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REGULARITY OF THE TIME CONSTANT FOR A SUPERCRITICAL BERNOULLI PERCOLATION*

Barbara Dembin**

Abstract. We consider an i.i.d. supercritical bond percolation on $\mathbb{Z}^d$, every edge is open with a probability $p > p_c(d)$, where $p_c(d)$ denotes the critical parameter for this percolation. We know that there exists almost surely a unique infinite open cluster $\mathcal{C}_p$. We are interested in the regularity properties of the chemical distance for supercritical Bernoulli percolation. The chemical distance between two points $x, y \in \mathcal{C}_p$ corresponds to the length of the shortest path in $\mathcal{C}_p$ joining the two points. The chemical distance between 0 and nx grows asymptotically like $n\mu_p(x)$. We aim to study the regularity properties of the map $p \to \mu_p$ in the supercritical regime. This may be seen as a special case of first passage percolation where the distribution of the passage time is $G_p = p\delta_1 + (1-p)\delta_\infty$, $p > p_c(d)$. It is already known that the map $p \to \mu_p$ is continuous.

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1. Introduction

The model of first passage percolation was first introduced by Hammersley and Welsh [9] as a model for the spread of a fluid in a porous medium. Let $d \geq 2$. We consider the graph $(\mathbb{Z}^d, \mathbb{E}^d)$ having for vertices $\mathbb{Z}^d$ and for edges $\mathbb{E}^d$ the set of pairs of nearest neighbors in $\mathbb{Z}^d$ for the Euclidean norm. To each edge $e \in \mathbb{E}^d$ we assign a random variable $t(e)$ with values in $\mathbb{R}^+$ so that the family $(t(e),\, e \in \mathbb{E}^d)$ is independent and identically distributed according to a given distribution G. The random variable $t(e)$ may be interpreted as the time needed for the fluid to cross the edge e. We can define a random pseudo-metric T on this graph: for any pair of vertices x, $y \in \mathbb{Z}^d$, the random variable $T(x, y)$ is the shortest time to go from x to y. Let $x \in \mathbb{Z}^d \setminus \{0\}$. One can ask what is the asymptotic behavior of the quantity $T(0, x)$ when $\|x\|$ goes to infinity. Under some assumptions on the distribution G, one can prove that asymptotically when n is large, the random variable $T(0, nx)$ behaves like $n \cdot \mu_G(x)$ where $\mu_G(x)$ is a deterministic constant depending only on the distribution G and the point x. The constant $\mu_G(x)$ corresponds to the limit of $T(0, nx)/n$ when n goes to infinity, when this limit exists. This result was proved by Cox and Durrett in [15] in dimension 2 under some integrability conditions on G, they also proved that μ_G is a semi-norm. Kesten extended this result to any dimension $d \geq 2$ in [11], and he proved that μ_G is a norm if and only if $G(\{0\}) < p_c(d)$. In the study of first passage percolation, μ_G is usually called

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Université de Paris, Sorbonne Université, CNRS, Laboratoire de Probabilités, Statistique et Modélisation (LPSM, UMR 8001), 75006 Paris, France.

** Corresponding author: bdembin@lpsm.paris

the time constant. The constant $\mu_G(x)$ may be seen as the inverse of the speed of spread of the fluid in the direction of x.

It is possible to extend this model by doing first passage percolation on a random environment. We consider an i.i.d. supercritical bond percolation on the graph $(\mathbb{Z}^d, \mathbb{E}^d)$. Every edge $e \in \mathbb{E}^d$ is open with a probability $p > p_c(d)$, where $p_c(d)$ denotes the critical parameter for this percolation. We know that there exists almost surely a unique infinite open cluster $\mathcal{C}_p$ [8]. We can define the model of first passage percolation on the infinite cluster $\mathcal{C}_p$. To do so, we consider a probability measure G on $[0, +\infty]$ such that $G([0, \infty[) = p$. In this setting, the p-closed edges correspond to the edges with an infinite value and so the cluster $\mathcal{C}_p$ made of the edges with finite passage time corresponds to the infinite cluster of a supercritical Bernoulli percolation of parameter p. The existence of a time constant for such distributions was first obtained in the context of stationary integrable ergodic field by Garet and Marchand in [4] and was later shown for an independent field without any integrability condition by Cerf and Théret in [2].

The question of the continuity of the map $G \to \mu_G$ started in dimension 2 with the article of Cox [14]. He showed the continuity of this map under the hypothesis of uniform integrability: if G_n weakly converges toward G and if there exists an integrable law F such that for all $n \in \mathbb{N}$, F stochastically dominates G_n, then $\mu_{G_n} \to \mu_G$. In [16], Cox and Kesten prove the continuity of this map in dimension 2 without any integrability condition. Their idea was to consider a geodesic for truncated passage times $\min(t(e), M)$, and along it to avoid clusters of p-closed edges, that is to say edges with a passage time larger than some $M > 0$, by bypassing them with a short path in the boundary of this cluster. Note that by construction, the edges of the boundary have passage time smaller than M. Thanks to combinatorial considerations, they were able to obtain a precise control on the length of these bypasses. This idea was later extended to all the dimensions $d \geq 2$ by Kesten in [11], by taking a M large enough such that the percolation of the edges with a passage time larger than M is highly subcritical: for such a M, the size of the clusters of p-closed edges can be controlled. However, this idea does not work anymore when we allow passage time to take infinite values. In [7], Garet, Marchand, Procaccia and Théret proved the continuity of the map $G \to \mu_G$ for general laws on $[0, +\infty]$ without any moment condition. More precisely, let $(G_n)_{n\in\mathbb{N}}$, and G probability measures on $[0, +\infty]$ such that G_n weakly converges toward G (we write $G_n \xrightarrow{d} G$), that is to say for all continuous bounded functions $f : [0, +\infty] \to [0, +\infty)$, we have

$$\lim_{n\to+\infty} \int_{[0,+\infty]} f \mathrm{d}G_n = \int_{[0,+\infty]} f \mathrm{d}G\,.$$

Equivalently, we say that $G_n \xrightarrow{d} G$ if and only if $\lim_{n\to+\infty} G_n([t, +\infty]) = G([t, +\infty])$ for all $t \in [0, +\infty]$ such that $x \to G([x, +\infty])$ is continuous at t. If moreover for all $n \in \mathbb{N}$, $G_n([0, +\infty)) > p_c(d)$ and $G([0, +\infty)) > p_c(d)$, then

$$\lim_{n\to\infty} \sup_{x\in\mathbb{S}^{d-1}} |\mu_{G_n}(x) - \mu_G(x)| = 0$$

where $\mathbb{S}^{d-1}$ is the unit sphere for the Euclidean norm.

In this paper, we focus on distributions of the form $G_p = p\delta_1 + (1 - p)\delta_\infty$, $p > p_c(d)$. We denote by $\mathcal{C}'_p$ be the subgraph of $\mathbb{Z}^d$ whose edges are open for the Bernoulli percolation of parameter p. The travel time given by a law G_p between two points x and $y \in \mathbb{Z}^d$ coincides with the so-called chemical distance that is the graph distance between x and y in $\mathcal{C}'_p$. Namely, for $x, y \in \mathbb{Z}^d$, we define the chemical distance $D^{\mathcal{C}'_p}(x, y)$ as the length of the shortest p-open path joining x and y. Note that if x and y are not in the same cluster of $\mathcal{C}'_p$, $D^{\mathcal{C}'_p}(x, y) = +\infty$. Actually, when x and y are in the same cluster, $D^{\mathcal{C}'_p}(x, y)$ is of order $\|y - x\|_1$. In [1], Antal and Pisztora obtained the following large deviation upper bound. There exists a constant $\rho > 0$ depending on p and d such that

$$\limsup_{\|y\|_1\to\infty} \frac{1}{\|y\|_1} \log \mathbb{P}(0 \leftrightarrow y, D^{\mathcal{C}'_p}(0, y) > \rho\|y\|_1) < 0\,.$$

This result implies that

$$\limsup_{\|y\|_1\to\infty} \frac{1}{\|y\|_1} D^{\mathcal{C}'_p}(0,y)\mathbb{1}_{0\leftrightarrow y} \leq \rho,\ \mathbb{P}_p \text{ a.s.}$$

These results were proved using renormalization arguments. They were improved later in [4] by Garet and Marchand, for the more general case of a stationary ergodic field. They proved that $D^{\mathcal{C}'_p}(0,x)$ grows linearly in $\|x\|_1$. More precisely, for each $y \in \mathbb{Z}^d \setminus \{0\}$, they proved the existence of a constant $\mu_p(y)$ such that

$$\lim_{\substack{n\to\infty \\ 0\leftrightarrow ny}} \frac{D^{\mathcal{C}'_p}(0,ny)}{n} = \mu_p(y),\ \mathbb{P}_p \text{ a.s.}\,.$$

The constant μ_p is called the time constant. The map $p \to \mu_p$ can be extended to $\mathbb{Q}^d$ by homogeneity and to $\mathbb{R}^d$ by continuity. It is a norm on $\mathbb{R}^d$. This convergence holds uniformly in all directions, this is equivalent of saying that an asymptotic shape emerges. Indeed, the set of points that are at a chemical distance from 0 smaller than n asymptotically looks like $n\mathcal{B}_{\mu_p}$, where $\mathcal{B}_{\mu_p}$ denotes the unit ball associated with the norm μ_p. In another paper [5], Garet and Marchand studied the fluctuations of $D^{\mathcal{C}'_p}(0,y)/\mu_p(y)$ around its mean and obtained the following large deviation result:

$$\forall \varepsilon > 0,\ \limsup_{\|x\|_1\to\infty} \frac{\ln \mathbb{P}_p\left(0 \leftrightarrow x, \frac{D^{\mathcal{C}'_p}(0,y)}{\mu_p(y)} \notin (1-\varepsilon, 1+\varepsilon)\right)}{\|x\|_1} < 0\,.$$

In the same paper, they showed another large deviation result that, as a corollary, proves the continuity of the map $p \to \mu_p$ in $p = 1$. In [6], Garet and Marchand obtained moderate deviations of the quantity $|D^{\mathcal{C}'_p}(0,y) - \mu_p(y)|$. As a corollary of the work of Garet, Marchand, Procaccia and Théret in [7] we obtain the continuity of the map $p \to \mu_p$ in $(p_c(d), 1]$. Our paper is a continuation of [7], our aim is to obtain better regularity properties for the map $p \to \mu_p$ than just continuity. We prove the following theorem.

Theorem 1.1 (Regularity of the time constant)**.** *Let $p_0 > p_c(d)$. There exists a constant κ_0 depending only on d and p_0, such that for all $p \leq q$ in $[p_0, 1]$*

$$\sup_{x\in\mathbb{S}^{d-1}} |\mu_p(x) - \mu_q(x)| \leq \kappa_0(q-p)|\log(q-p)|\,.$$

To study the regularity of the map $p \to \mu_p$, our aim is to control the difference between the chemical distance in the infinite cluster $\mathcal{C}_p$ of a Bernoulli percolation of parameter $p > p_c(d)$ with the chemical distance in $\mathcal{C}_q$ where $q \geq p$. The key part of the proof lies in the modification of a path. We couple the two percolations such that a p-open edge is also q-open but the converse does not necessarily hold. We consider a q-open path for some $q \geq p > p_c(d)$. Some of the edges of this path are p-closed, we want to build upon this path a p-open path by bypassing the p-closed edges. In order to bypass them, we use the idea of [7] and we build our bypasses at a macroscopic scale. This idea finds its inspiration in the works of Antal and Pisztora [13] and Cox and Kesten [16]. We have to consider an appropriate renormalization and we obtain a macroscopic lattice with good and bad sites. Good and bad sites correspond to boxes of size $2N$ in the microscopic lattice. We will do our bypasses using good sites at a macroscopic scale that will have good connectivity properties at a microscopic scale. The remainder of the proof consists in getting probabilistic estimates of the length of the bypass. In this article we improve the estimates obtained in [7]. We quantify the renormalization to be able to give quantitative bounds on continuity. Namely, we give an explicit expression of the appropriate size of a N-box. We use the idea of corridor that appeared in the work of Cox and Kesten [16] to have a better control on combinatorial terms and derive a more precise control of the length of the bypasses than the one obtained in [7].

We recall that $\mathcal{B}_{\mu_p}$ denotes the unit ball associated with the norm μ_p. From Theorem 1.1, we can easily deduce the following regularity of the asymptotic shape.

Corollary 1.2 (Regularity of the asymptotic shape)**.** *Let $p_0 > p_c(d)$. There exists a constant κ_0' depending only on d and p_0, such that for all $p \leq q$ in $[p_0, 1]$,*

$$d_{\mathcal{H}}(\mathcal{B}_{\mu_q}, \mathcal{B}_{\mu_p}) \leq \kappa_0'(q-p)|\log(q-p)|$$

where $d_{\mathcal{H}}$ is the Hausdorff distance between non-empty compact sets of $\mathbb{R}^d$.

Here is the structure of the paper. In Section 2, we introduce some definitions and preliminary results that are going to be useful in what follows. Section 3 presents the renormalization process and how we modify a q-open path to turn it into a p-open path and how we can control the length of the bypasses. In Sections 4 and 5, we get probabilistic estimates on the length of the bypasses. Finally, in Section 6 we prove the main Theorem 1.1 and its Corollary 1.2.

Remark 1.3. Section 3 is a simplified version of the renormalization process that was already present in [7]. The simplification comes from the fact that we are not interested in general distributions but only on distributions G_p for $p > p_c(d)$ which have the advantage of taking only two values 1 or $+\infty$. The original part of this work is the quantification of the renormalization and the combinatorial estimates of Section 5.

2. Definitions and preliminary results

Let $d \geq 2$. Let us recall the different distances in $\mathbb{R}^d$. Let $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we define

$$\|x\|_1 = \sum_{i=1}^{d} |x_i|, \quad \|x\|_2 = \sqrt{\sum_{i=1}^{d} x_i^2} \quad \text{and} \quad \|x\|_\infty = \max\{|x_i|, i = 1, \dots, d\}\,.$$

Let $\mathcal{G}$ be a subgraph of $(\mathbb{Z}^d, \mathbb{E}^d)$ and $x, y \in \mathcal{G}$. A path γ from x to y in $\mathcal{G}$ is a sequence $\gamma = (v_0, e_1, \dots, e_n, v_n)$ such that $v_0 = x$, $v_n = y$ and for all $i \in \{1, \dots, n\}$, the edge $e_i = \langle v_{i-1}, v_i \rangle$ belongs to $\mathcal{G}$. We say that x and y are connected in $\mathcal{G}$ if there exists such a path. We denote by $|\gamma| = n$ the length of γ. We define

$$D^{\mathcal{G}}(x, y) = \inf\{|r| : r \text{ is a path from } x \text{ to } y \text{ in } \mathcal{G}\}$$

the chemical distance between x and y in $\mathcal{G}$. If x and y are not connected in $\mathcal{G}$, $D^{\mathcal{G}}(x, y) = \infty$. In the following, $\mathcal{G}$ will be $\mathcal{C}_p'$ the subgraph of $\mathbb{Z}^d$ whose edges are open for the Bernoulli percolation of parameter $p > p_c(d)$. To get around the fact that the chemical distance can take infinite values we introduce a regularized chemical distance. Let $\mathcal{C} \subset \mathcal{C}_p'$ be a connected cluster, we define $\widetilde{x}^{\mathcal{C}}$ as the vertex of $\mathcal{C}$ which minimizes $\|x - \widetilde{x}^{\mathcal{C}}\|_1$ with a deterministic rule to break ties. As $\mathcal{C} \subset \mathcal{C}_p'$, we have

$$D^{\mathcal{C}_p'}(\widetilde{x}^{\mathcal{C}}, \widetilde{y}^{\mathcal{C}}) \leq D^{\mathcal{C}}(\widetilde{x}^{\mathcal{C}}, \widetilde{y}^{\mathcal{C}}) < \infty\,.$$

Typically, $\mathcal{C}$ is going to be the infinite cluster for Bernoulli percolation with a parameter $p_0 \leq p$ (thus $\mathcal{C}_{p_0} \subset \mathcal{C}_p'$).

We can define the regularized time constant as in [6] or as a special case of [2].

Proposition 2.1. *Let $p > p_c(d)$. There exists a deterministic function $\mu_p : \mathbb{Z}^d \to [0, +\infty)$, such that for every $p_0 \in (p_c(d), p]$:*

$$\forall x \in \mathbb{Z}^d \quad \lim_{n\to\infty} \frac{D^{\mathcal{C}_p}(\widetilde{0}^{\mathcal{C}_{p_0}}, \widetilde{nx}^{\mathcal{C}_{p_0}})}{n} = \mu_p(x) \text{ a.s. and in } L^1.$$

It is important to check that μ_p does not depend on p_0, *i.e.*, on the cluster $\mathcal{C}_{p_0}$ we use to regularize. This is done in Lemma 2.11 in [7]. As a corollary, we obtain the monotonicity of the map $p \to \mu_p$ which is non increasing, see Lemma 2.12 in [7].

Corollary 2.2. *For all $p_c(d) < p \leq q$ and for all $x \in \mathbb{Z}^d$,*

$$\mu_p(x) \geq \mu_q(x)\,.$$

We will also need this other definition of path that corresponds to the context of site percolation. Let $\mathcal{G}$ be a subset of $\mathbb{Z}^d$ and $x, y \in \mathcal{G}$. We say that the sequence $\gamma = (v_0, \ldots, v_n)$ is a $\mathbb{Z}^d$-path from x to y in $\mathcal{G}$ if $v_0 = x$, $v_n = y$ and for all $i \in \{1, \ldots, n\}$, $v_i \in \mathcal{G}$ and $\|v_i - v_{i-1}\|_1 = 1$.

3. Modification of a path

In this section we present the renormalization process. We are here at a macroscopic scale, we define good boxes to be boxes with useful properties to build our modified paths.

3.1. Definition of the renormalization process

Let $p > p_c(d)$ be the parameter of an i.i.d. Bernoulli percolation on the edges of $\mathbb{Z}^d$. For a large integer N, that will be chosen later, we set $B_N = [-N, N[^d \cap \mathbb{Z}^d$ and define the following family of N-boxes, for $\boldsymbol{i} \in \mathbb{Z}^d$,

$$B_N(\boldsymbol{i}) = \tau_{\boldsymbol{i}(2N+1)}(B_N)$$

where τ_b denotes the shift in $\mathbb{Z}^d$ with vector $b \in \mathbb{Z}^d$. $\mathbb{Z}^d$ is the disjoint union of this family: $\mathbb{Z}^d = \sqcup_{\boldsymbol{i} \in \mathbb{Z}^d} B_N(\boldsymbol{i})$. We need to introduce larger boxes that will help us to link N-boxes together. For $\boldsymbol{i} \in \mathbb{Z}^d$, we define

$$B'_N(\boldsymbol{i}) = \tau_{\boldsymbol{i}(2N+1)}(B_{3N}).$$

To define what a good box is, we have to list properties that a good box should have to ensure that we can build a modification of the path as we have announced in the introduction. We have to keep in mind that all the properties must occur with probability close to 1 when N goes to infinity. Before defining what a good box is, let us recall some definitions. A connected cluster C is crossing for a box B, if for all d directions, there is an open path in $C \cap B$ connecting the two opposite faces of B. We define the diameter of a finite cluster $\mathcal{C}$ as

$$\mathrm{Diam}(\mathcal{C}) := \max_{\substack{i=1,\ldots,d \\ x,y \in \mathcal{C}}} |x_i - y_i|\,.$$

Definition 3.1. Let $\beta > 0$ be a constant that will be defined later. We say that the macroscopic site $\boldsymbol{i}$ is p-good if the following events occur:

(i) There exists a unique p-cluster $\mathcal{C}$ in $B'_N(\boldsymbol{i})$ with diameter larger than N;
(ii) This p-cluster $\mathcal{C}$ is crossing for each of the 3^d N-boxes included in $B'_N(\boldsymbol{i})$;
(iii) For all $x, y \in B'_N(\boldsymbol{i})$, if x and y belong to $\mathcal{C}$ then $D^{\mathcal{C}'_p}(x, y) \leq 12\beta N$.

$\mathcal{C}$ is called the crossing p-cluster of the p-good box $B_N(\boldsymbol{i})$.

Let us define a percolation by site on the macroscopic grid given by the states of the boxes, *i.e.*, we say that a macroscopic site $\boldsymbol{i}$ is open if the box $B_N(\boldsymbol{i})$ is p-good, otherwise we say the site is closed. Note that the states of the boxes are not independent, there is a short range dependence.

On the macroscopic grid $\mathbb{Z}^d$, we consider the standard definition of closest neighbors, that is to say $\boldsymbol{i}$ and $\boldsymbol{j}$ are neighbors if $\|\boldsymbol{i} - \boldsymbol{j}\|_1 = 1$. Let C be a connected set of macroscopic sites, we define its exterior vertex

boundary

$$\partial_v C = \left\{ \begin{array}{c} \boldsymbol{i} \in \mathbb{Z}^d \setminus C : \boldsymbol{i} \text{ has a neighbor in } C \text{ and is connected} \\ \text{to infinity by a } \mathbb{Z}^d\text{-path in } \mathbb{Z}^d \setminus C \end{array} \right\}.$$

For a bad macroscopic site $\boldsymbol{i}$, let us denote by $C(\boldsymbol{i})$ the connected cluster of bad macroscopic sites containing $\boldsymbol{i}$. If $C(\boldsymbol{i})$ is finite, the set $\partial_v C(\boldsymbol{i})$ is not connected in the standard definition but it is with a weaker definition of neighbors. We say that two macroscopic sites $\boldsymbol{i}$ and $\boldsymbol{j}$ are $*$-neighbors if and only if $\|\boldsymbol{i}-\boldsymbol{j}\|_\infty = 1$. Therefore, $\partial_v C(\boldsymbol{i})$ is an $*$-connected set of good macroscopic sites, see for instance Lemma 2 in [17]. We adopt the convention that $\partial_v C(\boldsymbol{i}) = \{\boldsymbol{i}\}$ when $\boldsymbol{i}$ is a good site.

3.2. Construction of bypasses

Let us consider $p_c(d) < p_0 \leq p \leq q$. Let $\beta > 0$ that we will choose later depending on p_0. We fix N in this section and we build the renormalized process on the macroscopic N-boxes. Let us consider a q-open path γ. We couple the two percolation processes of parameters p and q in such a way that a p-open edge is necessarily q-open. Thus, some edges in γ might be p-closed. We denote by γ_o the set of p-open edges in γ, and by γ_c the set of p-closed edges in γ. Our aim is to build a bypass for each edge in γ_c using only p-open edges. The proof will follow the proof of Lemma 3.2 in [7] up to some adaptations.

As the bypasses are going to be made at a macroscopic scale, we need to consider the N-boxes that γ crosses. We denote by $\Gamma \subset \mathbb{Z}^d$ the connected set of all the N-boxes visited by γ. The set Γ is a lattice animal, *i.e.*, a connected finite set of macroscopic sites. We denote by *Bad* the random set of bad connected components on the macroscopic percolation given by the states of the N-boxes. The following Lemma states that we can bypass all the p-closed edges in γ and gives a control on the total size of these bypasses.

Lemma 3.2. *We fix some $\beta > 0$ and an integer N. Let us consider $y, z \in \mathcal{C}_p$ such that the N-boxes of y and z belong to an infinite cluster of p-good boxes. Let us consider a q-open path γ joining y to z. Then there exists a p-open path γ' between y and z that has the following properties:*

(1) $\gamma' \setminus \gamma$ is a set of disjoint self avoiding p-open paths that intersect $\gamma' \cap \gamma$ at their endpoints;

(2) $|\gamma' \setminus \gamma| \leq \rho_0 N \left(\sum_{C \in Bad : C \cap \Gamma \neq \emptyset} |C| + |\gamma_c| \right)$, where ρ_0 is a constant depending only on the dimension d and β.

Remark 3.3. Note that here we don't need to introduce a parameter p_0 and require that the bypasses are p_0 open as in [7]. Indeed, this condition was required because finite passage times of edges were not bounded. This is the reason why it was needed in [7] to bypass p-closed edges with p_0-open edges. These p_0-open edges were precisely edges with passage time smaller than some constant M_0. In our context, we can get rid of this technical aspect because passage times when finite may only take the value 1.

Before proving Lemma 3.2, we need to prove the following lemma that gives a control on the length of a path between two points in a $*$-connected set of good boxes.

Lemma 3.4. *Let $\mathcal{I}$ be a set of $n \in \mathbb{N}^*$ macroscopic sites such that $(B_N(\boldsymbol{i}))_{\boldsymbol{i} \in \mathcal{I}}$ is a $*$-connected set of p-good N-boxes. Let $x \in B_N(\boldsymbol{j})$ be in the p-crossing cluster of $B_N(\boldsymbol{j})$ with $\boldsymbol{j} \in \mathcal{I}$ and $y \in B_N(\boldsymbol{k})$ be in the p-crossing cluster of $B_N(\boldsymbol{k})$ with $\boldsymbol{k} \in \mathcal{I}$. Then, we can find a p-open path joining x and y of length at most $12\beta N n$ (with the same constant β as in Def. 3.1).*

Proof of Lemma 3.4. Since $\mathcal{I}$ is a $*$-connected set of macroscopic sites, there exists a self-avoiding macroscopic $*$-connected path $(\varphi_i)_{1 \leq i \leq r} \subset \mathcal{I}$ such that $\varphi_1 = \boldsymbol{j}$, $\varphi_r = \boldsymbol{k}$. Thus, we get that $r \leq |\mathcal{I}| = n$. As all the sites in $\mathcal{I}$ are good, all the N-boxes corresponding to the sites $(\varphi_i)_{1 \leq i \leq r}$ are good.

For each $2 \leq i \leq r-1$, we define x_i to be a point in the p-crossing cluster of the box $B_N(\varphi_i)$ chosen according to a deterministic rule. We define $x_1 = x$ and $x_r = y$. For each $1 \leq i < r$, x_i and x_{i+1} both belong to $B'_N(\varphi_i)$. Using property (iii) of a p-good box, we can build a p-open path $\gamma(i)$ from x_i to x_{i+1} of length at most $12\beta N$.

By concatenating the paths $\gamma(1),\dots,\gamma(r-1)$ in this order, we obtain a p-open path joining x to y of length at most $12\beta Nn$. □

Proof of Lemma 3.2. Let us consider $y,z\in\mathcal{C}_p$ such that the N-boxes of y and z belong to an infinite cluster of p-good boxes. Let γ be a q-open path joining y to z. The idea is the following. We want to bypass all the p-closed edges of γ. Let us consider an edge $e\in\gamma_c$ and $B_N(\boldsymbol{i})$ its associated N-box. There are two different cases:

- If $B_N(\boldsymbol{i})$ is a good box, we can build a p-open bypass of e at a microscopic scale by staying in a fixed neighborhood of $B_N(\boldsymbol{i})$. We will use the third property of good boxes to control the length of the bypass that will be at most $12\beta N$.
- If $B_N(\boldsymbol{i})$ is a bad box, we must build a p-open bypass at a macroscopic scale in the exterior vertex boundary $\partial_v C(\boldsymbol{i})$ that is an $*$-connected component of good boxes. We will use Lemma 3.4 to control the length of this bypass.

Let $\varphi_0=(\varphi_0(j))_{1\le j\le r_0}$ be the sequence of N-boxes γ visits. From the sequence φ_0, we can extract the sequence of N-boxes containing at least one p-closed edge of γ. We only keep the indices of the boxes containing the smallest extremity of a p-closed edge of γ for the lexicographic order. We obtain a sequence $\varphi_1=(\varphi_1(j))_{1\le j\le r_1}$. Notice that $r_1\le r_0$ and $r_1\le|\gamma_c|$. Before building our bypasses, we have to get rid of some pathological cases. We are going to proceed to further extractions. Note that two $*$-connected components of $(\partial_v C(\varphi_1(j)))_{1\le j\le r_1}$ can be $*$-connected together, in that case they count as a unique connected component. Namely, the set $E=\cup_{1\le j\le r_1}\partial_v C(\varphi_1(j))$ has at most r_1 $*$-connected components $(S_{\varphi_2(j)})_{1\le j\le r_2}$. Up to reordering, we can assume that the sequence $(S_{\varphi_2(j)})_{1\le j\le r_2}$ is ordered in such a way that $S_{\varphi_2(1)}$ is the first $*$-connected component of E visited by γ among the $(S_{\varphi_2(j)})_{1\le j\le r_2}$, $S_{\varphi_2(2)}$ is the second and so on. Next, we consider the case of nesting, that is to say when there exist $j\neq k$ such that $S_{\varphi_2(j)}$ is in the interior of $S_{\varphi_2(k)}$. In that case, we only keep the largest connected component $S_{\varphi_2(k)}$: we obtain another subsequence $(S_{\varphi_3(j)})_{1\le j\le r_3}$ with $r_3\le r_2$. Finally, we want to exclude a last case, when between the moment we enter for the first time in a given connected component and the last time we leave this connected component, we have explored other connected components of $(S_{\varphi_3(j)})_{1\le j\le r_3}$. That is to say we want to remove the macroscopic loops γ makes between different visits of the same $*$-connected components $S_{\varphi_3(j)}$ (see Fig. 1). We iteratively extract from $(S_{\varphi_3(j)})_{1\le j\le r_3}$ a sequence $(S_{\varphi_4(j)})_{1\le j\le r_4}$ in the following way: $S_{\varphi_4(1)}=S_{\varphi_3(1)}$, assume $(S_{\varphi_4(j)})_{1\le j\le k}$ is constructed $\varphi_4(k+1)$ is the smallest indice $\varphi_3(j)$ such that γ visits $S_{\varphi_3(j)}$ after its last visit to $S_{\varphi_4(k)}$. We stop the process when we cannot find such j. Of course, $r_4\le r_3$. The sequence $(S_{\varphi_4(j)})_{1\le j\le r_4}$ is a sequence of sets of good N-boxes that are all visited by γ.

Let us introduce some notations (see Fig. 2), we write $\gamma=(x_0,\dots,x_n)$. For all $k\in\{1,\dots,r_4\}$, we denote by $\Psi_{in}(k)$ (respectively $\Psi_{out}(k)$) the first moment that γ enters in $S_{\varphi_4(1)}$ (resp. last moment that γ exits from $S_{\varphi_4(1)}$). More precisely, we have

$$\Psi_{in}(1)=\min\left\{\,j\ge 1, x_j\in S_{\varphi_4(1)}\,\right\}$$

and

$$\Psi_{out}(1)=\max\left\{\,j\ge \Psi_{in}(1), x_j\in S_{\varphi_4(1)}\,\right\}.$$

Assume $\Psi_{in}(1),\dots,\Psi_{in}(k)$ and $\Psi_{out}(1),\dots,\Psi_{out}(k)$ are constructed then

$$\Psi_{in}(k+1)=\min\left\{\,j\ge \Psi_{out}(k), x_j\in S_{\varphi_4(k+1)}\,\right\}$$

and

$$\Psi_{out}(k+1)=\max\left\{\,j\ge \Psi_{in}(k+1), x_j\in S_{\varphi_4(k+1)}\,\right\}.$$

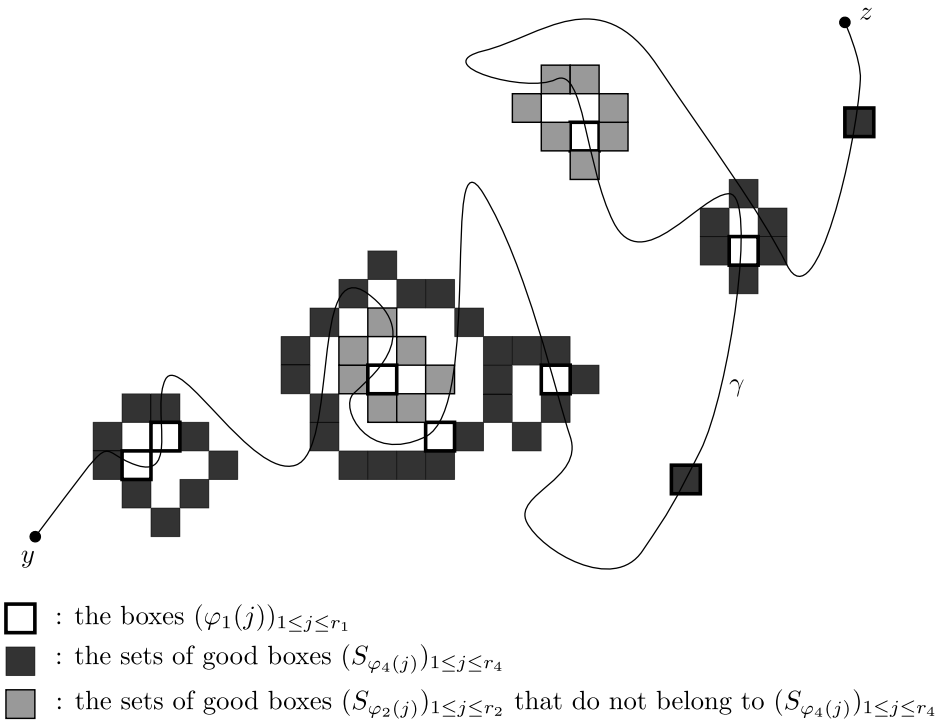

FIGURE 1. Construction of the path γ' – First step.

Let $B_{in}(j)$ be the N-box in $S_{\varphi_4(j)}$ containing $x_{\Psi_{in}(j)}$, $B_{out}(j)$ be the N-box in $S_{\varphi_4(j)}$ containing $x_{\Psi_{out}(j)}$. Let $\gamma(j)$ be the section of γ from $x_{\Psi_{out}(j)}$ to $x_{\Psi_{in}(j+1)}$ for $1 \leq j < r_4$, let $\gamma(0)$ (resp $\gamma(r_4)$) be the section of γ from y to $x_{\Psi_{in}(1)}$ (resp. from $x_{\Psi_{out}(r_4)}$ to z).

We have to study separately the beginning and the end of the path γ. Note that as the N-boxes of y and z both belong to an infinite cluster of good boxes, their box cannot be nested in a bigger $*$-connected components of good boxes of the collection $(S_{\varphi_4(j)})_{1\leq j\leq r_4}$. Thus, if $B_N(\boldsymbol{k})$, the N-box of y, contains a p-closed edge of γ, necessarily $S_{\varphi_4(1)}$ contains $B_N(\boldsymbol{k})$, $B_{in}(1) = B_N(\boldsymbol{k})$ and $x_{\Psi_{in}(1)} = y$. Similarly, if $B_N(\boldsymbol{l})$, the N-box of z, contains a p-closed edge of γ, necessarily $S_{\varphi_4(r_4)}$ contains $B_N(\boldsymbol{l})$, $B_{out}(r_4) = B_N(\boldsymbol{l})$ and $x_{\Psi_{out}(r_4)} = z$.

In order to apply Lemma 3.4, let us show that for every $j \in \{1, \dots, r_4\}$, $x_{\Psi_{in}(j)}$ (resp. $x_{\Psi_{out}(j)}$) belongs to the p-crossing cluster of $B_{in}(j)$ (resp. $B_{out}(j)$). Let us study separately the case of $x_{\Psi_{in}(1)}$ and $x_{\Psi_{out}(r_4)}$. If $x_{\Psi_{in}(1)} = y$ then $x_{\Psi_{in}(1)}$ belongs to the p-crossing cluster of $B_{in}(j)$. Suppose that $x_{\Psi_{in}(1)} \neq y$. As $y \in \mathcal{C}_p$ and y is connected to $x_{\Psi_{in}(1)}$ by a p-open path, $x_{\Psi_{in}(1)}$ is also in $\mathcal{C}_p$. By the property (i) of a good box applied to $B_{in}(1)$, we get that $x_{\Psi_{in}(1)}$ is in the p-crossing cluster of $B_{in}(1)$. We study the case of $x_{\Psi_{out}(r_4)}$ similarly. To study $x_{\Psi_{in}(j)}$ (resp. $x_{\Psi_{out}(j)}$)) for $j \in \{2, \dots, r_4 - 1\}$, we use the fact that by construction, thanks to the extraction φ_2, two different elements of $(S_{\varphi_4(j)})_{1\leq j\leq r_4}$ are not $*$-connected. Therefore, for $1 \leq j < r_4$, we have

$$\|x_{\Psi_{in}(j+1)} - x_{\Psi_{out}(j)}\|_1 \geq N$$

and so the section $\gamma(j)$ of γ from $x_{\Psi_{out}(j)}$ to $x_{\Psi_{in}(j+1)}$ has a diameter larger than N and contains only p-open edges. As $B_{out}(j)$ and $B_{in}(j+1)$ are good boxes, we obtain, using again property (i) of good boxes, that $x_{\Psi_{out}(j)}$ and $x_{\Psi_{in}(j+1)}$ belong to the p-crossing cluster of their respective boxes.

Finally, by Lemma 3.4, for every $j \in \{1, \dots, r_4\}$, there exists a p-open path $\gamma_{link}(j)$ joining $x_{\Psi_{in}(j)}$ and $x_{\Psi_{out}(j)}$ of length at most $12\beta N|S_{\varphi_4(j)}|$. We obtain a p-open path γ' joining y and z by concatenating $\gamma(0), \gamma_{link}(1), \gamma(1), \dots, \gamma_{link}(r_4), \gamma(r_4)$ in this order. Up to removing potential loops, we can suppose that each

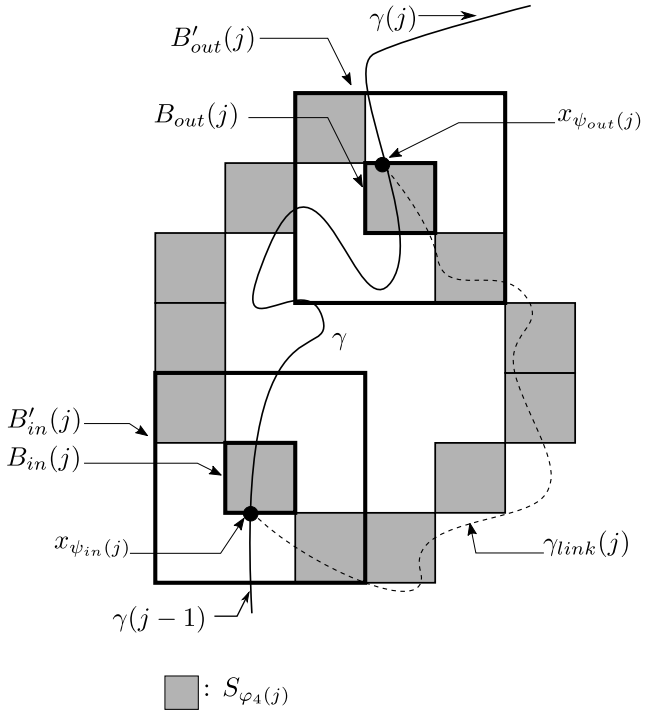

FIGURE 2. Construction of the path γ' – Second step.

$\gamma_{link}(j)$ is a self-avoiding path, that all the $\gamma_{link}(j)$ are disjoint and that each $\gamma_{link}(j)$ intersects only $\gamma(j-1)$ and $\gamma(j)$ at their endpoints. Let us estimate the quantity $|\gamma' \setminus \gamma|$, as $\gamma' \setminus \gamma \subset \cup_{i=1}^{r_4} \gamma_{link}(i)$, we obtain:

$$\begin{aligned} |\gamma' \setminus \gamma| &\leq \sum_{j=1}^{r_4} |\gamma_{link}(j)| \\ &\leq \sum_{j=1}^{r_4} 12\beta N |S_{\varphi_4(j)}| \\ &\leq 12\beta N |\gamma_c| + 12\beta N \sum_{C \in Bad : C \cap \Gamma \neq \emptyset} |\partial_v C| \end{aligned}$$

where the last inequality comes from the fact that each $S_{\varphi_4(j)}$ is the union of elements of $\{\partial_v C : C \in Bad; C \cap \Gamma \neq \emptyset\}$ and of good boxes that contain edges of γ_c. We conclude by noticing that $|\partial_v C| \leq 2d|C|$. □

3.3. Deterministic estimate

When $q - p$ is small, we want to control the probability that the total length of the bypasses $\gamma' \setminus \gamma$ of p-closed edges is large. We can notice in Lemma 3.2 that we need to control the bad connected components of the macroscopic site percolation. This will be done in Section 5. We will also need a deterministic control on $|\Gamma|$ which is the purpose of the following Lemma (this Lemma is an adaptation of Lemma 3.4 of [7]).

Lemma 3.5. *For every path γ of $\mathbb{Z}^d$, for every $N \in \mathbb{N}^*$, there exists a $*$-connected macroscopic path $\widetilde{\Gamma}$ such that*

$$\gamma \subset \bigcup_{\boldsymbol{i} \in \widetilde{\Gamma}} B'_N(\boldsymbol{i}) \text{ and } |\widetilde{\Gamma}| \leq 1 + \frac{|\gamma| + 1}{N} .$$

Proof. Let $\gamma = (x_i)_{1 \leq i \leq n}$ be a path of $\mathbb{Z}^d$ where x_i is the i-th vertex of γ. Let Γ be the set of N-boxes that γ visits. We are going to define iteratively the macroscopic path $\widetilde{\Gamma}$. Let $p(1) = 1$ and $\boldsymbol{i}_1$ be the macroscopic site such that $x_1 \in B_N(\boldsymbol{i}_1)$. We suppose that $\boldsymbol{i}_1, \dots, \boldsymbol{i}_k$ and $p(1), \dots, p(k)$ are constructed. Let us define

$$p(k+1) = \min \{ j > p(k) : x_j \notin B'_N(\boldsymbol{i}_k) \} .$$

If this set is not empty, we set $\boldsymbol{i}_{k+1}$ to be the macroscopic site such that

$$x_{p(k+1)} \in B_N(\boldsymbol{i}_{k+1}) .$$

Otherwise, we stop the process, and we get that for every $j \in \{p(k), \dots, n\}$, $x_j \in B'_N(\boldsymbol{i}_k)$. As n is finite, the process will eventually stop and the two sequences $(p(1), \dots, p(r))$ and $(\boldsymbol{i}_1, \dots, \boldsymbol{i}_r)$ are finite. Note that the $\boldsymbol{i}_j$ are not necessarily all different. We define $\widetilde{\Gamma} = (\boldsymbol{i}_1, \dots, \boldsymbol{i}_r)$. By construction,

$$\gamma \subset \bigcup_{\boldsymbol{i} \in \widetilde{\Gamma}} B'_N(\boldsymbol{i}) .$$

Notice that for every $1 \leq k < r$, $\|x_{p(k+1)} - x_{p(k)}\|_1 \geq N$, thus $p(k+1) - p(k) \geq N$. This leads to $N(r-1) \leq p(r) - p(1) \leq n$, and finally,

$$|\widetilde{\Gamma}| \leq 1 + \frac{|\gamma| + 1}{N} .$$

□

Remark 3.6. This Lemma implies that if Γ is the set of N-boxes that γ visits then

$$|\Gamma| \leq 3^d |\widetilde{\Gamma}| \leq 3^d \left(1 + \frac{|\gamma| + 1}{N} \right) .$$

4. Control of the probability that a box is good

We need in what follows to control the quantity $\sum |C|$ where the sum is over all $C \in Bad$ such that $C \cap \Gamma \neq \emptyset$. We would like to obtain a control which is uniform in the parameter of percolation p. To do so, we are going to introduce a parameter $p_0 > p_c(d)$ and obtain a uniform control in p for all $p \geq p_0$ of the probability that the box B_N is p-bad.

Theorem 4.1. *Let $p_0 > p_c(d)$. There exist positive constants $\beta(p_0) > 0$, $A(p_0)$ and $B(p_0)$ such that for all $p \geq p_0$ and for all $N \geq 1$*

$$\mathbb{P}(B_N \text{ is } p\text{-bad}) \leq A(p_0) \exp(-B(p_0) N) .$$

Note that the property (ii) of the definition of p-good box is a non-decreasing event in p. Thus, it will be easy to bound uniformly the probability that property (ii) is not satisfied by something depending only on p_0. However,

for properties (i) and (iii) a uniform bound is more delicate to obtain. Before proving Theorem 4.1, we need the two following lemmas that deal with properties (i) and (iii). Let $T_{m,N}(p)$ be the event that B_N has a p-crossing cluster and contains some other p-open cluster C having diameter at least m.

Lemma 4.2. *Let $p_0 > p_c(d)$, there exist $\nu = \nu(p_0) > 0$ and $\kappa = \kappa(p_0)$ such that for all $p \geq p_0$*

$$\mathbb{P}(T_{m,N}(p)) \leq \kappa N^{2d} \exp(-\nu m)\,. \tag{4.1}$$

The following Lemma is an improvement of the result of Antal and Pisztora in [1] that controls the probability that two connected points have a too large chemical distance. In the original result, the constants depend on p, we slightly modify its proof so that constants are the same for all $p \geq p_0$. This improvement is required to obtain a decay that is uniform in p.

Lemma 4.3. *Let $p_0 > p_c(d)$, there exist $\beta = \beta(p_0) > 0$, $\widehat{A} = \widehat{A}(p_0)$ and $\widehat{B} = \widehat{B}(p_0) > 0$ such that for all $p \geq p_0$*

$$\forall x \in \mathbb{Z}^d \quad \mathbb{P}(\beta\|x\|_1 \leq D^{\mathcal{C}'_p}(0,x) < +\infty) \leq \widehat{A}\exp(-\widehat{B}\|x\|_1)\,. \tag{4.2}$$

Remark 4.4. Note that this is not an immediate corollary of [1]. Although increasing the parameter of percolation p reduces the chemical distance, it also increases the probability that two vertices are connected. Therefore the event that we aim to control is neither non-increasing neither non-decreasing in p.

Before proving these two lemmas, we are first going to prove Theorem 4.1.

Proof of Theorem 4.1. Let us fix $p_0 > p_c(d)$. Let us denote by $(iii)'$ the property that for all $x, y \in B'_N(\boldsymbol{i})$, if $\|x-y\|_\infty \geq N$ and if x and y belong to the p-crossing cluster $\mathcal{C}$ then $D^{\mathcal{C}'_p}(x,y) \leq 6\beta N$. Note that properties (ii) and $(iii)'$ imply property (iii). Indeed, thanks to (ii), we can find $z \in \mathcal{C} \cap B'_N(\boldsymbol{i})$ such that $\|x-z\|_\infty \geq N$ and $\|y-z\|_\infty \geq N$. Therefore, by applying $(iii)'$,

$$\begin{aligned} D^{\mathcal{C}'_p}(x,y) &\leq D^{\mathcal{C}'_p}(x,z) + D^{\mathcal{C}'_p}(z,y) \\ &\leq 12\beta N\,. \end{aligned}$$

Thus, we can bound the probability that a N-box is bad by the probability that it does not satisfy one of the properties (i), (ii) or $(iii)'$. Since we want to control the probability of B_N being a p-bad box uniformly in p, we will emphasize the dependence of (i), (ii) and $(iii)'$ in p by writing $(i)_p$, $(ii)_p$ and $(iii)'_p$. First, let us prove that the probability that a N-box does not satisfy property $(ii)_p$, *i.e.*, the probability for a box not to have a p-crossing cluster, is decaying exponentially, see for instance Theorem 7.68 in [8]. There exist positive constants $\kappa_1(p_0)$ and $\kappa_2(p_0)$ such that for all $p \geq p_0$

$$\begin{aligned} \mathbb{P}(B_N \text{ does not satisfies } (ii)_p) &\leq \mathbb{P}(B_N \text{ does not satisfies } (ii)_{p_0}) \\ &\leq \kappa_1(p_0)\exp(-\kappa_2(p_0)N^{d-1})\,. \end{aligned} \tag{4.3}$$

Next, let us bound the probability that a N-box does not satisfy property $(iii)'_p$. Using Lemma 4.3, for $p \geq p_0$,

$$\begin{aligned} &\mathbb{P}(B_N \text{ does not satisfy } (iii)'_p) \\ &\quad\leq \sum_{x \in B'_N} \sum_{y \in B'_N} \mathbb{1}_{\|x-y\|_\infty \geq N} \mathbb{P}\left(6\beta N \leq D^{\mathcal{C}'_p}(x,y) < +\infty\right) \\ &\quad\leq \sum_{x \in B'_N} \sum_{y \in B'_N} \mathbb{1}_{\|x-y\|_\infty \geq N} \mathbb{P}\left(\beta\|x-y\|_\infty \leq D^{\mathcal{C}'_p}(x,y) < +\infty\right) \end{aligned}$$

$$\leq \sum_{x\in B'_N}\sum_{y\in B'_N} \mathbb{1}_{\|x-y\|_\infty\geq N}\widehat{A}\exp(-\widehat{B}N)$$
$$\leq (6N+1)^{2d}\widehat{A}\exp(-\widehat{B}N)\,.$$

Finally, by Lemma 4.2,

$$\begin{aligned}
&\mathbb{P}(B_N \text{ is } p\text{-bad})\\
&\quad\leq \mathbb{P}(B_N \text{ does not satisfies } (ii)_p) + \mathbb{P}(B_N \text{ satisfies } (ii)_p \text{ but not } (i)_p)\\
&\qquad + \mathbb{P}(B_N \text{ does not satisfy } (iii)'_p)\\
&\quad\leq \kappa_1\exp(-\kappa_2 N^{d-1}) + 3^d\kappa N^{2d}\exp\left(-\nu\frac{N}{3^d}\right) + (6N+1)^{2d}\widehat{A}\exp(-\widehat{B}N)\\
&\quad\leq A(p_0)e^{-B(p_0)N}\,.
\end{aligned}$$

For the second inequality, we used inequality (4.3) and the fact that the event that the 3^d N-boxes of B'_N are crossing and there exist another p-open cluster of diameter larger than N in B'_N is included in the event there exists a N-box in B'_N that has a crossing property and contains another p-open cluster of diameter at least $N/3^d$. The last inequality holds for $N \geq C_0(p_0)$, where $C_0(p_0)$, $A(p_0) > 0$ and $B(p_0) > 0$ depends only on p_0 and on the dimension d. □

Proof of Lemma 4.2. In dimension $d \geq 3$, we refer to the proof of Lemma 7.104 in [8]. The proof of Lemma 7.104 requires the proof of Lemma 7.78. The probability controlled in Lemma 7.78 is clearly non decreasing in the parameter p. Thus, if we choose $\delta(p_0)$ and $L(p_0)$ as in the proof of Lemma 7.78 for $p_0 > p_c(d)$, then these parameters can be kept unchanged for some $p \geq p_0$. Thanks to Lemma 7.104, we obtain

$$\begin{aligned}
\forall p\geq p_0,\ \mathbb{P}(T_{m,N}(p)) &\leq d(2N+1)^{2d}\exp\left(\left(\frac{m}{L(p_0)+1}-1\right)\log(1-\delta(p_0))\right)\\
&\leq \frac{d.3^d}{1-\delta(p_0)}N^{2d}\exp\left(-\frac{-\log(1-\delta(p_0))}{L(p_0)+1}m\right)\,.
\end{aligned}$$

We get the result with $\kappa = \frac{d.3^d}{1-\delta(p_0)}$ and $\nu = \frac{-\log(1-\delta(p_0))}{L(p_0)+1} > 0$.

In dimension 2, the result is obtained by Couronné and Messikh in the more general setting of FK-percolation in Theorem 9 in [3]. We proceed similarly as in dimension $d \geq 3$, the constant appearing in this theorem first appeared in Proposition 6. The probability of the event considered in this proposition is clearly increasing in the parameter of the underlying percolation, it is an event for the subcritical regime of the Bernoulli percolation. Let us fix a $p_0 > p_c(2) = 1/2$, then $1 - p_0 < p_c(2)$ and we can choose the parameter $c(1 - p_0)$ and keep it unchanged for some $1 - p \leq 1 - p_0$. In Theorem 9, we get the expected result with $c(1 - p_0)$ for a $p \geq p_0$ and $g(n) = n$. □

We explain now how to modify the proof of [1] to obtain the uniformity in p.

Proof of Lemma 4.3. Let $p_0 > p_c(d)$ and $p \geq p_0$. First note that the constant ρ appearing in [1] corresponds to our β. The proof of Lemma 2.3 in [1] can be adapted (as we did above in the proof of Lem. 4.2) to choose constants c_3, c_4, c_6 and c_7 that depend only on p_0 and d, we do not get into details again. Thanks to this, N may be chosen in the expression (4.47) of [1] such that it only depends on p_0 and d and so does ρ. This concludes the proof. □

5. Probabilistic estimates

We can now use the stochastic minoration by a field of independent Bernoulli variables to control the probability that the quantity $\sum |C|$ is big, where the sum is over all $C \in Bad$ such that $C \cap \Gamma \neq \emptyset$. The proof of the following Lemma is in the spirit of the work of Cox and Kesten in [16] and relies on combinatorial considerations. These combinatorial considerations were not necessary in [7].

We consider a path γ and its associated lattice animal Γ. We need in the proof of the following Lemma to define Γ as a path of macroscopic sites, that is to say a path $(\mathrm{i}_k)_{k\leq r}$ in the macroscopic grid such that $\{\boldsymbol{i}_k, k \leq r\} = \Gamma$ (this path may not be self-avoiding). We can choose for instance the sequence of macroscopic sites that γ visits. However, it is difficult to control the size of this sequence by the size of Γ. That is the reason why we consider the path of the macroscopic grid $\widetilde{\Gamma}$ that was introduced in Lemma 3.5.

Proposition 5.1. *Let $p_0 > p_c(d)$. Once p_0 is fixed, we choose $\beta > 0$ given by Theorem 4.1. Let $\varepsilon \in (0, 1-p_c(d))$. There exist a constant $C_\varepsilon \in (0,1)$ depending only on ε and a positive constant C_1 depending on p_0, d and β, such that for every $N \geq C_1|\log \varepsilon|$, then for all $p \geq p_0$, for every $n \in \mathbb{N}^*$*

$$\mathbb{P}\left(\exists \gamma \text{ starting from } 0 \text{ such that } \quad |\widetilde{\Gamma}| \leq n, \quad \sum_{C \in Bad : C \cap \Gamma \neq \emptyset} |C| \geq \varepsilon n\right) \leq C_\varepsilon^n$$

where Γ is the lattice animal associated with the path γ and $\widetilde{\Gamma}$ the macroscopic path given by Lemma 3.5.

Proof. Let $\varepsilon > 0$. Let $N \geq N(\varepsilon)$ where $N(\varepsilon)$ will be defined later. Let us consider a path γ starting from 0, its associated lattice animal Γ, *i.e.*, the set of boxes γ visits and its associated path on the macroscopic grid $\widetilde{\Gamma} = (\widetilde{\Gamma}(k))_{0\leq k\leq r}$ as defined in Lemma 3.5. We first want to include $\widetilde{\Gamma}$ in a subset of the macroscopic grid. Of course, $\widetilde{\Gamma}$ is included in the hypercube of side-length $2r$ centered at $\widetilde{\Gamma}(0)$, but we need to have a more precise control. We set K to be the unique integer such that

$$\frac{1}{\varepsilon} \leq K < \frac{1}{\varepsilon} + 1 \leq \frac{2}{\varepsilon}\,. \tag{5.1}$$

We recall that $\varepsilon < 1$. Let v be a site, we denote by $S(v)$ the hypercube of side-length $2K$ centered at v and by $\partial S(v)$ its inner boundary:

$$S(v) = \{w \in \mathbb{Z}^d : \|w - v\|_\infty \leq K\} \quad \text{and} \quad \partial S(v) = \{w \in \mathbb{Z}^d : \|w - v\|_\infty = K\}\,.$$

We define $v(0) = \widetilde{\Gamma}(0)$, $p_0 = 0$. If $p_0, \ldots, p_k$ and $v(0), \ldots, v(k)$ are constructed, we define if any

$$p_{k+1} = \min\left\{i \in \{p_k + 1, \ldots, r\} : \widetilde{\Gamma}(i) \in \partial S(v(k))\right\} \quad \text{and} \quad v(k+1) = \widetilde{\Gamma}(p_{k+1})\,.$$

If there is no such index we stop the process. Since $p_{k+1} - p_k \geq K$, there are at most $1 + r/K$ such p_k. Notice that $1 + r/K \leq 1 + n/K$ on the event $\{|\widetilde{\Gamma}| \leq n\}$. We define $\tau = 1 + n/K$. On the event $\{|\widetilde{\Gamma}| \leq n\}$, the macroscopic path $\widetilde{\Gamma}$ is contained in the union of those hypercubes:

$$D(v(0), \ldots, v(\tau)) = \bigcup_{i=0}^{\tau} S(v(i))\,.$$

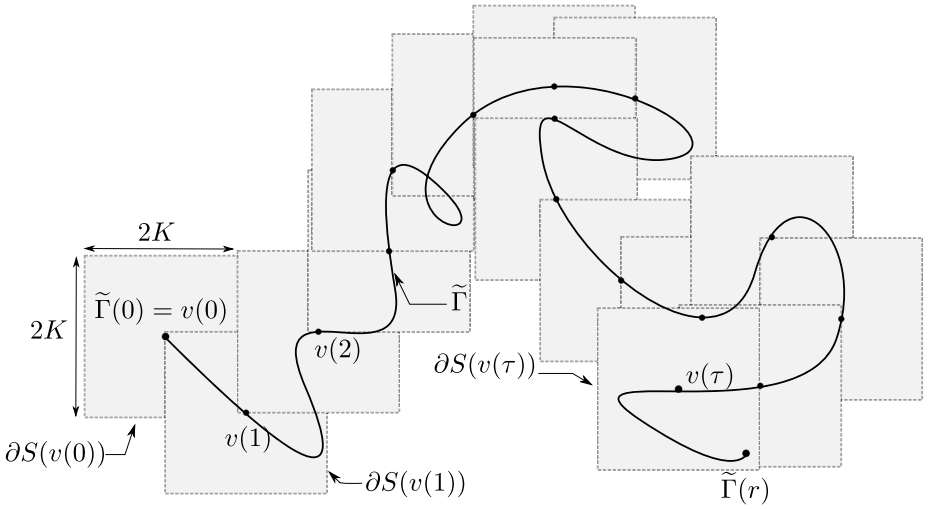

FIGURE 3. Construction of $v(0), \dots, v(\tau)$.

If we stop the process for a $k < \tau$, we artificially complete the sequence until attaining τ by setting for $k < j \leq \tau$, $v(j) = v(k)$. See Figure 3, the corridor $D(v(0), \dots, v(\tau))$ is represented by the grey section. By construction, for all $1 \leq k \leq r$, there exists a $j \leq \tau$ such that $\widetilde{\Gamma}(k)$ is in the strict interior of $S(v(j))$, so we have

$$\Gamma \subset \bigcup_{k=1}^{r} \left\{ \boldsymbol{j},\, \boldsymbol{j} \text{ is } *\text{-connected to } \widetilde{\Gamma}(k) \right\} \subset D(v(0), \dots, v(\tau))$$

where the first inclusion follows from Lemma 3.5.
Thus, we obtain

$$\begin{aligned}
&\mathbb{P}\left(\exists \gamma \text{ starting from 0 such that } \quad |\widetilde{\Gamma}| \leq n, \quad \sum_{C \in Bad : C \cap \Gamma \neq \emptyset} |C| \geq \varepsilon n \right) \\
&\leq \mathbb{P}\left(\bigcup_{v(0),\dots,v(\tau)} \left\{ \begin{array}{c} \exists \gamma \text{ starting from 0 such that} \\ \sum_{C \in Bad : C \cap \Gamma \neq \emptyset} |C| \geq \varepsilon n,\, \Gamma \subset D(v(0), \dots, v(\tau)) \end{array} \right\} \right) \\
&\leq \sum_{v(0),\dots,v(\tau)} \mathbb{P}\left(\begin{array}{c} \exists \gamma \text{ starting from 0 such that} \\ \sum_{C \in Bad : C \cap \Gamma \neq \emptyset} |C| \geq \varepsilon n,\, \Gamma \subset D(v(0), \dots, v(\tau)) \end{array} \right) \\
&\leq \sum_{v(0),\dots,v(\tau)} \mathbb{P}\left(\sum_{\substack{C \in Bad: \\ C \cap D(v(0),\dots,v(\tau)) \neq \emptyset}} |C| \geq \varepsilon n \right) \\
&\leq \sum_{v(0),\dots,v(\tau)} \sum_{j \geq \varepsilon n} \mathbb{P}\left(\sum_{\substack{C \in Bad: \\ C \cap D(v(0),\dots,v(\tau)) \neq \emptyset}} |C| = j \right)
\end{aligned}$$

where the first sum is over the sites $v(0),\dots,v(\tau)$ satisfying $v(0)=\Gamma(1)$ and for all $0\le k<\tau$, $v(k+1)\in \partial S(v(k))\cup\{v(k)\}$. Since $\partial S(v)\cup\{v\}$ contains at most $(c_dK)^{d-1}$ sites where $c_d\ge 1$ is a constant depending only on the dimension, the sum over the sites $v(0),\dots,v(\tau)$ contains at most

$$(c_dK)^{(d-1)\tau}\le (c_dK)^{\frac{2n(d-1)}{K}}:=C_2^n$$

terms for n large enough. For any fixed $v(0),\dots,v(\tau)$, $D(v(0),\dots,v(\tau))$ contains at most

$$(\tau+1)(2K+1)^d\le (n/K+2)(2K+1)^d\le 2n(3K)^d:=C_3n$$

macroscopic sites. The constants C_2 and C_3 only depend on ε and d. Let us recall that for a bad macroscopic site $\boldsymbol{i}$, $C(\boldsymbol{i})$ denotes the connected cluster of bad macroscopic sites containing $\boldsymbol{i}$. Let us notice that the following event

$$\left\{\sum_{\substack{C\in Bad:\\ C\cap D(v(0),\dots,v(\tau))\neq\emptyset}}|C|=j\right\}$$

is included in the event: there exist an integer $\rho\le C_3n$ and distinct bad macroscopic sites $\boldsymbol{i}_1,\dots,\boldsymbol{i}_\rho\in D(v(0),\dots,v(\tau))$, disjoint connected components $\bar C_1,\dots,\bar C_\rho$ such that for all $1\le k\le\rho$, $C(\boldsymbol{i}_k)=\bar C_k$ and $\sum_{k=1}^\rho|\bar C_k|=j$. Therefore, for any fixed $v(0),\dots,v(\tau)$,

$$\begin{aligned}&\mathbb{P}\left(\sum_{\substack{C\in Bad:\\ C\cap D(v(0),\dots,v(\tau))\neq\emptyset}}|C|=j\right)\\ &\le\sum_{\rho=1}^{C_3n}\sum_{\substack{\boldsymbol{i}_1\in D(v(0),\dots,v(\tau))\\ \dots\\ \boldsymbol{i}_\rho\in D(v(0),\dots,v(\tau))\\ \forall k\neq l,\boldsymbol{i}_k\neq\boldsymbol{i}_l}}\sum_{\substack{j_1,\dots,j_\rho\ge1\\ j_1+\dots+j_\rho=j}}\sum_{\substack{\bar C_1\in\mathrm{Animals}_{\boldsymbol{i}_1}^{j_1}\\ \dots\\ \bar C_\rho\in\mathrm{Animals}_{\boldsymbol{i}_\rho}^{j_\rho}}}\mathbb{P}\left(\begin{array}{c}\forall 1\le k\le\rho\\ C(\boldsymbol{i}_k)=\bar C_k\end{array}\right)\end{aligned}\tag{5.2}$$

where $\mathrm{Animals}_{\boldsymbol{v}}^k$ is the set of connected macroscopic sites of size k containing the site $\boldsymbol{v}$. We have $|\mathrm{Animals}_{\boldsymbol{v}}^k|\le(7^d)^k$ (see for instance Grimmett [8], p. 85). There are at most $\binom{C_3n}{\rho}$ ways of choosing the sites $\boldsymbol{i}_1,\dots,\boldsymbol{i}_\rho$. Thus, if we fix the sites $\boldsymbol{i}_1,\dots,\boldsymbol{i}_\rho$ the number of possible choices of the connected components $\bar C_1,\dots,\bar C_\rho$ such that for all $1\le k\le\rho$, $C(\boldsymbol{i}_k)=\bar C_k$ and $\sum_{k=1}^\rho|\bar C_k|=j$ is at most:

$$\sum_{\substack{j_1,\dots,j_\rho\ge1\\ j_1+\dots+j_\rho=j}}(7^d)^{j_1}\cdots(7^d)^{j_\rho}=(7^d)^j\sum_{\substack{j_1,\dots,j_\rho\ge1\\ j_1+\dots+j_\rho=j}}1\,.$$

Next we need to estimate, for given sites $\boldsymbol{i}_1,\dots,\boldsymbol{i}_\rho$ and disjoint connected components $\bar C_1,\dots,\bar C_\rho$, the probability that for all $1\le k\le\rho$, $C(\boldsymbol{i}_k)=\bar C_k$. For all sites $\boldsymbol{i}\in\cup_{k=1}^\rho\bar C_k$, the N-box $B_N(\boldsymbol{i})$ is bad. There is a short range of dependence between the state of the boxes. Indeed, by definition of a p-good box, the state of $B_N(\boldsymbol{i})$ only depends on boxes $B_N(\boldsymbol{j})$ such that $\|\boldsymbol{i}-\boldsymbol{j}\|_\infty\le 13\beta$. Thus, if $\|\boldsymbol{i}-\boldsymbol{j}\|_\infty\ge 27\beta$ the state of the boxes $B_N(\boldsymbol{i})$ and $B_N(\boldsymbol{j})$ are independent. We can deterministically extract from $\cup_{k=1}^\rho\bar C_k$ a set of macroscopic sites $\mathcal{E}$ such that $|\mathcal{E}|\ge j/(27\beta)^d$ and for any $\boldsymbol{i}\neq\boldsymbol{j}\in\mathcal{E}$, the state of the boxes $B_N(\boldsymbol{i})$ and $B_N(\boldsymbol{j})$ are independent. Therefore, we

have using Proposition 4.1

$$\begin{aligned}\mathbb{P}\left(\forall 1\leq k\leq \rho,\, C(\boldsymbol{i}_k)=\bar{C}_k\right) &\leq \mathbb{P}\left(\forall \boldsymbol{i}\in\mathcal{E},\, B_N(\boldsymbol{i}) \text{ is } p\text{-bad}\right)\\ &\leq \mathbb{P}(B_N(\mathbf{0}) \text{ is } p\text{-bad})^{j/(27\beta)^d}\\ &\leq \left(A(p_0)\exp(-B(p_0)N(\varepsilon))\right)^{j/(27\beta)^d}. \end{aligned} \tag{5.3}$$

In what follows, we set

$$\alpha=\alpha(\varepsilon)=\left(A(p_0)\exp(-B(p_0)N(\varepsilon))\right)^{1/(27\beta)^d} \tag{5.4}$$

in order to lighten the notations. We aim to find an expression of $\alpha(\varepsilon)$ such that we get the upper bound stated in the Proposition. The expression of $N(\varepsilon)$ will be determined by the choice of $\alpha(\varepsilon)$. Combining inequalities (5.2) and (5.3), we obtain

$$\mathbb{P}\left(\sum_{\substack{C\in Bad:\\ C\cap D(v(0),\dots,v(\tau))\neq\emptyset}}|C|=j\right)\leq\sum_{\rho=1}^{C_3n}\binom{C_3n}{\rho}(7^d\alpha)^j\sum_{\substack{j_1,\dots,j_\rho\geq 1\\ j_1+\cdots+j_\rho=j}}1$$

and so

$$\begin{aligned}\mathbb{P}\left(\exists\gamma \text{ starting from 0 such that } |\widetilde{\Gamma}|\leq n,\ \sum_{C\in Bad: C\cap\Gamma\neq\emptyset}|C|\geq\varepsilon n\right)\\ \leq C_2^n\sum_{j\geq\varepsilon n}(7^d\alpha)^j\sum_{\rho=1}^{C_3n}\binom{C_3n}{\rho}\sum_{\substack{j_1,\dots,j_\rho\geq 1\\ j_1+\cdots+j_\rho=j}}1.\end{aligned}$$

Notice that

$$\sum_{\rho=1}^{C_3n}\binom{C_3n}{\rho}\sum_{\substack{j_1,\dots,j_\rho\geq 1\\ j_1+\cdots+j_\rho=j}}1=\sum_{\substack{j_1,\dots,j_{C_3\cdot n}\geq 0\\ j_1+\cdots+j_{C_3\cdot n}=j}}1=\binom{C_3n+j-1}{j}.$$

To bound those terms we will need the following inequality, for $r\geq 3$, $N\in\mathbb{N}^*$ and a real z such that $0<ez(1+\frac{r}{N})<1$:

$$\sum_{j=N}^{\infty}z^j\binom{r+j-1}{j}\leq\nu\frac{(ez(1+\frac{r}{N}))^N}{1-ez(1+\frac{r}{N})} \tag{5.5}$$

where ν is an absolute constant. This inequality was present in [16] but without proof, for completeness we will give a proof of (5.5) at the end of the proof of Proposition 5.1. Using inequality (5.5) and assuming $0<e7^d\alpha(\varepsilon)(1+\frac{C_3}{\varepsilon})<1$, we get,

$$\mathbb{P}\left(\exists\gamma \text{ starting from 0 such that } |\widetilde{\Gamma}|\leq n,\ \sum_{C\in Bad: C\cap\Gamma\neq\emptyset}|C|\geq\varepsilon n\right)$$

$$\leq C_2^n \sum_{j\geq \varepsilon n} (7^d\alpha)^j \binom{C_3 n + j - 1}{j}$$
$$\leq \nu C_2^n \frac{\left[e7^d\alpha(\varepsilon)(1+\frac{C_3}{\varepsilon})\right]^{\varepsilon n}}{1 - e7^d\alpha(\varepsilon)(1+\frac{C_3}{\varepsilon})}\,.$$

Let us recall that $C_2 = (c_d K)^{2(d-1)/K}$, $C_3 = 2(3K)^d$ and that $K(\varepsilon)$ was defined in (5.1). We have to choose $\alpha(\varepsilon)$ and a constant $0 < C_\varepsilon < 1$ such that $C_2\left[e7^d\alpha(\varepsilon)(1+\frac{C_3}{\varepsilon})\right]^{\varepsilon} < C_\varepsilon$ that is to say

$$(c_d K)^{\frac{2(d-1)}{K}} \left[e7^d\alpha(\varepsilon)(1+\frac{2(3K)^d}{\varepsilon})\right]^{\varepsilon} < C_\varepsilon\,. \tag{5.6}$$

Note that the condition (5.6) implies the condition $0 < e7^d\alpha(\varepsilon)(1+\frac{C_3}{\varepsilon}) < 1$. Thus, using (5.1), we have

$$\begin{aligned}
&(c_d K)^{\frac{2(d-1)}{K}} \left[e7^d\alpha(\varepsilon)(1+\frac{2(3K)^d}{\varepsilon})\right]^{\varepsilon} \\
&\quad\leq (c_d K)^{\frac{2d}{K}} \left[e7^d\alpha(\varepsilon)\frac{4(3K)^d}{\varepsilon}\right]^{\varepsilon} \\
&\quad\leq \exp\left[\frac{2d}{K}\log(c_d K) + \varepsilon\log\left(e7^d\alpha(\varepsilon)\frac{4(3K)^d}{\varepsilon}\right)\right] \\
&\quad\leq \exp\left[2d\varepsilon\log\left(\frac{2c_d}{\varepsilon}\right) + \varepsilon\log\left(e7^d\alpha(\varepsilon)\frac{4(3\frac{2}{\varepsilon})^d}{\varepsilon}\right)\right] \\
&\quad\leq \exp\left[-2d\varepsilon\log\varepsilon + d\varepsilon\log(2c_d) + \varepsilon\log\left(4e(42)^d\alpha(\varepsilon)\frac{1}{\varepsilon^{d+1}}\right)\right].
\end{aligned}$$

We set

$$\alpha(\varepsilon) = \frac{\varepsilon^r}{(8ec_d(42))^d}$$

where $r = 3d + 2$. We obtain

$$\begin{aligned}
(c_d K)^{\frac{d}{K}} \left[e7^d\alpha(\varepsilon)(1+\frac{2(3K)^d}{\varepsilon})\right]^{\varepsilon} &\leq \exp((r-(3d+1))\varepsilon\log\varepsilon) \\
&\leq \exp(\varepsilon\log\varepsilon) < 1\,.
\end{aligned}$$

By definition of α in (5.4), it follows that there exists a positive constant C_1 depending on β, d, p_0 such that

$$N(\varepsilon) = C_1|\log\varepsilon|\,.$$

It remains now to prove inequality (5.5) to conclude. To show this inequality, we need a version of Stirling's formula with bounds: for all $n \in \mathbb{N}^*$, one has

$$\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} \leq n! \leq e\, n^{n+\frac{1}{2}} e^{-n}\,,$$

thus,

$$\begin{aligned}
\sum_{j=N}^{\infty} z^j \binom{r+j-1}{j} &= \sum_{j=N}^{\infty} z^j \frac{(r+j-1)!}{j!(r-1)!} \\
&\leq \sum_{j=N}^{\infty} z^j \frac{e\,(r+j-1)^{r+j-\frac{1}{2}} e^{-(r+j-1)}}{2\pi\, j^{j+\frac{1}{2}} (r-1)^{r-\frac{1}{2}} e^{-(r+j-1)}} \\
&= \sum_{j=N}^{\infty} \frac{e}{2\pi} z^j \left(\frac{r+j-1}{j}\right)^j \left(\frac{r+j-1}{r-1}\right)^{r-\frac{1}{2}} j^{-\frac{1}{2}} \\
&\leq \sum_{j=N}^{\infty} \frac{e}{2\pi} z^j \left(1+\frac{r}{N}\right)^j \left(1+\frac{j}{r-1}\right)^{r-1} \left(\frac{1}{j}+\frac{1}{r-1}\right)^{\frac{1}{2}} \\
&\leq \sum_{j=N}^{\infty} \frac{e}{2\pi} z^j \left(1+\frac{r}{N}\right)^j e^{(r-1)\log(1+j/(r-1))} \\
&\leq \sum_{j=N}^{\infty} \frac{e}{2\pi} (ez)^j \left(1+\frac{r}{N}\right)^j = \frac{e}{2\pi} \frac{(ez(1+\frac{r}{N}))^N}{1-ez(1+\frac{r}{N})}
\end{aligned}$$

where we use in the last inequality the fact that for all $x > 0$, $\log(1+x) \leq x$. □

6. Regularity of the time constant

In this section, we prove the main result Theorem 1.1 and its Corollary 1.2. Before proving this Theorem, we need to prove two lemmas. The following Lemma enables to control the number of p-closed edges $|\gamma_c|$ in a geodesic γ between two given points y and z in the infinite cluster $\mathcal{C}_p$. We denote by F_x the event that $0, x \in \mathcal{C}_p$ and the N-boxes containing 0 and x belong to an infinite cluster of p-good boxes.

Lemma 6.1. *Let $p_c(d) < p_0 \leq p \leq q$. We fix some $\beta > 0$ and let ρ_0 be the constant given in Lemma 3.2. Then for every integer N, for every $\delta > 0$ and $x \in \mathbb{Z}^d$, we have*

$$\mathbb{P}\left(F_x,\, D^{\mathcal{C}_p}(0,x) > D^{\mathcal{C}_q}(0,x)\left(1+\rho_0 N\left(\frac{q-p}{q}+\delta\right)\right) + \rho_0 N \sum_{\substack{C\in Bad:\\ C\cap\Gamma\neq\emptyset}} |C|\right) \leq e^{-2\delta^2\|x\|_1}.$$

where Γ is the lattice animal of N-boxes visited by an optimal path γ between 0 and x in $\mathcal{C}_q$.

Proof. On the event F_x, we have $0, x \in \mathcal{C}_p \subset \mathcal{C}_q$ so there exists a q-open path joining 0 to x, let γ be an optimal one. Necessarily, we have $|\gamma| \geq \|x\|_1$. We consider the modification γ' given by Lemma 3.2. As γ' is p-open,

$$\begin{aligned}
D^{\mathcal{C}_p}(0,x) < |\gamma'| &\leq |\gamma \cap \gamma'| + |\gamma' \setminus \gamma| \\
&\leq |\gamma| + \rho_0 \left(N|\gamma_c| + N \sum_{C\in Bad: C\cap\Gamma\neq\emptyset} |C|\right) \\
&\leq D^{\mathcal{C}_q}(0,x) + \rho_0 \left(N|\gamma_c| + N \sum_{C\in Bad: C\cap\Gamma\neq\emptyset} |C|\right). \qquad (6.1)
\end{aligned}$$

We want to control the size of γ_c. For that purpose, we want to introduce a coupling of the percolations q and p, such that if any edge is p-open then it is q-open, and we want the random path γ, which is an optimal q-open path between 0 and x, to be independent of the p-state of any edge, *i.e.*, any edge is p-open or p-closed independently of γ. This is not the case when we use the classic coupling with a unique uniform random variable for each edge. Here we introduce two sources of randomness to ease the computations by making the choice of γ independent from the p-state of its edges. We proceed in the following way: with each edge we associate two independent Bernoulli random variables V and Z of parameters respectively q and p/q. Then $W = Z\,V$ is also a Bernoulli random variable of parameter p. This implies

$$\mathbb{P}(W=0|V=1) = \mathbb{P}(Z=0|V=1) = \mathbb{P}(Z=0) = 1 - \frac{p}{q} = \frac{q-p}{q}\,.$$

Thus, we can now bound the following quantity by summing on all possible self-avoiding paths for γ. For short, we use the abbreviation s.a. for self-avoiding.

$$\begin{aligned}
&\mathbb{P}\left(|\gamma_c| \geq |\gamma|\left(\frac{q-p}{q}+\delta\right)\right)\\
&\quad= \sum_{k=\|x\|_1}^{\infty} \sum_{\substack{|r|=k\\ r \text{ s.a. path}}} \mathbb{P}\left(\gamma = r, |\gamma_c| \geq |\gamma|\left(\frac{q-p}{q}+\delta\right)\right)\\
&\quad= \sum_{k=\|x\|_1}^{\infty} \sum_{\substack{|r|=k\\ r \text{ s.a. path}}} \mathbb{P}\left(\gamma = r, |\{e\in r:\ e \text{ is } p\text{-closed}\}| \geq k\left(\frac{q-p}{q}+\delta\right)\right)\\
&\quad= \sum_{k=\|x\|_1}^{\infty} \sum_{\substack{|r|=k\\ r \text{ s.a. path}}} \mathbb{P}\left(\gamma = r, |\{e\in r : Z(e)=0\}| \geq k\left(\frac{q-p}{q}+\delta\right)\right)\\
&\quad= \sum_{k=\|x\|_1}^{\infty} \sum_{\substack{|r|=k\\ r \text{ s.a. path}}} \mathbb{P}\left(\gamma = r\right)\mathbb{P}\left(|\{e\in r : Z(e)=0\}| \geq k\left(\frac{q-p}{q}+\delta\right)\right)\\
&\quad\leq \sum_{k=\|x\|_1}^{\infty} \sum_{\substack{|r|=k\\ r \text{ s.a. path}}} \mathbb{P}\left(\gamma = r\right) e^{-2\delta^2 k} \leq e^{-2\delta^2\|x\|_1}
\end{aligned} \tag{6.2}$$

where we use Chernoff bound in the second to last inequality (see Thm. 1 in [10]). On the event $F_x \cap \left\{|\gamma_c| < |\gamma|\left(\frac{q-p}{q}+\delta\right)\right\}$, by (6.1), we get

$$\begin{aligned}
D^{\mathcal{C}_p}(0,x) &\leq D^{\mathcal{C}_q}(0,x) + \rho_0\left(N|\gamma|\left(\frac{q-p}{q}+\delta\right) + N\sum_{C\in Bad: C\cap\Gamma\neq\emptyset}|C|\right)\\
&= D^{\mathcal{C}_q}(0,x)\left(1+\rho_0 N\left(\frac{q-p}{q}+\delta\right)\right) + \rho_0 N\sum_{C\in Bad: C\cap\Gamma\neq\emptyset}|C|
\end{aligned}$$

and the conclusion follows. □

The proof of the following Lemma is the last step before proving Theorem 1.1.

Lemma 6.2. *Let $p_0 > p_c(d)$. Take $\beta > 0$ given by Theorem 4.1. Fix $\varepsilon \in (0, 1-p_0)$. There exist constants C_1' (depends on p_0 and d) and $\mathfrak{p}$ (depends on p_0, d and ε) such that if we set $N(\varepsilon) = C_1'|\log\varepsilon|$, for every $q \geq p \geq p_0$, for every $x \in \mathbb{Z}^d$ with $\|x\|_1$ large enough,*

$$\mathbb{P}\left(D^{\mathcal{C}_p}(\widetilde{0}^{\mathcal{C}_p}, \widetilde{x}^{\mathcal{C}_p}) \leq D^{\mathcal{C}_q}(\widetilde{0}^{\mathcal{C}_p}, \widetilde{x}^{\mathcal{C}_p})\left(1 + \rho_0\frac{q-p}{q}N(\varepsilon)\right) + \eta_0\varepsilon\|x\|_1\right) \geq \mathfrak{p}$$

where $\eta_0 > 0$ is a constant depending only on d and p_0.

Proof. Let us fix $\varepsilon > 0$. Let C_1 be the constant given by Proposition 5.1. Let $N \geq C_1|\log\varepsilon|$. Fix an $x \in \mathbb{Z}^d$ such that $\|x\|_1 \geq 3dN$. We denote by $\boldsymbol{i}_N(0)$ (respectively $\boldsymbol{i}_N(x)$) the $\boldsymbol{i} \in \mathbb{Z}^d$ such that $0 \in B_N(\boldsymbol{i})$ (resp. such that $x \in B_N(\boldsymbol{i})$). Let us define for $\boldsymbol{i} \in \mathbb{Z}^d$, $Y_{\boldsymbol{i}} = \mathbb{1}_{\{B_N(\boldsymbol{i}) \text{ is } p\text{-good}\}}$ and by $\underline{\mathcal{C}}_p$ the union of infinite cluster in the macroscopic site percolation $(Y_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$. We recall that

$$F_x = \left\{0 \in \mathcal{C}_p,\, x \in \mathcal{C}_p\right\} \cap \left\{\boldsymbol{i}_N(0) \in \underline{\mathcal{C}}_p,\, \boldsymbol{i}_N(x) \in \underline{\mathcal{C}}_p\right\}.$$

We have

$$\begin{aligned}
&\mathbb{P}\left(D^{\mathcal{C}_p}(\widetilde{0}^{\mathcal{C}_p}, \widetilde{x}^{\mathcal{C}_p}) \geq D^{\mathcal{C}_q}(\widetilde{0}^{\mathcal{C}_p}, \widetilde{x}^{\mathcal{C}_p})\left(1 + \rho_0\frac{q-p}{q}N\right) + 3\varepsilon\beta\rho_0\|x\|_1\right)\\
&\leq \mathbb{P}\left(F_x,\, D^{\mathcal{C}_p}(0,x) \geq D^{\mathcal{C}_q}(0,x)\left(1 + \rho_0\frac{q-p}{q}N\right) + 3\varepsilon\beta\rho_0\|x\|_1\right) + \mathbb{P}(F_x^c).
\end{aligned} \tag{6.3}$$

We have

$$\mathbb{P}(F_x^c) \leq \mathbb{P}\left(\{0 \in \mathcal{C}_p,\, x \in \mathcal{C}_p\}^c\right) + \mathbb{P}\left(\{\boldsymbol{i}_N(0) \in \underline{\mathcal{C}}_p,\, \boldsymbol{i}_N(x) \in \underline{\mathcal{C}}_p\}^c\right).$$

Using FKG inequality, we have

$$\mathbb{P}(0 \in \mathcal{C}_p,\, x \in \mathcal{C}_p) \geq \mathbb{P}(0 \in \mathcal{C}_p)\,\mathbb{P}(x \in \mathcal{C}_p) \geq \theta_{p_0}^2$$

where $\theta_{p_0} = \mathbb{P}(0 \in \mathcal{C}_{p_0})$. First note that the field $(Y_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ has a finite range of dependence that depends on β and d. Using the stochastic comparison in [12], for every $\mathfrak{p}_1$, there exists a positive constant α depending on β, d and $\mathfrak{p}_1$ such that if $\mathbb{P}(Y_{\mathbf{o}} = 0) \leq \alpha$ then the field $(Y_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ stochastically dominates a family of independent Bernoulli random variables with parameter $\mathfrak{p}_1$. Let us choose $\mathfrak{p}_1$ large enough such that

$$1 - \theta_{site,\mathfrak{p}_1}^2 \leq \frac{\theta_{p_0}^2}{2}$$

where $\theta_{site,\mathfrak{p}_1}$ denotes the probability for a site to belong to the infinite cluster of i.i.d. Bernoulli site percolation of parameter $\mathfrak{p}_1$. Thanks to Theorem 4.1, there exists a positive integer N_0 depending only on α, p_0 and d such that for every $N \geq N_0$,

$$\mathbb{P}(Y_{\mathbf{0}} = 0) \leq \alpha.$$

For every $\varepsilon \leq 1 - p_0$, we have $|\log\varepsilon| \geq |\log(1-p_0)|$. We set

$$C_1' = \max\left(\frac{N_0}{|\log(1-p_0)|}, C_1\right).$$

If we set $N(\varepsilon) = C_1' |\log \varepsilon|$, then we have $N(\varepsilon) \geq N_0$ so that using the stochastic domination and FKG we obtain

$$\mathbb{P}(\boldsymbol{i}_{N(\varepsilon)}(0) \in \underline{\mathcal{C}}_p,\, \boldsymbol{i}_{N(\varepsilon)}(x) \in \underline{\mathcal{C}}_p) \geq \theta^2_{site,\mathfrak{p}_1}\,.$$

Finally, we get

$$\mathbb{P}(F_x^c) \leq 1 - \theta^2_{p_0} + 1 - \theta^2_{site,\mathfrak{p}_1} \leq 1 - \frac{\theta^2_{p_0}}{2}\,. \tag{6.4}$$

On the event F_x, we have $0, x \in \mathcal{C}_p \subset \mathcal{C}_q$, we can consider γ a geodesic from 0 to x in $\mathcal{C}_q$, and let Γ be the set of $N(\varepsilon)$-boxes that γ visits.

By Lemma 6.1, we have for every $\delta > 0$

$$\begin{aligned}
&\mathbb{P}\Big(F_x,\, D^{\mathcal{C}_p}(0,x) \geq D^{\mathcal{C}_q}(0,x)\left(1 + \rho_0 \frac{q-p}{q} N(\varepsilon)\right) + 3\varepsilon\beta\rho_0\|x\|_1\Big)\\
\leq &\mathbb{P}\left(F_x,\, \rho_0 N(\varepsilon)\left(D^{\mathcal{C}_q}(0,x)\delta + \sum_{C \in Bad : C\cap\Gamma\neq\emptyset} |C|\right) \geq 3\varepsilon\beta\rho_0\|x\|_1\right)\\
&+ \mathbb{P}\left(\begin{array}{r} F_x,\, D^{\mathcal{C}_p}(0,x) > D^{\mathcal{C}_q}(0,x)\left(1 + \rho_0 N(\varepsilon)\left(\frac{q-p}{q} + \delta\right)\right)\\ +\rho_0 N(\varepsilon) \sum\limits_{\substack{C\in Bad:\\ C\cap\Gamma\neq\emptyset}} |C| \end{array}\right)\\
\leq &\mathbb{P}\left(F_x,\, |\gamma| \leq \beta\|x\|_1,\, \sum_{C\in Bad : C\cap\Gamma\neq\emptyset} |C| \geq \frac{3\varepsilon\beta\|x\|_1}{N(\varepsilon)} - \delta|\gamma|\right)\\
&+ \mathbb{P}\left(F_x,\, |\gamma| > \beta\|x\|_1\right) + e^{-2\delta^2\|x\|_1}\\
\leq &\mathbb{P}\left(F_x,\, |\gamma| \leq \beta\|x\|_1,\, \sum_{C\in Bad : C\cap\Gamma\neq\emptyset} |C| \geq \beta\|x\|_1\left(\frac{3\varepsilon}{N(\varepsilon)} - \delta\right)\right)\\
&+ \mathbb{P}\left(F_x,\, |\gamma| > \beta\|x\|_1\right) + e^{-2\delta^2\|x\|_1}\,.
\end{aligned} \tag{6.5}$$

We set $\delta = \varepsilon/N(\varepsilon)$. We know by Lemma 3.5 that $|\widetilde{\Gamma}| \leq 1 + (|\gamma|+1)/N(\varepsilon)$. Moreover as $|\gamma| \geq 3dN(\varepsilon)$, we have $|\widetilde{\Gamma}| \leq 2|\gamma|/N(\varepsilon)$. Using Proposition 5.1,

$$\begin{aligned}
&\mathbb{P}\left(F_x,\, |\gamma| \leq \beta\|x\|_1,\, \sum_{C\in Bad : C\cap\Gamma\neq\emptyset} |C| \geq \beta\|x\|_1\left(\frac{3\varepsilon}{N(\varepsilon)} - \delta\right)\right)\\
&\leq \mathbb{P}\left(\begin{array}{c} \exists\gamma \text{ starting from 0 such that } |\widetilde{\Gamma}| \leq \frac{2\beta\|x\|_1}{N(\varepsilon)},\\ \sum\limits_{C\in Bad : C\cap\Gamma\neq\emptyset} |C| \geq \varepsilon\frac{2\beta\|x\|_1}{N(\varepsilon)} \end{array}\right) \leq C_\varepsilon^{2\beta\|x\|_1/N(\varepsilon)}
\end{aligned} \tag{6.6}$$

where $C_\varepsilon < 1$. Moreover, by Lemma 4.3, we get

$$\mathbb{P}\left(F_x,\, |\gamma| > \beta\|x\|_1\right) \leq \mathbb{P}(\beta\|x\|_1 \leq D^{\mathcal{C}_q}(0,x) < +\infty) \leq \widehat{A}\exp(-\widehat{B}\|x\|_1)\,. \tag{6.7}$$

Finally, combining (6.3), (6.4), (6.5), (6.6) and (6.7), we obtain that

$$\begin{aligned}
&\mathbb{P}\left(D^{\mathcal{C}_p}(\widetilde{0}^{\mathcal{C}_p},\widetilde{x}^{\mathcal{C}_p}) \geq D^{\mathcal{C}_q}(\widetilde{0}^{\mathcal{C}_p},\widetilde{x}^{\mathcal{C}_p})\left(1+\rho_0\frac{q-p}{q}N(\varepsilon)\right)+3\varepsilon\beta\rho_0\|x\|_1\right)\\
&\leq 1-\frac{\theta_{p_0}^2}{2}+C_\varepsilon^{2\beta\|x\|_1/N(\varepsilon)}+\widehat{A}e^{-\widehat{B}\|x\|_1}+e^{-2\varepsilon^2\|x\|_1/N(\varepsilon)^2}\\
&\leq 1-\mathfrak{p}(\varepsilon,p_0)
\end{aligned}$$

for an appropriate choice of $\mathfrak{p}>0$ (depending on p_0, ε and d) and for every x such that $\|x\|_1$ is large enough. □

Proof of Theorem 1.1. Let $\varepsilon>0$, $\delta>0$, $p_0>p_c(d)$ and $x\in\mathbb{Z}^d$, consider $N(\varepsilon)=C_1'|\log\varepsilon|$ and $\mathfrak{p}$ as in Lemma 6.2 and $q\geq p\geq p_0$. With the convergence of the regularized times given by Proposition 2.1, we can choose n large enough such that

$$\mathbb{P}\left(\mu_p(x)-\delta\leq\frac{D^{\mathcal{C}_p}(\widetilde{0}^{\mathcal{C}_p},\widetilde{nx}^{\mathcal{C}_p})}{n}\right)\geq 1-\frac{\mathfrak{p}}{3}$$

$$\mathbb{P}\left(\frac{D^{\mathcal{C}_q}(\widetilde{0}^{\mathcal{C}_p},\widetilde{nx}^{\mathcal{C}_p})}{n}\leq\mu_q(x)+\delta\right)\geq 1-\frac{\mathfrak{p}}{3}$$

$$\mathbb{P}\left(D^{\mathcal{C}_p}(\widetilde{0}^{\mathcal{C}_p},\widetilde{nx}^{\mathcal{C}_p})\leq D^{\mathcal{C}_q}(\widetilde{0}^{\mathcal{C}_p},\widetilde{nx}^{\mathcal{C}_p})\left(1+\rho_0\frac{q-p}{q}N(\varepsilon)\right)+\eta_0\varepsilon n\|x\|_1\right)\geq\mathfrak{p}\,.$$

The intersection of these three events has positive probability, we obtain on this intersection

$$\mu_p(x)-\delta\leq(\mu_q(x)+\delta)\left(1+\rho_0\frac{q-p}{q}N(\varepsilon)\right)+\eta_0\varepsilon\|x\|_1\,.$$

By taking the limit when δ goes to 0 we get

$$\mu_p(x)\leq\mu_q(x)\left(1+\rho_0\frac{q-p}{q}N(\varepsilon)\right)+\eta_0\varepsilon\|x\|_1\,.$$

By Corollary 2.2, we know that the map $p\to\mu_p$ is non-increasing. We also know that $\mu_p(x)\leq\|x\|_1\mu_p(e_1)$ for $e_1=(1,0,\dots,0)$, for any $p>p_c(d)$ and any $x\in\mathbb{Z}^d$. Thus, for every $\varepsilon>0$,

$$\begin{aligned}
\mu_p(x)-\mu_q(x)&\leq\mu_q(x)\rho_0\frac{q-p}{q}N(\varepsilon)+\eta_0\varepsilon\|x\|_1\\
&\leq\mu_{p_0}(e_1)\|x\|_1\rho_0\frac{q-p}{p_c(d)}N(\varepsilon)+\eta_0\varepsilon\|x\|_1\\
&\leq\eta_0'(p_0)\|x\|_1(N(\varepsilon)(q-p)+\varepsilon)
\end{aligned}$$

where η_0' is a constant depending on d and p_0. Using the expression of $N(\varepsilon)$ stated in Lemma 6.2, we obtain

$$\mu_p(x)-\mu_q(x)\leq\eta_0'\|x\|_1\left(C_1'|\log\varepsilon|(q-p)+\varepsilon\right)\,. \tag{6.8}$$

By setting $\varepsilon = q - p$ in the inequality, we get

$$\mu_p(x) - \mu_q(x) \leq \eta_0'' \|x\|_1 (q-p) |\log(q-p)|$$

where $\eta_0'' > 0$ depends only on p_0 and d. Thanks to Corollary 2.2, we have $\mu_p(x) - \mu_q(x) \geq 0$, so that

$$|\mu_p(x) - \mu_q(x)| \leq \eta_0'' \|x\|_1 (q-p) |\log(q-p)| \,. \tag{6.9}$$

By homogeneity, (6.9) also holds for all $x \in \mathbb{Q}^d$. Let us recall that for all $x, y \in \mathbb{R}^d$ and $p \geq p_c(d)$,

$$|\mu_p(x) - \mu_p(y)| \leq \mu_p(e_1) \|x - y\|_1 \,, \tag{6.10}$$

see for instance Theorem 1 in [2]. Moreover, there exists a finite set $(y_1, \ldots, y_m)$ of rational points of $\mathbb{S}^{d-1}$ such that

$$\mathbb{S}^{d-1} \subset \bigcup_{i=1}^{m} \left\{ x \in \mathbb{S}^{d-1} : \|y_i - x\|_1 \leq (q-p) |\log(q-p)| \right\}.$$

Let $x \in \mathbb{S}^{d-1}$ and y_i such that $\|y_i - x\|_1 \leq (q-p)|\log(q-p)|$. Using inequality (6.10), we get

$$\begin{aligned}
&|\mu_p(x) - \mu_q(x)| \\
&\quad \leq |\mu_p(x) - \mu_p(y_i)| + |\mu_p(y_i) - \mu_q(y_i)| + |\mu_q(y_i) - \mu_q(x)| \\
&\quad \leq \mu_p(e_1) \|y_i - x\|_1 + \eta_0'' \|y_i\|_1 (q-p) |\log(q-p)| + \mu_q(e_1) \|y_i - x\|_1 \\
&\quad \leq \left(2\mu_{p_0}(e_1) + \eta_0''\right) (q-p) |\log(q-p)| \,.
\end{aligned}$$

This yields the result. □

Proof of Corollary 1.2. Let $p_0 > p_c(d)$. We consider the constant κ_0 appearing in the Theorem 1.1. Let $p \leq q$ in $[p_0, 1]$. We recall the following definition of the Hausdorff distance between two subsets E and F of $\mathbb{R}^d$:

$$d_{\mathcal{H}}(E, F) = \inf \left\{ r \in \mathbb{R}^+ : E \subset F^r \text{ and } F \subset E^r \right\}$$

where $E^r = \{y : \exists x \in E, \|y - x\|_2 \leq r\}$. Thus, we have

$$d_{\mathcal{H}}(\mathcal{B}_{\mu_p}, \mathcal{B}_{\mu_q}) \leq \sup_{y \in \mathbb{S}^{d-1}} \left\| \frac{y}{\mu_p(y)} - \frac{y}{\mu_q(y)} \right\|_2 .$$

Note that $y/\mu_p(y)$ (resp. $y/\mu_q(y)$) is in the unit sphere for the norm μ_p (resp. μ_q). Let us define $\mu_p^{min} = \inf_{x \in \mathbb{S}^{d-1}} \mu_p(x)$. As the map $p \to \mu_p$ is uniformly continuous on the sphere $\mathbb{S}^{d-1}$ (see Thm. 1.2 in [7],) the map $p \to \mu_p^{min}$ is also continuous and $\mu^{min} = \inf_{p \in [p_0, 1]} \mu_p^{min} > 0$. Finally

$$\begin{aligned}
d_{\mathcal{H}}(\mathcal{B}_{\mu_p}, \mathcal{B}_{\mu_q}) &\leq \sup_{y \in \mathbb{S}^{d-1}} \left| \frac{1}{\mu_p(y)} - \frac{1}{\mu_q(y)} \right| \\
&\leq \sup_{y \in \mathbb{S}^{d-1}} \frac{1}{\mu_q(y)\mu_p(y)} |\mu_p(y) - \mu_q(y)| \\
&\leq \sup_{y \in \mathbb{S}^{d-1}} \frac{1}{(\mu^{min})^2} |\mu_p(y) - \mu_q(y)|
\end{aligned}$$

$$\leq \frac{\kappa_0}{(\mu^{min})^2}(q-p)|\log(q-p)|\,.$$

This yields the result. □

Remark 6.3. At this stage, we were not able to obtain Lipschitz continuity for $p \to \mu_p$. The difficulty comes from the fact that we do not know the correlation between γ and the state of the boxes that γ visits. At first sight, it may seem that the renormalization is responsible for the appearance of the log terms in Theorem 1.1. However, when p is very close to 1, we can avoid renormalization and bypass p-closed edges at a microscopic scale as in [15] but even in that case, we cannot obtain Lipschitz continuous regularity with the kind of combinatorial computations made in Section 5. A similar issue arises, it is hard to deal with the correlation between p-closed edges of γ and the length of the microscopic bypasses.

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