}_{\{N_T=n\}}\Big]\\&\qquad\leq\sum_{n\geq0}C^{n+1}\sum_{k=1}^{n+1}\mathbb{E}\Big[(1-F(T-\zeta_n))^{-1}(f(\zeta_k-\zeta_{k-1}))^{-1}\prod_{i=1,i\neq k}^{n}(f(\zeta_i-\zeta_{i-1}))^{-1}(\zeta_i-\zeta_{i-1})^{-\frac12}\mathbf{1}_{\{N_T=n\}}\Big]\\&\qquad=\sum_{n\geq0}C^{n+1}\sum_{k=1}^{n+1}\int_{\Delta_n(T)}\prod_{i=1,i\neq k}^{n}(s_i-s_{i-1})^{-1/2}\,\mathrm{d}\mathbf{s}_n\\&\qquad\leq\sum_{n\geq0}(n+1)C^{n+1}T^{(n+1)/2}\frac{\Gamma^n(1/2)}{\Gamma(1+n/2)}<\infty.\end{aligned}$$

From similar arguments that we omit, it follows

$$\begin{aligned}&\Big|(\zeta_k-\zeta_{k-1})\sum_{j=1}^{k}\mathbb{E}\Big[h(\bar{X}_{n+1},\bar{Y}_{n+1})\Big(\overrightarrow{\theta}_j^{C_j^{n+1}}+\overrightarrow{\theta}_j^{\mathcal{I}_k^{(1),n+1}}\Big)\Big|\tau^{n+1}\Big]\Big|\\&\quad\leq C^{n+1}(\zeta_k-\zeta_{k-1})\sum_{j=1}^{k}(1-F(T-\zeta_n))^{-1}\prod_{i=1}^{n}(f(\zeta_i-\zeta_{i-1}))^{-1}(\zeta_i-\zeta_{i-1})^{-1/2}[1+\mathbf{1}_{\{i=k\}}(\zeta_i-\zeta_{i-1})^{-1/2}].\end{aligned}$$

so that using again the identity (2.9)

$$\begin{aligned}&\sum_{n\geq0}\mathbb{E}\Big[\sum_{k=1}^{n+1}\Big|(\zeta_k-\zeta_{k-1})\sum_{j=1}^{k}\mathbb{E}\Big[h(\bar{X}_{n+1},\bar{Y}_{n+1})\Big(\overrightarrow{\theta}_j^{C_j^{n+1}}+\overrightarrow{\theta}_j^{\mathcal{I}_k^{(1),n+1}}\Big)\Big|\tau^{n+1}\Big]\Big|\mathbf{1}_{\{N_T=n\}}\Big]\\&\leq\sum_{n\geq0}C^{n+1}\sum_{k=1}^{n+1}\mathbb{E}\big[(\zeta_k-\zeta_{k-1})\sum_{j=1}^{k}(1-F(T-\zeta_n))^{-1}\prod_{i=1}^{n}(f(\zeta_i-\zeta_{i-1}))^{-1}(\zeta_i-\zeta_{i-1})^{-1/2}[1+\mathbf{1}_{\{i=k\}}(\zeta_i-\zeta_{i-1})^{-1/2}]\mathbf{1}_{\{N_T=n\}}\big]\\&\leq\sum_{n\geq0}C^{n+1}(n+1)(n+2)T^{(n+1)/2}\frac{\Gamma^n(1/2)}{\Gamma(1+n/2)}<\infty.\end{aligned}$$

The preceding estimates combined with (4.7) and the Lebesgue dominated convergence theorem allows to conclude that $y_0 \mapsto \mathbb{E}[h(X_T, Y_T)]$ is continuously differentiable with

$$\begin{aligned}
T\partial_{y_0}\mathbb{E}[h(X_T,Y_T)] &= T\partial_{y_0}\mathbb{E}\Big[h(\bar{X}_{N_T+1},\bar{Y}_{N_T+1})\prod_{i=1}^{N_T+1}\theta_i\Big]\\
&= \sum_{n\geq 0}\mathbb{E}\Big[T\partial_{y_0}\mathbb{E}\Big[h(\bar{X}_{n+1},\bar{Y}_{n+1})\prod_{i=1}^{n+1}\theta_i\Big|\tau^{n+1}\Big]\,\mathbf{1}_{\{N_T=n\}}\Big]\\
&= \sum_{n\geq 0}\mathbb{E}\Big[\mathbb{E}\Big[h(\bar{X}_{n+1},\bar{Y}_{n+1})\sum_{k=1}^{n+1}(\zeta_k-\zeta_{k-1})\Big(\overrightarrow{\theta}^{\mathcal{I}_k^{(2),n+1}}+\sum_{j=1}^{k}\Big(\overrightarrow{\theta}^{C_j^{n+1}}+\overrightarrow{\theta}_j^{\mathcal{I}_k^{(1),n+1}}\Big)\Big)\Big|\tau^{n+1}\Big]\,\mathbf{1}_{\{N_T=n\}}\Big]\\
&= \mathbb{E}\Big[h(\bar{X}_{N_T+1},\bar{Y}_{N_T+1})\sum_{k=1}^{N_T+1}(\zeta_k-\zeta_{k-1})\Big(\overrightarrow{\theta}^{\mathcal{I}_k^{(2),N_T+1}}+\sum_{j=1}^{k}\Big(\overrightarrow{\theta}^{C_j^{n+1}}+\overrightarrow{\theta}_j^{\mathcal{I}_k^{(1),N_T+1}}\Big)\Big)\Big],
\end{aligned}$$

where we used Fubini's theorem for the last equality. This completes the proof of the IBP formula (4.6) for $h \in \mathcal{C}_b^1(\mathbb{R}^2)$.

Step 2: Extension to $h \in \mathcal{B}_\gamma(\mathbb{R}^2)$ for some positive γ.

We now extend the two IBP formulae that we have established in the previous step to the case of a test function $h \in \mathcal{B}_\gamma(\mathbb{R}^2)$ for some sufficiently small $\gamma > 0$.

It follows from (A.10) that the random variable (X_T, Y_T) admits a density $p(T, x_0, y_0, x, y)$ satisfying the probabilistic representation

$$\begin{aligned}
p(T,x_0,y_0,x,y) &= \mathbb{E}\Big[\bar{p}(\zeta_{N_T},T,\bar{X}_{N_T},\bar{Y}_{N_T},x,y)\prod_{i=1}^{N_T+1}\theta_i\Big]\\
&= \sum_{n\geq 0}\mathbb{E}\Big[\mathbb{E}\Big[\bar{p}(\zeta_n,T,\bar{X}_n,\bar{Y}_n,x,y)\prod_{i=1}^{n+1}\theta_i|\tau^{n+1}\Big]\,\mathbf{1}_{\{N_T=n\}}\Big].
\end{aligned}$$

We then proceed in a completely analogous way as in the previous step. Namely, we prove that the map $y_0 \mapsto p(T, x_0, y_0, x, y)$ is continuously differentiable and satisfies

$$\begin{aligned}
&T\partial_{y_0}p(T,x_0,y_0,x,y)\\
&\quad= \mathbb{E}\Big[\bar{p}(\zeta_{N_T},T,\bar{X}_{N_T},\bar{Y}_{N_T},x,y)\sum_{k=1}^{N_T+1}(\zeta_k-\zeta_{k-1})\Big(\overrightarrow{\theta}^{\mathcal{I}_k^{(2),N_T+1}}+\sum_{j=1}^{k}\Big(\overrightarrow{\theta}^{C_j^{n+1}}+\overrightarrow{\theta}_j^{\mathcal{I}_k^{(1),N_T+1}}\Big)\Big)\Big].
\end{aligned}\tag{4.16}$$

The only non-trivial point that must be justified is to provide a sharp upper-bound for the quantity

$$\begin{aligned}
&T\partial_{y_0}\mathbb{E}\Big[p(\zeta_n,T,\bar{X}_{n+1},\bar{Y}_{n+1},x,y)\prod_{i=1}^{n+1}\theta_i\Big|\,\tau^{n+1}\Big]\\
&= \mathbb{E}\Big[\bar{p}(\zeta_n,T,\bar{X}_{n+1},\bar{Y}_{n+1},x,y)\sum_{k=1}^{n+1}(\zeta_k-\zeta_{k-1})\Big(\overrightarrow{\theta}^{\mathcal{I}_k^{(2),n+1}}+\sum_{j=1}^{k}\Big(\overrightarrow{\theta}^{C_j^{n+1}}+\overrightarrow{\theta}_j^{\mathcal{I}_k^{(1),n+1}}\Big)\Big)\Big|\tau^{n+1}\Big].
\end{aligned}$$

Note that since $f(\zeta_i-\zeta_{i-1})\theta_i, f(\zeta_i-\zeta_{i-1})\overrightarrow{\theta}^{e,Y}_i \in \mathbb{M}_{i-1,n}(\bar{X},\bar{Y},-1/2)$ and $f(\zeta_k-\zeta_{k-1})(\zeta_k-\zeta_{k-1})\mathcal{I}^{(2)}_k(\overrightarrow{\theta}^{e,Y}_k) \in \mathbb{M}_{k-1,n}(\bar{X},\bar{Y},0)$, for any $c>4\kappa$, it holds

$$\begin{aligned}
&\mathbb{E}\Big[\bar{p}(\zeta_n,T,\bar{X}_n,\bar{Y}_n,x,y)\sum_{k=1}^{n+1}(\zeta_k-\zeta_{k-1})\Big|\overrightarrow{\theta}^{\mathcal{I}^{(2),n+1}_k}\Big|\Big|\tau^{n+1}\Big]\\
&\quad\le C^{n+1}\int_{(\mathbb{R}^2)^n}\bar{q}_c(\zeta_n,T,x_n,y_n,x,y)\sum_{k=1}^{n+1}(1-F(T-\zeta_n))^{-1}\prod_{i=k+1}^{n}(f(\zeta_i-\zeta_{i-1}))^{-1}(\zeta_i-\zeta_{i-1})^{-1/2}\\
&\qquad\times(f(\zeta_k-\zeta_{k-1}))^{-1}\prod_{i=1}^{k-1}(f(\zeta_i-\zeta_{i-1}))^{-1}(\zeta_i-\zeta_{i-1})^{-1/2}\prod_{i=1}^{n}\bar{q}_c(\zeta_{i-1},\zeta_i,x_{i-1},y_{i-1},x_i,y_i)\,\mathrm{d}\mathbf{x}_n\mathrm{d}\mathbf{y}_n\\
&\quad\le C^{n+1}\bar{q}_{c'}(T,x_0,y_0,x,y)\sum_{k=1}^{n+1}(1-F(T-\zeta_n))^{-1}\prod_{i=1}^{n}(f(\zeta_i-\zeta_{i-1}))^{-1}\prod_{i=1,i\neq k}^{n}(\zeta_i-\zeta_{i-1})^{-1/2},
\end{aligned}\tag{4.17}$$

where, for the first inequality we used the upper-estimate (2.7) and for the last inequality we used Lemma B.3 and set $c':=(C')^2c$. From similar arguments, one gets

$$\begin{aligned}
&\mathbb{E}\Big[\bar{p}(\zeta_n,T,\bar{X}_n,\bar{Y}_n,x,y)\sum_{k=1}^{n+1}(\zeta_k-\zeta_{k-1})\sum_{j=1}^{k}\Big(\Big|\overrightarrow{\theta}^{C^{n+1}_j}\Big|+\Big|\overrightarrow{\theta}^{\mathcal{I}^{(1),N_T+1}_k}_j\Big|\Big)\Big|\tau^{n+1}\Big]\\
&\le C^{n+1}\bar{q}_{c'}(T,x_0,y_0,x,y)\sum_{k=1}^{n+1}(\zeta_k-\zeta_{k-1})\sum_{j=1}^{k}(1-F(T-\zeta_n))^{-1}\prod_{i=1}^{n}(f(\zeta_i-\zeta_{i-1}))^{-1}(\zeta_i-\zeta_{i-1})^{-1/2}[1+\mathbf{1}_{\{i=k\}}(\zeta_i-\zeta_{i-1})^{-1/2}].
\end{aligned}\tag{4.18}$$

Now, from the upper-bounds (4.17) and (4.18) as well as the identity (2.9), we conclude

$$\begin{aligned}
&\sum_{n\ge0}\mathbb{E}\Big[\mathbb{E}\Big[\bar{p}(\zeta_n,T,\bar{X}_n,\bar{Y}_n,x,y)\sum_{k=1}^{n+1}(\zeta_k-\zeta_{k-1})\Big(\Big|\overrightarrow{\theta}^{\mathcal{I}^{(2),n+1}_k}\Big|+\sum_{j=1}^{k}\Big(\Big|\overrightarrow{\theta}^{C^{n+1}_j}\Big|+\Big|\overrightarrow{\theta}^{\mathcal{I}^{(1),n+1}_k}_j\Big|\Big)\Big)\Big|\tau^{n+1}\Big]\mathbf{1}_{\{N_T=n\}}\Big]\\
&\quad\le\bar{q}_c(T,x_0,y_0,x,y)\sum_{n\ge0}C^{n+1}\mathbb{E}\Big[\Big(\sum_{k=1}^{n+1}(1-F(T-\zeta_n))^{-1}\prod_{i=1}^{n}(f(\zeta_i-\zeta_{i-1}))^{-1}\prod_{i=1,i\neq k}^{n}(\zeta_i-\zeta_{i-1})^{-1/2}\\
&\qquad+\sum_{k=1}^{n+1}(\zeta_k-\zeta_{k-1})\sum_{j=1}^{k}(1-F(T-\zeta_n))^{-1}\prod_{i=1}^{n}(f(\zeta_i-\zeta_{i-1}))^{-1}(\zeta_i-\zeta_{i-1})^{-1/2}[1+\mathbf{1}_{\{i=k\}}(\zeta_i-\zeta_{i-1})^{-1/2}]\Big)\mathbf{1}_{\{N_T=n\}}\Big]\\
&\quad\le\bar{q}_{c'}(T,x_0,y_0,x,y)\sum_{n\ge0}C^{n+1}[(n+1)+(n+1)(n+2)/2]T^{(n+1)/2}\frac{\Gamma^n(1/2)}{\Gamma(1+n/2)}\\
&\quad=CT^{1/2}\bar{q}_{c'}(T,x_0,y_0,x,y).
\end{aligned}\tag{4.19}$$

From the preceding inequality and Fubini's theorem, we thus get

$$\begin{aligned}
&\Big|\mathbb{E}\Big[\bar{p}(\zeta_{N_T},T,\bar{X}_{N_T},\bar{Y}_{N_T},x,y)\sum_{k=1}^{N_T+1}(\zeta_k-\zeta_{k-1})\Big(\overrightarrow{\theta}^{\mathcal{I}^{(2),N_T+1}_k}+\sum_{j=1}^{k}\Big(\overrightarrow{\theta}^{C^{N_T+1}_j}+\overrightarrow{\theta}^{\mathcal{I}^{(1),N_T+1}_k}_j\Big)\Big)\Big]\Big|\\
&\le CT^{1/2}\bar{q}_{c'}(T,x_0,y_0,x,y),
\end{aligned}\tag{4.20}$$

for some positive constant $C := C(T)$ such that $T \mapsto C(T)$ is non-decreasing. Hence, combining (4.16) with Lebesgue's differentiation theorem

$$\begin{aligned}
&T\partial_{y_0}\mathbb{E}[h(X_T, Y_T)] \\
&\quad = \int_{\mathbb{R}^2} h(x,y)\, T\partial_{y_0} p(T, x_0, y_0, x, y)\, \mathrm{d}x\mathrm{d}y \\
&\quad = \int_{\mathbb{R}^2} h(x,y)\, \mathbb{E}\Big[\bar{p}(T-\zeta_{N_T}, \bar{X}_{N_T}, \bar{Y}_{N_T}, x, y) \sum_{k=1}^{N_T+1} (\zeta_k - \zeta_{k-1})\Big(\overrightarrow{\theta}^{\mathcal{I}_k^{(2),N_T+1}} + \sum_{j=1}^{k}\Big(\overrightarrow{\theta}^{C_j^{N_T+1}} + \overrightarrow{\theta}^{\mathcal{I}_k^{(1),N_T+1}}_{j}\Big)\Big)\Big]\, \mathrm{d}x\mathrm{d}y,
\end{aligned}$$

for any $h \in \mathcal{C}_b^1(\mathbb{R}^2)$. A monotone class argument allows to conclude that the preceding identity is still valid for any bounded and measurable map h defined over $\mathbb{R}^2$ and a standard approximation argument allows to extend it to $h \in \mathcal{B}_\gamma(\mathbb{R}^2)$ for any $0 \leq \gamma < (2c'T)^{-1} = (2(C')^2cT)^{-1}$, for any $c > 4\kappa$. We eventually conclude from the preceding identity, (4.20) combined with Fubini's theorem that

$$T\partial_{y_0}\mathbb{E}[h(X_T, Y_T)] = \mathbb{E}\Big[h(\bar{X}_{N_T+1}, \bar{Y}_{N_T+1}) \sum_{k=1}^{N_T+1} (\zeta_k - \zeta_{k-1})\Big(\overrightarrow{\theta}^{\mathcal{I}_k^{(2),N_T+1}} + \sum_{j=1}^{k}\Big(\overrightarrow{\theta}^{C_j^{N_T+1}} + \overrightarrow{\theta}^{\mathcal{I}_k^{(1),N_T+1}}_{j}\Big)\Big)\Big],$$

for any $h \in \mathcal{B}_\gamma(\mathbb{R}^2)$ such that $0 \leq \gamma < (2(C')^2cT)^{-1}$.

Step 3: $L^p(\mathbb{P})$-moments for a renewal process with Beta jump times.

From the above formula, the proof of the $L^p(\mathbb{P})$-moment estimate when N is a renewal process with Beta jump times follows by similar arguments as those employed at step 3 of the proof of Theorem 3.1. We omit the remaining technical details.

□

5. Numerical results

In this section, as a proof of concept, we provide some simple numerical results for the unbiased Monte Carlo algorithm that stems from the probabilistic representation formula established in Theorem 3.1 and the Bismut-Elworthy-Li formulae of Theorem 4.2 for the couple (S_T, Y_T) that allows to compute the Delta and the Vega related to the option price of the vanilla option with payoff $h(S_T)$. As already mentioned in the introduction, we believe that one needs to study numerical issues and to compare our algorithm with other existing methods to compute Greeks in more details. However, this is beyond the scope of the current paper and is left to future research.

We here consider the unique strong solution associated to the SDE (1.1) for three different models corresponding to three different diffusion coefficient function σ_S and two different options, namely Call and digital Call options with maturity T and strike K, with payoff functions $h(x,y) = (\exp(x) - K)_+$ and $h(x,y) = \mathbf{1}_{\{\exp(x)\geq K\}}$ respectively. For these three models, the drift function of the volatility process is defined by $b_Y(x) = \lambda_Y(\mu - x)$ and we fix the parameters as follows: $T = 0.5$, $r = 0.03$, $K = 1.5$, $x_0 = \ln(s_0) = 0.4$, $Y_0 = 0.2$, $\sigma_Y(.) \equiv \sigma_Y = 0.2$, $\lambda_Y = 0.5$, $\mu = 0.3$ and $\rho = 0.6$. We also consider two type of renewal process N: a Poisson process with intensity parameter $\lambda = 0.5$ and a renewal process with $Beta(1-\alpha, 1)$ jump times with parameters $\alpha = 0.5$ and $\bar{\tau} = 2$. A crude Monte Carlo estimator gives that $\mathbb{E}[N_T] = 1.25$ for Exponential sampling (which is inline with the theoretical value $1 + \lambda T$) and $\mathbb{E}[N_T] = 1.79$ for Beta sampling.

The total time for the computation of the price, Delta and Vega are about 8 seconds for the Monte Carlo method with Euler scheme and about 10 seconds for the unbiased Monte Carlo method with Exponential and Beta sampling. Generally speaking, we observe that the variance of the unbiased Monte Carlo estimators is larger than the variance of the Monte Carlo estimator with Euler-Maruyama discretization scheme. This should not come as a big surprise since this fact is reminiscent of unbiased Monte Carlo methods. However, the Monte Carlo method with Euler scheme is also affected by its inherent bias.

5.1. Black-Scholes model

We first consider the simple (toy) example corresponding to the Black-Scholes dynamics

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma_S S_t\,\mathrm{d}W_t, \quad \mathrm{d}Y_t = b_Y(Y_t)\,\mathrm{d}t + \sigma_Y(Y_t)dB_t, \quad d\langle B, W\rangle_t = \rho\mathrm{d}t,\ \rho \in (-1,1).$$

with constant diffusion coefficient function $\sigma_S(.) \equiv \sigma_S > 0$. The law of (S_T, Y_T) can be computed explicitly so that analytical formulas are available for the price, Delta and Vega. Note that the discount factor e^{-rT} has been added in our probabilistic representation formula for comparison purposes. In this example, we importantly remark that the dynamics of the Euler scheme writes

$$\begin{cases} \bar{X}_{i+1} &= \bar{X}_i + \left(r - \frac{1}{2}a_{S,i}\right)(\zeta_{i+1}-\zeta_i) + \sigma_{S,i}\sqrt{\zeta_{i+1}-\zeta_i}Z^1_{i+1}, \\ \bar{Y}_{i+1} &= m_i + \sigma_{Y,i}\sqrt{\zeta_{i+1}-\zeta_i}\left(\rho_i Z^1_{i+1} + \sqrt{1-\rho_i^2}Z^2_{i+1}\right), \end{cases} \tag{5.1}$$

with $m_i = m_{\zeta_{i+1}-\zeta_i}(\bar{Y}_i) = \mu + (\bar{Y}_i - \mu)e^{-\lambda(\zeta_{i+1}-\zeta_i)}$. Also, the weights $(\theta_i)_{1\le i\le N_T+1}$ in the probabilistic representation (3.2) of Theorem 3.1 greatly simplifies, namely

$$\theta_i = (f(\zeta_i - \zeta_{i-1}))^{-1}\mathcal{I}_i^{(2)}(b_Y^i),\ 1 \le i \le N_T, \quad \text{and} \quad \theta_{N_T+1} = (1 - F(T - \zeta_{N_T}))^{-1}.$$

We perform $M_1 = 2.56 \times 10^7$ for the unbiased Monte Carlo method with Exponential sampling and $M_1 = 1.79 \times 10^7$ in the case of Beta sampling to approximate the price as well as the two Greeks so that the (average) computational cost (up to a constant multiplicative factor) is given by $\mathbb{E}[N_T] \times M_1 = 3.2 \times 10^7$ in both cases. We compare them with the corresponding values obtained using the standard Monte Carlo method combined with an Euler-Maruyama approximation scheme for the dynamics (1.1) with $M_2 = 160000$ Monte Carlo simulations paths and mesh size $\delta = T/n$ where $n = 200$. Its computational complexity (up to a constant multiplicative factor) is given by $n \times M_2 = 3.2 \times 10^7$. Hence, both Monte Carlo estimators have comparable computational complexity though their computational time are slightly different in practical implementation. The Delta and Vega are obtained using the Monte Carlo finite difference approach combined with the Euler-Maruyama discretization scheme, that is, denoting by $E^n_{M_2}(s_0, y_0)$ the Monte Carlo estimator associated to the Euler-Maruyama scheme, we compute $(E^n_{M_2}(s_0+\varepsilon, y_0) - E^n_{M_2}(s_0, y_0))/\varepsilon$ and $(E^n_{M_2}(s_0, y_0+\varepsilon) - E^n_{M_2}(s_0, y_0))/\varepsilon$ respectively with $\varepsilon = 10^{-2}$. The numerical results for the three different quantities are summarized in Tables 1–3 respectively. The first column provides the value of the parameter σ_S. The second column stands for the value of the price, Delta or Vega obtained by the corresponding Black-Scholes formula. The third, fourth and fifth columns correspond to the value obtained by the Monte Carlo estimator using Euler-Maruyama discretization scheme together with its half-width 95% confidence interval and its empirical variance. The sixth, seventh and eighth (resp. the ninth, tenth and eleventh) columns provide the estimated value with its halfwidth 95% confidence interval and empirical variance by our method in the case of Exponential sampling (resp. Beta sampling). Note that though the variance of the Monte Carlo estimator in the case of Exponential sampling may explode, we compute it for sake of completeness. Indeed, in our numerical experiences, we observed that the variance of the Monte Carlo estimator in the Exponential sampling case slightly increases with respect to M_1. Nevertheless, we observe a good behaviour of the unbiased estimators for all three quantities and for all the values of the parameter σ_S.

TABLE 1. Comparison between the unbiased Monte Carlo estimation and the Monte Carlo Euler-Maruyama scheme for the price of a Call option in the Black-Scholes model for different values of σ_S.

σ_S	B-S formula	Euler Scheme			Exponential sampling			Beta sampling		
		Price	Half-width	Variance	Price	Half-width	Variance	Price	Half-width	Variance
0.25	0.111804	0.111853	0.000860286	0.0308244	0.112196	0.000124112	0.102648	0.112199	0.000154064	0.110598
0.3	0.132621	0.132808	0.0010515	0.0460493	0.133193	0.000152038	0.15404	0.133036	0.000187336	0.163524
0.4	0.174152	0.173559	0.00144315	0.0867423	0.174754	0.000208983	0.291037	0.174711	0.000257441	0.308813
0.6	0.256572	0.255388	0.00235625	0.231233	0.257287	0.000334903	0.747423	0.256978	0.0004127	0.793617

TABLE 2. Comparison between the unbiased Monte Carlo estimation and the Monte Carlo Euler-Maruyama scheme for the Delta of a Call option in the Black-Scholes model for different values of σ_S.

σ_S	B-S formula	Euler Scheme			Exponential sampling			Beta sampling		
		Delta	Half-width	Variance	Delta	Half-width	Variance	Delta	Half-width	Variance
0.25	0.556589	0.55675	0.00280539	0.327789	0.554992	0.000895101	5.33915	0.555192	0.00114054	6.0612
0.3	0.560018	0.560534	0.00290622	0.351775	0.558285	0.000923515	5.6835	0.557974	0.00116621	6.33719
0.4	0.569512	0.570228	0.00311011	0.402864	0.567568	0.000978965	6.38649	0.567091	0.00123	7.04938
0.6	0.592743	0.590041	0.00358714	0.535925	0.589	0.0010899	7.91588	0.587681	0.00137469	8.80548

TABLE 3. Comparison between the unbiased Monte Carlo estimation for the Vega of a Call option in the Black-Scholes model for different values of σ_S.

σ_S	B-S formula	Exponential sampling			Beta sampling		
		Vega	Half-width	Variance	Vega	Half-width	Variance
0.25	0	0.000690222	0.00115103	8.82877	−0.000559242	0.00128448	7.68766
0.3	0	0.00182175	0.00137953	12.6821	0.000500579	0.00156401	11.3978
0.4	0	−0.00163321	0.00189888	24.0283	−0.000817515	0.00215655	21.6701
0.6	0	−0.000830748	0.00300346	60.1136	−0.001055	0.00340386	53.9862

5.2. A Stein-Stein type model

In this second example, we consider a Stein-Stein type model where the diffusion coefficient function for the spot price is an affine function, namely $\sigma_S(x) = \sigma_1 x + \sigma_2$ where σ_1 and σ_2 are two positive constants. Note carefully that σ_S is not uniformly elliptic and bounded so that **(AR)** and **(ND)** are not satisfied. However, we heuristically choose σ_1 and σ_2 so that $\sigma_S(Y_t)$ is bounded and strictly positive with high probability. Also,

TABLE 4. Comparison between the unbiased Monte Carlo estimation for the price of a Call option in the Stein-Stein type model for different values of the parameters σ_1 and σ_2.

σ_1	σ_2	Euler Scheme			Exponential sampling			Beta sampling		
		Price	Half-width	Variance	Price	Half-width	Variance	Price	Half-width	Variance
0.1	0.15	0.0788438	0.000591655	0.0145796	0.0785159	0.000184533	0.226921	0.0787702	0.000127826	0.076134
0.2	0.25	0.129	0.0010556	0.0464096	0.129024	0.000391611	1.02197	0.128967	0.000238248	0.264483
0.3	0.4	0.200983	0.00179584	0.13432	0.200121	0.000570515	2.16901	0.200039	0.000388453	0.703103
0.4	0.5	0.250972	0.00240492	0.240884	0.249897	0.000762565	3.87509	0.249507	0.000539937	1.3584

TABLE 5. Comparison between the unbiased Monte Carlo estimation for the Delta of a Call option in the Stein-Stein type model for different values of the parameters σ_1 and σ_2.

σ_1	σ_2	Euler Scheme			Exponential sampling			Beta sampling		
		Delta	Half-width	Variance	Delta	Half-width	Variance	Delta	Half-width	Variance
0.1	0.15	0.547988	0.00265342	0.293238	0.538724	0.00165166	18.179	0.539677	0.00137889	8.85936
0.2	0.25	0.54942	0.00289865	0.349943	0.539137	0.00211916	29.9265	0.538131	0.00152902	10.8935
0.3	0.4	0.566344	0.00328048	0.448211	0.556553	0.00254168	43.0495	0.556605	0.00162791	12.3481
0.4	0.5	0.580157	0.00359471	0.53819	0.567956	0.0026141	45.5377	0.567737	0.00184814	15.9151

TABLE 6. Comparison between the unbiased Monte Carlo estimation for the Vega of a Call option in the Stein-Stein type model for different values of the parameters σ_1 and σ_2.

σ_1	σ_2	Euler Scheme			Exponential sampling			Beta sampling		
		Vega	Half-width	Variance	Vega	Half-width	Variance	Vega	Half-width	Variance
0.1	0.15	0.0369128	0.000309236	0.0039828	0.0325922	0.00150912	15.1766	0.0349527	0.00104028	5.04243
0.2	0.25	0.0733673	0.000689904	0.0198237	0.0671736	0.00333565	74.1462	0.0680856	0.00190913	16.9829
0.3	0.4	0.109991	0.00121245	0.0612259	0.0990781	0.004919	161.243	0.0954942	0.00311339	45.1655
0.4	0.5	0.145413	0.0018251	0.138734	0.129808	0.00689955	317.226	0.122975	0.00443729	91.7436

analytical expressions for the coefficients are available, namely

$$
\begin{aligned}
a_{S,i} &= \int_0^{\zeta_{i+1}-\zeta_i} \Big[\sigma_1\big(\mu + (\bar{Y}_i - \mu)e^{-\lambda_Y s}\big) + \sigma_2\Big]^2 \mathrm{d}s, \\
&= (\sigma_1\mu + \sigma_2)^2(\zeta_{i+1} - \zeta_i) + \sigma_1^2(\bar{Y}_i - \mu)^2 \frac{1 - e^{-2\lambda_Y(\zeta_{i+1}-\zeta_i)}}{2\lambda_Y} + 2\sigma_1(\sigma_1\mu + \sigma_2)(\bar{Y}_i - \mu)\frac{1 - e^{-\lambda_Y(\zeta_{i+1}-\zeta_i)}}{\lambda_Y}, \\
a'_{S,i} &= \sigma_1^2(\bar{Y}_i - \mu)\frac{1 - e^{-2\lambda_Y(\zeta_{i+1}-\zeta_i)}}{\lambda_Y} + 2\sigma_1(\sigma_1\mu + \sigma_2)\frac{1 - e^{-\lambda_Y(\zeta_{i+1}-\zeta_i)}}{\lambda_Y}, \\
\rho_i &= \rho\frac{\int_0^{\zeta_{i+1}-\zeta_i}\Big[\alpha\big(\mu + (\bar{Y}_i - \mu)e^{-\lambda s}\big) + C\Big]\mathrm{d}s}{\sigma_{S,i}\sqrt{\zeta_{i+1} - \zeta_i}} = \rho\frac{\alpha(\bar{Y}_i - \mu)(1 - e^{-\lambda(\zeta_{i+1}-\zeta_i)})/\lambda + (\sigma_1\mu + \sigma_2)(\zeta_{i+1} - \zeta_i)}{\sigma_{S,i}\sqrt{\zeta_{i+1} - \zeta_i}}, \\
\rho'_i &= \rho\frac{\sigma_{S,i}\big(\sigma_1(1 - e^{-\lambda_Y(\zeta_{i+1}-\zeta_i)})/\lambda_Y\big) - \sigma'_{S,i}\big(\sigma_1(\bar{Y}_i - \mu)(1 - e^{-\lambda_Y(\zeta_{i+1}-\zeta_i)})/\lambda_Y + (\sigma_1\mu + \sigma_2)(\zeta_{i+1} - \zeta_i)\big)}{a_{S,i}\sqrt{\zeta_{i+1} - \zeta_i}}.
\end{aligned}
$$

TABLE 7. Comparison between the unbiased Monte Carlo estimation for the price of a digital Call option in the Stein-Stein type model for different values of the parameters σ_1 and σ_2.

σ_1	σ_2	Euler Scheme			Exponential sampling			Beta sampling		
		Price	Half-width	Variance	Price	Half-width	Variance	Price	Half-width	Variance
0	0.3	0.468101	0.00241055	0.242013	0.468695	0.000363387	0.879968	0.46889	0.000466078	1.01218
0.1	0.15	0.490844	0.00241351	0.242608	0.48959	0.000706916	3.33015	0.489924	0.000534225	1.32981
0.2	0.25	0.458089	0.00240761	0.241423	0.458472	0.000780893	4.0636	0.458395	0.000536405	1.34069
0.3	0.4	0.430371	0.00239421	0.238744	0.428881	0.000840943	4.71261	0.429222	0.000513132	1.22687
0.4	0.5	0.410102	0.00237947	0.235813	0.409966	0.000737924	3.6287	0.409407	0.000510215	1.21296

TABLE 8. Comparison between the unbiased Monte Carlo estimation for the Delta of a digital Call option in the Stein-Stein type model for different values of the parameters σ_1 and σ_2.

σ_1	σ_2	Euler Scheme			Exponential sampling			Beta sampling		
		Delta	Half-width	Variance	Delta	Half-width	Variance	Delta	Half-width	Variance
0	0.3	1.23092	0.0306499	39.126	1.2445	0.00157472	16.5247	1.24456	0.00199696	18.5815
0.1	0.15	2.20347	0.0403758	67.8968	2.17998	0.0049492	163.229	2.18349	0.00400828	74.8611
0.2	0.25	1.27673	0.0311925	40.5237	1.27004	0.00456344	138.776	1.27093	0.00239161	26.6516
0.3	0.4	0.79344	0.0247765	25.5675	0.793619	0.0023143	35.6918	0.792876	0.00144989	9.79521
0.4	0.5	0.617625	0.0219193	20.0107	0.623268	0.00168461	18.9116	0.622453	0.00115744	6.24217

The parameters for the unbiased Monte Carlo method and the Monte Carlo method combined with an Euler-Maruyama approximation scheme are chosen as in the first example. The numerical results related to the price, Delta and Vega are provided in Tables 4–6 respectively for the Call option and in Tables 7–9 for the digital Call option. In spite of the fact that the main assumptions are not satisfied, we again observe a good performance of the unbiased estimators for all three quantities and for all the values of the parameters σ_1, σ_2, except for the computation of the Vega of a Call option for large values of σ_1 and σ_2.

5.3. A model with a periodic diffusion coefficient function

In our last example, the volatility of spot price takes the following form $\sigma_S(x) = \sigma_1 \cos(x) + \sigma_2$ where σ_1 and σ_2 are two positive constants such that $\sigma_2 - \sigma_1 > 0$ in order to ensure that **(ND)** is satisfied. Here, the coefficients appearing in the dynamics (2.10) write

$$
\begin{aligned}
a_{S,i} &= \int_0^{\zeta_{i+1}-\zeta_i} \Big[\sigma_1 \cos\big(\mu + (\bar{Y}_i - \mu)e^{-\lambda_Y s}\big) + \sigma_2\Big]^2 \mathrm{d}s, \\
a'_{S,i} &= -2\alpha \int_0^{\zeta_{i+1}-\zeta_i} e^{-\lambda_Y s} \sin\big(\mu + (\bar{Y}_i - \mu)e^{-\lambda_Y s}\big) \Big[\sigma_1 \cos\big(\mu + (\bar{Y}_i - \mu)e^{-\lambda_Y s}\big) + \sigma_2\Big] \mathrm{d}s, \\
\rho_i &= \rho \frac{\int_0^{\zeta_{i+1}-\zeta_i} \Big[\sigma_1 \cos\big(\mu + (\bar{Y}_i - \mu)e^{-\lambda_Y s}\big) + \sigma_2\Big] \mathrm{d}s}{\sigma_{S,i}\sqrt{\zeta_{i+1}-\zeta_i}}, \\
\rho'_i &= -\rho \frac{\sigma_1 \sigma_{S,i} \int_0^{\zeta_{i+1}-\zeta_i} e^{-\lambda_Y s} \sin\big(\mu + (\bar{Y}_i - \mu)e^{-\lambda_Y s}\big)\mathrm{d}s + \sigma'_{S,i} \int_0^{\zeta_{i+1}-\zeta_i} \Big[\sigma_1 \cos\big(\mu + (\bar{Y}_i - \mu)e^{-\lambda_Y s}\big) + \sigma_2\Big] \mathrm{d}s}{a_{S,i}\sqrt{\zeta_{i+1}-\zeta_i}}
\end{aligned}
$$

TABLE 9. Comparison between the unbiased Monte Carlo estimation for the Vega of a digital Call option in the Stein-Stein type model for different values of the parameters σ_1 and σ_2.

σ_1	σ_2	Euler Scheme			Exponential sampling			Beta sampling		
		Vega	Half-width	Variance	Vega	Half-width	Variance	Vega	Half-width	Variance
0	0.3	0	0	0	−0.000131092	0.00343628	78.6873	0.000348826	0.00401465	75.0995
0.1	0.15	−0.0369417	0.0147786	9.09656	−0.0279754	0.00583723	227.061	−0.0278411	0.00446809	93.0219
0.2	0.25	−0.0292455	0.0138245	7.95983	−0.0324527	0.00764528	389.506	−0.0368005	0.00445648	92.539
0.3	0.4	−0.0415594	0.015675	10.2334	−0.0437127	0.00764653	389.633	−0.0405366	0.0042496	84.1467
0.4	0.5	−0.0538733	0.0178463	13.2649	−0.0526566	0.00643334	275.8048	−0.0546736	0.00423566	83.5954

TABLE 10. Comparison between the unbiased Monte Carlo estimation for the price of a Call option in the model with $\sigma_S(x) = \sigma_1 \cos(x) + \sigma_2$ for different values of the parameters σ_1 and σ_2.

σ_1	σ_2	Euler Scheme			Exponential sampling			Beta sampling		
		Price	Half-width	Variance	Price	Half-width	Variance	Price	Half-width	Variance
0.1	0.15	0.110563	0.000847678	0.0299274	0.111364	0.000124637	0.10352	0.111372	0.000153086	0.109198
0.2	0.25	0.193444	0.00164016	0.112042	0.193513	0.000243912	0.396457	0.193538	0.000291832	0.396832
0.3	0.4	0.294835	0.00281222	0.329386	0.295101	0.000416276	1.15476	0.295277	0.000496958	1.15075
0.4	0.5	0.372503	0.0039339	0.644546	0.373974	0.000648198	2.79991	0.374822	0.000693144	2.23866

TABLE 11. Comparison between the unbiased Monte Carlo estimation for the Delta of a Call option in the model with $\sigma_S(x) = \sigma_1 \cos(x) + \sigma_2$ for different values of the parameters σ_1 and σ_2.

σ_1	σ_2	Euler Scheme			Exponential sampling			Beta sampling		
		Delta	Half-width	Variance	Delta	Half-width	Variance	Delta	Half-width	Variance
0.1	0.15	0.556212	0.00279964	0.326447	0.558216	0.000913553	5.56155	0.558105	0.00114048	6.06062
0.2	0.25	0.576577	0.00321369	0.430147	0.573738	0.00101499	6.86522	0.574758	0.00125848	7.37962
0.3	0.4	0.608348	0.0038355	0.612705	0.60132	0.00118321	9.32936	0.601943	0.00145509	9.86555
0.4	0.5	0.629084	0.00444972	0.824655	0.623976	0.00135463	12.2284	0.625204	0.001653	12.7317

and no analytical expressions are available. However, a simple numerical integration method can be employed for the computation of the above integrals. We here use Simpson's 3/8 rule which for a real-valued $\mathcal{C}^4([0,T])$ function g writes as follows

$$\forall t \in [0,T], \quad \int_0^t g(s)\mathrm{d}s \approx \frac{t}{8}\left(g(0) + 3g\left(\frac{t}{3}\right) + 3g\left(\frac{2t}{3}\right) + g(t)\right)$$

with an error given by $g^{(4)}(t')T^5/6480$ for some $t' \in [0,T]$.

Table 12. Comparison between the unbiased Monte Carlo estimation for the Vega of a Call option in the model with $\sigma_S(x) = \sigma_1 \cos(x) + \sigma_2$ for different values of the parameters σ_1 and σ_2.

σ_1	σ_2	Euler Scheme			Exponential sampling			Beta sampling		
		Vega	Half-width	Variance	Vega	Half-width	Variance	Vega	Half-width	Variance
0.1	0.15	−0.00775169	9.12481e−05	0.000346781	−0.00651665	0.00112781	8.47614	−0.00739643	0.00128018	7.63629
0.2	0.25	−0.0156966	0.000218523	0.00198885	−0.0142195	0.00220569	32.4202	−0.015778	0.00245716	28.1324
0.3	0.4	−0.0233796	0.000412417	0.00708403	−0.0174822	0.00373758	93.0911	−0.0179794	0.00413541	79.6854
0.4	0.5	−0.0307742	0.000670215	0.0187084	−0.0311582	0.00565561	213.151	−0.0304437	0.0042962	153.747

Table 13. Comparison between the unbiased Monte Carlo estimation for the price of a digital Call option in the model with $\sigma_S(x) = \sigma_1 \cos(x) + \sigma_2$ for different values of the parameters σ_1 and σ_2.

σ_1	σ_2	Euler Scheme			Exponential sampling			Beta sampling		
		Price	Half-width	Variance	Price	Half-width	Variance	Price	Half-width	Variance
0	0.3	0.466531	0.00241015	0.241934	0.468532	0.000363404	0.880049	0.468702	0.000466623	1.01455
0.1	0.15	0.481467	0.00241291	0.242488	0.481189	0.000371395	0.91918	0.481203	0.000469696	1.02795
0.2	0.25	0.442993	0.00240127	0.240155	0.445142	0.000368266	0.903758	0.445054	0.000456271	0.970033
0.3	0.4	0.406075	0.00237603	0.235133	0.407653	0.000357256	0.850523	0.407567	0.000441207	0.907039
0.4	0.5	0.377704	0.00234699	0.22942	0.380003	0.000346223	0.798802	0.380009	0.000429336	0.858886

Table 14. Comparison between the unbiased Monte Carlo estimation for the Delta of a digital Call option in the model with $\sigma_S(x) = \sigma_1 \cos(x) + \sigma_2$ for different values of the parameters σ_1 and σ_2.

σ_1	σ_2	Euler Scheme			Exponential sampling			Beta sampling		
		Delta	Half-width	Variance	Delta	Half-width	Variance	Delta	Half-width	Variance
0	0.3	1.23339	0.0306795	39.2017	1.24309	0.00156929	16.411	1.24507	0.0019971	18.5841
0.1	0.15	1.51796	0.0338824	47.8142	1.51053	0.00193524	24.9572	1.51051	0.00242614	27.4265
0.2	0.25	0.816965	0.0251319	26.3063	0.834766	0.00107561	7.70968	0.834635	0.00132928	8.23333
0.3	0.4	0.52951	0.0203232	17.2025	0.527783	0.000676488	3.04964	0.527829	0.000838507	3.27608
0.4	0.5	0.389601	0.0174702	12.7117	0.403047	0.000518279	1.79001	0.403017	0.0006438	1.93127

The parameters of the unbiased Monte Carlo method and the Monte Carlo Euler-Maruyama scheme remain unchanged. The numerical results related to the price, Delta and Vega are provided in Tables 10–12 respectively for the Call option and in Tables 13–15 for the digital Call option. Here again, the unbiased estimators perform very well for all range of values of the parameters.

TABLE 15. Comparison between the unbiased Monte Carlo estimation for the Vega of a digital Call option in the model with $\sigma_S(x) = \sigma_1 \cos(x) + \sigma_2$ for different values of the parameters σ_1 and σ_2.

σ_1	σ_2	Euler Scheme			Exponential sampling			Beta sampling		
		Vega	Half-width	Variance	Vega	Half-width	Variance	Vega	Half-width	Variance
0	0.3	0	0	0	0.000525633	0.00341519	77.7246	−0.00121724	0.00401747	75.2048
0.1	0.15	0.0107747	0.00798181	2.65345	0.00924064	0.00351717	82.4356	0.00920282	0.00407047	77.2024
0.2	0.25	0.0138531	0.00905046	3.41153	0.0129711	0.00349226	81.2719	0.0125995	0.00391887	71.5586
0.3	0.4	0.0123139	0.00853288	3.03249	0.0117377	0.00336349	75.389	0.0119138	0.00375579	65.7269
0.4	0.5	0.0153924	0.00953999	3.79057	0.0170944	0.00319472	68.0131	0.0168796	0.00363443	61.548

APPENDIX A. PROOF OF THEOREM 3.1 AND LEMMA 4.1

The strategy to establish a probabilistic representation formula for the couple (X_T, Y_T) follows similar lines to the one implemented in [2, 3, 6, 9, 12]. The central argument indeed consists in a perturbation argument of the Markov semigroup associated to the original process $(X_t, Y_t)_{t\geq 0}$ around the one generated by an approximation process $(\bar{X}_t, \bar{Y}_t)_{t\geq 0}$. As previously mentioned, the main difference here with respect to the aforementioned references lies exactly in the choice of this approximation process around which this perturbation argument is performed. Though it is still Gaussian, as in [2, 3, 6, 12], it is here crucial to take into account the transport of the initial condition by the ODE, see Section 2.2 for its definition. This leads in turn to a specification of the weights $(\theta_i)_{1\leq i\leq N_T+1}$, see (3.3)–(3.4), that is different from the previous works. The main difficulty then consists in proving that the conditional $L^1(\mathbb{P})$-moment of the weights θ_i are of the correct order, that is, they do not lead to a non-integrable time singularity as hinted in the estimate (2.24) (with $p = 1$) of Definition 2.6. Roughly speaking, these weights are given by Malliavin IBP operators of order 1 or 2 applied to the difference of the coefficients appearing in the dynamics of $(X_t, Y_t)_{t\geq 0}$ and $(\bar{X}_t, \bar{Y}_t)_{t\geq 0}$. As discussed right after the Definition 2.6, the Malliavin IBP operator $\mathcal{I}^{(1,1)}_{i+1}(1) \in \mathbb{M}_{i,n}(\bar{X}, \bar{Y}, -1)$ so that it generates a non-integrable time singularity of order one and the same conclusion holds true for $\mathcal{I}^{(2,2)}_{i+1}(1)$ and $\mathcal{I}^{(1,2)}_{i+1}(1)$. However, the coefficients c^i_S, c^i_Y, b^i_Y, $c^i_{Y,S}$ appearing inside these Malliavin IBP operators, which write as the difference of the coefficients evaluated along the dynamics (2.10) between two consecutive times, allow to remove this time singularity. This is where the transport of the initial condition by the ODE appearing in the second component of (2.10) plays a key role since it allows precisely to remove part of this time singularity. We refer the reader to the technical Lemma B.2 for a rigorous proof of this claim.

A.1 Proof of Theorem 3.1

The proof is divided into three steps. In the first part, we establish the probabilistic representation for a bounded and continuous function h. We then provide the extension to measurable maps satisfying the growth condition 3.1. We eventually conclude by establishing the L^p-moments when the jump times are distributed according to the Beta law.

Denote by $\mathcal{L}_s$ and $\bar{\mathcal{L}}^{(\tau,\xi)}_s$ the infinitesimal generators of $(P_{s,t})_{t\geq s}$ and $(\bar{P}^{(\tau,\xi)}_{s,t})_{t\geq s}$ respectively given by

$$\begin{aligned}\mathcal{L}_s f(x,y) &= (r - \frac{1}{2}a_S(s,y))\partial_x f(x,y) + \frac{1}{2}a_S(s,y)\partial^2_x f(x,y) + b_Y(s,y)\partial_y f(x,y)\\ &\quad + \frac{1}{2}a_Y(s,y)\partial^2_y f(x,y) + \rho(\sigma_S\sigma_Y)(s,y)\partial^2_{x,y} f(x,y),\end{aligned}$$

$$\bar{\mathcal{L}}_s^{(\tau,\xi)} f(x,y) = (r - \frac{1}{2}a_S(s, m_{s,\tau}(\xi)))\partial_x f(x,y) + \frac{1}{2}a_S(s, m_{s,\tau}(\xi))\partial_x^2 f(x,y) + b_Y(s, m_{s,\tau}(\xi))\partial_y f(x,y) + \frac{1}{2}a_Y(s, m_{s,\tau}(\xi))\partial_y^2 f(x,y) + \rho(\sigma_S\sigma_Y)(s, m_{s,\tau}(\xi))\partial_{x,y}^2 f(x,y),$$

for any $f \in \mathcal{C}_b^2(\mathbb{R}^2)$.

Step 1: Probabilistic representation for a bounded and continuous map h

We establish a first order expansion of the Markov semigroup $P_{s,t}$ around $\bar{P}_{s,t}$. We apply Itô's rule to the map $[s,t] \times \mathbb{R}^2 \ni (u,x,y) \mapsto P_{u,t}h(x,y) \in \mathcal{C}_b^{1,2}([s,t]\times\mathbb{R}^2)$ for $h \in \mathcal{C}_b^\infty(\mathbb{R}^2)$, observing that $\partial_s P_{s,t}h(x,y) = -\mathcal{L}_s P_{s,t}h(x,y)$. We obtain

$$\begin{aligned} h(\bar{X}_t^{s,x,(\tau,\xi)}, \bar{Y}_t^{s,y,(\tau,\xi)}) &= P_{s,t}h(x,y) + \int_s^t \Big(\partial_u P_{u,t} h(\bar{X}_u^{s,x,(\tau,\xi)}, \bar{Y}_u^{s,y,(\tau,\xi)}) + \bar{\mathcal{L}}_u P_{s,t} h(\bar{X}_u^{s,x,(\tau,\xi)}, \bar{Y}_u^{s,y,(\tau,\xi)})\Big)\,\mathrm{d}u + M_t \\ &= P_{s,t}h(x,y) + \int_s^t (\bar{\mathcal{L}}_u^{(\tau,\xi)} - \mathcal{L}_u) P_{s,t} h(\bar{X}_u^{s,x}, \bar{Y}_u^{s,y})\,\mathrm{d}u + M_t, \end{aligned}$$

where $M := (M_t)_{t\geq s}$ is a square integrable martingale. We then take expectation in the previous expression, take $(\tau,\xi) = (s,y)$ and make use of Fubini's theorem so that

$$\begin{aligned} &P_{s,T}h(x,y) \\ &= \bar{P}_{s,T}h(x,y) + \int_s^T \mathbb{E}[(\mathcal{L}_u - \bar{\mathcal{L}}_u)P_{u,T}h(\bar{X}_u^{s,x}, \bar{Y}_u^{s,y})]\,\mathrm{d}u \\ &= \mathbb{E}[h(\bar{X}_T^{s,x}, \bar{Y}_T^{s,y})] + \int_s^T \mathbb{E}\Big[\frac{1}{2}(a_S(u, \bar{Y}_u^{s,y}) - a_S(u, m_{u,s}(y)))[\partial_x^2 P_{u,T}h(\bar{X}_u^{s,x}, \bar{Y}_u^{s,y}) - \partial_x P_{u,T}h(\bar{X}_u^{s,x}, \bar{Y}_u^{s,y})]\Big]\,\mathrm{d}u \\ &+ \int_s^T \mathbb{E}\Big[\frac{1}{2}(a_Y(u, \bar{Y}_u^{s,y}) - a_Y(u, m_{u,s}(y)))\partial_y^2 P_{u,T}h(\bar{X}_u^{s,x}, \bar{Y}_u^{s,y}) + (b_Y(u, \bar{Y}_u^{s,y}) - b_Y(u, m_{u,s}(y)))\partial_y P_{u,T}h(\bar{X}_u^{s,x}, \bar{Y}_u^{s,y})\Big]\,\mathrm{d}u \\ &+ \int_s^T \mathbb{E}\Big[\rho((\sigma_S\sigma_Y)(u, \bar{Y}_u^{s,x}) - (\sigma_S\sigma_Y)(u, m_{u,s}(y)))\partial_{x,y}^2 P_{u,T}h(\bar{X}_u^{s,x}, \bar{Y}_u^{s,y})\Big]\,\mathrm{d}u. \end{aligned} \tag{A.1}$$

We now take $s = 0$ in the previous first order expansion and express it using the Markov chain $(\bar{X}_i, \bar{Y}_i)_{0\leq i\leq N_T+1}$ and the renewal process N. From the previous identity, the definition of θ_{N_T+1} in (3.4) and the identity (2.9), we directly obtain

$$\begin{aligned} P_T h(x_0,y_0) &= \mathbb{E}[h(\bar{X}_{N_T+1}, \bar{Y}_{N_T+1})\theta_{N_T+1}\mathbf{1}_{\{N_T=0\}}] \\ &\quad + \mathbb{E}\Bigg[((1-F(T-\zeta_1))f(\zeta_1))^{-1}\mathbf{1}_{\{N_T=1\}}\Bigg[\frac{1}{2}(a_S(\zeta_1, \bar{Y}_1) - a_S(\zeta_1, m_0))\mathcal{D}_1^{(1,1)} P_{\zeta_1,T}h(\bar{X}_1, \bar{Y}_1) \\ &\quad - \frac{1}{2}(a_S(\zeta_1, \bar{Y}_1) - a_S(\zeta_1, m_0))\mathcal{D}_1^{(1)} P_{\zeta_1,T}h(\bar{X}_1, \bar{Y}_1) + \frac{1}{2}(a_Y(\zeta_1, \bar{Y}_1) - a_Y(\zeta_1, m_0))\mathcal{D}_1^{(2,2)} P_{\zeta_1,T}h(\bar{X}_1, \bar{Y}_1) \\ &\quad + (b_Y(\zeta_1, \bar{Y}_1) - b_Y(\zeta_1, m_0))\mathcal{D}_1^{(2)} P_{\zeta_1,T}h(\bar{X}_1, \bar{Y}_1) + \rho((\sigma_S\sigma_Y)(\zeta_1, \bar{Y}_1) - (\sigma_S\sigma_Y)(\zeta_1, m_0))\mathcal{D}_1^{(1,2)} P_{\zeta_1,T}h(\bar{X}_1, \bar{Y}_1)\Bigg]\Bigg] \\ &= \mathbb{E}[h(\bar{X}_{N_T+1}, \bar{Y}_{N_T+1})\theta_{N_T+1}\mathbf{1}_{\{N_T=0\}}] + \mathbb{E}\Big[((1-F(T-\zeta_1))f(\zeta_1))^{-1}\mathbf{1}_{\{N_T=1\}}\Big[c_S^1\mathcal{D}_1^{(1,1)} P_{\zeta_1,T}h(\bar{X}_1, \bar{Y}_1) \\ &\quad - c_S^1\mathcal{D}_1^{(1)} P_{\zeta_1,T}h(\bar{X}_1, \bar{Y}_1) + c_Y^1\mathcal{D}_1^{(2,2)} P_{\zeta_1,T}h(\bar{X}_1, \bar{Y}_1) + b_Y^1\mathcal{D}_1^{(2)} P_{\zeta_1,T}h(\bar{X}_1, \bar{Y}_1) + c_{S,Y}^1\mathcal{D}_1^{(1,2)} P_{\zeta_1,T}h(\bar{X}_1, \bar{Y}_1)\Big]\Big]. \end{aligned} \tag{A.2}$$

Next, we apply the IBP formula (2.17) with respect to the random vector $(\bar{X}_1, \bar{Y}_1)$ in the above expression. In order to do that rigorously, one first has to take the conditional expectation $\mathbb{E}_{0,1}[.]$ in the second term of the above equality. We thus obtain

$$\mathbb{E}_{0,1}\Big[c_S^1\mathcal{D}_1^{(1,1)}P_{\zeta_1,T}h(\bar X_1,\bar Y_1)-c_S^1\mathcal{D}_1^{(1)}P_{\zeta_1,T}h(\bar X_1,\bar Y_1)+c_Y^1\mathcal{D}_1^{(2,2)}P_{\zeta_1,T}h(\bar X_1,\bar Y_1)+b_Y^1\mathcal{D}_1^{(2)}P_{\zeta_1,T}h(\bar X_1,\bar Y_1)$$
$$+c_{S,Y}^1\mathcal{D}_1^{(1,2)}P_{\zeta_1,T}h(\bar X_1,\bar Y_1)\Big]=\mathbb{E}_{0,1}\Big[\Big[\mathcal{I}_1^{(1,1)}(c_S^1)-\mathcal{I}_1^{(1)}(c_S^1)+\mathcal{I}_1^{(2,2)}(c_Y^1)+\mathcal{I}_1^{(2)}(b_Y^1)+\mathcal{I}_1^{(1,2)}(c_{S,Y}^1)\Big]P_{\zeta_1,T}h(\bar X_1,\bar Y_1)\Big]. \tag{A.3}$$

From Lemma B.2 and the estimate (2.27), we get

$$\mathbb{E}_{0,1}\Big[\Big|\Big[\mathcal{I}_1^{(1,1)}(c_S^1)-\mathcal{I}_1^{(1)}(c_S^1)+\mathcal{I}_1^{(2,2)}(c_Y^1)+\mathcal{I}_1^{(2)}(b_Y^1)+\mathcal{I}_1^{(1,2)}(c_{S,Y}^1)\Big]\Big|\Big|P_{\zeta_1,T}h(\bar X_1,\bar Y_1)\Big|\Big]\le C_T|h|_\infty\zeta_1^{-1/2}, \tag{A.4}$$

for some positive constant C_T such that $T\mapsto C_T$ is non-decreasing. The previous estimate yields an integrable time singularity. Indeed, from the previous estimate and (2.9), one directly gets

$$\begin{aligned}\mathbb{E}\Big[&((1-F(T-\zeta_1))f(\zeta_1))^{-1}\mathbf{1}_{\{N_T=1\}}\Big|\mathbb{E}_{0,1}\Big[\Big[\mathcal{I}_1^{(1,1)}(c_S^1)-\mathcal{I}_1^{(1)}(c_S^1)+\mathcal{I}_1^{(2,2)}(c_Y^1)+\mathcal{I}_1^{(2)}(b_Y^1)+\mathcal{I}_1^{(1,2)}(c_{S,Y}^1)\Big]P_{\zeta_1,T}h(\bar X_1,\bar Y_1)\Big]\Big|\Big]\\
&\le C\mathbb{E}[((1-F(T-\zeta_1))f(\zeta_1))^{-1}\zeta_1^{-1/2}\mathbf{1}_{\{N_T=1\}}]\\
&=C\int_0^T s_1^{-1/2}\,\mathrm{d}s_1<\infty.\end{aligned}$$

Coming back to (A.2) and using (A.3), we thus derive

$$\begin{aligned}P_Th(x_0,y_0)&=\mathbb{E}[h(\bar X_{N_T+1},\bar Y_{N_T+1})\theta_{N_T+1}\mathbf{1}_{\{N_T=0\}}]\\
&\quad+\mathbb{E}\Big[((1-F(T-\zeta_1))f(\zeta_1))^{-1}\mathbf{1}_{\{N_T=1\}}\\
&\qquad\times\Big[\mathcal{I}_1^{(1,1)}(c_S^1)-\mathcal{I}_1^{(1)}(c_S^1)+\mathcal{I}_1^{(2,2)}(c_Y^1)+\mathcal{I}_1^{(2)}(b_Y^1)+\mathcal{I}_1^{(1,2)}(c_{S,Y}^1)\Big]P_{\zeta_1,T}h(\bar X_1,\bar Y_1)\Big]\\
&=\mathbb{E}[h(\bar X_{N_T+1},\bar Y_{N_T+1})\theta_{N_T+1}\mathbf{1}_{\{N_T=0\}}]+\mathbb{E}\Big[P_{\zeta_1,T}h(\bar X_1,\bar Y_1)\theta_2\theta_1\mathbf{1}_{\{N_T=1\}}\Big].\end{aligned} \tag{A.5}$$

Our aim now is to iterate the above first order expansion. We prove by induction the following formula: for any positive integer n, one has

$$P_Th(x_0,y_0)=\sum_{j=0}^{n-1}\mathbb{E}\left[h(\bar X_{N_T+1},\bar Y_{N_T+1})\prod_{i=1}^{N_T+1}\theta_i\mathbf{1}_{\{N_T=j\}}\right]+\mathbb{E}\left[P_{\zeta_n,T}h(\bar X_n,\bar Y_n)\prod_{i=1}^{n+1}\theta_i\mathbf{1}_{\{N_T=n\}}\right]. \tag{A.6}$$

The case $n=1$ corresponds to (A.5). We thus assume that (A.6) holds at step n. We expand the last term appearing in the right-hand side of the previous equality using again (A.1), by then applying Lemma B.1 and by finally performing IBPs as before.

To be more specific, using the notations introduced in Section 2.2, from (A.1) and a change of variable, for any (deterministic) $\zeta\in[0,T]$, one has

$$\begin{aligned}&P_{\zeta,T}h(x,y)\\
&\quad=\mathbb{E}[h(\bar X_T^{\zeta,x},\bar Y_T^{\zeta,y})]\\
&\quad+\int_\zeta^T\mathbb{E}\Big[\frac12(a_S(u,\bar Y_u^{\zeta,y})-a_S(u,m_{u,\zeta}(y)))[\partial_x^2P_{u,T}h(\bar X_u^{\zeta,x},\bar Y_u^{\zeta,y})-\partial_xP_{u,T}h(\bar X_u^{\zeta,x},\bar Y_u^{\zeta,y})]\Big]\,\mathrm{d}u\\
&\quad+\int_\zeta^T\mathbb{E}\Big[\frac12(a_Y(u,\bar Y_u^{\zeta,y})-a_Y(u,m_{u,\zeta}(y)))\partial_y^2P_{u,T}h(\bar X_s^{\zeta,x},\bar Y_s^{\zeta,y})+(b_Y(u,\bar Y_u^{\zeta,y})-b_Y(u,m_{u,\zeta}(y)))\partial_yP_{u,T}h(\bar X_u^{\zeta,x},\bar Y_u^{\zeta,y})\Big]\,\mathrm{d}u\end{aligned}$$

$$+\int_{\zeta}^{T}\mathbb{E}\Big[\rho((\sigma_S\sigma_Y)(u,\bar{Y}_u^{\zeta,y})-(\sigma_S\sigma_Y)(u,m_{u,\zeta}(y)))\partial^2_{x,y}P_{u,T}h(\bar{X}_u^{\zeta,x},\bar{Y}_u^{\zeta,y})\Big]\,\mathrm{d}u.$$

We take $\zeta=\zeta_n$, $(x,y)=(\bar{X}_{N_T},\bar{Y}_{N_T})$ in the previous equality, then multiply it by $\prod_{i=1}^{n+1}\theta_i\mathbf{1}_{\{N_T=n\}}$ and finally take expectation. We obtain

$$\begin{aligned}
&\mathbb{E}\Big[P_{\zeta_n,T}h(\bar{X}_n,\bar{Y}_n)\prod_{i=1}^{n+1}\theta_i\mathbf{1}_{\{N_T=n\}}\Big]\\
&=\mathbb{E}\Big[h(\bar{X}_T^{\zeta_n,\bar{X}_n},\bar{Y}_T^{\zeta_n,\bar{Y}_n})\prod_{i=1}^{n+1}\theta_i\mathbf{1}_{\{N_T=n\}}\Big]\\
&+\mathbb{E}\Big[\prod_{i=1}^{n+1}\theta_i\mathbf{1}_{\{N_T=n\}}\int_{\zeta_n}^{T}\frac{1}{2}(a_S(u,\bar{Y}_u^{\zeta_n,\bar{Y}_n})-a_S(u,m_{u,\zeta_n}(\bar{Y}_n)))[\partial_x^2P_{u,T}h(\bar{X}_u^{\zeta_n,\bar{X}_n},\bar{Y}_u^{\zeta_n,\bar{Y}_n})-\partial_xP_{u,T}h(\bar{X}_u^{\zeta_n,\bar{X}_n},\bar{Y}_u^{\zeta_n,\bar{Y}_n})]\,\mathrm{d}u\Big]\\
&+\mathbb{E}\Big[\prod_{i=1}^{n+1}\theta_i\mathbf{1}_{\{N_T=n\}}\int_{\zeta_n}^{T}\frac{1}{2}(a_Y(u,\bar{Y}_u^{\zeta_n,\bar{Y}_n})-a_Y(u,m_{u,\zeta_n}(\bar{Y}_n)))\partial_y^2P_{u,T}h(\bar{X}_u^{\zeta_n,\bar{X}_n},\bar{Y}_u^{\zeta_n,\bar{Y}_n})\,\mathrm{d}u\Big]\\
&+\mathbb{E}\Big[\prod_{i=1}^{n+1}\theta_i\mathbf{1}_{\{N_T=n\}}\int_{\zeta_n}^{T}(b_Y(u,\bar{Y}_u^{\zeta_n,\bar{Y}_n})-b_Y(u,m_{u,\zeta_n}(\bar{Y}_n)))\partial_yP_{u,T}h(\bar{X}_u^{\zeta_n,\bar{X}_n},\bar{Y}_u^{\zeta_n,\bar{Y}_n})\,\mathrm{d}u\Big]\\
&+\mathbb{E}\Big[\prod_{i=1}^{n+1}\theta_i\mathbf{1}_{\{N_T=n\}}\int_{\zeta_n}^{T}\rho((\sigma_S\sigma_Y)(u,\bar{Y}_u^{\zeta_n,\bar{Y}_n})-(\sigma_S\sigma_Y)(u,m_{u,\zeta_n}(\bar{Y}_n)))\partial^2_{x,y}P_{u,T}h(\bar{X}_u^{\zeta_n,\bar{X}_n},\bar{Y}_u^{\zeta_n,\bar{Y}_n})\,\mathrm{d}u\Big].
\end{aligned}\tag{A.7}$$

Now, from the very definition of the Markov chain $(\bar{X}_i,\bar{Y}_i)_{0\le i\le N_T+1}$ and of the weight sequence $(\theta_i)_{1\le i\le N_T+1}$ of Theorem 3.1, the first term of the above equality can be written as

$$\mathbb{E}\Big[h(\bar{X}_T^{\zeta_n,\bar{X}_n},\bar{Y}_T^{\zeta_n,\bar{Y}_n})\prod_{i=1}^{n+1}\theta_i\mathbf{1}_{\{N_T=n\}}\Big]=\mathbb{E}\Big[h(\bar{X}_{N_T+1},\bar{Y}_{N_T+1})\prod_{i=1}^{N_T+1}\theta_i\mathbf{1}_{\{N_T=n\}}\Big].\tag{A.8}$$

We now look at the second, third, fourth and fifth terms. Let us deal with the third and fourth terms. The others are treated in a similar manner and we will omit some technical details. We first take its conditional expectation w.r.t $\{\zeta_1=t_1,\dots,\zeta_n=t_n,N_T=n\}$ and introduce the measurable function

$$\begin{aligned}
G(t_1,\dots,t_n,u,T):=\mathbb{E}\Big[\prod_{i=1}^{n+1}\theta_i\Big[&\frac{1}{2}(a_Y(u,\bar{Y}_u^{\zeta_n,\bar{Y}_n})-a_Y(u,m_{u,\zeta_n}(\bar{Y}_n)))\partial_y^2P_{u,T}h(\bar{X}_u^{\zeta_n,\bar{X}_n},\bar{Y}_u^{\zeta_n,\bar{Y}_n})\\
&+(b_Y(u,\bar{Y}_u^{\zeta_n,\bar{Y}_n})-b_Y(u,m_{u,\zeta_n}(\bar{Y}_n)))\partial_yP_{u,T}h(\bar{X}_u^{\zeta_n,\bar{X}_n},\bar{Y}_u^{\zeta_n,\bar{Y}_n})\Big]\Big|\zeta_1=t_1,\dots,\zeta_n=t_n,N_T=n\Big],
\end{aligned}$$

which satisfies

$$\begin{aligned}
|G(t_1,\dots,t_n,u,T)|&\le C\mathbb{E}\Big[\prod_{i=1}^{n+1}|\theta_i|\Big(1+\Big|\int_{t_n}^{u}\sigma_Y(s,m_{s,t_n})\,\mathrm{d}B_s\Big|\Big)\Big|\zeta_1=t_1,\dots,\zeta_n=t_n,N_T=n\Big]\\
&\le C\mathbb{E}\Big[\prod_{i=1}^{n+1}|\theta_i|\big|\zeta_1=t_1,\dots,\zeta_n=t_n,N_T=n\Big],
\end{aligned}$$

where we used the boundedness of a_Y, the Lipschitz regularity of b_Y, the inequalities $\sup_{0\le t\le T}|\partial_y^{\ell}P_th|_\infty\le C$ for $\ell=1$, 2 and, for the last inequality, the inequality $\mathbb{E}[|\int_{t_n}^{u}\sigma_Y(s,m_{s,t_n})\,\mathrm{d}B_s||\mathcal{G}_n,\zeta_1=t_1,\dots,\zeta_n=t_n,N_T=n]\le$

$C|\sigma_Y|_\infty(u-t_n)^{1/2} \leq CT^{1/2}$. Recall now that $\mathbb{P}(N_T = n, \zeta_1 \in dt_1, \ldots, \zeta_n \in dt_n) = (1-F(T-t_n))\prod_{j=0}^{n-1} f(t_{j+1} - t_j)\mathrm{d}t_1, \ldots, \mathrm{d}t_n$ on the set $\Delta_n(T)$ so that from Lemma B.2 and the estimate (2.27), we obtain

$$\mathbb{E}\Big[\int_{\zeta_n}^T |G(\zeta_1,\ldots,\zeta_n,u,T)|\mathbf{1}_{\{N_T=n\}}\mathrm{d}u\Big] \leq C\int_{t_n}^T \int_{\Delta_n(T)} (T-t_n)\prod_{i=1}^n (t_i - t_{i-1})^{-1/2}\mathrm{d}t_1\cdots \mathrm{d}t_n\mathrm{d}t_{n+1} < \infty.$$

Hence, by Lemma B.1, it holds

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\{N_T=n\}}\prod_{i=1}^{n+1}\theta_i \int_{\zeta_n}^T \Big[\frac{1}{2}(a_Y(s,\bar{Y}_s^{\zeta_n,\bar{Y}_n}) - a_Y(s,m_{s,\zeta_n}(\bar{Y}_n)))\partial_y^2 P_{s,T}h(\bar{X}_s^{\zeta_n,\bar{X}_n},\bar{Y}_s^{\zeta_n,\bar{Y}_n})\\
&\quad + (b_Y(s,\bar{Y}_s^{\zeta_n,\bar{Y}_n}) - b_Y(s,m_{s,\zeta_n}(\bar{Y}_n)))\partial_y P_{s,T}h(\bar{X}_s^{\zeta_n,\bar{X}_n},\bar{Y}_s^{\zeta_n,\bar{Y}_n})\Big]\,\mathrm{d}s\Big]\\
&= \mathbb{E}\Big[\mathbf{1}_{\{N_T=n\}}\int_{\zeta_n}^T G(\zeta_1,\ldots,\zeta_n,s,T)\,\mathrm{d}s\Big]\\
&= \mathbb{E}\Big[\prod_{i=1}^n \theta_i(1-F(T-\zeta_{n+1}))^{-1}(f(\zeta_{n+1}-\zeta_n))^{-1}\Big[\frac{1}{2}(a_Y(\zeta_{n+1},\bar{Y}_{n+1}) - a_Y(\zeta_{n+1},m_n))\mathcal{D}_{n+1}^{(2,2)}P_{\zeta_{n+1},T}h(\bar{X}_{n+1},\bar{Y}_{n+1})\\
&\quad + (b_Y(\zeta_{n+1},\bar{Y}_{n+1}) - b_Y(\zeta_{n+1},m_n))\mathcal{D}_{n+1}^{(2)}P_{\zeta_{n+1},T}h(\bar{X}_{n+1},\bar{Y}_{n+1})\Big]\mathbf{1}_{\{N_T=n+1\}}\Big].
\end{aligned}$$

Finally, we take the conditional expectation $\mathbb{E}_{n,n+1}[.]$ inside the above expectation and then employ the IBP formula (2.17), two times w.r.t. the diffusion coefficient and one time w.r.t the drift coefficient as done before. We obtain

$$\begin{aligned}
&\mathbb{E}\Big[\prod_{i=1}^n \theta_i(1-F(T-\zeta_{n+1}))^{-1}(f(\zeta_{n+1}-\zeta_n))^{-1}\Big[\frac{1}{2}(a_Y(\zeta_{n+1},\bar{Y}_{n+1}) - a_Y(\zeta_{n+1},m_n))\mathcal{D}_{n+1}^{(2,2)}P_{\zeta_{n+1},T}h(\bar{X}_{n+1},\bar{Y}_{n+1})\\
&\quad + (b_Y(\zeta_{n+1},\bar{Y}_{n+1}) - b_Y(\zeta_{n+1},m_n))\mathcal{D}_{n+1}^{(2)}P_{\zeta_{n+1},T}h(\bar{X}_{n+1},\bar{Y}_{n+1})\Big]\mathbf{1}_{\{N_T=n+1\}}\Big]\\
&= \mathbb{E}\Big[\prod_{i=1}^n \theta_i(1-F(T-\zeta_{n+1}))^{-1}(f(\zeta_{n+1}-\zeta_n))^{-1}[\mathcal{I}_{n+1}^{(2,2)}(c_Y^{n+1}) + \mathcal{I}_{n+1}^{(2)}(b_Y^{n+1})]P_{\zeta_{n+1},T}h(\bar{X}_{n+1},\bar{Y}_{n+1})\mathbf{1}_{\{N_T=n+1\}}\Big].
\end{aligned}$$

In a completely analogous manner, we derive

$$\begin{aligned}
&\mathbb{E}\Big[\prod_{i=1}^{n+1}\theta_i\mathbf{1}_{\{N_T=n\}}\int_{\zeta_n}^T \frac{1}{2}(a_S(s,\bar{Y}_s^{\zeta_n,\bar{Y}_n}) - a_S(s,m_{s,\zeta_n}(\bar{Y}_n)))[\partial_x^2 P_{s,T}h(\bar{X}_s^{\zeta_n,\bar{X}_n},\bar{Y}_s^{\zeta_n,\bar{Y}_n}) - \partial_x P_{s,T}h(\bar{X}_s^{\zeta_n,\bar{X}_n},\bar{Y}_s^{\zeta_n,\bar{Y}_n})]\,\mathrm{d}s\Big]\\
&= \mathbb{E}\Big[\prod_{i=1}^n \theta_i(1-F(T-\zeta_{n+1}))^{-1}(f(\zeta_{n+1}-\zeta_n))^{-1}[\mathcal{I}_{n+1}^{(1,1)}(c_S^{n+1}) - \mathcal{I}_{n+1}^{(1)}(c_S^{n+1})]P_{\zeta_{n+1},T}h(\bar{X}_{n+1},\bar{Y}_{n+1})\mathbf{1}_{\{N_T=n+1\}}\Big],
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\Big[\prod_{i=1}^{n+1}\theta_i\mathbf{1}_{\{N_T=n\}}\int_{\zeta_n}^T \rho((\sigma_S\sigma_Y)(s,\bar{Y}_s^{\zeta_n,\bar{Y}_n}) - (\sigma_S\sigma_Y)(s,m_{s,\zeta_n}(\bar{Y}_n)))\partial_{x,y}^2 P_{s,T}h(\bar{X}_s^{\zeta_n,\bar{X}_n},\bar{Y}_s^{\zeta_n,\bar{Y}_n})\,\mathrm{d}s\Big]\\
&= \mathbb{E}\Big[\prod_{i=1}^n \theta_i(1-F(T-\zeta_{n+1}))^{-1}(f(\zeta_{n+1}-\zeta_n))^{-1}\mathcal{I}_{n+1}^{(1,2)}(c_{Y,S}^{n+1})P_{\zeta_{n+1},T}h(\bar{X}_{n+1},\bar{Y}_{n+1})\mathbf{1}_{\{N_T=n+1\}}\Big].
\end{aligned}$$

Summing the three previous identities, we obtain that the sum of the second, third, fourth and fifth term in the right-hand side of (A.7) is equal to

$$\begin{aligned}
&\mathbb{E}\Big[\prod_{i=1}^{n}\theta_i(1-F(T-\zeta_{n+1}))^{-1}(f(\zeta_{n+1}-\zeta_n))^{-1}\\
&\qquad\times\Big[\mathcal{I}^{(1,1)}_{n+1}(a^{n+1}_S)-\mathcal{I}^{(1)}_{n+1}(c^{n+1}_S)+\mathcal{I}^{(2,2)}_{n+1}(c^{n+1}_Y)+\mathcal{I}^{(2)}_{n+1}(b^{n+1}_Y)+\mathcal{I}^{(1,2)}_{n+1}(c^{n+1}_{Y,S})\Big]P_{\zeta_{n+1},T}h(\bar{X}_{n+1},\bar{Y}_{n+1})\mathbf{1}_{\{N_T=n+1\}}\Big]\\
&\quad=\mathbb{E}\Big[\prod_{i=1}^{n+2}\theta_i P_{\zeta_{n+1},T}h(\bar{X}_{n+1},\bar{Y}_{n+1})\mathbf{1}_{\{N_T=n+1\}}\Big],
\end{aligned}$$

where we used the very definitions (3.3) and (3.4) of the weights $(\theta_i)_{1\leq i\leq N_T+1}$ on the set $\{N_T=n+1\}$. This concludes the proof of (A.6) at step $n+1$.

To conclude it remains to prove the absolute convergence of the first sum and the convergence to zero of the last term in (A.6). These two results follow directly from the boundedness of h and the general estimates on the product of weights established in Lemma B.2.

Indeed, from Lemma B.2, the estimate (2.27), the tower property of conditional expectation and the identity (2.9), we obtain

$$\begin{aligned}
\mathbb{E}\Big[\big|h(\bar{X}_{N_T+1},\bar{Y}_{N_T+1})\big|\prod_{i=1}^{N_T+1}|\theta_i|\,\mathbf{1}_{\{N_T=j\}}\Big]&\leq C^j|h|_\infty\mathbb{E}\Big[(1-F(T-\zeta_j))^{-1}\prod_{i=1}^{j}(f(\zeta_i-\zeta_{i-1}))^{-1}(\zeta_i-\zeta_{i-1})^{-\frac{1}{2}}\mathbf{1}_{\{N_T=j\}}\Big]\\
&=C^j|h|_\infty\int_{\Delta_j(T)}\prod_{i=1}^{j}(s_i-s_{i-1})^{-\frac{1}{2}}\,\mathrm{d}\mathbf{s}_j\\
&=C^j|h|_\infty T^{\frac{j}{2}}\frac{(\Gamma(1/2))^j}{\Gamma(1+j/2)},
\end{aligned}$$

which in turn yields

$$\sum_{j=0}^{n-1}\mathbb{E}\Big[\big|h(\bar{X}_{N_T+1},\bar{Y}_{N_T+1})\big|\prod_{i=1}^{N_T+1}|\theta_i|\,\mathbf{1}_{\{N_T=j\}}\Big]\leq|h|_\infty\sum_{n\geq0}\frac{(CT^{1/2})^n}{\Gamma(1+n/2)}=|h|_\infty E_{1/2,1}(CT^{1/2}),$$

so that the series converge absolutely. Similarly,

$$\Big|\mathbb{E}\Big[P_{\zeta_n,T}h(\bar{X}_n,\bar{Y}_n)\prod_{i=1}^{n+1}\theta_i\mathbf{1}_{\{N_T=n\}}\Big]\Big|\leq C^n|h|_\infty T^{\frac{n}{2}}\frac{\Gamma^n(1/2)}{\Gamma(1+n/2)},$$

so that the remainder indeed vanishes as n goes to infinity. We thus conclude

$$P_Th(x_0,y_0)=\sum_{n\geq0}\mathbb{E}\Big[h(\bar{X}_{N_T+1},\bar{Y}_{N_T+1})\prod_{i=1}^{N_T+1}\theta_i\mathbf{1}_{\{N_T=n\}}\Big]=\mathbb{E}\Big[h(\bar{X}_{N_T+1},\bar{Y}_{N_T+1})\prod_{i=1}^{N_T+1}\theta_i\Big],\tag{A.9}$$

for any $h\in\mathcal{C}^2_b(\mathbb{R}^2)$. We eventually extend the above representation formula to any bounded and continuous function h using a standard approximation argument. The remaining technical details are omitted.

Step 2: Extension to measurable maps satisfying the growth assumption (3.1)

We first extend the previous result to any bounded and measurable h. This follows from a monotone class argument that we now detail.

Let us first consider $h \in \mathcal{C}_b(\mathbb{R}^2)$. From Fubini's theorem, it holds

$$\mathbb{E}\Big[h(\bar{X}_{n+1},\bar{Y}_{n+1})\prod_{i=1}^{n+1}\theta_i\Big|N_T=n,\zeta^{n+1}\Big]$$
$$=\int_{\mathbb{R}^2}h(x,y)\,\mathbb{E}\Big[\bar{p}(\zeta_n,T,\bar{X}_n,\bar{Y}_n,x,y)\prod_{i=1}^{n+1}\theta_i(\bar{X}_{i-1},\bar{Y}_{i-1},\bar{X}_i,\bar{Y}_i,\zeta^{n+1})\Big|N_T=n,\zeta^{n+1}\Big]\,\mathrm{d}x\mathrm{d}y,$$

which can be justified as follows. From the upper-bound estimate (2.7), Lemma B.2 and Lemma B.3, it holds

$$\begin{aligned}
&\Big|\mathbb{E}\Big[\bar{p}(\zeta_n,T,\bar{X}_n,\bar{Y}_n,x,y)\prod_{i=1}^{n+1}\theta_i(\bar{X}_{i-1},\bar{Y}_{i-1},\bar{X}_i,\bar{Y}_i,\zeta^{n+1})\Big|N_T=n,\zeta^{n+1}\Big]\Big|\\
&\quad\le(1-F(T-\zeta_n))^{-1}\int_{(\mathbb{R}^2)^n}\bar{p}(\zeta_n,T,x_n,y_n,x,y)\prod_{i=1}^{n}|\theta_i(x_{i-1},y_{i-1},x_i,y_i,\zeta^{n+1})|\bar{p}(\zeta_{i-1},\zeta_i,x_{i-1},y_{i-1},x_i,y_i)\,\mathrm{d}\mathbf{x}_n\mathrm{d}\mathbf{y}_n\\
&\quad\le(C_T)^{n+1}(1-F(T-\zeta_n))^{-1}\int_{(\mathbb{R}^2)^n}\bar{q}_c(\zeta_n,T,x_n,y_n,x,y)\prod_{i=1}^{n}(f(\zeta_i-\zeta_{i-1}))^{-1}(\zeta_i-\zeta_{i-1})^{-\frac12}\\
&\qquad\times\bar{q}_c(\zeta_{i-1},\zeta_i,x_{i-1},y_{i-1},x_i,y_i)\,\mathrm{d}\mathbf{x}_n\mathrm{d}\mathbf{y}_n\\
&\quad\le(C_T)^{n+1}(1-F(T-\zeta_n))^{-1}\prod_{i=1}^{n}(f(\zeta_i-\zeta_{i-1}))^{-1}(\zeta_i-\zeta_{i-1})^{-\frac12}\,\bar{q}_{c'}(T,x_0,y_0,x,y),
\end{aligned}$$

where we recall that $c':=(C')^2c$ for any $c>4\kappa$, recalling that C' is defined in (B.2). Hence, from (A.9) and again Fubini's theorem, justified by the previous estimate and the fact that $\mathbb{E}\Big[(C_T)^{N_T+1}(1-F(T-\zeta_{N_T}))^{-1}\prod_{i=1}^{N_T}(f(\zeta_i-\zeta_{i-1}))^{-1}(\zeta_i-\zeta_{i-1})^{-\frac12}\Big]<\infty$, one has

$$P_Th(x_0,y_0)=\int_{\mathbb{R}^2}h(x,y)\,\mathbb{E}\Big[\bar{p}(\zeta_{N_T},T,\bar{X}_{N_T},\bar{Y}_{N_T},x,y)\prod_{i=1}^{N_T+1}\theta_i\Big]\,\mathrm{d}x\mathrm{d}y,\tag{A.10}$$

for any $h\in\mathcal{C}_b(\mathbb{R}^2)$. Moreover, from the previous computations, the following upper-bound holds

$$\begin{aligned}
&\Big|\mathbb{E}\Big[\bar{p}(T-\zeta_{N_T},\bar{X}_{N_T},\bar{Y}_{N_T},x,y)\prod_{i=1}^{N_T+1}\theta_i\Big]\Big|\\
&=\Big|\sum_{n\ge0}\int_{\Delta_n(T)}\mathbb{E}\Big[\bar{p}(s_n,T,\bar{X}_n,\bar{Y}_n,x,y)\prod_{i=1}^{n+1}\theta_i\,\Big|N_T=n,\zeta^{n+1}=(0,s_1,\ldots,s_n,T)\Big]\,(1-F(T-s_n))\prod_{i=1}^{n}f(s_i-s_{i-1})\,\mathrm{d}\mathbf{s}_n\Big|\\
&\le\Big(\sum_{n\ge0}\int_{\Delta_n(T)}(C_T)^{n+1}\prod_{i=1}^{n}(s_i-s_{i-1})^{-\frac12}\,\mathrm{d}\mathbf{s}_n\Big)\bar{q}_c(T,x_0,y_0,x,y)\\
&=CE_{1/2,1}(CT^{1/2})\bar{q}_{c'}(T,x_0,y_0,x,y).
\end{aligned}\tag{A.11}$$

It now follows from (A.10) and a monotone class argument that the probabilistic representation formula (3.2) as well as (A.10) are valid for any real-valued bounded and measurable map h defined over $\mathbb{R}^2$. Since (A.10) is valid for any bounded and measurable h, a density function, denoted by $(x,y)\mapsto p_T(x_0,y_0,x,y)$, can be associated to the semigroup P_T. Namely, $P_T h(x_0,y_0) = \int_{\mathbb{R}^2} h(x,y)\, p_T(x_0,y_0,x,y)\, \mathrm{d}x\mathrm{d}y$ for any bounded and measurable h defined on $\mathbb{R}^2$. Moreover, it holds

$$p_T(x_0,y_0,x,y) = \mathbb{E}\Big[\bar{p}(\zeta_{N_T}, T, \bar{X}_{N_T}, \bar{Y}_{N_T}, x, y) \prod_{i=1}^{N_T+1} \theta_i\Big],$$

with

$$p_T(x_0,y_0,x,y) \le C E_{1/2,1}(CT^{1/2}) \bar{q}_{c'}(T, x_0, y_0, x, y),$$

recalling (A.11).

The extension to any measurable map h satisfying the growth assumption $|h(x,y)| \le C \exp(\gamma(|x|^2+|y|^2))$ for any $0 \le \gamma < (2(C')^2 cT)^{-1}$, where c is any constant strictly greater than 4κ, now follows from the above integral representation combined with a standard approximation argument. Indeed, let's consider such a function h and introduce $h_K(x,y) := -K \vee h(x,y) \wedge K$, for $K \ge 0$. Since $\int_{\mathbb{R}^2} \exp(\gamma(|x|^2+|y|^2))\, \bar{q}_{c'}(T,x_0,y_0,x,y)\, \mathrm{d}x\mathrm{d}y < \infty$, the dominated convergence theorem guarantees that

$$P_T h(x_0,y_0) = \lim_{K\uparrow\infty} P_T h_K(x_0,y_0) = \int_{\mathbb{R}^2} h(x,y)\, \mathbb{E}\Big[\bar{p}(\zeta_{N_T}, T, \bar{X}_{N_T}, \bar{Y}_{N_T}, x, y) \prod_{i=1}^{N_T+1} \theta_i\Big]\, \mathrm{d}x\mathrm{d}y,$$

so that the probabilistic representation formula (3.2) is valid for any $h \in \mathcal{B}_\gamma(\mathbb{R}^2)$. In particular, the random variable appearing inside the expectation in the right-hand side of (3.2) is integrable.

Step 3: Finite $L^p(\mathbb{P})$-moment for the probabilistic representation

If N is a renewal process with $Beta(\alpha,1)$ jump times then $f(s_i - s_{i-1}) = \frac{1-\alpha}{\bar{\tau}^{1-\alpha}} \frac{1}{(s_i-s_{i-1})^\alpha} \mathbf{1}_{[0,\bar{\tau}]}(s_i - s_{i-1})$ and $1 - F(T-s_n) = 1 - \left(\frac{T-s_n}{\bar{\tau}}\right)^{1-\alpha} \ge 1 - (\frac{T}{\bar{\tau}})^{1-\alpha}$, similarly to step 2, by Fubini's theorem, we get

$$\begin{aligned}
&\mathbb{E}\Big[|h(\bar{X}_{n+1}, \bar{Y}_{n+1})|^p \prod_{i=1}^{n+1} |\theta_i|^p \Big| N_T = n, \zeta^{n+1}\Big] \\
&\qquad = \int_{\mathbb{R}^2} |h(x,y)|^p\, \mathbb{E}\Big[\bar{p}(\zeta_n, T, \bar{X}_n, \bar{Y}_n, x, y) \prod_{i=1}^{n+1} |\theta_i(\bar{X}_{i-1}, \bar{Y}_{i-1}, \bar{X}_i, \bar{Y}_i, \zeta^{n+1})|^p \Big| N_T = n, \zeta^{n+1}\Big]\, \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

The above formula is justified by Lemma B.2 and Lemma B.3 which yield

$$\begin{aligned}
&\mathbb{E}\Big[\bar{p}(\zeta_n, T, \bar{X}_n, \bar{Y}_n, x, y) \prod_{i=1}^{n+1} |\theta_i(\bar{X}_{i-1}, \bar{Y}_{i-1}, \bar{X}_i, \bar{Y}_i, \zeta^n)|^p \Big| N_T = n, \zeta^{n+1}\Big] \\
&\quad \le C(1-F(T-\zeta_n))^{-1} \int_{(\mathbb{R}^2)^n} \bar{p}(\zeta_n, T, x_n, y_n, x, y) \prod_{i=1}^{n} |\theta_i(x_{i-1}, y_{i-1}, x_i, y_i, \zeta^{n+1})|^p \bar{p}(\zeta_{i-1}, \zeta_i, x_{i-1}, y_{i-1}, x_i, y_i)\, \mathrm{d}\mathbf{x}_n \mathrm{d}\mathbf{y}_n \\
&\quad \le C^{n+1}(1-F(T-\zeta_n))^{-1} \int_{(\mathbb{R}^2)^n} \bar{q}_c(\zeta_n, T, x_n, y_n, x, y) \\
&\qquad \times \prod_{i=1}^{n} (f(\zeta_i - \zeta_{i-1}))^{-p} (\zeta_i - \zeta_{i-1})^{-\frac{p}{2}} \bar{q}_c(\zeta_{i-1}, \zeta_i, x_{i-1}, y_{i-1}, x_i, y_i)\, \mathrm{d}\mathbf{x}_n \mathrm{d}\mathbf{y}_n
\end{aligned}$$

$$\leq C^{n+1}(1-F(T-\zeta_n))^{-1}\prod_{i=1}^{n}(f(\zeta_i-\zeta_{i-1}))^{-1}(\zeta_i-\zeta_{i-1})^{-\frac{p}{2}+\alpha p-\alpha}\bar{q}_{c'}(T,x_0,y_0,x,y),$$

where we recall that $c' := (C')^2 c$ for any $c > 4\kappa$. Now, using the fact that $\mathbb{E}\Big[C^{N_T+1}(1-F(T-\zeta_{N_T}))^{-1}\prod_{i=1}^{N_T}(f(\zeta_i-\zeta_{i-1}))^{-1}(\zeta_i-\zeta_{i-1})^{-\frac{p}{2}+\alpha p-\alpha}\Big] < \infty$ as soon as $p(\frac{1}{2}-\alpha) < 1-\alpha$ and that $h \in \mathcal{B}_\gamma(\mathbb{R}^2)$, from the previous computation, we obtain

$$\begin{aligned}
&\mathbb{E}[|h(\bar{X}_{N_T+1},\bar{Y}_{N_T+1})|^p\prod_{i=1}^{N_T+1}|\theta_i|^p]\\
&=\sum_{i=0}^{+\infty}\mathbb{E}\Big[\mathbb{E}\Big[|h(\bar{X}_{n+1},\bar{Y}_{n+1})|^p\prod_{i=1}^{n+1}|\theta_i|^p\Big|N_T,\zeta^{n+1}\Big]\mathbf{1}_{N_T=n}\Big]\\
&\leq \mathbb{E}\Big[C^{N_T+1}(1-F(T-\zeta_{N_T}))^{-1}\prod_{i=1}^{N_T}(f(\zeta_i-\zeta_{i-1}))^{-1}(\zeta_i-\zeta_{i-1})^{-\frac{p}{2}+\alpha p+\alpha}\Big]\int_{\mathbb{R}^2}e^{\gamma p(|x|^2+|y|^2)}\bar{q}_{c'}(T,x_0,y_0,x,y)\,\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

To conclude the proof, it suffices to note that the above space integral is finite as soon as $0 \leq \gamma p < (2(C')^2 cT)^{-1}$ for any $c > 4\kappa$.

A.2 Proof of Lemma 4.1

Since $h \in \mathcal{C}^1_p(\mathbb{R}^2)$ and $\mathbb{E}_{i,n}\Big[|\bar{X}_{i+1}|^q+|\bar{Y}_{i+1}|^q\Big] < \infty$ a.s., for any $q \geq 1$, under **(AR)**, one may differentiate under the (conditional) expectation and deduce that $(x,y)\mapsto \mathbb{E}_{i,n}\Big[h(\bar{X}_{i+1},\bar{Y}_{i+1})\theta_{i+1}|(\bar{X}_i,\bar{Y}_i)=(x,y)\Big] \in \mathcal{C}^1_p(\mathbb{R}^2)$ for any $i \in \{0,\dots,n\}$ *a.s.* The rest of the proof is divided into three parts.

Step 1: proofs of (4.1) *and* (4.2)

The transfer of derivatives formulae (4.1) and (4.2) are easily obtained by differentiating under expectation (which is allowed by the polynomial growth at infinity of h) noting from the definition of the Markov chain $\bar{X}$ that $\partial_{s_0}\bar{X}_1 = \partial_{s_0}\ln(s_0) = \frac{1}{s_0}$ and $\partial_{\bar{X}_i}h(\bar{X}_{i+1},\bar{Y}_{i+1}) = \partial_{\bar{X}_{i+1}}h(\bar{X}_{i+1},\bar{Y}_{i+1})\partial_{\bar{X}_i}\bar{X}_{i+1} = \partial_{\bar{X}_{i+1}}h(\bar{X}_{i+1},\bar{Y}_{i+1})$. Observe as well that from (2.18), the fact that $\partial_{\bar{X}_i}c_S^{i+1} = \partial_{\bar{X}_i}c_Y^{i+1} = \partial_{\bar{X}_i}b_Y^{i+1} = \partial_{\bar{X}_i}c_{Y,S}^{i+1} = \partial_{\bar{X}_i}\mathcal{I}^{(1)}_{i+1}(1) = \partial_{\bar{X}_i}\mathcal{I}^{(2)}_{i+1}(1) = 0$ and the very definition of the random variables $(\theta_i)_{1\leq i\leq n+1}$, one has $\partial_{\bar{X}_i}\theta_{i+1} = 0$. This gives the identities (4.1) and (4.2).

Step 2: proofs of (4.3) *and* (4.4)

The proofs of (4.3) and (4.4) are more involved. Let us prove (4.3). We proceed by considering the difference between the term appearing on the left-hand side and the first two terms appearing on the right-hand side of (4.3). On the one hand, using the IBP formula (2.17) and (2.11), we get

$$\begin{aligned}
\partial_{\bar{Y}_i}\mathbb{E}_{i,n}\Big[h(\bar{X}_{i+1},\bar{Y}_{i+1})\theta_{i+1}\Big] &= \mathbb{E}_{i,n}\Big[\partial_{\bar{X}_{i+1}}h(\bar{X}_{i+1},\bar{Y}_{i+1})\partial_{\bar{Y}_i}\bar{X}_{i+1}\theta_{i+1}\Big] + \mathbb{E}_{i,n}\Big[\partial_{\bar{Y}_{i+1}}h(\bar{X}_{i+1},\bar{Y}_{i+1})\partial_{\bar{Y}_i}\bar{Y}_{i+1}\theta_{i+1}\Big]\\
&\quad + \mathbb{E}_{i,n}\Big[h(\bar{X}_{i+1},\bar{Y}_{i+1})\partial_{\bar{Y}_i}\theta_{i+1}\Big]\\
&= \mathbb{E}_{i,n}\Big[h(\bar{X}_{i+1},\bar{Y}_{i+1})\Big[\mathcal{I}^{(1)}_{i+1}(\partial_{\bar{Y}_i}\bar{X}_{i+1}\theta_{i+1}) + \mathcal{I}^{(2)}_{i+1}(m_i'\theta_{i+1})\Big]\Big]\\
&\quad + \mathbb{E}_{i,n}\Big[h(\bar{X}_{i+1},\bar{Y}_{i+1})\mathcal{I}^{(2)}_{i+1}\Big(\sigma'_{Y,i}\Big(\rho_i Z^1_{i+1} + \sqrt{1-\rho_i^2}Z^2_{i+1}\Big)\theta_{i+1}\Big)\Big]
\end{aligned}$$

$$+\,\mathbb{E}_{i,n}\Big[h(\bar{X}_{i+1},\bar{Y}_{i+1})\mathcal{I}^{(2)}_{i+1}\Big(\sigma_{Y,i}\frac{\rho'_i}{\sqrt{1-\rho_i^2}}\Big(\sqrt{1-\rho_i^2}Z^1_{i+1}-\rho_i Z^2_{i+1}\Big)\theta_{i+1}\Big)\Big]$$
$$+\,\mathbb{E}_{i,n}\Big[h(\bar{X}_{i+1},\bar{Y}_{i+1})\partial_{\bar{Y}_i}\theta_{i+1}\Big].$$

On the other hand, again from the IBP formula (2.17), we obtain

$$\mathbb{E}_{i,n}\Big[\partial_{\bar{X}_{i+1}}h(\bar{X}_{i+1},\bar{Y}_{i+1})\overrightarrow{\theta}^{e,X}_{i+1}\Big]+\mathbb{E}_{i,n}\Big[\partial_{\bar{Y}_{i+1}}h(\bar{X}_{i+1},\bar{Y}_{i+1})\overrightarrow{\theta}^{e,Y}_{i+1}\Big]+\mathbb{E}_{i,n}\Big[h(\bar{X}_{i+1},\bar{Y}_{i+1})\overrightarrow{\theta}^{c}_{i+1}\Big]$$
$$=\mathbb{E}_{i,n}\Big[h(\bar{X}_{i+1},\bar{Y}_{i+1})\Big[\mathcal{I}^{(1)}_{i+1}(\overrightarrow{\theta}^{e,X}_{i+1})+\mathcal{I}^{(2)}_{i+1}(\overrightarrow{\theta}^{e,Y}_{i+1})+\overrightarrow{\theta}^{c}_{i+1}\Big]\Big].$$

Combining the two previous identities, we see that the difference

$$\partial_{\bar{Y}_i}\mathbb{E}_{i,n}\Big[h(\bar{X}_{i+1},\bar{Y}_{i+1})\theta_{i+1}\Big]-\Big(\mathbb{E}_{i,n}\Big[\partial_{\bar{X}_{i+1}}h(\bar{X}_{i+1},\bar{Y}_{i+1})\overrightarrow{\theta}^{e,X}_{i+1}\Big]+\mathbb{E}_{i,n}\Big[\partial_{\bar{Y}_{i+1}}h(\bar{X}_{i+1},\bar{Y}_{i+1})\overrightarrow{\theta}^{e,Y}_{i+1}\Big]+\mathbb{E}_{i,n}\Big[h(\bar{X}_{i+1},\bar{Y}_{i+1})\overrightarrow{\theta}^{c}_{i+1}\Big]\Big)$$

can be written as

$$\begin{aligned}
&\mathbb{E}_{i,n}\Big[h(\bar{X}_{i+1},\bar{Y}_{i+1})\mathcal{I}^{(2)}_{i+1}\Big(m'_i\theta_{i+1}-\overrightarrow{\theta}^{e,Y}_{i+1}\Big)\Big]+\mathbb{E}_{i,n}\Big[h(\bar{X}_{i+1},\bar{Y}_{i+1})\partial_{\bar{Y}_i}\theta_{i+1}\Big]-\mathbb{E}_{i,n}\Big[h(\bar{X}_{i+1},\bar{Y}_{i+1})\mathcal{I}^{(1)}_{i+1}(\overrightarrow{\theta}^{e,X}_{i+1})\Big]\\
&+\mathbb{E}_{i,n}\Big[h(\bar{X}_{i+1},\bar{Y}_{i+1})\mathcal{I}^{(1)}_{i+1}(\partial_{\bar{Y}_i}\bar{X}_{i+1}\theta_{i+1})\Big]+\mathbb{E}_{i,n}\Big[h(\bar{X}_{i+1},\bar{Y}_{i+1})\mathcal{I}^{(2)}_{i+1}\Big(\sigma'_{Y,i}\Big(\rho_i Z^1_{i+1}+\sqrt{1-\rho_i^2}Z^2_{i+1}\Big)\theta_{i+1}\Big)\Big]\\
&+\mathbb{E}_{i,n}\Big[h(\bar{X}_{i+1},\bar{Y}_{i+1})\mathcal{I}^{(2)}_{i+1}\Big(\sigma_{Y,i}\frac{\rho'_i}{\sqrt{1-\rho_i^2}}\Big(\sqrt{1-\rho_i^2}Z^1_{i+1}-\rho_i Z^2_{i+1}\Big)\theta_{i+1}\Big)\Big]\\
&-\mathbb{E}_{i,n}\Big[h(\bar{X}_{i+1},\bar{Y}_{i+1})\overrightarrow{\theta}^{c}_{i+1}\Big].
\end{aligned}\tag{A.12}$$

Before proceeding, we provide the explicit expression for the quantity $\partial_{\bar{Y}_i}\theta_{i+1}$. Using the chain rule formula of Lemma 2.5, after some standard but cumbersome computations, we obtain

$$\begin{aligned}
\partial_{\bar{Y}_i}\theta_{i+1}=(f(\zeta_{i+1}-\zeta_i))^{-1}\Big[&\mathcal{I}^{(1,1)}_{i+1}(\partial_{\bar{Y}_i}c^{i+1}_S)-\mathcal{I}^{(1)}_{i+1}(\partial_{\bar{Y}_i}c^{i+1}_S)+\mathcal{I}^{(2,2)}_{i+1}(\partial_{\bar{Y}_i}c^{i+1}_Y)+\mathcal{I}^{(2)}_{i+1}(\partial_{\bar{Y}_i}b^{i+1}_Y)+\mathcal{I}^{(1,2)}_{i+1}(\partial_{\bar{Y}_i}c^{i+1}_{Y,S})\\
&-\Big(\frac{\sigma'_{S,i}}{\sigma_{S,i}}\Big(2\mathcal{I}^{(1,1)}_{i+1}(c^{i+1}_S)-\mathcal{I}^{(1)}_{i+1}(c^{i+1}_S)+\mathcal{I}^{(1,2)}_{i+1}(c^{i+1}_{Y,S})\Big)+\Big(\frac{\sigma'_{Y,i}}{\sigma_{Y,i}}-\frac{\rho'_i\rho_i}{1-\rho_i^2}\Big)\Big(2\mathcal{I}^{(2,2)}_{i+1}(c^{i+1}_Y)+\mathcal{I}^{(2)}_{i+1}(b^{i+1}_Y)+\mathcal{I}^{(1,2)}_{i+1}(c^{i+1}_{Y,S})\Big)\Big)\\
&-\frac{\rho'_i}{1-\rho_i^2}\frac{\sigma_{Y,i}}{\sigma_{S,i}}\Big(\mathcal{I}^{(1,2)}_{i+1}(c^{i+1}_S)+\mathcal{I}^{(2,1)}_{i+1}(c^{i+1}_S)-\mathcal{I}^{(2)}_{i+1}(c^{i+1}_S)+\mathcal{I}^{(2,2)}_{i+1}(c^{i+1}_{Y,S})\Big)\Big].
\end{aligned}$$

Also, after some simple algebraic simplifications using the definitions of $\overrightarrow{\theta}^{e,Y}_{i+1}$ and $\overrightarrow{\theta}^{e,X}_{i+1}$ in (4.3), one obtains

$$\mathcal{I}^{(2)}_{i+1}\Big(m'_i\theta_{i+1}-\overrightarrow{\theta}^{e,Y}_{i+1}\Big)=-(f(\zeta_{i+1}-\zeta_i))^{-1}\Big[\mathcal{I}^{(2,2)}_{i+1}\Big(\partial_{\bar{Y}_i}c^{i+1}_Y\Big)+\mathcal{I}^{(1,2)}_{i+1}\Big(\partial_{\bar{Y}_i}c^{i+1}_{Y,S}\Big)\Big]$$

and

$$\mathcal{I}^{(1)}_{i+1}(\overrightarrow{\theta}^{e,X}_{i+1})=(f(\zeta_{i+1}-\zeta_i))^{-1}\mathcal{I}^{(1,1)}_{i+1}\Big(\partial_{\bar{Y}_i}c^{i+1}_S\Big).$$

Combining the three previous identities and gathering similar terms, we obtain

$$\mathcal{I}^{(2)}_{i+1}\Big(m'_i\theta_{i+1}-\overrightarrow{\theta}^{e,Y}_{i+1}\Big)+\partial_{\bar{Y}_i}\theta_{i+1}-\mathcal{I}^{(1)}_{i+1}(\overrightarrow{\theta}^{e,X}_{i+1})=(f(\zeta_{i+1}-\zeta_i))^{-1}\Big[-\mathcal{I}^{(1)}_{i+1}(\partial_{\bar{Y}_i}c^{i+1}_S)+\mathcal{I}^{(2)}_{i+1}(\partial_{\bar{Y}_i}b^{i+1}_Y)$$

$$
- \Big(\frac{\sigma'_{S,i}}{\sigma_{S,i}}\Big(2\mathcal{I}^{(1,1)}_{i+1}(c^{i+1}_S) - \mathcal{I}^{(1)}_{i+1}(c^{i+1}_S) + \mathcal{I}^{(1,2)}_{i+1}(c^{i+1}_{Y,S})\Big) + \Big(\frac{\sigma'_{Y,i}}{\sigma_{Y,i}} - \frac{\rho'_i\rho_i}{1-\rho_i^2}\Big)\Big(2\mathcal{I}^{(2,2)}_{i+1}(c^{i+1}_Y) + \mathcal{I}^{(2)}_{i+1}(b^{i+1}_Y) + \mathcal{I}^{(1,2)}_{i+1}(c^{i+1}_{Y,S})\Big)\Big)
$$
$$
- \frac{\rho'_i}{1-\rho_i^2}\frac{\sigma_{Y,i}}{\sigma_{S,i}}\Big(\mathcal{I}^{(1,2)}_{i+1}(c^{i+1}_S) + \mathcal{I}^{(2,1)}_{i+1}(c^{i+1}_S) - \mathcal{I}^{(2)}_{i+1}(c^{i+1}_S) + \mathcal{I}^{(2,2)}_{i+1}(c^{i+1}_{Y,S})\Big)\Big]. \tag{A.13}
$$

The previous identity will be used in the next step of the proof. Coming back to (A.12) and using the definition of the weight $\overrightarrow{\theta}^{c}_{i+1}$ allows to conclude the proof of the identity (4.3).

Step 3: The weight sequences $(\overrightarrow{\theta}^{e,Y}_i)_{1\leq i\leq n+1}$, $(\overrightarrow{\theta}^{e,X}_i)_{1\leq i\leq n+1}$ and $(\overrightarrow{\theta}^{c}_i)_{1\leq i\leq n+1}$ and the related spaces $\mathbb{M}_{i,n}(\bar{X},\bar{Y},\ell/2)$, $\ell\in\mathbb{Z}$.

In this last step, we prove the last statement of Lemma 4.1 concerning the weight sequences $(\overrightarrow{\theta}^{e,Y}_i)_{1\leq i\leq n+1}$, $(\overrightarrow{\theta}^{e,X}_i)_{1\leq i\leq n+1}$ and $(\overrightarrow{\theta}^{c}_i)_{1\leq i\leq n+1}$.

Following similar lines of reasonings as those used in the proof of Lemma B.2, namely using the fact that d^{i+1}_S, d^{i+1}_Y, $d^{i+1}_{Y,S}$, $e^{Y,i+1}_S$, $e^{Y,i+1}_Y \in \mathbb{M}_{i,n}(\bar{X},\bar{Y},1/2)$ and $\mathcal{D}^{(1)}_{i+1}d^{i+1}_S$, $\mathcal{D}^{(1,1)}_{i+1}d^{i+1}_S$, $\mathcal{D}^{(2)}_{i+1}d^{i+1}_Y$, $\mathcal{D}^{(2,2)}_{i+1}d^{i+1}_Y$, $\mathcal{D}^{(1)}_{i+1}d^{i+1}_{Y,S}$, $\mathcal{D}^{(2)}_{i+1}d^{i+1}_{Y,S}$, $\mathcal{D}^{(1,2)}_{i+1}d^{i+1}_{Y,S}$, $\mathcal{D}^{(1)}_{i+1}e^{Y,i+1}_S$, $\mathcal{D}^{(2)}_{i+1}e^{Y,i+1}_Y \in \mathbb{M}_{i,n}(\bar{X},\bar{Y},0)$ as well as Lemma 2.7, we conclude

$$
f(\zeta_{i+1}-\zeta_i)\,\overrightarrow{\theta}^{e,Y}_{i+1} \in \mathbb{M}_{i,n}(\bar{X},\bar{Y},-1/2), \quad i\in\{0,\ldots,n-1\}.
$$

Note also that

$$
\begin{aligned}
e^{X,i+1}_S &= \partial_{\bar{Y}_i} c^{i+1}_S \\
&= \frac{1}{2}(a'_S(\bar{Y}_{i+1})\partial_{\bar{Y}_i}\bar{Y}_{i+1} - a'_S(m_i)m'_i) \\
&= \frac{1}{2}(a'_S(\bar{Y}_{i+1}) - a'_S(m_i))\partial_{\bar{Y}_i}\bar{Y}_{i+1} + \frac{1}{2}a'_S(m_i)(\partial_{\bar{Y}_i}\bar{Y}_{i+1} - m'_i)
\end{aligned}
$$

so that, using on the one hand the Lipschitz regularity of a'_S and on the other hand (2.11), from similar arguments as those used in the proof of Lemma B.2, we conclude that

$$
\frac{1}{2}(a'_S(\bar{Y}_{i+1}) - a'_S(m_i))\partial_{\bar{Y}_i}\bar{Y}_{i+1}, \quad \frac{1}{2}a'_S(m_i)(\partial_{\bar{Y}_i}\bar{Y}_{i+1} - m'_i) \in \mathbb{M}_{i,n}(\bar{X},\bar{Y},1/2)
$$

which in turn implies that $e^{X,i+1}_S \in \mathbb{M}_{i,n}(\bar{X},\bar{Y},1/2)$. Moreover, standard computations that we omit show that $\mathcal{D}^{(1)}_{i+1}e^{X,i+1}_S \in \mathbb{M}_{i,n}(\bar{X},\bar{Y},0)$ so that by Lemma 2.7 we deduce

$$
f(\zeta_{i+1}-\zeta_i)\,\overrightarrow{\theta}^{e,X}_{i+1} \in \mathbb{M}_{i,n}(\bar{X},\bar{Y},0).
$$

We now prove that $f(\zeta_{i+1}-\zeta_i)\,\overrightarrow{\theta}^{c}_{i+1} \in \mathbb{M}_{i,n}(\bar{X},\bar{Y},-1/2)$ for any $i\in\{0,\ldots,n-1\}$. We use the decomposition

$$
\begin{aligned}
f(\zeta_{i+1}-\zeta_i)\,\overrightarrow{\theta}^{c}_{i+1} &= f(\zeta_{i+1}-\zeta_i)\Big(\mathcal{I}^{(2)}_{i+1}\Big(m'_i\theta_{i+1} - \overrightarrow{\theta}^{e,Y}_{i+1}\Big) + \partial_{\bar{Y}_i}\theta_{i+1} - \mathcal{I}^{(1)}_{i+1}(\overrightarrow{\theta}^{e,X}_{i+1})\Big) + \mathcal{I}^{(1)}_{i+1}\Big(\partial_{\bar{Y}_i}\bar{X}_{i+1}f(\zeta_{i+1}-\zeta_i)\theta_{i+1}\Big) \\
&\quad + \mathcal{I}^{(2)}_{i+1}\Big(\Big(\sigma'_{Y,i}\Big(\rho_i Z^1_{i+1} + \sqrt{1-\rho_i^2}Z^2_{i+1}\Big) + \sigma_{Y,i}\frac{\rho'_i}{\sqrt{1-\rho_i^2}}\Big(\sqrt{1-\rho_i^2}Z^1_{i+1} - \rho_i Z^2_{i+1}\Big)\Big)f(\zeta_{i+1}-\zeta_i)\theta_{i+1}\Big).
\end{aligned}
$$

We first prove that $f(\zeta_{i+1}-\zeta_i)\Big(\mathcal{I}^{(2)}_{i+1}\Big(m_i'\theta_{i+1}-\overrightarrow{\theta}^{e,Y}_{i+1}\Big)+\partial_{\bar{Y}_i}\theta_{i+1}-\mathcal{I}^{(1)}_{i+1}(\overrightarrow{\theta}^{e,X}_{i+1})\Big)\in\mathbb{M}_{i,n}(\bar{X},\bar{Y},-1/2)$. We investigate each term appearing on the right-hand side of (A.13).

In particular, we first use the fact that c_S^{i+1}, c_Y^{i+1}, b_Y^{i+1}, $c_{Y,S}^{i+1}\,\partial_{\bar{Y}_i}c_S^{i+1},\partial_{\bar{Y}_i}b_Y^{i+1}\in\mathbb{M}_{i,n}(\bar{X},\bar{Y},1/2)$ and the fact that when one applies the differential operators $\mathcal{D}^{(\alpha_1)}_{i+1}$, $\mathcal{D}^{(\alpha_1,\alpha_2)}_{i+1}$ to these elements the resulting random variables belong to $\mathbb{M}_{i,n}(\bar{X},\bar{Y},0)$ for any $(\alpha_1,\alpha_2)\in\{1,2\}^2$. From Lemma 2.7, we thus conclude that the elements $\mathcal{I}^{(1,1)}_{i+1}(c_S^{i+1})$, $\mathcal{I}^{(1,2)}_{i+1}(c_S^{i+1})$, $\mathcal{I}^{(2,1)}_{i+1}(c_S^{i+1})$, $\mathcal{I}^{(2,2)}_{i+1}(c_Y^{i+1})$, $\mathcal{I}^{(1,2)}_{i+1}(c_{Y,S}^{i+1})$, $\mathcal{I}^{(2,2)}_{i+1}(c_{Y,S}^{i+1})$ belong to $\mathbb{M}_{i,n}(\bar{X},\bar{Y},-1/2)$ and that $\mathcal{I}^{(1)}_{i+1}(c_S^{i+1}),\mathcal{I}^{(2)}_{i+1}(b_Y^{i+1})$, $\mathcal{I}^{(1)}_{i+1}(\partial_{\bar{Y}_i}c_S^{i+1}),\mathcal{I}^{(2)}_{i+1}(\partial_{\bar{Y}_i}b_Y^{i+1})$ belong to $\mathbb{M}_{i,n}(\bar{X},\bar{Y},0)$. Moreover, using **(ND)**, one gets that there exists $C>0$ such that for any $i\in\{0,\dots,n-1\}$, $|\sigma'_{S,i}/\sigma_{S,i}|+|\sigma'_{Y,i}/\sigma_{Y,i}|+|\sigma_{Y,i}/\sigma_{S,i}|+|\rho_i'/(1-\rho_i^2)|+|\rho_i'\rho_i/(1-\rho_i^2)|\le C$. We thus conclude that $f(\zeta_{i+1}-\zeta_i)\Big(\mathcal{I}^{(2)}_{i+1}\Big(m_i'\theta_{i+1}-\overrightarrow{\theta}^{e,Y}_{i+1}\Big)+\partial_{\bar{Y}_i}\theta_{i+1}-\mathcal{I}^{(1)}_{i+1}(\overrightarrow{\theta}^{e,X}_{i+1})\Big)\in\mathbb{M}_{i,n}(\bar{X},\bar{Y},-1/2)$.

It thus suffices to prove $\mathcal{I}^{(1)}_{i+1}\Big(\partial_{\bar{Y}_i}\bar{X}_{i+1}f(\zeta_{i+1}-\zeta_i)\theta_{i+1}\Big)$, $\mathcal{I}^{(2)}_{i+1}\Big(\sigma'_{Y,i}(\rho_iZ^1_{i+1}+\sqrt{1-\rho_i^2}Z^2_{i+1})f(\zeta_{i+1}-\zeta_i)\theta_{i+1}\Big)$ and $\mathcal{I}^{(2)}_{i+1}\Big(\sigma_{Y,i}\frac{\rho_i'}{\sqrt{1-\rho_i^2}}\Big(\sqrt{1-\rho_i^2}Z^1_{i+1}-\rho_iZ^2_{i+1}\Big)\Big)f(\zeta_{i+1}-\zeta_i)\theta_{i+1}\Big)$ belong to $\mathbb{M}_{i,n}(\bar{X},\bar{Y},-1/2)$. In order to do this, we remark that

$$
\begin{aligned}
&\partial_{\bar{Y}_i}\bar{X}_{i+1}=-\frac{1}{2}a'_{S,i}+\sigma'_{S,i}Z^1_{i+1}\in\mathbb{M}_{i,n}(\bar{X},\bar{Y},1/2),\\
&\mathcal{D}^{(1)}_{i+1}(\partial_{\bar{Y}_i}\bar{X}_{i+1})=\frac{\sigma'_{S,i}}{\sigma_{S,i}}\in\mathbb{M}_{i,n}(\bar{X},\bar{Y},0),\\
&\sigma'_{Y,i}(\rho_iZ^1_{i+1}+\sqrt{1-\rho_i^2}Z^2_{i+1})\in\mathbb{M}_{i,n}(\bar{X},\bar{Y},1/2),\\
&\mathcal{D}^{(2)}_{i+1}\Big(\sigma'_{Y,i}(\rho_iZ^1_{i+1}+\sqrt{1-\rho_i^2}Z^2_{i+1}\Big)=\frac{\sigma'_{Y,i}}{\sigma_{Y,i}}\in\mathbb{M}_{i,n}(\bar{X},\bar{Y},0),\\
&\sigma_{Y,i}\frac{\rho_i'}{\sqrt{1-\rho_i^2}}\Big(\sqrt{1-\rho_i^2}Z^1_{i+1}-\rho_iZ^2_{i+1}\Big)\in\mathbb{M}_{i,n}(\bar{X},\bar{Y},1/2),\\
&\mathcal{D}^{(2)}_{i+1}\Big(\sigma_{Y,i}\frac{\rho_i'}{\sqrt{1-\rho_i^2}}\Big(\sqrt{1-\rho_i^2}Z^1_{i+1}-\rho_iZ^2_{i+1}\Big)\Big)=-\frac{\rho_i\rho_i'}{1-\rho_i^2}\in\mathbb{M}_{i,n}(\bar{X},\bar{Y},0)
\end{aligned}
$$

and from Lemma B.2, $f(\zeta_{i+1}-\zeta_i)\theta_{i+1}\in\mathbb{M}_{i,n}(\bar{X},\bar{Y},-1/2)$. From Lemma 2.7, it follows that $f(\zeta_{i+1}-\zeta_i)\theta_{i+1}\partial_{\bar{Y}_i}\bar{X}_{i+1}$, $f(\zeta_{i+1}-\zeta_i)\theta_{i+1}\sigma'_{Y,i}(\rho Z^1_{i+1}+\sqrt{1-\rho^2}Z^2_{i+1})$, $f(\zeta_{i+1}-\zeta_i)\theta_{i+1}\sigma_{Y,i}\frac{\rho_i'}{\sqrt{1-\rho_i^2}}\Big(\sqrt{1-\rho_i^2}Z^1_{i+1}-\rho_iZ^2_{i+1}\Big)\in\mathbb{M}_{i,n}(\bar{X},\bar{Y},0)$. Now following similar computations as those employed in the proof of Lemma B.2 and omitting some technical details we obtain $\mathcal{D}^{(\alpha)}_{i+1}(f(\zeta_{i+1}-\zeta_i)\theta_{i+1})\in\mathbb{M}_{i,n}(\bar{X},\bar{Y},-1)$ so that from the chain rule formula and Lemma 2.7, the random variables $\mathcal{D}^{(\alpha)}_{i+1}(f(\zeta_{i+1}-\zeta_i)\theta_{i+1}\partial_{\bar{Y}_i}\bar{X}_{i+1})$, $\mathcal{D}^{(\alpha)}_{i+1}(\sigma'_{Y,i}(\rho_iZ^1_{i+1}+\sqrt{1-\rho_i^2}Z^2_{i+1})f(\zeta_{i+1}-\zeta_i)\theta_{i+1})$ and $\mathcal{D}^{(\alpha)}_{i+1}(\sigma_{Y,i}\frac{\rho_i'}{\sqrt{1-\rho_i^2}}(\sqrt{1-\rho_i^2}Z^1_{i+1}-\rho_iZ^2_{i+1})f(\zeta_{i+1}-\zeta_i)\theta_{i+1})$ belong to $\mathbb{M}_{i,n}(\bar{X},\bar{Y},-1/2)$. From Lemma 2.7, we thus conclude that $\mathcal{I}^{(1)}_{i+1}\Big(\partial_{\bar{Y}_i}\bar{X}_{i+1}f(\zeta_{i+1}-\zeta_i)\theta_{i+1}\Big)$, $\mathcal{I}^{(2)}_{i+1}\Big(\sigma'_{Y,i}(\rho_iZ^1_{i+1}+\sqrt{1-\rho_i^2}Z^2_{i+1})f(\zeta_{i+1}-\zeta_i)\theta_{i+1}\Big)$ and $\mathcal{I}^{(2)}_{i+1}\Big(\sigma_{Y,i}\frac{\rho_i'}{\sqrt{1-\rho_i^2}}\ (\sqrt{1-\rho_i^2}Z^1_{i+1}-\rho_iZ^2_{i+1})f(\zeta_{i+1}-\zeta_i)\theta_{i+1}\Big)$ belong to $\mathbb{M}_{i,n}(\bar{X},\bar{Y},-1/2)$. From the preceding arguments, we eventually deduce that $f(\zeta_{i+1}-\zeta_i)\overrightarrow{\theta}^{c}_{i+1}\in\mathbb{M}_{i,n}(\bar{X},\bar{Y},-1/2)$ for any $i\in\{0,\dots,n-1\}$.

Finally, from the very definition of the weights on the last time interval $\overrightarrow{\theta}^{e,Y}_{n+1}$ and $\overrightarrow{\theta}^{e,X}_{n+1}$ one directly gets that

$$(1-F(T-\zeta_n))\,\overrightarrow{\theta}^{e,Y}_{n+1} = m'_n + \sigma'_{Y,n}\left(\rho_n Z^1_{n+1} + \sqrt{1-\rho_n^2}Z^2_{n+1}\right) + \sigma_{Y,n}\frac{\rho'_n}{\sqrt{1-\rho_n^2}}\left(\sqrt{1-\rho_n^2}Z^1_{n+1} - \rho_n Z^2_{n+1}\right)$$

belongs to $\mathbb{M}_{n,n}(\bar{X},\bar{Y},0)$ and that

$$(1-F(T-\zeta_n))\,\overrightarrow{\theta}^{e,X}_{n+1} = -\frac{1}{2}a'_{S,n} + \sigma'_{S,n}Z^1_{n+1}$$

belongs to $\mathbb{M}_{n,n}(\bar{X},\bar{Y},1/2)$. The proof is now complete.

Appendix B. Some technical results

B.1 Emergence of jumps in the renewal process N

The next result is used in the proof of the probabilistic representation in Theorem 3.1 to express that time integrals add jumps to the renewal process N. In what follows, N is a renewal process in the sense of Definition 2.2.

Lemma B.1. *Let $n \in \mathbb{N}$ and $G : \{(t_1,\dots,t_{n+2}) : 0 < t_1 < \cdots < t_{n+1} < t_{n+2} := T\} \to \mathbb{R}$ be a measurable function such that $\mathbb{E}\left[\int_{\zeta_n}^T |G(\zeta_1,\dots,\zeta_n,s,T)|\mathbf{1}_{\{N_T=n\}}\mathrm{d}s\right] < \infty$. Then, it holds*

$$\begin{aligned}
&\mathbb{E}\Big[\int_{\zeta_n}^T G(\zeta_1,\dots,\zeta_n,s,T)\mathbf{1}_{\{N_T=n\}}\mathrm{d}s\Big]\\
&\quad= \mathbb{E}\Big[G(\zeta_1,\dots,\zeta_n,\zeta_{n+1},T)(1-F(T-\zeta_{n+1}))^{-1}(1-F(T-\zeta_n))(f(\zeta_{n+1}-\zeta_n))^{-1}\mathbf{1}_{\{N_T=n+1\}}\Big].
\end{aligned}$$

Proof. The proof follows by rewriting the above expectations using (2.9). We rewrite the expectation on the right-hand side in integral form. By Fubini's theorem, we obtain

$$\begin{aligned}
&\mathbb{E}\Big[G(\zeta_1,\dots,\zeta_n,\zeta_{n+1},T)(1-F(T-\zeta_{n+1}))^{-1}(1-F(T-\zeta_n))(f(\zeta_{n+1}-\zeta_n))^{-1}\mathbf{1}_{\{N_T=n+1\}}\Big]\\
&= \int_{\Delta_{n+1}(T)} G(s_1,\dots,s_{n+1},T)(1-F(T-s_{n+1}))^{-1}(1-F(T-s_n))(f(s_{n+1}-s_n))^{-1}\\
&\quad\times (1-F(T-s_{n+1}))\prod_{j=0}^{n} f(s_{j+1}-s_j)\,d\mathbf{s}_{n+1}\\
&= \int_{\Delta_n(T)}\int_{s_n}^T G(s_1,\dots,s_{n+1},T)\,\mathrm{d}s_{n+1}\,(1-F(T-s_n))\prod_{j=0}^{n-1} f(s_{j+1}-s_j)\,d\mathbf{s}_n.
\end{aligned}$$

This completes the proof. □

Lemma B.2. *Let $n \in \mathbb{N}$. On the set $\{N_T = n\}$, the sequence of weights $(\theta_i)_{1\le i\le n+1}$ defined by (3.3) and (3.4) satisfy:*

$$\forall i \in \{1,\dots,n\},\ f(\zeta_i-\zeta_{i-1})\theta_i \in \mathbb{M}_{i-1,n}(\bar{X},\bar{Y},-1/2),\quad (1-F(T-\zeta_n))\theta_{n+1} \in \mathbb{M}_{n,n}(\bar{X},\bar{Y},0). \tag{B.1}$$

Proof. We investigate each term appearing in the definition of $f(\zeta_i - \zeta_{i-1})\theta_i \in \mathbb{S}_{i-1,n}(\bar{X}, \bar{Y})$, recalling (3.3) and (3.4), and seek to apply Lemma 2.7. From the Lipschitz property of a_S and the space-time inequality (1.3), the map $(x_{i-1}, y_{i-1}, x_i, y_i, \mathbf{s}_{n+1}) \mapsto c^i_S(x_{i-1}, y_{i-1}, x_i, y_i, \mathbf{s}_{n+1})$ satisfies for any $c > 0$ and any $c' > c$

$$\begin{aligned} c^i_S(x_{i-1}, y_{i-1}, x_i, y_i, \mathbf{s}_{n+1})|^p \bar{q}_c(s_i - s_{i-1}, x_{i-1}, y_{i-1}, x_i, y_i) &\leq C|y_i - m_{i-1}(y_{i-1})|^p \bar{q}_c(s_i - s_{i-1}, x_{i-1}, y_{i-1}, x_i, y_i) \\ &\leq C(s_i - s_{i-1})^{p/2} \bar{q}_{c'}(s_i - s_{i-1}, x_{i-1}, y_{i-1}, x_i, y_i), \end{aligned}$$

so that, the random variables $c^i_S \in \mathbb{M}_{i-1,n}(\bar{X}, \bar{Y}, 1/2)$, for any $i \in \{1, \ldots, n+1\}$. Moreover, since c^i_S does not depend on $\bar{X}_i$ and $\partial_{\bar{X}_i} \bar{Y}_i = 0$, one has

$$\mathcal{D}^{(1)}_i c^i_S = \mathcal{D}^{(1,1)}_i c^i_S = 0.$$

From Lemma 2.7, we thus conclude

$$\mathcal{I}^{(1)}_i(c^i_S) \in \mathbb{M}_{i-1,n}(\bar{X}, \bar{Y}, 0) \quad \text{and} \quad \mathcal{I}^{(1,1)}_i(c^i_S) \in \mathbb{M}_{i-1,n}(\bar{X}, \bar{Y}, -1/2), \quad i \in \{1, \ldots, n\}.$$

In a completely analogous manner, omitting some technical details, we derive

$$\mathcal{I}^{(2)}_i(b^i_Y) \in \mathbb{M}_{i-1,n}(\bar{X}, \bar{Y}, 0), \quad \text{and} \quad \mathcal{I}^{(1,2)}_i(c^i_{Y,S}), \mathcal{I}^{(2,2)}_i(c^i_Y) \in \mathbb{M}_{i-1,n}(\bar{X}, \bar{Y}, -1/2).$$

Hence, we obtain $f(\zeta_i - \zeta_{i-1})\theta_i \in \mathbb{M}_{i-1,n}(\bar{X}, \bar{Y}, -1/2)$, for any $i \in \{1, \ldots, n\}$. We finally observe that $(1 - F(T - \zeta_n))\theta_{n+1} = 1 \in \mathbb{M}_{n,n}(\bar{X}, \bar{Y}, 0)$. The proof is now complete. $\square$

Lemma B.3. *Let $T > 0$ and n a positive integer. For any $\mathbf{s}_n = (s_1, \ldots, s_n) \in \Delta_n(T)$, any $(x, y) \in \mathbb{R}^2$ and any positive constant c there exist two positive constants C and $C' := C'(T) \geq 1$ such that the transition density $(t, x, y) \mapsto \bar{q}_c(t, x_0, y_0, x, y)$ defined by (2.8) satisfies the following semigroup property:*

$$\begin{aligned} \int_{(\mathbb{R}^2)^n} \bar{q}_c(s_n, T, x_n, y_n, x, y) \times \bar{q}_c(s_{n-1}, s_n, x_{n-1}, y_{n-1}, x_n, y_n) \times \cdots \times \bar{q}_c(0, s_1, x_0, y_0, x_1, y_1)\, \mathrm{d}\mathbf{x}_n \mathrm{d}\mathbf{y}_n \\ \leq C^n \bar{q}_{c'}(T, x_0, y_0, x, y), \end{aligned}$$

where $c' := (C')^2 c$.

Proof. The $\mathrm{d}x_1 \cdots \mathrm{d}x_n$ integrals are treated using the standard semigroup property of Gaussian kernels so that from the very definition of $\bar{q}_c$, it directly follows

$$\begin{aligned} &\int_{(\mathbb{R}^2)^n} \bar{q}_c(s_n, T, x_n, y_n, x, y) \times \bar{q}_c(s_{n-1}, s_n, x_{n-1}, y_{n-1}, x_n, y_n) \times \cdots \times \bar{q}_c(0, s_1, x_0, y_0, x_1, y_1)\, \mathrm{d}\mathbf{x}_n \mathrm{d}\mathbf{y}_n \\ &= \frac{1}{\sqrt{2\pi c T}} e^{-\frac{(x-x_0)^2}{2cT}} \int_{\mathbb{R}^n} \frac{1}{\sqrt{2\pi c(T-s_n)}} e^{-\frac{(y - m_{T,s_n}(y_n))^2}{2c(T-s_n)}} \times \cdots \times \frac{1}{\sqrt{2\pi c s_1}} e^{-\frac{(y_1 - m_{s_1}(y_0))^2}{2cs_1}}\, \mathrm{d}\mathbf{y}_n. \end{aligned}$$

We now provide an upper-bound for the integral appearing in the right-hand side of the above identity. We perform the change of variables $y_1 = m_{s_1}(z_1), y_2 = m_{s_2}(z_2), \ldots, y_n = m_{s_n}(z_n)$. Observe that since b_Y admits a bounded first order derivative, the determinants of the Jacobians $J_{s_1}(z_1) := m'_{s_1}(z_1), \ldots, J_{s_n}(z_n) = m'_{s_n}(z_n)$ are uniformly bounded for any $(s_1, \ldots, s_n) \in \Delta_n(T)$. Remark also that from the semigroup property

$m_{s_{i+1}-s_i}(m_{s_i}(z_i)) = m_{s_{i+1}}(z_i)$, for $1 \le i \le n$ with the convention $s_{n+1} = T$. Hence, for some positive constants C and $c' > c$ that may change from line to line, we get

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi cT}} e^{-\frac{(x-x_0)^2}{2cT}} \int_{\mathbb{R}^n} \frac{1}{\sqrt{2\pi c(T-s_n)}} e^{-\frac{(y-m_{T,s_n}(y_n))^2}{2c(T-s_n)}} \times \cdots \times \frac{1}{\sqrt{2\pi c s_1}} e^{-\frac{(y_1-m_{s_1}(y_0))^2}{2cs_1}} \,\mathrm{d}\mathbf{y}_n \\
&\le C^n \frac{1}{\sqrt{2\pi cT}} e^{-\frac{(x-x_0)^2}{2cT}} \int_{\mathbb{R}^n} \frac{1}{\sqrt{2\pi c(T-s_n)}} e^{-\frac{(y-m_T(z_n))^2}{2c(T-s_n)}} \times \cdots \times \frac{1}{\sqrt{2\pi c s_1}} e^{-\frac{(m_{s_1}(z_1)-m_{s_1}(y_0))^2}{2cs_1}} \,\mathrm{d}\mathbf{z}_n \\
&\le C^n \frac{1}{\sqrt{2\pi cT}} e^{-\frac{(x-x_0)^2}{2cT}} \int_{\mathbb{R}^n} \frac{1}{\sqrt{2\pi c(T-s_n)}} e^{-(C')^{-1}\frac{(m_T^{-1}(y)-z_n)^2}{2c(T-s_n)}} \times \cdots \times \frac{1}{\sqrt{2\pi c s_1}} e^{-(C')^{-1}\frac{(z_1-y_0)^2}{2cs_1}} \,\mathrm{d}\mathbf{z}_n \\
&= C^n \frac{1}{2\pi cT\sqrt{C'}} e^{-\frac{(x-x_0)^2}{2cT}} e^{-\frac{(m_T^{-1}(y)-y_0)^2}{2cC'T}} \\
&\le C^n \bar{q}_{c'}(T, x_0, y_0, x, y),
\end{aligned}$$

for $c' := c(C')^2$, where we first used the bi-Lipschitz property of the flow $\mathbb{R}_+ \times \mathbb{R} \ni (s,x) \mapsto m_s(x)$, namely, for any $T > 0$, any $t \in [0,T]$ and any $x, z \in \mathbb{R}$

$$(C')^{-1}|x-z| \le |m_t(x) - m_t(z)| \le C'|x-z|, \quad C' := \exp([b_Y]_T T), \tag{B.2}$$

recalling that $[b_Y]_T$ is defined by (2.1), then the semigroup property satisfied by Gaussian kernels and again the bi-Lipschitz property of the flow. This completes the proof. □

Appendix C. Some useful formulas

We here provide some useful formulas in order to device the unbiased Monte Carlo algorithms based on Theorem 3.1 and Theorem 4.2. Their proofs follow from standard computations as those used in Section 2.4 and are omitted.

The following formulae are required in order to compute the weights $(\theta_i)_{1 \le i \le N_T}$ appearing in the identity (3.2). Note that in our examples since $a_Y(.)$ is constant, one has $c^i_Y(.) \equiv 0$ for $i \in \{1, \dots, N_T\}$. Hence, for $i \in \{1, \dots, N_T\}$, one has:

$$\begin{aligned}
\mathcal{I}^{(1)}_i(c^i_S) &= c^i_S \mathcal{I}^{(1)}_i(1), \\
\mathcal{I}^{(1,1)}_i(c^i_S) &= c^i_S((\mathcal{I}^{(1)}_i(1))^2 - \mathcal{D}^{(1)}_i \mathcal{I}^{(1)}_i(1)), \\
\mathcal{I}^{(2)}_i(b^i_Y) &= b^i_Y \mathcal{I}^{(2)}_i(1) - b'_Y(\bar{Y}_i), \\
\mathcal{I}^{(1,2)}_i(c^i_{Y,S}) &= \mathcal{I}^{(2)}_i(c^i_{Y,S}\mathcal{I}^{(1)}_i(1)) = c^i_{Y,S}\mathcal{I}^{(1)}_i(1)\mathcal{I}^{(2)}_i(1) - \mathcal{I}^{(1)}_i(1)\mathcal{D}^{(2)}_i(c^i_{Y,S}) - c^i_{Y,S}\mathcal{D}^{(2)}_i\mathcal{I}^{(1)}_i(1).
\end{aligned}$$

The following formulae are needed in order to compute the weights for the Delta appearing in the identity (4.5), for $i \in \{1, \dots, N_T\}$ one has:

$$\begin{aligned}
\mathcal{D}^{(1)}_i(\mathcal{I}^{(1,1)}_i(c^i_S)) &= 2c^i_S \mathcal{I}^{(1)}_i(1)\mathcal{D}^{(1)}_i\mathcal{I}^{(1)}_i(1), \\
\mathcal{D}^{(1)}_i(\mathcal{I}^{(1)}_i(c^i_S)) &= c^i_S \mathcal{D}^{(1)}_i\mathcal{I}^{(1)}_i(1), \\
\mathcal{D}^{(1)}_i(\mathcal{I}^{(2)}_i(b^i_Y)) &= b^i_Y \mathcal{D}^{(1)}_i\mathcal{I}^{(2)}_i(1), \\
\mathcal{D}^{(1)}_i(\mathcal{I}^{(1,2)}_i(c^i_{Y,S})) &= c^i_{Y,S}\mathcal{D}^{(1)}_i\mathcal{I}^{(1)}_i(1)\mathcal{I}^{(2)}_i(1) + c^i_{Y,S}\mathcal{D}^{(1)}_i\mathcal{I}^{(2)}_i(1)\mathcal{I}^{(1)}_i(1) - \mathcal{D}^{(1)}_i\mathcal{I}^{(1)}_i(1)\mathcal{D}^{(2)}_i(c^i_{Y,S}),
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}^{(1)}_i\theta_i &= (f(\zeta_i - \zeta_{i-1}))^{-1}\Big[\mathcal{D}^{(1)}_i\mathcal{I}^{(1,1)}_i(c^i_S) - \mathcal{D}^{(1)}_i\mathcal{I}^{(1)}_i(c^i_S) + \mathcal{D}^{(1)}_i\mathcal{I}^{(2)}_i(b^i_Y) + \mathcal{D}^{(1)}_i\mathcal{I}^{(1,2)}_i(c^i_{Y,S})\Big], \\
\mathcal{I}^{(1)}_k(\theta_k) &= \mathcal{I}^{(1)}_k(1)\theta_k - \mathcal{D}^{(1)}_k\theta_k, \; k \le N_T,
\end{aligned}$$

$$\mathcal{I}^{(1)}_{N_T+1}(\theta_{N_T+1}) = \theta_{N_T+1}\mathcal{I}^{(1)}_{N_T+1}(1) - \mathcal{D}^{(1)}_{N_T+1}\theta_{N_T+1} = \theta_{N_T+1}\mathcal{I}^{(1)}_{N_T+1}(1).$$

The following formulae are required for the computation of the weights for the Vega appearing in the identity (4.6), for $i \in \{1, \ldots, N_T\}$ it holds:

$$\begin{aligned}
\mathcal{I}^{(1,1)}_{i+1}(d^{i+1}_S) &= m'_i \mathcal{I}^{(1,1)}_{i+1}(c^{i+1}_S),\\
\mathcal{I}^{(1)}_{i+1}(e^{Y,i+1}_S) &= -m'_i \mathcal{I}^{(1)}_{i+1}(c^{i+1}_S) + \mathcal{D}^{(2)}_i(c^{i+1}_{Y,S})\mathcal{I}^{(1)}_{i+1}(1) - \mathcal{D}^{(1)}_{i+1}\mathcal{D}^{(2)}_i(c^{i+1}_{Y,S}),\\
\mathcal{I}^{(2)}_{i+1}(e^{Y,i+1}_Y) &= m'_i \mathcal{I}^{(2)}_{i+1}(b^{i+1}_Y),\\
\mathcal{I}^{(1,2)}_{i+1}(d^{i+1}_{Y,S}) &= m'_i \mathcal{I}^{(1,2)}_{i+1}(c^{i+1}_{Y,S}),\\
\mathcal{I}^{(1)}_{i+1}(e^{X,i+1}_S) &= e^{X,i+1}_S \mathcal{I}^{(1)}_{i+1}(1) - \mathcal{D}^{(1)}_{i+1} e^{X,i+1}_S = \mathcal{D}^{(2)}_i(c^{i+1}_S)\mathcal{I}^{(1)}_{i+1}(1) - \mathcal{D}^{(1)}_{i+1}\mathcal{D}^{(2)}_i(c^{i+1}_S),
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}^{(2)}_i(\mathcal{I}^{(1,1)}_i(c^i_S)) &= \mathcal{D}^{(2)}_i(c^i_S)(\mathcal{I}^{(1)}_i(1))^2 + 2c^i_S\mathcal{I}^{(1)}_i(1)\mathcal{D}^{(2)}_i\mathcal{I}^{(1)}_i(1) - \mathcal{D}^{(2)}_i(c^i_S)\mathcal{D}^{(1)}_i\mathcal{I}^{(1)}_i(1),\\
\mathcal{D}^{(2)}_i(\mathcal{I}^{(1)}_i(c^i_S)) &= \mathcal{D}^{(2)}_i(c^i_S)\mathcal{I}^{(1)}_i(1) + c^i_S\mathcal{D}^{(2)}_i\mathcal{I}^{(1)}_i(1),\\
\mathcal{D}^{(2)}_i(\mathcal{I}^{(2)}_i(b^i_Y)) &= b'_Y(\bar{Y}_i)\mathcal{I}^{(2)}_i(1) + b^i_Y\mathcal{D}^{(2)}_i\mathcal{I}^{(2)}_i(1) - b''_Y(\bar{Y}_i),\\
\mathcal{D}^{(2)}_i(\mathcal{I}^{(1,2)}_i(c^i_{Y,S})) &= \mathcal{D}^{(2)}_i(c^i_{Y,S})\mathcal{I}^{(1)}_i(1)\mathcal{I}^{(2)}_i(1) + c^i_{Y,S}\mathcal{D}^{(2)}_i\mathcal{I}^{(2)}_i(1)\mathcal{I}^{(1)}_i(1) + c^i_{Y,S}\mathcal{I}^{(2)}_i(1)\mathcal{D}^{(2)}_i\mathcal{I}^{(1)}_i(1)\\
&\quad - \mathcal{I}^{(1)}_i(1)\mathcal{D}^{(2,2)}_i(c^i_{Y,S}) - 2\mathcal{D}^{(2)}_i(c^i_{Y,S})\mathcal{D}^{(2)}_i\mathcal{I}^{(1)}_i(1),
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}^{(2)}_{i-1}(\mathcal{I}^{(1,1)}_i(c^i_S)) &= \mathcal{D}^{(2)}_{i-1}(c^i_S)(\mathcal{I}^{(1)}_i(1))^2 + 2c^i_S\mathcal{I}^{(1)}_i(1)\mathcal{D}^{(2)}_{i-1}\mathcal{I}^{(1)}_i(1)\\
&\quad - \mathcal{D}^{(2)}_{i-1}(c^i_S)\mathcal{D}^{(1)}_i\mathcal{I}^{(1)}_i(1) - c^i_S\mathcal{D}^{(2)}_{i-1}(\mathcal{D}^{(1)}_i\mathcal{I}^{(1)}_i(1)),\\
\mathcal{D}^{(2)}_{i-1}(\mathcal{I}^{(1)}_i(c^i_S)) &= \mathcal{D}^{(2)}_{i-1}(c^i_S)\mathcal{I}^{(1)}_i(1) + c^i_S\mathcal{D}^{(2)}_{i-1}\mathcal{I}^{(1)}_i(1),\\
\mathcal{D}^{(2)}_{i-1}(\mathcal{I}^{(2)}_i(b^i_Y)) &= b^i_Y\mathcal{D}^{(2)}_{i-1}\mathcal{I}^{(2)}_i(1) + \mathcal{I}^{(2)}_i(1)(b'_Y(\bar{Y}_i)\partial_{\bar{Y}_{i-1}}\bar{Y}_i - b'_Y(m_{i-1})m'_{i-1}) - b''_Y(\bar{Y}_i)\partial_{\bar{Y}_{i-1}}\bar{Y}_i,\\
\mathcal{D}^{(2)}_{i-1}(\mathcal{I}^{(1,2)}_i(c^i_{Y,S})) &= \mathcal{D}^{(2)}_{i-1}(c^i_{Y,S})\mathcal{I}^{(1)}_i(1)\mathcal{I}^{(2)}_i(1) + c^i_{Y,S}(\mathcal{I}^{(2)}_i(1)\mathcal{D}^{(2)}_{i-1}\mathcal{I}^{(1)}_i(1) + \mathcal{I}^{(1)}_i(1)\mathcal{D}^{(2)}_{i-1}\mathcal{I}^{(2)}_i(1))\\
&\quad - \mathcal{I}^{(1)}_i(1)\mathcal{D}^{(2)}_{i-1}(\mathcal{D}^{(2)}_i(c^i_{Y,S})) - \mathcal{D}^{(2)}_{i-1}(\mathcal{I}^{(1)}_i(1))\mathcal{D}^{(2)}_i(c^i_{Y,S}) - c^i_{Y,S}\mathcal{D}^{(2)}_{i-1}\mathcal{D}^{(2)}_i\mathcal{I}^{(1)}_i(1)\\
&\quad - \mathcal{D}^{(2)}_{i-1}(c^i_{Y,S})\mathcal{D}^{(2)}_i\mathcal{I}^{(1)}_i(1).
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}^{(2)}_i\theta_i =& (f(\zeta_i - \zeta_{i-1}))^{-1}\Big[\mathcal{D}^{(2)}_i(\mathcal{I}^{(1,1)}_i(c^i_S)) - \mathcal{D}^{(2)}_i(\mathcal{I}^{(1)}_i(c^i_S)) + \mathcal{D}^{(2)}_i(\mathcal{I}^{(2)}_i(b^i_Y)) + \mathcal{D}^{(2)}_i(\mathcal{I}^{(1,2)}_i(c^i_{Y,S}))\Big],\\
\mathcal{D}^{(2)}_{i-1}\theta_i =& (f(\zeta_i - \zeta_{i-1}))^{-1}\Big[\mathcal{D}^{(2)}_{i-1}(\mathcal{I}^{(1,1)}_i(c^i_S)) - \mathcal{D}^{(2)}_{i-1}(\mathcal{I}^{(1)}_i(c^i_S)) + \mathcal{D}^{(2)}_{i-1}(\mathcal{I}^{(2)}_i(b^i_Y)) + \mathcal{D}^{(2)}_{i-1}(\mathcal{I}^{(1,2)}_i(c^i_{Y,S}))\Big],
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{\theta}^{e,Y}_i &= m'_{i-1}\theta_i + (f(\zeta_i - \zeta_{i-1}))^{-1}(\mathcal{I}^{(1)}_i(1)\mathcal{D}^{(2)}_{i-1}(c^i_{Y,S}) - \mathcal{D}^{(1)}_i\mathcal{D}^{(2)}_{i-1}(c^i_{Y,S})),\\
\mathcal{D}^{(2)}_i\overrightarrow{\theta}^{e,Y}_i &= m'_{i-1}\mathcal{D}^{(2)}_i\theta_i + (f(\zeta_i - \zeta_{i-1}))^{-1}\Big[\mathcal{D}^{(2)}_i(\mathcal{D}^{(2)}_{i-1}(c^i_{Y,S}))\mathcal{I}^{(1)}_i(1)\\
&\qquad + \mathcal{D}^{(2)}_{i-1}(c^i_{Y,S})\mathcal{D}^{(2)}_i\mathcal{I}^{(1)}_i(1) - \mathcal{D}^{(2)}_i\mathcal{D}^{(1)}_i\mathcal{D}^{(2)}_{i-1}(c^i_{Y,S})\Big],\\
\mathcal{I}^{(2)}_i(\overrightarrow{\theta}^{e,Y}_i) &= \overrightarrow{\theta}^{e,Y}_i\mathcal{I}^{(2)}_i(1) - \mathcal{D}^{(2)}_i\overrightarrow{\theta}^{e,Y}_i,\\
\mathcal{I}^{(2)}_i(m'_{i-1}\theta_i - \overrightarrow{\theta}^{e,Y}_i) &= -(f(\zeta_i - \zeta_{i-1}))^{-1}\mathcal{I}^{(1,2)}_i(\mathcal{D}^{(2)}_{i-1}(c^i_{Y,S}))\\
&= -(f(\zeta_i - \zeta_{i-1}))^{-1}\Big(\mathcal{D}^{(2)}_{i-1}(c^i_{Y,S})\mathcal{I}^{(1)}_i(1)\mathcal{I}^{(2)}_i(1) - \mathcal{I}^{(1)}_i(1)\mathcal{D}^{(2)}_i\mathcal{D}^{(2)}_{i-1}(c^i_{Y,S})\\
&\qquad - \mathcal{D}^{(2)}_{i-1}(c^i_{Y,S})\mathcal{D}^{(1)}_i\mathcal{I}^{(2)}_i(1) - \mathcal{D}^{(1)}_i\mathcal{D}^{(2)}_{i-1}(c^i_{Y,S})\mathcal{I}^{(2)}_i(1) + \mathcal{D}^{(2)}_i\mathcal{D}^{(1)}_i\mathcal{D}^{(2)}_{i-1}(c^i_{Y,S})\Big),
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{\theta}^{e,X}_i &= (f(\zeta_i - \zeta_{i-1}))^{-1}\mathcal{I}^{(1)}_i(e^{X,i}_S) = (f(\zeta_i - \zeta_{i-1}))^{-1}(\mathcal{I}^{(1)}_i(1)\mathcal{D}^{(2)}_{i-1}(c^i_S) - \mathcal{D}^{(1)}_i\mathcal{D}^{(2)}_{i-1}(c^i_S)),\\
\mathcal{D}^{(1)}_i\overrightarrow{\theta}^{e,X}_i &= (f(\zeta_i - \zeta_{i-1}))^{-1}e^{X,i}_S\mathcal{D}^{(1)}_i\mathcal{I}^{(1)}_i(1),\\
\mathcal{I}^{(1)}_i(\overrightarrow{\theta}^{e,X}_i) &= \overrightarrow{\theta}^{e,X}_i\mathcal{I}^{(1)}_i(1) - \mathcal{D}^{(1)}_i\overrightarrow{\theta}^{e,X}_i,
\end{aligned}$$

$$\mathcal{I}_i^{(1)}(\theta_i\mathcal{D}_{i-1}^{(2)}\bar{X}_i) = (\theta_i\mathcal{I}_i^{(1)}(1) - \mathcal{D}_i^{(1)}\theta_i)\mathcal{D}_{i-1}^{(2)}\bar{X}_i - \mathcal{D}_i^{(1)}(\mathcal{D}_{i-1}^{(2)}\bar{X}_i)\theta_i,$$

$$\mathcal{I}_i^{(2)}\Big((\sqrt{1-\rho_{i-1}^2}Z_i^1 - \rho_{i-1}Z_i^2)\theta_i\Big) = \Big(\sqrt{1-\rho_{i-1}^2}Z_i^1 - \rho_{i-1}Z_i^2\Big)\,(\mathcal{I}_i^{(2)}(1)\theta_i - \mathcal{D}_i^{(2)}\theta_i) + \frac{\rho_{i-1}\theta_i}{\sigma_{Y,i-1}\sqrt{1-\rho_{i-1}^2}},$$

$$\begin{aligned}\overrightarrow{\theta}_i^{c} =& \mathcal{I}_i^{(2)}(m'_{i-1}\theta_i - \overrightarrow{\theta}_i^{e,Y}) + \sigma_{Y,i-1}\frac{\rho'_{i-1}}{\sqrt{1-\rho_{i-1}^2}}\mathcal{I}_i^{(2)}\Big((\sqrt{1-\rho_{i-1}^2}Z_i^1 - \rho_{i-1}Z_i^2)\theta_i\Big)\\ &+ \mathcal{I}_i^{(1)}(\theta_i\mathcal{D}_{i-1}^{(2)}\bar{X}_i) - \mathcal{I}_i^{(1)}(\overrightarrow{\theta}_i^{e,X}) + \mathcal{D}_{i-1}^{(2)}\theta_i,\end{aligned}$$

$$\begin{aligned}\overrightarrow{\theta}_{N_T+1}^{e,Y} =& \theta_{N_T+1} = \Big(m'_{N_T} + \sigma_{Y,N_T}\frac{\rho'_{N_T}}{\sqrt{1-\rho_{N_T}^2}}(\sqrt{1-\rho_{N_T}^2}Z_{N_T+1}^1 - \rho_{N_T}Z_{N_T+1}^2)\Big),\\ \overrightarrow{\theta}_{N_T+1}^{e,X} =& \theta_{N_T+1} = \Big(-\frac{1}{2}a'_{S,N_T} + \sigma'_{S,N_T}Z_{N_T+1}^1\Big),\\ \overrightarrow{\theta}_{N_T+1}^{c} =& 0,\end{aligned}$$

$$\mathcal{I}_{N_T+1}^{(2)}(\overrightarrow{\theta}_{N_T+1}^{e,Y}) = \overrightarrow{\theta}_{N_T+1}^{e,Y}\mathcal{I}_{N_T+1}^{(2)}(1) - \mathcal{D}_{N_T+1}^{(2)}\overrightarrow{\theta}_{N_T+1}^{e,Y} = \overrightarrow{\theta}_{N_T+1}^{e,Y}\mathcal{I}_{N_T+1}^{(2)}(1) + \theta_{N_T+1}\frac{\rho'_{N_T}\rho_{N_T}}{1-\rho_{N_T}^2},$$

$$\mathcal{I}_{N_T+1}^{(2)}(\overrightarrow{\theta}_{N_T+1}^{e,X}) = \overrightarrow{\theta}_{N_T+1}^{e,X}\mathcal{I}_{N_T+1}^{(2)}(1) - \mathcal{D}_{N_T+1}^{(2)}\overrightarrow{\theta}_{N_T+1}^{e,X} = \overrightarrow{\theta}_{N_T+1}^{e,X}\mathcal{I}_{N_T+1}^{(2)}(1) - \theta_{N_T+1}\frac{\sigma'_{S,N_T}}{\sigma_{S,N_T}}.$$

References

[1] A. Agarwal and E. Gobet, Finite variance unbiased estimation of stochastic differential equations. *Proceedings of the 2017 Winter Simulation Conference* (2017) 1950–1961.

[2] P. Andersson and A. Kohatsu-Higa, Unbiased simulation of stochastic differential equations using parametrix expansions. *Bernoulli* **23** (2017) 2028–2057.

[3] V. Bally and A. Kohatsu-Higa, A probabilistic interpretation of the parametrix method. *Ann. Appl. Probab.* **25** (2015) 3095–3138.

[4] G. Bernis, E. Gobet and A. Kohatsu-Higa, Monte Carlo evaluation of Greeks for multidimensional barrier and lookback options. Applications of Malliavin Calculus in Finance (Rocquencourt, 2001). *Math. Finance* **13** (2003) 99–113.

[5] F. Delarue, Estimates of the Solutions of a System of Quasi-linear PDEs. A Probabilistic Scheme. Springer Berlin Heidelberg, Berlin, Heidelberg (2003) 290–332.

[6] M. Doumbia, N. Oudjane and X. Warin, Unbiased Monte Carlo estimate of stochastic differential equations expectations. *ESAIM: PS* **21** (2017) 56–87.

[7] E. Fournié, J.-M. Lasry, J. Lebuchoux and P.-L. Lions, Applications of Malliavin calculus to Monte-Carlo methods in finance. II. *Finance Stoch.* **5** (2001) 201–236.

[8] E. Fournié, J.-M. Lasry, J. Lebuchoux, P.-L. Lions and N. Touzi, Applications of Malliavin calculus to Monte Carlo methods in finance. *Finance Stoch.* **3** (1999) 391–412.

[9] N. Frikha, A. Kohatsu-Higa and L. Li, Integration by parts formula for killed processes: a point of view from approximation theory. *Electr. J. Probab.* **24** (2019) 1–44.

[10] E. Gobet, Revisiting the Greeks for European and American Options, Stochastic Processes and Applications to Mathematical Finance (2004) 53–71. 10.1142/9789812702852_0003

[11] E. Gobet and A. Kohatsu-Higa, Computation of Greeks for barrier and look-back options using Malliavin calculus. *Electr. Comm. Probab.* **8** (2003) 51–62.

[12] P. Henry-Labordère, X. Tan and N. Touzi, Unbiased simulation of stochastic differential equations. *Ann. Appl. Probab.* **27** (2017) 3305–3341.

[13] S. Kusuoka and D. Stroock, Applications of the Malliavin Calculus, Part I, in K. Itô (editor), Stochastic Analysis. vol. 32 of *North-Holland Mathematical Library*. Elsevier (1984) 271–306.

[14] S. Kusuoka and D. Stroock, Applications of the Malliavin calculus, Part II. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **32** (1985) 1–76.

[15] S. Kusuoka and D. Stroock, Applications of the Malliavin calculus, Part III. *J. Faculty Sci. Univ. Tokyo. Section IA. Math.* **34** (1987) 391–442.

[16] P. Malliavin and A. Thalmaier, Stochastic Calculus of Variations in Mathematical Finance. Springer Finance, Springer Berlin Heidelberg (2005).

[17] D. Nualart, The Malliavin calculus and related topics, *Probability and its Applications* (New York), 2nd edn., Springer-Verlag, Berlin (2006).