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REDUNDANCY IN GAUSSIAN RANDOM FIELDS

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Abstract. In this paper, we introduce a notion of spatial redundancy in Gaussian random fields. This study is motivated by applications of the *a contrario* method in image processing. We define similarity functions on local windows in random fields over discrete or continuous domains. We derive explicit Gaussian asymptotics for the distribution of similarity functions when computed on Gaussian random fields. Moreover, for the special case of the squared L^2 norm, we give non-asymptotic expressions in both discrete and continuous periodic settings. Finally, we present fast and accurate approximations of these non-asymptotic expressions using moment methods and matrix projections.

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1. Introduction

Stochastic geometry [3, 12, 52] aims at describing the arrangement of random structures based on the knowledge of the distribution of geometrical elementary patterns (point processes, random closed sets, etc.). When the considered patterns are functions over some topological space, we can study the geometry of the associated random field. For example, centering a kernel function at each point of a Poisson point process gives rise to the notion of shot-noise random field [18, 50, 51]. We can then study the perimeter or the Euler-Poincaré characteristic of the excursion sets among other properties [2, 4]. In the present work we will focus on the geometrical notion of redundancy of local windows in random fields. We say that a local window in is redundant if it is "similar" to other local windows in the same random field. The similarity of two local windows is defined as the output of some similarity function computed over these local windows. The lower is the output, the more similar the local windows are.

Identifying such spatial redundancy is a fundamental task in the field of image processing. For instance, in the context of denoising, Buades *et al.* in [9], propose the Non-Local means algorithm in which a noisy patch is replaced by a weighted mean over all similar patches. Other examples can be found in the domains of inpainting [17] and video coding [34]. Spatial redundancy is also of crucial importance in exemplar-based texture synthesis, where we aim at sampling images with the same perceptual properties as an input exemplar texture. If Gaussian

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random fields [26, 40, 55, 57] give good visual results for input textures with no, or few, spatial redundancy, they fail when it comes to sampling structured textures (brick walls, fabric with repeated patterns, etc.). In this case, more elaborated models are needed [20, 27, 43]. In this work, we derive explicit probability distribution functions for the random variables associated with the output of similarity functions computed on local windows of random fields. The knowledge of such functions allows us to conduct rigorous statistical testing on the spatial redundancy in natural images.

In order to compute these explicit distributions we will consider specific random fields over specific topological spaces. First, the random fields will be defined either over $\mathbb{R}^2$ (or $\mathbb{T}^2$, where $\mathbb{T}^2$ is the 2-dimensional torus, when considering periodicity assumptions on the field), or over $\mathbb{Z}^2$ (or $(\mathbb{Z}/(M\mathbb{Z}))^2$, with $M \in \mathbb{N}$ when considering periodicity assumptions on the field). Each of these spaces is embedded with its classical topology. The first case is the *continuous setting*, whereas the second one is the *discrete setting*. In image processing, the most common framework is the finite discrete setting. The discrete setting ($\mathbb{Z}^2$) can be used to define asymptotic properties when the size of images grows or when their resolution increases [8], whereas continuous settings are needed in specific applications where, for instance, rotation invariant models are required [54]. All the considered random fields will be Gaussian. This assumption will allow us to explicitly derive moments of some similarity functions computed on local windows of the random field. Once again, another reason for this restriction comes from image processing. Indeed, given an input image, we can compute its first and second-order statistics. Sampling from the associated Gaussian random field gives examples of images which preserve the covariance structure but lose the global arrangement of the input image. Investigating redundancy of such fields is a first step towards giving a mathematical description of this lack of structure.

Finding measurements which correspond to the ones of our visual system is a long-standing problem in image processing. It was considered in the early days of texture synthesis and analyzed by Julesz [36, 37, 58] who formulated the conjecture that textures with similar first-order statistics (first conjecture) or that textures with similar first and second-order statistics (second conjecture) could not be discriminated by the human eye. Even if both conjectures were disproved [22], the work of Gatys *et al.* [27] suggests that second-order statistics of image features are enough to characterize a broad range of textures. To compute features on images we embed them in a higher dimensional space. This operation can be conducted using linear filtering [47] or convolutional neural networks [27] for instance. Some recent works examine the response of convolutional neural network to elementary geometrical pattern [45], giving insight about the perceptual properties of such a lifting. In the present work, we focus on another embedding given by considering a square neighborhood, called a patch, around each pixel. This embedding, is exploited in many image processing tasks such as inpainting [30], denoising [9, 39], texture synthesis [23, 24, 41, 49], *etc.*

In the special case where the similarity functions are given by the L^2 norm, explicit distributions can be inferred even in the non-asymptotic case. Calculating this distribution exactly is demanding since it requires the knowledge of some covariance matrix eigenvalues as well as an efficient method to compute cumulative distribution functions of quadratic forms of Gaussian random variables. We propose an efficient algorithm to approximate this distribution. In [7], this algorithm is applied to denoising and periodicity detection problems in an *a contrario* framework.

The paper is organized as follows. We recall basic notions of Gaussian random fields in general settings in Section 2.1. Similarity functions to be evaluated on these random fields, as well as their statistical properties, are described in Section 2.2. We give the asymptotic properties of these similarity functions in Gaussian random fields in the discrete setting in Section 3.1 and in the continuous setting in Section 3.2. It is shown in Section 3.3 that the Gaussian asymptotic approximation is valid only for large patches. In order to overcome this problem we consider an explicit formulation of the probability distribution function for a particular similarity function: the square L^2 norm. The computations are conducted in the finite discrete case in Section 4.1. We also derive an efficient algorithm to compute these probability distribution functions. Similar non-asymptotic expressions are given in the continuous case in Section 4.2. Technical proofs and additional results on multidimensional central limit theorems are presented in the Appendices.

2. Similarity functions and random fields

2.1. Gaussian random fields

Let $(\mathcal{A}, \mathcal{F}, \mathbb{P})$ be a probability space. Following [1], a random field over a topological space Ω is defined as a measurable mapping $U : \mathcal{A} \to \mathbb{R}^\Omega$. Thus, for all a in $\mathcal{A}$, $U(a)$ is a function over Ω and, for any $a \in \mathcal{A}$ and any $\mathbf{x} \in \Omega$, $U(a)(\mathbf{x})$ is a real number. For the sake of clarity we will omit a in what follows.

We say that a random field U is of order $r > 0$ if for any finite sequence $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \Omega^n$ with $n \in \mathbb{N}$, the vector $V = (U(\mathbf{x}_1), \dots, U(\mathbf{x}_n))$ satisfy $\mathbb{E}[\|V\|_r^r] < +\infty$. Assuming that U is a second-order random field, we define the mean function of U, $m : \Omega \to \mathbb{R}$ as well as its covariance function, $C : \Omega^2 \to \mathbb{R}$ for any $\mathbf{x}, \mathbf{y} \in \Omega^2$ by

$$m(\mathbf{x}) = \mathbb{E}[U(\mathbf{x})] \qquad \text{and} \quad C(\mathbf{x}, \mathbf{y}) = \mathbb{E}[(U(\mathbf{x}) - m(\mathbf{x}))(U(\mathbf{y}) - m(\mathbf{y}))].$$

A random field U is said to be stationary if for any finite sequence $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \Omega^n$ with $n \in \mathbb{N}$ and $\mathbf{t} \in \Omega$, the vector $(U(\mathbf{x}_1), \dots, U(\mathbf{x}_n))$ and $(U(\mathbf{x}_1 + \mathbf{t}), \dots, U(\mathbf{x}_n + \mathbf{t}))$ have same distribution. A second-order random field U over a topological vector field is said to be stationary in the weak sense if its mean function is constant and if for all $\mathbf{x}, \mathbf{y} \in \Omega$, $C(\mathbf{x}, \mathbf{y}) = C(\mathbf{x} - \mathbf{y}, \mathbf{0})$. In this case the covariance of U is fully characterized by its auto-covariance function $\Gamma : \Omega \to \mathbb{R}$ given for any $\mathbf{x} \in \Omega$ by

$$\Gamma(\mathbf{x}) = C(\mathbf{x}, 0) .$$

A random field U is said to be a Gaussian random field if, for any finite sequence $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \Omega^n$ with $n \in \mathbb{N}$, the vector $(U(\mathbf{x}_1), \dots, U(\mathbf{x}_n))$ is a n-dimensional Gaussian random vector. The distribution of a Gaussian random field is entirely characterized by its mean and covariance functions. As a consequence, the notions of stationarity and weak stationarity coincide for Gaussian random fields.

Since the applications we are interested in are image processing tasks, we consider the case where $\Omega = \mathbb{R}^2$ (in the continuous setting) and $\Omega = \mathbb{Z}^2$ (in the discrete setting). In Section 2.2 we will consider Lebesgue integrals of random fields and thus need integrability condition for U over compact sets. Let $K = [a, b] \times [c, d]$ be a compact rectangular domain in $\mathbb{R}^2$. Continuity requirements on the function C imply that $\int_K g(\mathbf{x})U(\mathbf{x})\mathrm{d}\mathbf{x}$ is well-defined as the quadratic mean limit for real-valued functions g over Ω such that $\int_{K \times K} g(\mathbf{x})g(\mathbf{y})C(\mathbf{x}, \mathbf{y})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}$ is finite, see [42]. However, we are interested in almost sure quantities and thus we want the integral to be defined almost surely over rectangular windows. Imposing the existence of a continuous modification of a random field, ensures the almost sure existence of Riemann integrals over rectangular windows. The following assumptions will ensure continuity almost surely, see Lemma 2.3 whose proof can be found in ([1], Thm. 1.4.1) and ([48], Lem. 4.2, Lem. 4.3, Thm. 4.5). We define $D : \Omega \times \Omega \to \mathbb{R}$ such that for any $\mathbf{x}, \mathbf{y} \in \Omega$

$$D(\mathbf{x}, \mathbf{y}) = \mathbb{E}\left[(U(\mathbf{x}) - U(\mathbf{y}))^2\right] = C(\mathbf{x}, \mathbf{x}) + C(\mathbf{y}, \mathbf{y}) - 2C(\mathbf{x}, \mathbf{y}) + (m(\mathbf{x}) - m(\mathbf{y}))^2 .$$

Assumption 2.1 (A2.1)**.** *U is a second-order random field and there exist $M, \eta, \alpha > 0$ such that for any $\mathbf{x} \in \Omega$ and $\mathbf{y} \in \mathrm{B}(\mathbf{x}, \eta) \cap \Omega$ with $\mathbf{y} \neq \mathbf{x}$ we have*

$$D(\mathbf{x}, \mathbf{y}) \leqslant \frac{M\|\mathbf{x} - \mathbf{y}\|_2^2}{|\log(\|\mathbf{x} - \mathbf{y}\|_2)|^{2+\alpha}} .$$

This assumption can be considerably weakened in the case of a stationary Gaussian random field.

Assumption 2.2 (A2.2)**.** *U is a stationary Gaussian random field and there exist $M, \eta, \alpha > 0$ such that for any $\mathbf{x} \in \Omega$ and $\mathbf{y} \in \mathrm{B}(\mathbf{x}, \eta) \cap \Omega$ with $\mathbf{y} \neq \mathbf{x}$ we have*

$$D(\mathbf{x}, \mathbf{y}) \leqslant \frac{M}{|\log(\|\mathbf{x} - \mathbf{y}\|_2)|^{1+\alpha}}.$$

Lemma 2.3 (Sample path continuity)**.** *Assume* (A2.1) *or* (A2.2)*. In addition, assume that for any* $\mathbf{x} \in \Omega$*,* $m(\mathbf{x}) = 0$*. Then there exists a modification of* U*, i.e. a random field* $\widetilde{U}$ *such that for any* $\mathbf{x} \in \Omega$*,* $\mathbb{P}[U(\mathbf{x}) = \widetilde{U}(\mathbf{x})] = 1$*, and for any* $a \in \mathcal{A}$*,* $\widetilde{U}(a)$ *is continuous over* Ω*.*

In the rest of the paper we always replace U by its continuous modification $\widetilde{U}$. Note that in the discrete case all random fields are continuous with respect to the discrete topology.

In Sections 3 and 4, we will suppose that U is a stationary Gaussian random field with zero mean. Asymptotic theorems derived in the next section remain true in broader frameworks, however restricting ourselves to stationary Gaussian random fields allows for explicit computations of asymptotic quantities in order to numerically assess the rate of convergence.

2.2. Similarity functions

In order to evaluate redundancy in random fields, we first need to derive a criterion for comparing random fields. We introduce similarity functions which take rectangular restrictions of random fields as inputs.

When comparing local windows of random fields (patches), two cases can occur. We can compare a patch with a patch extracted from the same image. We call this situation *internal matching*. Applications can be found in denoising [9] or inpainting [17] where the information of the image itself is used to perform the image processing task. On the other hand, we can compare a patch with a patch extracted from another image. We call this situation *template matching*. An application can be found in the non-parametric exemplar-based texture synthesis algorithm proposed by Efros and Leung [23].

The L_2 norm is the usual way to measure the similarity between patches [39] but many other measurements exist, corresponding to different structural properties, see Figure 1.

Definition 2.4. Let $P, Q \in \mathbb{R}^\omega$ with $\omega \subset \mathbb{R}^2$ or $\omega \subset \mathbb{Z}^2$. When it is defined we introduce

(a) the L^p-similarity, $s_p(P,Q) = \|P-Q\|_p = \left(\int_{\mathbf{x}\in\omega} |P(\mathbf{x}) - Q(\mathbf{x})|^p \mathrm{d}\mathbf{x}\right)^{1/p}$, with $p \in (0,+\infty)$;
(b) the L^∞-similarity, $s_\infty(P,Q) = \sup_{\mathbf{x}\in\omega}(|P(\mathbf{x}) - Q(\mathbf{x})|)$;
(c) the p-th power of the L^p-similarity, $s_{p,p}(P,Q) = s_p(P,Q)^p$, with $p \in (0,+\infty)$;
(d) the scalar product similarity, $s_{sc}(P,Q) = -\langle P, Q\rangle = \frac{1}{2}\left(s_{2,2}(P,Q) - \|P\|_2^2 - \|Q\|_2^2\right)$;
(e) the cosine similarity, $s_{\cos}(P,Q) = \frac{s_{sc}(P,Q)}{\|P\|_2\|Q\|_2}$, if $\|P\|_2\|Q\|_2 \neq 0$.

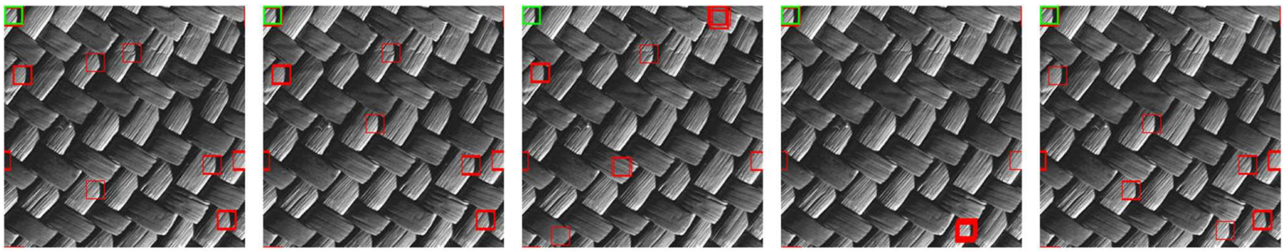
Depending on the case $\mathrm{d}\mathbf{x}$ is either the Lebesgue measure or the discrete measure over ω.

The locality of the measurements is ensured by the fact that these functions are defined on patches, *i.e.* local windows. Following conditions (1) and (3) in [21] one can check that similarity functions (a), (c) and (e) satisfy the following properties

- (Symmetry) $s(P,Q) = s(Q,P)$;
- (Maximal self-similarity) $s(P,P) \leqslant s(P,Q)$;
- (Equal self-similarities) $s(P,P) = s(Q,Q)$.

Note that since s_{sc}, the scalar product similarity, is homogeneous in P, maximal self-similarity and equal self-similarity properties are not satisfied. All introduced similarities satisfy the symmetry condition and s_∞ satisfies the maximal self-similarity property. In [21], the authors present many other similarity functions all relying on statistical properties such as likelihood ratios, joint likelihood criteria and mutual information kernels. In the present paper we focus only on similarity functions defined directly in the spatial domain.

Definition 2.5 (Auto-similarity and template similarity)**.** Let u and v be two functions defined over a domain $\Omega \subset \mathbb{R}^2$ or $\mathbb{Z}^2$. Let $\omega \subset \Omega$ be a patch domain. We introduce $P_\omega(u) = u|_\omega$, the restriction of u to the patch domain ω. When it is defined we introduce the auto-similarity with patch domain ω and offset $\mathbf{t} \in \mathbb{R}^2$ or $\mathbb{Z}^2$



s_2 s_1 s_∞ s_{sc} $s_{\cos}$

FIGURE 1. Structural properties of similarity functions. In this experiment the image size is 512×512 and the patch size is 20×20. We show the 20 closest patches (*red squares*) to the upper-left patch (*green square*) among all patches for different similarity functions. All introduced similarity functions, see Definition 2.4, correctly identify the structure of the patch, *i.e.* a large clear part with diagonal textures and a dark ray on the right side of the patch, except for s_∞ which is too sensitive to outliers. Similarities s_2, s_1 and $s_{\cos}$ have analogous behaviors and identify correct regions. Similarity s_{sc} is too sensitive to contrast and, as it selects a correct patch, it gives too much importance to illumination.

such that $\mathbf{t} + \omega \subset \Omega$ by

$$\mathcal{AS}_i(u, \mathbf{t}, \omega) = s_i \left(P_{\mathbf{t}+\omega}(u), P_\omega(u) \right),$$

where s_i corresponds to s_p with $p \in (0, +\infty]$, $s_{p,p}$ with $p \in (0, +\infty)$, s_{sc} or s_{cos}. In the same way, when it is defined, we introduce the template similarity with patch ω and offset $\mathbf{t}$ by

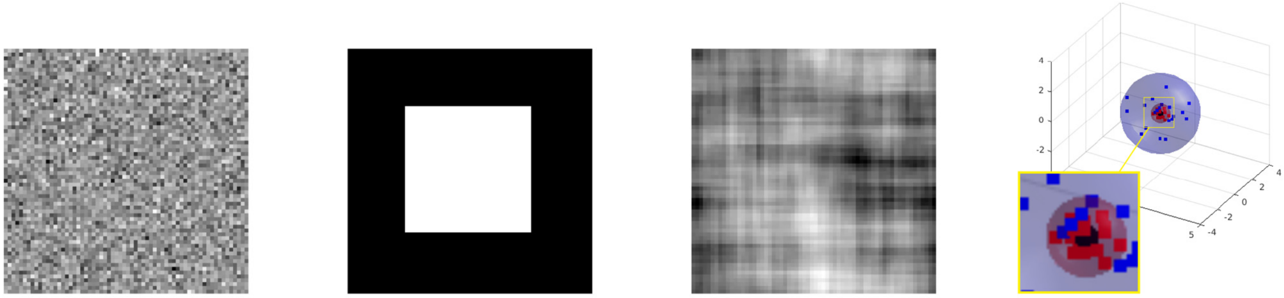
$$\mathcal{TS}_i(u, v, \mathbf{t}, \omega) = s_i \left(P_{\mathbf{t}+\omega}(u), P_\omega(v) \right).$$

Note that in the finite discrete setting, *i.e.* $\Omega = (\mathbb{Z}/(M\mathbb{Z}))^2$ with $M \in \mathbb{N}$, the definition of $\mathcal{AS}$ and $\mathcal{TS}$ can be extended to any patch domain $\omega \subset \mathbb{Z}^2$ by replacing u by $\dot{u}$, its periodic extension to $\mathbb{Z}^2$. A similar extension can be derived in the finite continuous setting, *i.e.* $\Omega = \mathbb{T}^2$.

Suppose we evaluate the scalar product auto-similarity $\mathcal{AS}_{sc}(U, \mathbf{t}, \omega)$ with U a random field. Then the auto-similarity function is a random variable and its expectation depends on the second-order statistics of U. In the template case, the expectation of $\mathcal{TS}_{sc}(U, v, \mathbf{t}, \omega)$ depends on the first-order statistics of U. This shows that auto-similarity and template similarity can exhibit very different behaviors even for the same similarity functions.

In the discrete case, it is well-known that, due to the curse of dimensionality, the L^2 norm does not behave well in large-dimensional spaces and is a poor measure of structure. Thus, considering u and v two images, $s_2(u, v)$, the L^2 template similarity on full images, does not yield interesting information about the perceptual differences between u and v. The template similarity $\mathcal{TS}_2(u, v, \mathbf{0}, \omega)$ avoids this effect by considering patches which reduces the dimension of the data (if the cardinality of ω, denoted $|\omega|$, is small) and also allows for fast computation of similarity mappings, see Figure 1 for a comparison of the different similarity functions on a natural image.

We extract patches from images as follow. For each position in the image we consider a square ω centered around this position. This operation is called patch lifting. In Figure 2, we investigate the behavior of patch lifting on different Gaussian random fields. Roughly speaking, patches are said to be similar if they are clustered in the patch space. Using Principal Component Analysis we illustrate that patches are more scattered in Gaussian white noise than in the Gaussian random field $U = f * W$ (with periodic convolution, *i.e.* $f * W(\mathbf{x}) = \sum_{\mathbf{y} \in \Omega} W(\mathbf{y}) \dot{f}(\mathbf{x} - \mathbf{y})$ where $\dot{f}$ is the periodic extension of f to $\mathbb{Z}^2$), where W is a Gaussian white noise over Ω (a finite discrete grid) and f is the indicator function of a rectangle non reduced to a single pixel.



(A) (B) (C) (D)

FIGURE 2. Gaussian models and spatial redundancy. In this experiment we illustrate the notion of spatial redundancy in two models. In (A), we present a 64×64 Gaussian white noise. (B) Shows an indicator function f . In (C), we present a realization of the Gaussian random field defined by $f * W$ (with periodic convolution) where W is a Gaussian white noise over Ω (domain of size 64×64). Note that f was chosen so that the two Gaussian random fields (A) and (C) have the same gray-level distribution for each pixel. To each pixel position in (A) and (C) we associate the surrounding patch, with patch domain ω (of size 3×3). Hence, for each image (A) and (C) we obtain $64 \times 64 = 5096$ vectors each of size $3 \times 3 = 9$. These 9-dimensional vectors are projected in a 3-dimensional space using Principal Component Analysis. In the subfigure (D), we display the 20 vectors closest to 0 in each case: Gaussian white noise model (in blue) and the Gaussian random field (C) (in red). The radius of the blue, respectively red, sphere represents the maximal L^2 norm of these 20 vectors in the Gaussian white noise model, respectively in model (C). Since the radius of the blue sphere is larger than the red one the points are more scattered in the patch space of (A) than in the patch space of (B). This implies that there is more spatial redundancy in (C) than in (A), which is expected.

We continue this investigation in Figure 3 in which we present the closest patches (of size 10×10), for the L^2 norm, in two Gaussian random fields $U = f * W$ (where the convolution is periodic) for different functions f called spots, [25]. The more regular f is, the more similar the patches are. Limit cases are $f = 0$ (all patches are constant and thus all the patches are similar) and $f = \delta_0$, *i.e.* $U = W$.

We introduce the notion of autocorrelation. Let $f \in L^2(\mathbb{Z}^2)$. We denote by Γ_f the autocorrelation of f, *i.e.* $\Gamma_f = f * \check{f}$ where for any $\mathbf{x} \in \mathbb{Z}^2$, $\check{f}(\mathbf{x}) = f(-\mathbf{x})$ and define the associated random field to a square-integrable function f as the stationary Gaussian random field U such that for any $\mathbf{x} \in \Omega$

$$\mathbb{E}\left[U(\mathbf{x})\right] = 0 \qquad \text{and} \quad \Gamma(\mathbf{x}) = \Gamma_f(\mathbf{x}).$$

In Figure 4, we compare the patch spaces of natural images and the ones of their associated random fields. Since the associated Gaussian random fields lose all global structures, most of the spatial information is discarded. This situation can be observed in the patch space. In the natural images, patches containing the same highly spatial information (such as a white diagonal) are close for the L^2 norm. In Gaussian random field since this highly spatial information is lost, close patches for the L^2 norm are not necessarily perceptually close.

3. Asymptotic results

In this section we aim at giving explicit asymptotic expressions for the probability distribution functions of the auto-similarity and the template similarity in both discrete and continuous settings. Using general versions of the law of large numbers and central limit theorems we will derive Gaussian asymptotic approximations.

Additional assumptions are required in the case of template matching since we use an exemplar input image v to compute $\mathcal{TS}_i(U, v, \mathbf{t}, \omega)$. Let $v \in \mathbb{R}^\Omega$, where Ω is $\mathbb{R}^2$ or $\mathbb{Z}^2$. We denote by $(v_k)_{k \in \mathbb{N}}$ the sequence of the restriction

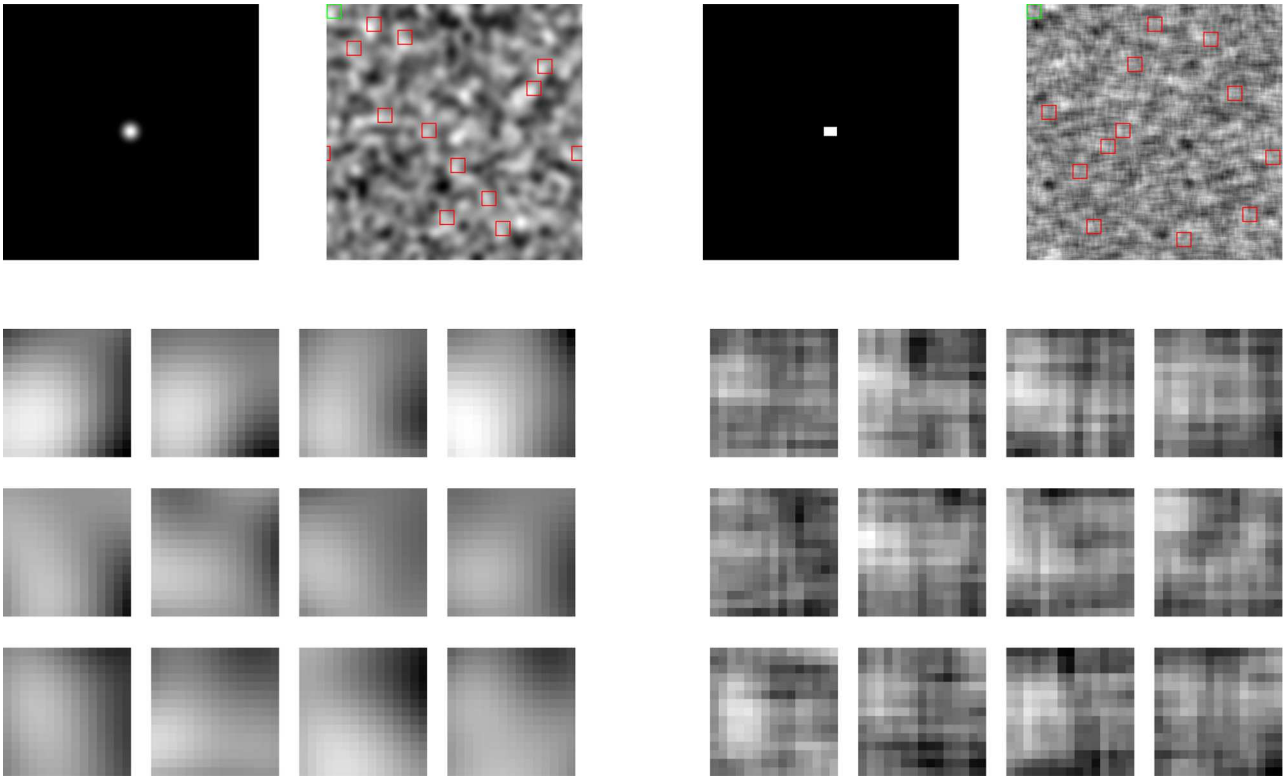


FIGURE 3. Patch similarity in Gaussian random fields. In this figure we show two examples of Gaussian random fields in the discrete periodic case. On the left of the first row we show a Gaussian spot f and a realization of the Gaussian random field $U = f * W$, where the convolution is periodic and W is a Gaussian white noise. The associated random field is smooth and isotropic. The random field $U = f * W$ associated with a rectangular plateau f is no longer smooth nor isotropic. Images are displayed on the right of their respective spot. For each setting (Gaussian spot or rectangular spot) we present 12 patches of size 15×15. In each case the top-left patch is the top-left patch in the presented realization of the random field, shown in green. Following from the top to the bottom and from the left to the right are the closest patches in the patch space for the L^2 norm. We discard patches which are spatially too close (if ω_1 and ω_2 are two patch domains we impose $\sup_{\mathbf{x},\mathbf{y}} \|\mathbf{x}-\mathbf{y}\|_\infty \geqslant 10$).

of v to ω_k, extended to $\mathbb{Z}^2$ (or $\mathbb{R}^2$) by zero-padding, *i.e.* $v_k(\mathbf{x}) = 0$ for $\mathbf{x} \notin \omega_k$. We suppose that $\lim_{k\to+\infty} |\omega_k| = +\infty$, where $|\omega_k|$ is the Lebesgue measure, respectively the cardinality, of ω_k if $\Omega = \mathbb{R}^2$, respectively $\Omega = \mathbb{Z}^2$. Note that the following assumptions are well-defined for both continuous and discrete settings.

Assumption 3.1 (A3.1)**.** The function v is bounded on Ω.

The following assumption ensures the existence of spatial moments of any order for the function v.

Assumption 3.2 (A3.2)**.** For any $m, n \in \mathbb{N}$, there exist $\beta_m \in \mathbb{R}\backslash\{0\}$ and $\gamma_{m,n} \in \mathbb{R}^\Omega$ such that

(a) $\lim_{k\to+\infty} |\omega_k|^{1/2} \left(|\omega_k|^{-1} \int_{\omega_k} v_k^{2m}(\mathbf{x})\mathrm{d}\mathbf{x} - \beta_m\right) = 0$;

(b) for any $K \subset \Omega$ compact, $\lim_{k\to+\infty} \sup_{\mathbf{x}\in K} |\omega_k|^{-1} \int_{\mathbf{y}\in\omega_k} v_k^{2m}(\mathbf{y}) v_k^{2n}(\mathbf{x}+\mathbf{y})\mathrm{d}\mathbf{y} - \gamma_{m,n}(\mathbf{x})| = 0$.

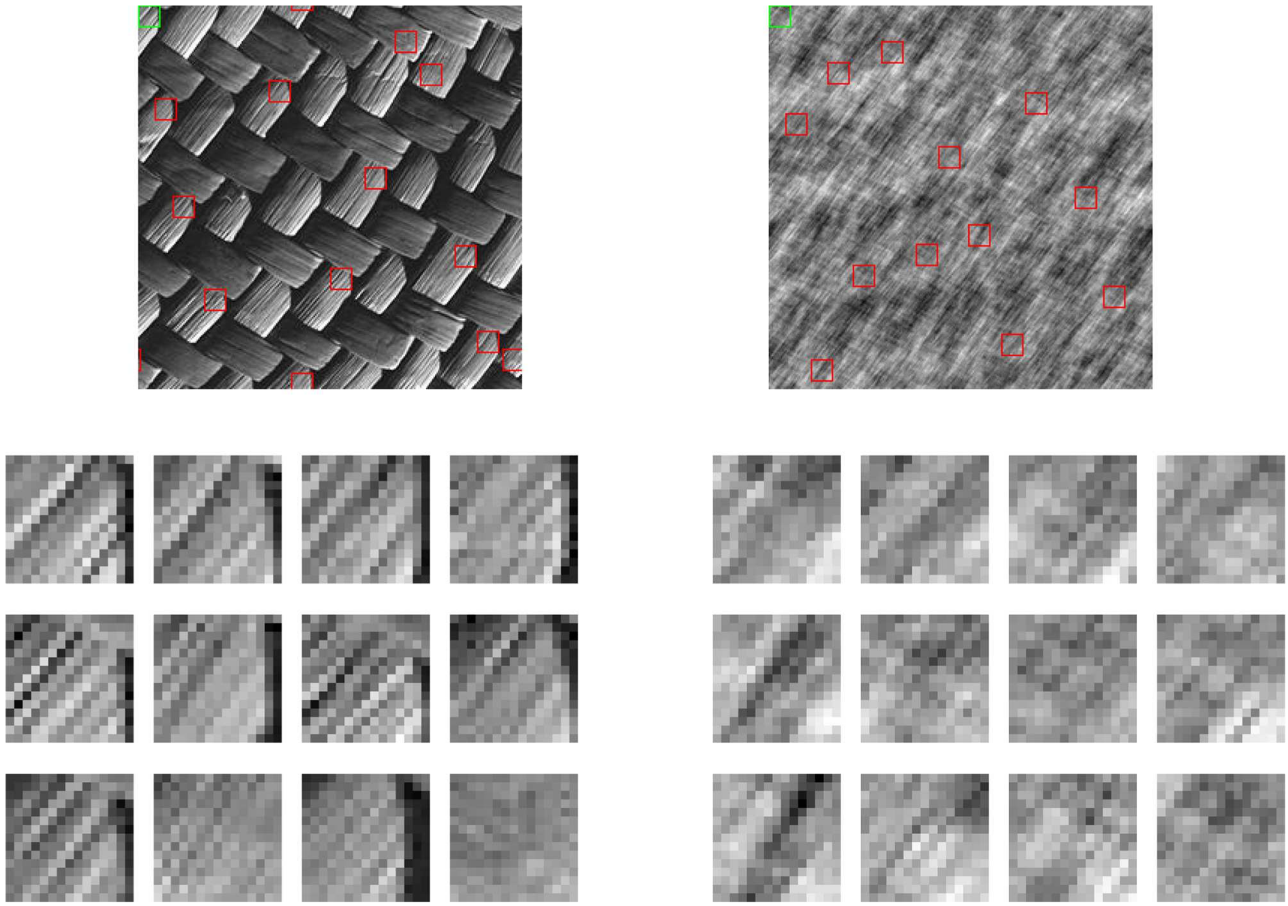


FIGURE 4. Natural images and Gaussian random fields. In this experiment we present the same image, f, which was used in Figure 1 and the associated Gaussian random field $U = f * W$, where the convolution is periodic and W is a Gaussian white noise. As in Figure 3 we present under each image the top-left patch (of size 15×15 and shown in green in the original images) and its 11 closest matches for the ℓ_2 similarity. We discard patches which are spatially too close (if ω_1 and ω_2 are two patch domains we impose $\sup_{\mathbf{x},\mathbf{y}} \|\mathbf{x}-\mathbf{y}\|_\infty \geqslant 10$). Note that if a structure is clearly identified in the real image (black and white diagonals) and is retrieved in every patch, it is not as clear in the Gaussian random field.

Note that in the case where Ω is discrete, the uniform convergence on compact sets introduced in (b) is equivalent to the pointwise convergence.

Assumption 3.3 (A3.3)**.** There exists $\gamma \in \mathbb{R}^\Omega$ with for any $K \subset \Omega$ compact, $\lim_{k\to+\infty} \sup_{\mathbf{x}\in K} |\omega_k|^{-1} \int_{\mathbf{y}\in\omega_k} v_k(\mathbf{y}) v_k(\mathbf{x}+\mathbf{y})\mathrm{d}\mathbf{y} - \gamma(\mathbf{x})| = 0$.

3.1. Discrete case

In the discrete case, we consider a random field U over $\mathbb{Z}^2$ and compute local similarity measurements. The asymptotic approximation is obtained when the patch size grows to infinity. In Theorems 3.4 and 3.6 we obtain Gaussian asymptotic probability distribution in the auto-similarity case and in the template similarity case. In

Propositions 3.5 and 3.7 we give explicit mean and variance for the Gaussian approximations. We recall that $\mathcal{N}(\mu, \sigma^2)$ is the probability distribution of a Gaussian real random variable with mean μ and variance σ^2.

Theorem 3.4 (Discrete case – asymptotic auto-similarity results)**.** *Let $(m_k)_{k\in\mathbb{N}}, (n_k)_{k\in\mathbb{N}}$ be two positive increasing integer sequences and $(\omega_k)_{k\in\mathbb{N}}$ be the sequence of subsets defined for any $k \in \mathbb{N}$ by, $\omega_k = [\![0, m_k]\!] \times [\![0, n_k]\!]$. Let $f \in \mathbb{R}^{\mathbb{Z}^2}$, $f \neq 0$ with finite support, W a Gaussian white noise over $\mathbb{Z}^2$ and $U = f * W$. For $i = \{p, (p,p), sc, \cos\}$ with $p \in (0, +\infty)$ there exist μ_i, σ_i^2, real valued functions on $\mathbb{Z}^2$, and $(\alpha_{i,k})_{k\in\mathbb{N}}$ a positive sequence such that for any $\mathbf{t} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ we get*

1. $\lim_{k\to+\infty} \frac{1}{\alpha_{i,k}} \mathcal{AS}_i(U, \mathbf{t}, \omega_k) \underset{a.s}{=} \mu_i(\mathbf{t})$;
2. $\lim_{k\to+\infty} |\omega_k|^{\frac{1}{2}} \left(\frac{1}{\alpha_{i,k}} \mathcal{AS}_i(U, \mathbf{t}, \omega_k) - \mu_i(\mathbf{t}) \right) \underset{\mathcal{L}}{=} \mathcal{N}\left(0, \sigma_i^2(\mathbf{t})\right)$.

The asymptotics derived in Theorem 3.4 can be extended to vectors of autosimilarities, *i.e.* selecting $(\mathbf{t}_j)_{j\in\{1\dots N\}}$ a finite number of shifts the results of Theorem 3.4 hold for the sequence $((\mathcal{AS}_i(U, \mathbf{t}_j, \omega_k))_{j\in\{1\dots N\}})_{k\in\mathbb{N}}$. Note that in Theorem 3.4 if $\mathbf{t}$ varies with k such that for any $k \in \mathbb{N}$, $(\omega_k + \mathbf{t}_k) \cap \omega_k = \varnothing$ then similar results can be obtained with the usual law of large numbers and central limit theorem since true independence hold.

Proof. The proof is divided into three parts. First we show 1 and 2 for $i = p, p$ and extends the result to $i = p$. Then we show 1 and 2 for $i = sc$. Finally, we show 1 and 2 for $i = \cos$.

1. Let $p \in (0, +\infty)$, $\mathbf{t} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ and define $V_{p,\mathbf{t}}$ for any $\mathbf{x} \in \mathbb{Z}^2$ by, $V_{p,\mathbf{t}}(\mathbf{x}) = |U(\mathbf{x}) - U(\mathbf{x} + \mathbf{t})|^p$. We remark that for any $k \in \mathbb{N}$ we have
$$\mathcal{AS}_{p,p}(U, \mathbf{t}, \omega_k) = \sum_{\mathbf{x}\in\omega_k} V_{p,\mathbf{t}}(\mathbf{x}).$$
We first notice that U is R-independent with $R > 0$, see Lemma B.5. Since for any $\mathbf{x} \in \mathbb{Z}^2$, $V_{p,\mathbf{t}}(\mathbf{x})$ depends only on $U(\mathbf{x})$ and $U(\mathbf{x} + \mathbf{t})$ we have that $V_{p,\mathbf{t}}$ is $R_{\mathbf{t}} = R + \|\mathbf{t}\|_\infty$-independent. Since U is stationary, so is $V_{p,\mathbf{t}}$. The random field $V_{p,\mathbf{t}}$ admits moments of every order since it is the pth power of the absolute value of a Gaussian random field. Thus $V_{p,\mathbf{t}}$ is a $R_{\mathbf{t}}$-independent second-order stationary random field. We can apply Lemma B.6 and we get
   (a) $\lim_{k\to+\infty} \frac{1}{|\omega_k|} \mathcal{AS}_{p,p}(U, \mathbf{t}, \omega_k) \underset{a.s.}{=} \mu_{p,p}(\mathbf{t})$;
   (b) $\lim_{k\to+\infty} |\omega_k|^{\frac{1}{2}} \left(\frac{1}{|\omega_k|} \mathcal{AS}_{p,p}(U, \mathbf{t}, \omega_k) - \mu_{p,p}(\mathbf{t}) \right) \underset{\mathcal{L}}{=} \mathcal{N}\left(0, \sigma_{p,p}^2(\mathbf{t})\right)$.
   with $\mu_{p,p}(\mathbf{t}) = \mathbb{E}[V_{p,\mathbf{t}}(\mathbf{0})]$ and $\sigma_{p,p}(\mathbf{t})^2 = \sum_{\mathbf{x}\in\mathbb{Z}^2} \mathrm{Cov}[V_{p,\mathbf{t}}(\mathbf{x}), V_{p,\mathbf{t}}(\mathbf{0})]$. By continuity of the p-th root over $[0, +\infty)$ we get 1 for $i = p$ with
$$\alpha_{p,k} = |\omega_k|^{1/p}, \qquad \mu_p(\mathbf{t}) = \mu_{p,p}(\mathbf{t})^{1/p}.$$
By Lemma B.7 we get that $\mathbb{E}\left[(U(\mathbf{0}) - U(\mathbf{t}))^2\right] = 2(\Gamma_f(\mathbf{0}) - \Gamma_f(\mathbf{t})) > 0$ thus $\mu_{p,p}(\mathbf{t}) = \mathbb{E}[V_{p,\mathbf{t}}(\mathbf{0})] > 0$. Since the pth root is continuously differentiable on $(0, +\infty)$ we can apply the Delta method, see [14], and we get 2 for $i = p$ with
$$\alpha_{p,k} = |\omega_k|^{1/p}, \qquad \mu_p(\mathbf{t}) = \mu_{p,p}(\mathbf{t})^{1/p}, \qquad \sigma_p(\mathbf{t})^2 = \frac{1}{p^2} \sigma_{p,p}(\mathbf{t})^2 \mu_{p,p}(\mathbf{t})^{2/p-2}. \tag{3.1}$$

2. We now prove the theorem for $i = sc$. Let $\mathbf{t} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ and define $V_{sc,\mathbf{t}}$ for any $\mathbf{x} \in \mathbb{Z}^2$, $V_{sc,\mathbf{t}}(\mathbf{x}) = -U(\mathbf{x})U(\mathbf{x} + \mathbf{t})$. We remark that for any $k \in \mathbb{N}$ we have
$$\mathcal{AS}_{sc}(U, \mathbf{t}, \omega_k) = \sum_{\mathbf{x}\in\omega_k} V_{sc,\mathbf{t}}(\mathbf{x}).$$

Since for any $\mathbf{x} \in \mathbb{Z}^2$, $V_{sc,\mathbf{t}}(\mathbf{x})$ depends only on $U(\mathbf{x})$ and $U(\mathbf{x}+\mathbf{t})$, we have that $V_{sc,\mathbf{t}}$ is $R_{\mathbf{t}} = R + \|\mathbf{t}\|_\infty$-independent. Since U is stationary, so is $V_{sc,\mathbf{t}}$. The random field $V_{sc,\mathbf{t}}$ admits moments of every order since it is a product of Gaussian random fields. Thus $V_{sc,\mathbf{t}}$ is a $R_{\mathbf{t}}$-independent second-order stationary random field. We can again apply Lemma B.6 and we get

(a) $\lim_{k \to +\infty} \frac{1}{|\omega_k|} \mathcal{AS}_{sc}(U, \mathbf{t}, \omega_k) \underset{a.s.}{=} \mu_{sc}(\mathbf{t})$;

(b) $\lim_{k \to +\infty} |\omega_k|^{\frac{1}{2}} \left(\frac{1}{|\omega_k|} \mathcal{AS}_{sc}(U, \mathbf{t}, \omega_k) - \mu_{sc}(\mathbf{t}) \right) \underset{\mathcal{L}}{=} \mathcal{N}\left(0, \sigma_{sc}^2(\mathbf{t})\right)$,

with $\mu_{sc}(\mathbf{t}) = \mathbb{E}\left[V_{sc,\mathbf{t}}(\mathbf{0})\right]$ and $\sigma_{sc}(\mathbf{t})^2 = \sum_{\mathbf{x} \in \mathbb{Z}^2} \mathrm{Cov}\left[V_{sc,\mathbf{t}}(\mathbf{x}), V_{sc,\mathbf{t}}(\mathbf{0})\right]$, which concludes the proof.

3. Finally, we consider the case $i = \cos$. Let $\mathbf{t} \in \mathbb{Z}^2 \backslash \{\mathbf{0}\}$ and define $V_{\cos,\mathbf{t}}$ for any $\mathbf{x} \in \mathbb{Z}^2$,

$$V_{\cos,\mathbf{t}}(\mathbf{x}) = \begin{pmatrix} -U(\mathbf{x})U(\mathbf{x}+\mathbf{t}) \\ U(\mathbf{x})^2 \\ U(\mathbf{x}+\mathbf{t})^2 \end{pmatrix}. \tag{3.2}$$

We remark that for any $k \in \mathbb{N}$ we have

$$\mathcal{AS}_{s_{\cos}}(U, \mathbf{t}, \omega_k) = h\left(|\omega_k|^{-1} \sum_{\mathbf{x} \in \omega_k} V_{\cos,\mathbf{t}}(\mathbf{x}) \right), \tag{3.3}$$

with $h(x, y, z) = xy^{-1/2}z^{-1/2}$. Since U is stationary, so is $V_{\cos,\mathbf{t}}$. The random field $V_{\cos,\mathbf{t}}$ admits moments of every order since it is a vector of products of Gaussian random fields. Thus $V_{\cos,\mathbf{t}}$ is a $R_{\mathbf{t}}$-independent second-order stationary random field. We can apply Lemma B.6 and there exist $\tilde{\mu}_{\cos}(\mathbf{t})$ and $\tilde{C}_{\cos}(\mathbf{t})$ such that

(a) $\lim_{k \to +\infty} \frac{1}{|\omega_k|} V_{\cos,\mathbf{t}} \underset{a.s.}{=} \tilde{\mu}_{\cos}(\mathbf{t})$;

(b) $\lim_{k \to +\infty} |\omega_k|^{\frac{1}{2}} \left(\frac{1}{|\omega_k|} V_{\cos,\mathbf{t}} - \tilde{\mu}_{\cos}(\mathbf{t}) \right) \underset{\mathcal{L}}{=} \mathcal{N}\left(0, \tilde{C}_{\cos}(\mathbf{t})\right)$.

We conclude the proof using the multivariate Delta method, [14].

□

In the following proposition we give explicit values for the constants involved in the law of large numbers and the central limit theorem derived in Theorem 3.4. We introduce the following quantities for $k, \ell \in \mathbb{N}$ and $j \in [\![0, k \wedge \ell]\!]$, where $k \wedge \ell = \min(k, \ell)$,

$$q_\ell = \frac{(2\ell)!}{\ell!\, 2^\ell}, \qquad r_{j,k,\ell} = q_{k-j} q_{\ell-j} \binom{2k}{2j} \binom{2\ell}{2j} (2j)!\,. \tag{3.4}$$

We also denote $r_{j,\ell} = r_{j,\ell,\ell}$. Note that for all $\ell \in \mathbb{N}$, $r_{0,\ell} = q_\ell^2$ and $\sum_{j=0}^{\ell} r_{j,\ell} = q_{2\ell}$. We also introduce the following functions:

$$\Delta_f(\mathbf{t}, \mathbf{x}) = 2\Gamma_f(\mathbf{x}) - \Gamma_f(\mathbf{x}+\mathbf{t}) - \Gamma_f(\mathbf{x}-\mathbf{t})\,, \quad \widetilde{\Delta}_f(\mathbf{t}, \mathbf{x}) = \Gamma_f(\mathbf{x})^2 + \Gamma_f(\mathbf{x}+\mathbf{t})\Gamma_f(\mathbf{x}-\mathbf{t}). \tag{3.5}$$

Note that Δ_f is a second-order statistic on the Gaussian field $U = f * W$ with W a Gaussian white noise over $\mathbb{Z}^2$, whereas $\widetilde{\Delta}_f$ is a fourth-order statistic on the same random field.

Proposition 3.5 (Explicit constants – Auto-similarity)**.** *In Theorem 3.4 we have the following constants for any* $\mathbf{t} \in \mathbb{Z}^2 \backslash \{\mathbf{0}\}$.

(i) If $i = p$ with $p = 2\ell$ and $\ell \in \mathbb{N}$, then for all $k \in \mathbb{N}$, we get that $\alpha_{p,k} = |\omega_k|^{1/(2\ell)}$ and

$$\mu_p(\mathbf{t}) = q_\ell^{1/(2\ell)} \Delta_f(\mathbf{t},\mathbf{0})^{1/2} \quad \text{and} \quad \sigma_p(\mathbf{t})^2 = \frac{q_\ell^{1/\ell-2}}{(2\ell)^2}\sum_{j=1}^{\ell} r_{j,\ell}\left(\frac{\|\Delta_f(\mathbf{t},\cdot)\|_{2j}}{\Delta_f(\mathbf{t},\mathbf{0})}\right)^{2j}\Delta_f(\mathbf{t},\mathbf{0}),$$

where $(r_{i,jk})_{i,j,k\in\mathbb{N}}$ and $(q_k)_{k\in\mathbb{N}}$ are given in (3.4).

(ii) If $i = sc$, then for all $k \in \mathbb{N}$, we get that $\alpha_{sc,k} = |\omega_k|$ and

$$\mu_{sc}(\mathbf{t}) = \Gamma_f(\mathbf{t}) \quad \text{and} \quad \sigma_{sc}(\mathbf{t})^2 = \sum_{\mathbf{x}\in\mathbb{Z}^2} \widetilde{\Delta}_f(\mathbf{t},\mathbf{x}).$$

(iii) if $i = \cos$, then for all $k \in \mathbb{N}$, we get that $\alpha_{\cos,k} = 1$ and

$$\mu_{\cos}(\mathbf{t}) = \Gamma_f(\mathbf{t})/\Gamma_f(\mathbf{0}) \;\; \text{and} \;\; \sigma_{\cos}(\mathbf{t})^2 = \Gamma_f(\mathbf{0})^{-2}\left\{\|\Gamma_f\|_2^2\left(1+2\frac{\Gamma_f(\mathbf{t})^2}{\Gamma_f(\mathbf{0})^2}\right) - 4\frac{\Gamma_f(\mathbf{t})}{\Gamma_f(\mathbf{0})}\Gamma_f * \check{\Gamma}_f(\mathbf{t}) + \Gamma_f * \check{\Gamma}_f(2\mathbf{t})\right\}.$$

Proof. The proof is postponed to Appendix D. □

For example we have

$$\begin{aligned}
&\mu_2(\mathbf{t}) = \Delta_f(\mathbf{t},\mathbf{0})^{1/2}, && \mu_4(\mathbf{t}) = 3^{1/4}\Delta_f(\mathbf{t},\mathbf{0})^{1/2}\\
&\sigma_2^2(\mathbf{t}) = \frac{1}{2}\frac{\|\Delta_f(\mathbf{t},\cdot)\|_2^2}{\Delta_f(\mathbf{t},\mathbf{0})}, && \sigma_4^2(\mathbf{t}) = 2\sqrt{3}\frac{\|\Delta_f(\mathbf{t},\cdot)\|_2^2}{\Delta_f(\mathbf{t},\mathbf{0})} + \frac{\sqrt{3}}{6}\frac{\|\Delta_f(\mathbf{t},\cdot)\|_4^4}{\Delta_f(\mathbf{t},\mathbf{0})^3}.
\end{aligned} \tag{3.6}$$

We now derive similar asymptotic properties in the template similarity case.

Theorem 3.6 (Discrete case – asymptotic template similarity results)**.** *Let $(m_k)_{k\in\mathbb{N}}, (n_k)_{k\in\mathbb{N}}$ be two positive increasing integer sequences and $(\omega_k)_{k\in\mathbb{N}}$ be the sequence of subsets defined for any $k \in \mathbb{N}$, $\omega_k = [\![0, m_k]\!] \times [\![0, n_k]\!]$. Let $f \in \mathbb{R}^{\mathbb{Z}^2}$, $f \neq 0$ with finite support, W a Gaussian white noise over $\mathbb{Z}^2$, $U = f * W$ and let v, a real valued function on $\mathbb{Z}^2$. For $i = \{p, (p,p), sc, \cos\}$ with $p = 2\ell$ and $\ell \in \mathbb{N}$, if $i = p$ or (p,p) assume (A3.1) and (A3.2), if $i = sc$ assume (A3.1) and (A3.3) and if $i = \cos$ assume (A3.1), (A3.2) and (A3.3). Then there exist $\mu_i, \sigma_i^2 \in \mathbb{R}$ and $(\alpha_{i,k})_{k\in\mathbb{N}}$ a positive sequence such that for any $\mathbf{t} \in \mathbb{Z}^2$ we get*

1. $\lim_{k\to+\infty} \frac{1}{\alpha_{i,k}} \mathcal{TS}_i(U, v, \mathbf{t}, \omega_k) \underset{a.s}{=} \mu_i$;
2. $\lim_{k\to+\infty} |\omega_k|^{\frac{1}{2}} \left(\frac{1}{\alpha_{i,k}}\mathcal{TS}_i(U, v, \mathbf{t}, \omega_k) - \mu_i(\mathbf{t})\right) \underset{\mathcal{L}}{=} \mathcal{N}\left(0, \sigma_i^2\right)$.

Note that contrarily to Theorem 3.4 we could not obtain such a result for all $p \in (0, +\infty)$ but only for even integers. Indeed, in the general case the convergence of the sequence $\left(|\omega_k|^{-1}\mathbb{E}\left[\mathcal{TS}_{p,p}(U, v, \mathbf{t}, \omega_k)\right]\right)_{k\in\mathbb{N}}$, which is needed in order to apply Theorem B.3, is not trivial. Assuming that v is bounded it is easy to show that $\left(|\omega_k|^{-1}\mathbb{E}\left[\mathcal{TS}_{p,p}(U, v, \mathbf{t}, \omega_k)\right]\right)_{k\in\mathbb{N}}$ is also bounded and we can deduce the existence of a convergent subsequence. In the general case, for Theorem 3.6 to hold with any $p \in (0, +\infty)$, we must verify that for any $\mathbf{t} \in \Omega$, there exist $\mu_{p,p}(\mathbf{t}) > 0$ and $\sigma_{p,p}^2(\mathbf{t}) \geqslant 0$ such that

(a) $\lim_{k\to+\infty} |\omega_k|^{\frac{1}{2}} \left(\frac{1}{|\omega_k|}\mathbb{E}\left[\mathcal{TS}_{p,p}(U, v, \mathbf{t}, \omega_k)\right] - \mu_{p,p}(\mathbf{t})\right) = 0$;

(b) $\lim_{k\to+\infty} \frac{1}{|\omega_k|} \operatorname{Var}\left[\mathcal{TS}_{p,p}(U, v, \mathbf{t}, \omega_k)\right] = \sigma_{p,p}^2(\mathbf{t})$.

We now turn to the proof of Theorem 3.6.

Proof. As for the proof of Theorem 3.4, the proof is divided into three parts. First we show 1 and 2 for $i = (p,p)$ and extends the result to $i = p$. Then we show 1 and 2 for $i = sc$. Finally, we show 1 and 2 for $i = \cos$.

1. Let $p \in (0, +\infty)$, $\mathbf{t} \in \mathbb{Z}^2$ and define $V_{p,\mathbf{t}}$ the random field on $\mathbb{Z}^2$ for any $\mathbf{x} \in \mathbb{Z}^2$, by $V_{p,\mathbf{t}}(\mathbf{x}) = |v(\mathbf{x}) - U(\mathbf{x}+\mathbf{t})|^p$. We remark that for any $k \in \mathbb{N}$ we have

$$\mathcal{TS}_{p,p}(U, v, \mathbf{t}, \omega_k) = \sum_{\mathbf{x} \in \omega_k} V_{p,\mathbf{t}}(\mathbf{x}).$$

By Lemma B.5, U is R-independent with $R > 0$. Since for any $\mathbf{x} \in \mathbb{Z}^2$ we have that $V_{p,\mathbf{t}}(\mathbf{x})$ depends only on $U(\mathbf{x}+\mathbf{t})$ we also have that $V_{p,\mathbf{t}}$ is R-independent. We define the random field $V_{p,\mathbf{t}}^{\infty}$ for any $\mathbf{x} \in \mathbb{Z}^2$, $V_{p,\mathbf{t}}^{\infty}(\mathbf{x}) = (\sup_{\mathbb{Z}^2} |v| + U(\mathbf{x}+\mathbf{t}))^p$. We have that $V_{p,\mathbf{t}}^{\infty}(\mathbf{x}) + \mathbb{E}\left[V_{p,\mathbf{t}}^{\infty}(\mathbf{0})\right]$ uniformly almost surely dominates $V_{p,\mathbf{t}}(\mathbf{x}) - \mathbb{E}\left[V_{p,\mathbf{t}}(\mathbf{x})\right]$. The random field $V_{p,\mathbf{t}}^{\infty}$ admits moments of every order since it is the pth power of the absolute value of a Gaussian random field and is stationary because U is. Thus $V_{p,\mathbf{t}}$ is a $R_{\mathbf{t}}$-independent random field and $V_{p,\mathbf{t}}(\mathbf{x}) - \mathbb{E}\left[V_{p,\mathbf{t}}(\mathbf{x})\right]$ is uniformly stochastically dominated by $V_{p,\mathbf{t}}^{\infty}(\mathbf{x}) + \mathbb{E}\left[V_{p,\mathbf{t}}^{\infty}(\mathbf{0})\right]$, a second-order stationary random field. Using (A3.2) and Lemma B.8, we can apply Theorem B.3 and B.4 and we get

(a) $\lim_{k \to +\infty} \frac{1}{|\omega_k|} \mathcal{TS}_{p,p}(U, v, \mathbf{t}, \omega_k) \underset{a.s.}{=} \mu_{p,p}(\mathbf{t})$;

(b) $\lim_{k \to +\infty} |\omega_k|^{\frac{1}{2}} \left(\frac{1}{|\omega_k|} \mathcal{TS}_{p,p}(U, v, \mathbf{t}, \omega_k) - \mu_{p,p}(\mathbf{t}) \right) \underset{\mathcal{L}}{=} \mathcal{N}\left(0, \sigma_{p,p}^2(\mathbf{t})\right)$.

Note that since U is stationary we have for any $\mathbf{t} \in \mathbb{Z}^2$, $\mu_{p,p} = \mu_{p,p}(\mathbf{0}) = \mu_{p,p}(\mathbf{t})$ and $\sigma_{p,p}^2 = \sigma_{p,p}^2(\mathbf{0}) = \sigma_{p,p}^2(\mathbf{t})$. By continuity of the pth root over $[0, +\infty)$ we get 1 for $i = p$ with

$$\alpha_{p,k} = |\omega_k|^{1/p}, \qquad \mu_p = \mu_{p,p}^{1/p}.$$

By Lemma B.8, we have that $\mu_{p,p} > 0$. Since the pth root is continuously differentiable on $(0, +\infty)$ we can apply the Delta method and we get 2 for $i = p$ with

$$\alpha_{p,k} = |\omega_k|^{1/p}, \qquad \mu_p = \mu_{p,p}^{1/p}, \qquad \sigma_p^2 = \sigma_{p,p}^2 \mu_{p,p}^{2/p-2} / p^2 . \tag{3.7}$$

2. We now prove the theorem for $i = sc$. Let $\mathbf{t} \in \mathbb{Z}^2$ and define $V_{sc,\mathbf{t}}$ the random field on $\mathbb{Z}^2$ such that for any $\mathbf{x} \in \mathbb{Z}^2$, $V_{sc,\mathbf{t}}(\mathbf{x}) = -v(\mathbf{x})U(\mathbf{x}+\mathbf{t})$. We remark that for any $k \in \mathbb{N}$ we have

$$\mathcal{TS}_{sc}(U, v, \mathbf{t}, \omega_k) = \sum_{\mathbf{x} \in \omega_k} V_{sc,\mathbf{t}}(\mathbf{x}).$$

It is clear that for any $k \in \mathbb{N}$, $\mathcal{TS}_{sc}(U, v, \mathbf{t}, \omega_k)$ is a R-independent Gaussian random variable with $\mathbb{E}\left[\mathcal{TS}_{sc}(U, v, \mathbf{t}, \omega_k)\right] = 0$ and

$$\operatorname{Var}\left[\mathcal{TS}_{sc}(U, v, \mathbf{t}, \omega_k)\right] = \sum_{\mathbf{x},\mathbf{y} \in \omega_k} \mathbb{E}\left[V_{sc,\mathbf{t}}(\mathbf{x}) V_{sc,\mathbf{t}}(\mathbf{y})\right] = \sum_{\mathbf{x},\mathbf{y} \in \omega_k} v(\mathbf{x}) v(\mathbf{y}) \Gamma_f(\mathbf{x}-\mathbf{y}) = \sum_{\mathbf{x} \in \mathbb{Z}^2} \Gamma_f(\mathbf{x}) v_k * \check{v}_k(\mathbf{x}) ,$$

where we recall that v_k is the restriction of v to ω_k. The last sum is finite since $\operatorname{Supp}(f)$ finite implies that $\operatorname{Supp}(\Gamma_f)$ is finite. Using (A3.3) we obtain that for any $k \in \mathbb{N}$,

$$\sum_{\mathbf{x} \in \omega_k} \left(\mathbb{E}\left[V_{sc,\mathbf{t}}\right](\mathbf{x}) - \mu_{sc}\right) = 0 , \qquad \lim_{k \to +\infty} |\omega_k|^{-1} \sum_{\mathbf{x},\mathbf{y} \in \omega_k} \operatorname{Cov}\left[V_{sc,\mathbf{t}}(\mathbf{x}), V_{sc,\mathbf{t}}(\mathbf{y})\right] = \sigma_{sc}^2 , \tag{3.8}$$

with $\mu_{sc} = 0$ and $\sigma_{sc}^2 = \sum_{\mathbf{x} \in \mathbb{Z}^2} \Gamma_f(\mathbf{x}) \gamma(\mathbf{x})$, where γ is given in (A3.3). Since $V_{sc,\mathbf{t}}$ is a R-independent second-order random field using (3.8) we can apply Theorems B.3 and B.4 to conclude.

3. We now consider the case $i = \cos$. First, notice that

$$\mathcal{TS}_{\cos}(U, v, \mathbf{t}, \omega_k) = \frac{|\omega_k|^{-1} \mathcal{TS}_{sc}(U, v, \mathbf{t}, \omega_k)}{\left(|\omega_k|^{-1} \sum_{\mathbf{x} \in \omega_k} v(\mathbf{x})^2\right)^{1/2} \left(|\omega_k|^{-1} \sum_{\mathbf{x} \in \omega_k} U(\mathbf{x})^2\right)^{1/2}}. \tag{3.9}$$

Using that $\lim_{k\to+\infty}|\omega_k|^{-1}\mathcal{TS}_{sc}(U,v,\mathbf{t},\omega_k)=0$, $\lim_{k\to+\infty}|\omega_k|^{-1}\sum_{\mathbf{x}\in\omega_k}U(\mathbf{x})^2=\Gamma_f(\mathbf{0})$ by Lemma B.6 and $\lim_{k\to+\infty}|\omega_k|^{-1}\sum_{\mathbf{x}\in\omega_k}v(\mathbf{x})^2=v_1\neq 0$ by (A3.2), we get that

$$\lim_{k\to+\infty}\mathcal{TS}_{\cos}(U,v,\mathbf{t},\omega_k)=0.$$

In addition, using Slutsky's theorem and the fact that $\lim_{k\to+\infty}|\omega_k|^{-1/2}\mathcal{TS}_{sc}(U,v,\mathbf{t},\omega_k)=\mathcal{N}(0,\sigma_{sc}^2)$ we obtain that $\lim_{k\to+\infty}|\omega_k|^{-1/2}\mathcal{TS}_{cos}(U,v,\mathbf{t},\omega_k)=\mathcal{N}(0,\sigma_{cos}^2)$ with

$$\sigma_{cos}^2=\frac{\langle\gamma,\Gamma_f\rangle}{v_1\Gamma_f(\mathbf{0})}.$$

□

Proposition 3.7 (Explicit constants – template similarity)**.** *In Theorem 3.6 we have the following constants for any* $\mathbf{t}\in\mathbb{Z}^2$.

(i) If $i=p$ *with* $p=2\ell$ *and* $\ell\in\mathbb{N}$*, then we get that* $\alpha_{p,k}=|\omega_k|^{\frac{1}{p}}$*, and*

$$\mu_p=\left(\sum_{j=0}^{\ell}\binom{2\ell}{2j}q_{\ell-j}\Gamma_f(\mathbf{0})^{-j}\beta_j\right)^{1/2\ell}\Gamma_f(\mathbf{0})^{1/2},$$

$$\sigma_p^2=\left(\sum_{i,j=0}^{\ell}\binom{2\ell}{2i}\binom{2\ell}{2j}\sum_{m=1}^{\ell-i\wedge\ell-j}r_{m,\ell-i,\ell-j}\Gamma_f(\mathbf{0})^{-(i+j+2m)}\left\langle\Gamma_f^{2m},\gamma_{i,j}\right\rangle\right)\left(\sum_{j=0}^{\ell}\binom{2\ell}{2j}q_{\ell-j}\Gamma_f(\mathbf{0})^{-j}\beta_j\right)^{1/\ell-2}\frac{\Gamma_f(\mathbf{0})}{(2\ell)^2},$$

where $(\beta_j)_{j\in\mathbb{N}}$, $(\gamma_{i,j})_{i,j\in\mathbb{N}}$ *are given in (A3.2) and* $(r_{i,jk})_{i,j,k\in\mathbb{N}}$ *and* $(q_k)_{k\in\mathbb{N}}$ *are given in* (3.4).

(ii) If $i=sc$ *then for all* $k\in\mathbb{N}$*, we get that* $\alpha_{sc,k}=|\omega_k|$ *and*

$$\mu_{sc}=0,\quad \sigma_{sc}^2=\langle\gamma,\Gamma_f\rangle.$$

(iii) If $i=s_{\cos}$ *then for all* $k\in\mathbb{N}$*, we get that* $\alpha_{s_{\cos},k}=1$ *and*

$$\mu_{s_{\cos}}=0,\quad \sigma_{s_{\cos}}^2=\frac{\langle\gamma,\Gamma_f\rangle}{v_1\Gamma_f(\mathbf{0})}.$$

Proof. The proof is postponed to Appendix D. □

For example we have

$$\begin{aligned}
\mu_2&=(2\Gamma_f(\mathbf{0})+\beta_1)^{1/2},\quad \mu_4=(3\Gamma_f(\mathbf{0})^2+12\Gamma_f(\mathbf{0})^3\beta_1+\beta_2)^{1/4},\\
\sigma_2^2&=\frac{1}{4}\frac{\|\Gamma_f\|_2^2}{\Gamma_f(\mathbf{0})}(2+\Gamma_f(\mathbf{0})^{-1}\beta_1)^{-1},\\
\sigma_4^2&=\frac{1}{16}\left(288\Gamma_f(\mathbf{0})^{-1}\|\Gamma_f\|_2^2+144\Gamma_f(\mathbf{0})^{-2}\langle\Gamma_f^2,\gamma_{0,1}\rangle+24\Gamma_f(\mathbf{0})^{-3}\|\Gamma_f\|_4^4+\Gamma_f(\mathbf{0})^{-3}\langle\Gamma_f^2,\gamma\rangle\right)\\
&\quad\times\left(3+12\Gamma_f(\mathbf{0})^{-1}\beta_1+\Gamma_f(\mathbf{0})^{-2}\beta_2\right)^{-3/2}.
\end{aligned}\tag{3.10}$$

Note that the limit mean and standard deviation do not depend on the offset anymore. Indeed, template similarity functions are stationary in $\mathbf{t}$. If v has finite support then (A3.2) holds with $\beta_i=0$ and $\gamma_{i,j}=0$ as soon as $i \neq 0$ or $j \neq 0$. Remarking that $\beta_0 = 1$ and $\gamma_{0,0} = 1$ we obtain that

$$\mu_p = q_\ell^{1/(2\ell)} \Gamma_f(\mathbf{0})^{1/2}\,, \quad \sigma_p^2 = \frac{q_\ell^{1/\ell-2}}{(2\ell)^2} \sum_{j=1}^{\ell} r_{j,\ell} \left(\frac{\|\Gamma_f\|_{2j}}{\Gamma_f(\mathbf{0})} \right)^{2j} \Gamma_f(\mathbf{0}).$$

3.2. Continuous case

We now turn to the continuous setting. Theorem 3.8, respectively Theorem 3.10, is the continuous counterpart of Theorem 3.4, respectively Theorem 3.6.

Theorem 3.8 (Continuous case – asymptotic auto-similarity results)**.** *Let $(m_k)_{k\in\mathbb{N}}, (n_k)_{k\in\mathbb{N}}$ be two positive increasing integer sequences and $(\omega_k)_{k\in\mathbb{N}}$ be the sequence of subsets defined for any $k \in \mathbb{N}$ by, $\omega_k = [0, m_k] \times [0, n_k]$. Let U be a zero-mean Gaussian random field over $\mathbb{R}^2$ with covariance function Γ. Assume (A2.2) and that Γ has finite support. For $i \in \{p, (p,p), sc, \cos\}$ with $p \in (0, +\infty)$ there exist μ_i, σ_i^2, real valued functions on $\mathbb{R}^2$, and $(\alpha_{i,k})_{k\in\mathbb{N}}$ a positive sequence such that for any $\mathbf{t} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ we get*

1. $\lim_{k\to+\infty} \frac{1}{\alpha_{i,k}} \mathcal{AS}_i(U, \mathbf{t}, \omega_k) \underset{a.s}{=} \mu_i(\mathbf{t})$;
2. $\lim_{k\to+\infty} |\omega_k|^{\frac{1}{2}} \left(\frac{1}{\alpha_{i,k}} \mathcal{AS}_i(U, \mathbf{t}, \omega_k) - \mu_i(\mathbf{t}) \right) \underset{\mathcal{L}}{=} \mathcal{N}\left(0, \sigma_i^2(\mathbf{t})\right)$.

Proof. The proof is the same as the one of Theorem 3.4 replacing Lemmas B.6 and B.7 by Lemma C.3 and Lemma C.4. □

Proposition 3.9 (Explicit constants – Continuous auto-similarity)**.** *Constants given in Proposition 3.5 apply to Theorem 3.8 provided that Γ_f is replaced by Γ in* (3.5)*.*

Proof. The proof is the same as the one of Proposition 3.5. □

Theorem 3.10 (Continuous case – asymptotic template similarity results)**.** *Let $(m_k)_{k\in\mathbb{N}}, (n_k)_{k\in\mathbb{N}}$ be two positive increasing integer sequences and $(\omega_k)_{k\in\mathbb{N}}$ be the sequence of subsets defined for any $k \in \mathbb{N}$ by, $\omega_k = [0, m_k] \times [0, n_k]$. Let U be a zero-mean Gaussian random field over $\mathbb{R}^2$ with covariance function Γ. Assume (A2.2) and that Γ has finite support. For $i \in \{p, (p,p), sc, \cos\}$ with $p \in (0, +\infty)$, if $i = p$ or (p,p) assume (A3.1) and (A3.2), if $i = sc$ assume (A3.1) and (A3.3) and if $i = \cos$ assume (A3.1), (A3.2) and (A3.3). Then there exist $\mu_i, \sigma_i^2 \in \mathbb{R}$ and $(\alpha_{i,k})_{k\in\mathbb{N}}$ a positive sequence such that for any $\mathbf{t} \in \mathbb{R}^2$ we get*

1. $\lim_{k\to+\infty} \frac{1}{\alpha_{i,k}} \mathcal{TS}_i(U, v, \mathbf{t}, \omega_k) \underset{a.s.}{=} \mu_i$;
2. $\lim_{k\to+\infty} |\omega_k|^{\frac{1}{2}} \left(\frac{1}{\alpha_{i,k}} \mathcal{TS}_i(U, v, \mathbf{t}, \omega_k) - \mu_i(\mathbf{t}) \right) \underset{\mathcal{L}}{=} \mathcal{N}\left(0, \sigma_i^2\right)$.

Proof. The proof is the same as the one of Theorem 3.6. □

Proposition 3.11 (Explicit constants – Continuous auto-similarity)**.** *Constants given in Proposition 3.7 apply to Theorem 3.10 provided that Γ_f is replaced by Γ in* (3.5)*.*

Proof. The proof is similar to the one of Proposition 3.7. □

3.3. Speed of convergence

In the discrete setting, Theorem 3.4 justifies the use of a Gaussian approximation to compute $\mathcal{AS}_i(U, \mathbf{t}, \omega)$. However this asymptotic behavior strongly relies on the increasing size of the patch domains. We define the patch size to be $|\omega|$, the cardinality of ω, and the spot size $|\mathrm{Supp}\,(f)\,|$ to be the cardinality of the support of the spot f. The quantity of interest is the ratio $r = \frac{\text{patch size}}{\text{spot size}}$. If $r \gg 1$ then the Gaussian random field associated to f can be well approximated by a Gaussian white noise from the patch perspective. If $r \approx 1$ this approximation is not valid and the Gaussian approximation is no longer accurate, see Figure 5. We say that an offset $\mathbf{t}$ is *detected* in a Gaussian random field if $\mathcal{AS}_i(U, \mathbf{t}, \omega) \leqslant a(\mathbf{t})$ for some threshold $a(\mathbf{t})$. In the experiments presented in Figure 6

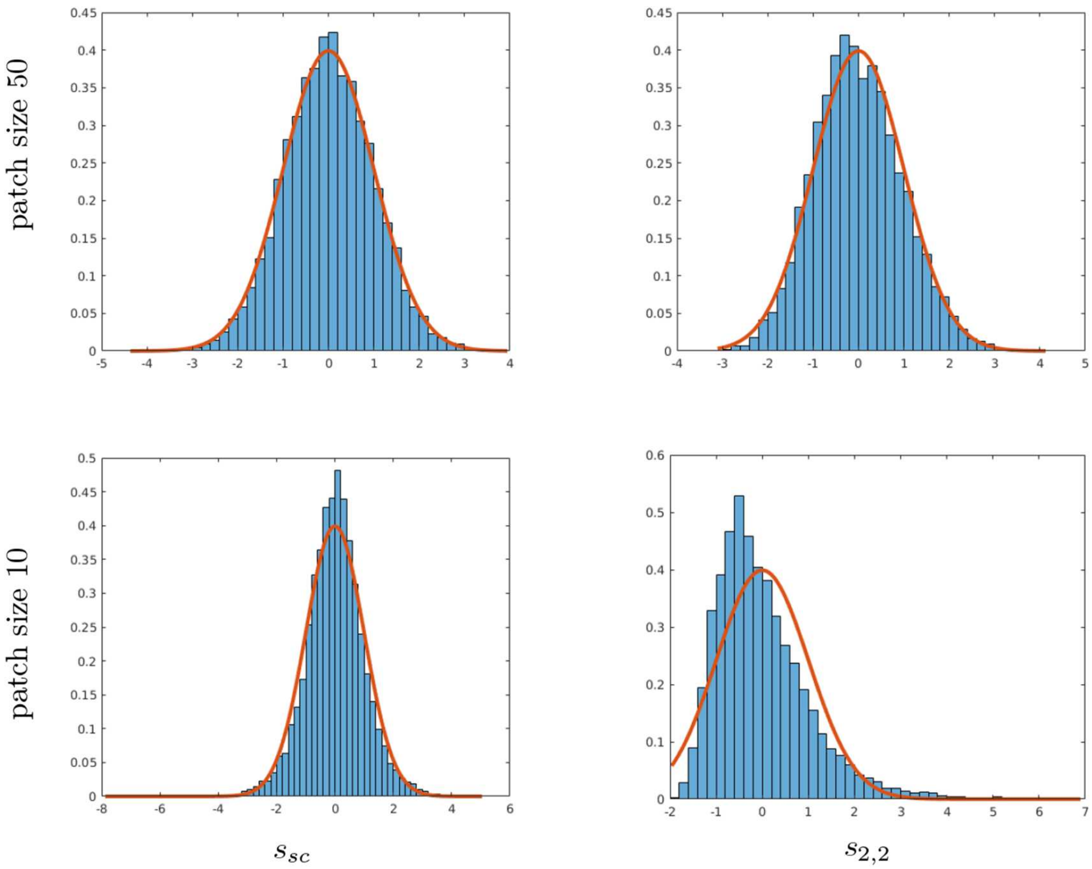


FIGURE 5. Gaussian moment matching. In this experiment, 10^4 samples of 128×128 Gaussian images are computed with a spot of size 5×5 (the spot is the indicator of this square). Scalar product auto-similarities and squared L^2 auto-similarities are computed for a fixed offset $(70, 100)$. We then plot the normalized histogram of these values. The red curve corresponds to the standard Gaussian $\mathcal{N}(0,1)$. On the top row $r = 100 \gg 1$ and the Gaussian approximation is valid. On the bottom row $r \approx 1$ and the Gaussian approximation is not valid.

and Table 1 the threshold is given by the asymptotic Gaussian inverse cumulative distribution function evaluated at some quantile. The parameters of the Gaussian random variable are given by Proposition 3.5. We find that except for small spot sizes and large patches, *i.e.* $r \gg 1$, the approximation is not valid. More precisely, let $U = f * W$ with f a finitely supported function over $\mathbb{Z}^2$ and W a Gaussian white noise over $\mathbb{Z}^2$. Let $\omega \subset \mathbb{Z}^2$ and let Ω_0 be a finite subset of $\mathbb{Z}^2$. We compute $\sum_{\mathbf{t} \in \Omega_0} \mathbb{1}_{\mathcal{AS}_i(U,\mathbf{t},\omega) \leqslant a(\mathbf{t})}$, with $a(\mathbf{t})$ defined by the inverse cumulative distribution function of quantile $10/|\Omega_0|$ for the Gaussian $\mathcal{N}(\mu, \sigma^2)$ where μ, σ^2 are given by Theorem 3.4 and Proposition 3.5. Note that $a(\mathbf{t})$ would satisfy $\mathbb{P}\left[\mathcal{AS}_i(U,\mathbf{t},\omega) \leqslant a(\mathbf{t})\right] \approx 10/|\Omega_0|$ if the approximation for the cumulative distribution function was correct. In other words, if the Gaussian asymptotic was always valid, we would have a number of detections equal to 10 independently of r. This is clearly not the case in Table 1. One way to interpret this is by looking at the left tail of the approximated distribution for $s_{2,2}$ and s_{sc} on Figure 5. For s_{sc} the histogram is above the estimated curve, see (a) in Figure 6 for example. Whereas for $s_{2,2}$ the histogram is under the estimated curve. Thus for s_{sc} we expect to obtain more detections than what is predicted whereas we will observe the opposite behavior for $s_{2,2}$. This situation is also illustrated for similarities

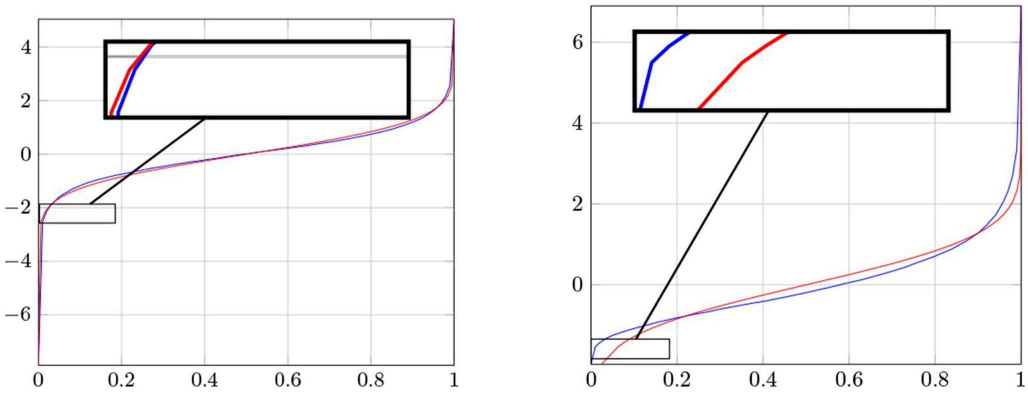


FIGURE 6. Theoretical and empirical cumulative distribution Function. This experiment illustrates the non-Gaussianity in Figure 5. In both cases, the red curve is the inverse cumulative distribution function of the standard Gaussian and the blue curve is the empirical inverse cumulative distribution function of normalized auto-similarity functions computed with 10^4 samples of Gaussian models. We present auto-similarity results obtained for $\mathbf{t} = (70, 100)$ and similarity function s_{sc} (*on the left*) and s_2 (*on the right*). We note that for rare events, see the magnified region, the theoretical inverse cumulative distribution function is above the empirical inverse cumulative distribution function. The opposite behavior is observed for similarity s_2. These observations are in accordance with the findings of Table 1.

TABLE 1. Asymptotic properties. Number of detections with different patch domains from 5×5 to 70×70 and spot domains from 1×1 to 25×25 for the $s_{2,2}$ (*left table*) or s_{sc} (*right table*) auto-similarity function. We only consider patch domains larger than spot domains. We generate 5000 Gaussian random field images of size 256×256 for each setting (with spot the indicator of the spot domain). We set $\alpha = 10/256^2$. For each setting we compute $a(\mathbf{t})$ the inverse cumulative distribution function of $\mathcal{N}(\mu_i(\mathbf{t}), \sigma_i^2(\mathbf{t}))$ evaluated at quantile α, with μ_i and σ_i^2 given by Proposition 3.5. For each pair of patch size and spot size we compute $\sum_{\mathbf{t}\in\Omega} \mathbb{1}_{\mathcal{AS}_i(u,\mathbf{t},\omega)\leqslant a(\mathbf{t})}$, namely the number of detections, for all the 5000 random fields samples. The empirical averages are displayed in the table. If $\mathcal{AS}_i(u, \mathbf{t}, \omega)$ had Gaussian distribution with parameters given by Proposition 3.5 then the number in each cell would be $\sum_{\mathbf{t}\in\Omega} \mathbb{P}\left[\mathcal{AS}_i(U, \mathbf{t}, \omega) \leqslant a(\mathbf{t})\right] \approx 10$.

	5	10	15	20	40	70
1	0.3	1.4	3.2	4.6	7.4	9.0
2	0.3	0.4	1.2	2.2	5.8	8.5
5	0.3	0.4	0.4	0.5	1.3	4.1
10		0.4	0.5	0.5	0.4	1.4
15			0.5	0.5	0.5	0.5
20				0.5	0.5	0.5
25					0.5	0.5

	5	10	15	20	40	70
1	18.1	11.6	10.9	10.4	10.1	10.0
2	34.2	16.5	12.8	11.5	10.4	9.9
5	93.9	49.3	30.8	20.9	13.2	11.5
10		86.7	57.6	46.0	19.7	14.5
15			83.9	63.8	30.0	18.2
20				79.5	36.7	24.7
25					51.5	26.6

s_2 and s_{sc} in Figure 6 in which we compare the asymptotic cumulative distribution function with the empirical one.

In the next section we address this problem by studying non-asymptotic cases for the $s_{2,2}$ auto-similarity function in both continuous and discrete settings.

4. A non-asymptotic case: internal Euclidean matching

4.1. Discrete periodic case

In this section Ω is a finite rectangular domain in $\mathbb{Z}^2$. We fix $\omega \subset \Omega$. We also define f a function over Ω. We consider the Gaussian random field $U = f * W$ (we consider the periodic convolution) with W a Gaussian white noise over Ω.

In the previous section, we derived asymptotic properties for similarity functions. However, a necessary condition for the asymptotic Gaussian approximation to be valid is for the spot size to be very small when compared to the patch size. This condition is not often met and non-asymptotic techniques must be developed. For instance it should be noted that the distribution of the s_{sc} template similarity, $\mathcal{TS}_{sc}(U, v, \mathbf{t}, \omega)$, is Gaussian for every ω. We might also derive a non-asymptotic expression for the template similarity in the cosine case if the Gaussian model is a white noise model. In what follows we restrict ourselves to the auto-similarity framework and consider the square of the L^2 norm auto-similarity function, *i.e.* $\mathcal{AS}_{2,2}(U, \mathbf{t}, \omega)$. In this case we present an efficient method to compute the cumulative distribution function of the auto-similarity function even in the non-asymptotic case.

Proposition 4.1 (Squared L^2 auto-similarity function exact probability distribution function)**.** *Let $\Omega = (\mathbb{Z}/M\mathbb{Z})^2$ with $M \in \mathbb{N}$, $\omega \subset \Omega$, $f \in \mathbb{R}^\Omega$ and $U = f * W$ where W is a Gaussian white noise over Ω. The following equality holds for any $\mathbf{t} \in \Omega$ up to a change of the underlying probability space*

$$\mathcal{AS}_{2,2}(U, \mathbf{t}, \omega) \underset{a.s}{=} \sum_{k=0}^{|\omega|-1} \lambda_k(\mathbf{t}, \omega) Z_k \ , \tag{4.1}$$

with Z_k independent chi-square random variables with parameter 1 and $\lambda_k(\mathbf{t}, \omega)$ the eigenvalues of the covariance matrix $C_\mathbf{t}$ associated with function $\Delta_f(\mathbf{t}, \cdot)$, see equation (3.5), restricted to ω, i.e. for any $\mathbf{x_1}, \mathbf{x_2} \in \omega$, $C_\mathbf{t}(\mathbf{x_1}, \mathbf{x_2}) = \Delta_f(\mathbf{t}, \mathbf{x_1} - \mathbf{x_2})$.

Proof. Let $\mathbf{t} \in \Omega$ and $V_\mathbf{t}$ be given for any $\mathbf{x} \in \Omega$ by $V_\mathbf{t}(\mathbf{x}) = U(\mathbf{x}) - U(\mathbf{x} + \mathbf{t})$. It is a Gaussian vector with mean 0 and covariance matrix C_V given for any $\mathbf{x_1}, \mathbf{x_2} \in \Omega$ by

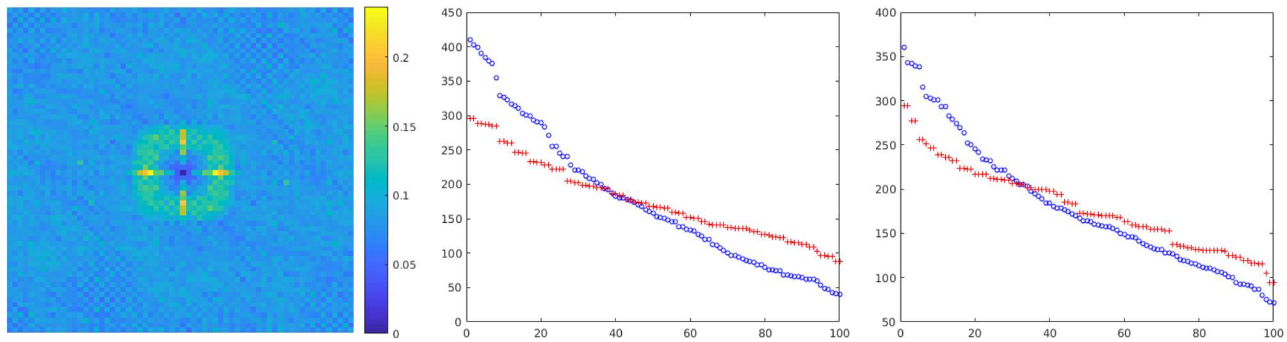
$$C_V(\mathbf{x_1}, \mathbf{x_2}) = 2\Gamma_f(\mathbf{x_1} - \mathbf{x_2}) - \Gamma_f(\mathbf{x_1} - \mathbf{x_2} - \mathbf{t}) - \Gamma_f(\mathbf{x_1} - \mathbf{x_2} + \mathbf{t}) = \Delta_f(\mathbf{t}, \mathbf{x_1} - \mathbf{x_2}).$$

The covariance of the random field $P_\omega(V_\mathbf{t})$, the restriction of $V_\mathbf{t}$ to ω, is given by the restriction of C_V to ω. This new covariance matrix, $C_\mathbf{t}$, is symmetric and the spectral theorem ensures that there exists an orthonormal basis $\mathcal{B}$ such that $C_\mathbf{t}$ is diagonal when expressed in $\mathcal{B}$. Thus we obtain that $P_\omega(V_\mathbf{t}) = \sum_{e_k \in \mathcal{B}} \langle P_\omega(V_\mathbf{t}), e_k \rangle e_k$. It is clear that, for any $k \in [\![0, |\omega| - 1]\!]$, $\langle P_\omega(V_\mathbf{t}), e_k \rangle$ is a Gaussian random variable with mean 0 and variance $e_k^T C_\mathbf{t} e_k = \lambda_k(\mathbf{t}, \omega) \geqslant 0$. We set $K = \{k \in [\![0, |\omega| - 1]\!],\ \lambda_k(\mathbf{t}, \omega) \neq 0\}$ and define X a random vector in $\mathbb{R}^{|\omega|}$ such that

$$X_k = \lambda_k(\mathbf{t}, \omega)^{-1/2} \langle P_\omega(V_\mathbf{t}), e_k \rangle, \text{ if } k \in K \ , \qquad \text{and } X_{K_-} = Y \ ,$$

where X_{K_-} is the restriction of X to the indices of $K_- = [\![0, |\omega| - 1]\!] \backslash K$ and Y is a standard Gaussian random vector on $\mathbb{R}^{|K_-|}$ independent from the sigma field generated by $\{(X_k), k \in K\}$. By construction we have $\mathbb{E}[X_k X_\ell] = 0$ if $\ell \in K$ and $k \in K_-$, or $\ell \in K_-$ and $k \in K_-$. Suppose now that $k, \ell \in K$. We obtain that

$$\mathbb{E}[X_k X_\ell] = \lambda_k(\mathbf{t}, \omega)^{-1/2} \lambda_\ell^{-1/2}(\mathbf{t}, \omega) \mathbb{E}\left[e_k^T C_\mathbf{t} e_\ell\right] = 0.$$



(A) (B) (C)

FIGURE 7. Eigenvalues approximation. We consider a Gaussian random field generated with $f * W$ with W a Gaussian white noise and f is a fixed sample of an independent Gaussian white noise over Ω. We consider patches of size 10×10 and study the approximation of the eigenvalues for the covariance matrix of the random field restricted to a domain of size 10×10, similarly to Proposition 4.1. (A) shows the Normalized Root-Mean Square Deviation between the eigenvalues computed with standard routines and the ones given by the approximation for each offset, see (4.2). Offset zero is at the center of the image. (B) and (C) illustrate the properties of Proposition 4.2. Blue circles correspond to the 100 eigenvalues computed with MATLAB routine for offset $(5, 5)$ in (B), respectively $(10, 20)$ in (C), and red crosses correspond to the 100 approximated eigenvalues for the same offsets. Note that a standard routine takes $273s$ for 10×10 patches on 256×256 images whereas it only takes $1.11s$ when approximating the eigenvalues using the discrete Fourier transform.

Thus X is a standard Gaussian random vector and we have $P_\omega(V_t) = \sum_{k=0}^{|\omega|-1} \lambda_k^{1/2}(\mathbf{t}, \omega) X_k e_k$, where the equality holds almost surely. We get that

$$\mathcal{AS}_{2,2}(U, \mathbf{t}, \omega) = \|P_\omega(V_\mathbf{t})\|_2^2 = \sum_{e_k \in \mathcal{B}} \langle P_\omega(V_\mathbf{t}), e_k \rangle^2 = \sum_{k=0}^{|\omega|-1} \lambda_k(\mathbf{t}, \omega) X_k^2.$$

Setting $Z_k = X_k^2$ concludes the proof. □

Note that if $\omega = \Omega$ then we obtain that the covariance matrix $C_\mathbf{t}$ is block-circulant with circulant blocks and the eigenvalues are given by the discrete Fourier transform.

In order to compute the true cumulative distribution function of the auto-similarity square L^2 norm we need to: (1) compute the eigenvalues of a covariance matrix in $\mathcal{M}_{|\omega|}(\mathbb{R})$; (2) compute the cumulative distribution function of a positive-weighted sum of independent chi-square random variable with weights given by the computed eigenvalues. Storing all covariance matrices for each offset $\mathbf{t}$ is not feasible. For instance considering a patch of size 10×10 and an image of size 512×512 we have approximately 2.6×10^9 coefficients to store, *i.e.* 10.5GB in float precision. In the rest of the section we suppose that $\mathbf{t}$ and ω are fixed and we denote by $C_\mathbf{t}$ the covariance matrix associated to the restriction of $\Delta_f(\mathbf{t}, \cdot)$ to $\omega + (-\omega)$. In Proposition 4.2 we propose a method to approximate the eigenvalues of $C_\mathbf{t}$ by using its specific structure. Indeed, as a covariance matrix, $C_\mathbf{t}$ is symmetric and positive and, since its associated Gaussian random field is stationary, it is block-Toeplitz with Toeplitz blocks, *i.e.* is block-diagonally constant and each block has constant diagonals. In the one-dimensional case these properties translate into symmetry, positivity and Toeplitz properties of the covariance matrix. Proposition 4.2 is stated in the one-dimensional case for the sake of simplicity but two-dimensional analogous can be derived. Note that this approximation is not always sharp as shown in Figure 7.

We recall that the Frobenius norm of a matrix of size $n \times n$ is the L^2 norm of the associated vector of size n^2.

Proposition 4.2 (Eigenvalues approximation)**.** *Let b be a function defined over $[\![-(n-1), n-1]\!]$ with $n \in \mathbb{N}\setminus\{0\}$. We define $T_b(j,\ell) = b(j-\ell)$ for $j,\ell \in [\![0, n-1]\!]$. The matrix T_b is a circulant matrix if and only if b is n-periodic. T_b is symmetric if and only if b is symmetric. Let b be symmetric, defining $\Pi(T_b)$ the projection of T_b onto the set of symmetric circulant matrix for the Frobenius product, we obtain that*

1. *the projection satisfies $\Pi(T_b) = T_c$ with $c(j) = \left(1 - \frac{j}{n}\right) b(j) + \frac{j}{n} b(n-j)$ for all $j \in [\![0, n-1]\!]$ and c is extended by n-periodicity to $\mathbb{Z}$;*
2. *the eigenvalues of $\Pi(T_b)$ are given by $\left(2\,\mathrm{Re}(\hat{d}(j)) - b(0)\right)_{j\in[\![0,n-1]\!]}$ with $d(j) = \left(1 - \frac{j}{n}\right) b(j)$, and $\hat{d}$ is the discrete Fourier transform over $[\![0, n-1]\!]$;*
3. *let $(\lambda_j)_{j\in[\![1,n]\!]}$ be the sorted eigenvalues of T_b and $(\tilde{\lambda}_j)_{j\in[\![1,n]\!]}$ the sorted eigenvalues of $\Pi(T_b)$ (in the same order). For any $j \in [\![1, n]\!]$, we have $|\lambda_j - \tilde{\lambda}_j| \leqslant \|T_b - \Pi(T_b)\|_{\mathrm{Fr}}$;*
4. *if T_b is positive-definitive then $\Pi(T_b)$ is positive-definite.*

Proof. (1) Let T_c be an element of the symmetric circulant matrices set. Minimizing $\|T_b - T_c\|^2_{\mathrm{Fr}}$ in $c(j)_{j\in[\![0,n-1]\!]}$ we get that $c(j)$ satisfies for any $j \in [\![0, n-1]\!]$

$$c(j) = \underset{s\in\mathbb{R}}{\operatorname{argmin}} \left(2(n-j)(s - b(j))^2 + 2j(s - b(n-j))^2\right) ,$$

which gives the result.

(2) Since $T_c = \Pi(T_b)$ is circulant, its eigenvalues are given by the discrete Fourier transform of c. We have that if $i \neq 0$ then $c(i) = \dot{d}(j) + \dot{d}(-j)$ with $d(j) = \left(1 - \frac{j}{n}\right) b(j)$ and $\dot{d}$ its extension to $\mathbb{Z}$ by n-periodicity. We also have $c(0) = b(0)$. We conclude the proof by taking the discrete Fourier transform of c.

(3) The proof of the Lipschitz property on the sorted eigenvalues of symmetric matrices with respect to the L^2 matricial norm can be found in [13]. We conclude using the fact that the L^2 matricial norm is upper-bounded by the Frobenius norm.

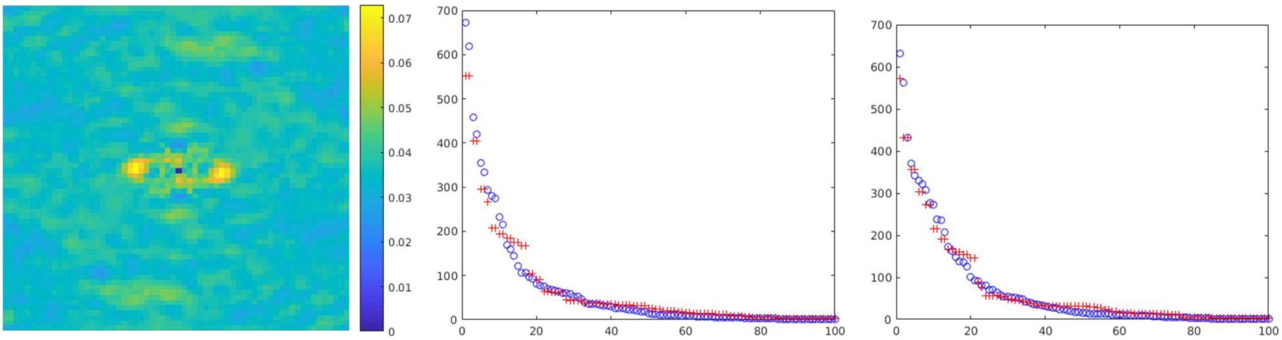
(4) This result is a special case of the spectrum contraction property of the projection proved in Theorem 2 of [11]. ☐

In Figure 7 we display the behavior of the projection for the eigenvalues in the two-dimensional case. The measure we consider is the Normalized Root Mean Square Deviation

$$\mathrm{NRMSD} = \frac{\left(\frac{1}{|\omega|}\sum_{k=0}^{|\omega|-1} |\tilde{\lambda}_k(\mathbf{t},\omega) - \lambda_k(\mathbf{t},\omega)|^2\right)^{1/2}}{\max\left(\lambda_k(\mathbf{t},\omega)\right)_{k\in[\![0,|\omega|-1]\!]} - \min\left(\lambda_k(\mathbf{t},\omega)\right)_{k\in[\![0,|\omega|-1]\!]}} , \tag{4.2}$$

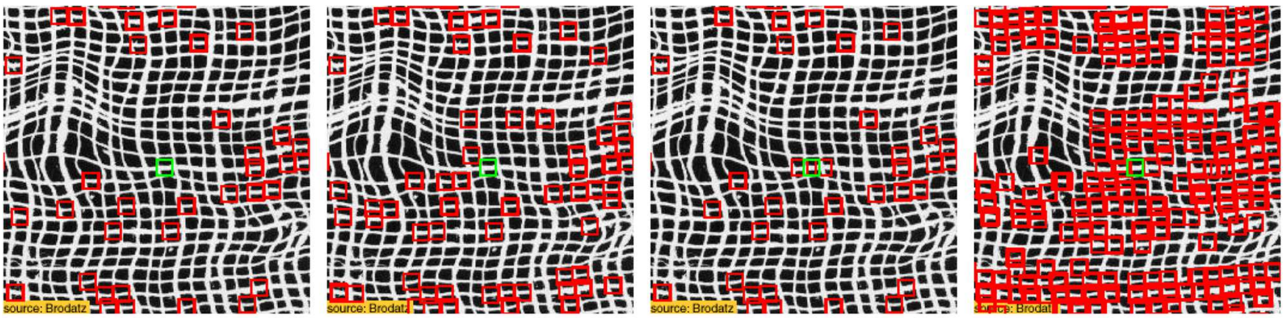
with $\tilde{\lambda}_k(\mathbf{t},\omega)$ the approximation of the eigenvalues, for every possible offset in the image and $\lambda_k(\mathbf{t},\omega)$ the true eigenvalues, for every possible offset. Computing the eigenvalues of the projection is done via Fast Fourier Transform (FFT) which is faster than standard routines for computing eigenvalues of Toeplitz matrices. The major cons of using such an approximation is that it may not be valid for small offsets $\mathbf{t} \in \Omega$ as shown in Figure 7. However, in most cases the random field is smooth and in this case, see Figure 8, the approximation is satisfactory. We also highlight that for similarity detection purposes, see Figure 9, the level of precision achieved by our approximation is satisfactory, see [7].

Suppose the approximation of the eigenvalues is valid, we need an efficient algorithm to compute the distribution of the associated positive-weighted sum of chi-square random variables in equation (4.1). Exact computation has been derived by Imhof in [31] but requires to compute heavy integrals. This exact method, named Imhof method in the following, will be used as a baseline for other algorithms. Numerous methods such as differential



(A) (B) (C)

FIGURE 8. Eigenvalues approximation. Same study as the one conducted in Figure 7 with $f = \mathbb{1}_{\{1,2,3\}^2}$. Note that in this case the approximation is better than the one presented in Figure 7.



(A) $1513s, n_d = 52$ (B) $200s, n_d = 116$ (C) $4.77s, n_d = 50$ (D) $4.64s, n_d = 671$

FIGURE 9. Similarity detection. In this figure we illustrate the accuracy of the different proposed approximations of the cumulative distribution function of $\mathcal{AS}_{2,2}(U, \mathbf{t}, \omega)$. We say that an offset $\mathbf{t}$ is *detected* in an image if $\mathcal{AS}_{2,2}(u, \mathbf{t}, \omega) \leqslant a(\mathbf{t})$ for some threshold $a(\mathbf{t}) \in \mathbb{R}$. In every image, in green we display the patch domain ω (in the center of the image) and in red we display the shifted patch domain for detected offsets with function $a(\mathbf{t})$ such that for any $\mathbf{t} \in \Omega$, $\mathbb{P}\left[\mathcal{AS}_{2,2}(U, \mathbf{t}, \omega) \leqslant a(\mathbf{t})\right] = 1/256^2$, where U is given by the Gaussian random field $f * W$ where f is the original image of fabric and W is a Gaussian white noise over $\Omega = 256 \times 256$. Approximations of the cumulative distribution function of $\mathcal{AS}_{2,2}(U, \mathbf{t}, \omega)$ lead to approximations of $a(\mathbf{t})$. The most precise approximation is given in (A) where the eigenvalues are computed using a MATLAB routine and the cumulative distribution function is given by the Imhof method. In (B) we approximate the eigenvalues using the projection described in Proposition 4.2 and still use the Imhof method. It yields twice as many detections. In (C) Wood F method is used instead of Imhof's yielding less detections but performing seven times faster. Interestingly errors seem to compensate and the obtained result with Wood F method is very close to the results obtained with the baseline algorithm in (A). In (D) HBE method is used instead of Imhof's, in this case we obtain too many detections, *i.e.* the approximation of the cumulative distribution function is not valid.

equations [19], series truncation [38], negative binomial mixtures [46] approaches were later introduced but all require stopping criteria such as truncation criteria which can be hard to set. We focus on cumulant methods which generalize and refine the Gaussian approximations used in Section 3. These methods rely on computing moments of the original distribution and then fitting a known probability distribution function to the objective distribution using these moments. Bodenham *et al.* in [6] show that the following methods can be efficiently computed:

- Gaussian approximation (discarded due to its poor results for small patches as illustrated in Sect. 3);
- Hall-Buckley-Eagleson [10, 29] (HBE), (three moments fitted Gamma distribution);
- Wood F [56] (three moments fitted Fischer-Snedecor distribution).

Other methods such as the Lindsay-Pilla-Basak-4 method, which relies on the computation of eight moments, are slower than HBE by a factor 350 at least, see [6], and are thus discarded. In Figure 9 we investigate the trade-off between computational speed and accuracy of these methods for the task of detection.

The experiments conducted in Figure 9 show that the HBE approximation does not give good results when evaluating the probability of rare events. This was already noticed by Bodenham *et al.* in [6] who stated that "Hall–Buckley–Eagleson method is recommended for most practitioners [...]. However, [...], for very small probability values, either the Wood F or the Lindsay–Pilla-Basak method should be used".

4.2. Continuous periodic case

To conclude we show that a similar non-asymptotic study can be conducted in continuous settings.

Proposition 4.3 (Squared L^2 continuous auto-similarity function exact probability distribution function)**.** *Let $\Omega = \mathbb{T}^2$, $\omega \subset \Omega$ and let U be a zero-mean Gaussian random field on Ω with covariance function Γ. Assume (A2.2), then the following equality holds for any $\mathbf{t} \in \Omega$ up to a change of the underlying probability space*

$$\mathcal{AS}_{2,2}(U, \mathbf{t}, \omega) \underset{a.s}{=} \sum_{k \in \mathbb{N}} \lambda_k(\mathbf{t}, \omega) Z_k,$$

with Z_k independent chi-square random variables with parameter 1 and $\lambda_k(\mathbf{t}, \omega)$ the eigenvalues of the kernel $C_{\mathbf{t}}$ associated with function $\Delta(\mathbf{t}, \cdot) = 2\Gamma(\mathbf{t}) - \Gamma(\cdot + \mathbf{t}) - \Gamma(\cdot - \mathbf{t})$ restricted to ω, i.e. for any $\mathbf{x_1}, \mathbf{x_2} \in \omega$, $C_{\mathbf{t}}(\mathbf{x_1}, \mathbf{x_2}) = \Delta(\mathbf{t}, \mathbf{x_1} - \mathbf{x_2})$.

Proof. We consider the stationary Gaussian random field $P_\omega(V_{\mathbf{t}})$ over ω defined by the restriction to ω where for any $\mathbf{x} \in \Omega$ by $V_{\mathbf{t}}(\mathbf{x}) = U(\mathbf{x}) - U(\mathbf{x} + \mathbf{t})$. The Karhunen-Loeve theorem [28] ensures the existence of $(\lambda_k(\mathbf{t}, \omega))_{k \in \mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$, $(X_k)_{k \in \mathbb{N}}$ a sequence of independent normal Gaussian random variables and $(e_k)_{k \in \mathbb{N}}$ a sequence of orthonormal function over $L^2(\omega)$ such that

$$\lim_{n \to +\infty} \sup_{\mathbf{x} \in \omega} \mathbb{E}\left[\left|P_\omega(V_{\mathbf{t}})(\mathbf{x}) - \sum_{k=0}^{n} \sqrt{\lambda_k(\mathbf{t}, \omega)} e_k(\mathbf{x}) X_k\right|^2\right] = 0\,, \tag{4.3}$$

We define the sequence $(I_n)_{n \in \mathbb{N}} = (\int_\omega \left(\sum_{k=0}^{n} \sqrt{\lambda_k(\mathbf{t}, \omega)} e_k(\mathbf{x}) X_k\right)^2 \mathrm{d}\mathbf{x})_{n \in \mathbb{N}}$. We have, using the Cauchy-Schwarz inequality on $L^2(\mathcal{A} \times \omega)$ and (4.3)

$$\begin{aligned}
\mathbb{E}\left[|\mathcal{AS}_{2,2}(U, \mathbf{t}, \omega) - I_n|\right] &\leqslant \mathbb{E}\left[\int_\omega \left|P_\omega(V_{\mathbf{t}})^2(\mathbf{x}) - \left(\sum_{k=0}^{n} \sqrt{\lambda_k(\mathbf{t}, \omega)} e_k(\mathbf{x}) X_k\right)^2\right| \mathrm{d}\mathbf{x}\right] \\
&\leqslant \mathbb{E}\left[\int_\omega (P_\omega(V_{\mathbf{t}})(\mathbf{x}) - \sum_{k=0}^{n} \sqrt{\lambda_k(\mathbf{t}, \omega)} e_k(\mathbf{x}) X_k)^2 \mathrm{d}\mathbf{x}\right]^{1/2} \mathbb{E}\left[\int_\omega (P_\omega(V_{\mathbf{t}})(\mathbf{x}) + \sum_{k=0}^{n} \sqrt{\lambda_k(\mathbf{t}, \omega)} e_k(\mathbf{x}) X_k)^2 \mathrm{d}\mathbf{x}\right]^{1/2} \\
&\leqslant 2\mathbb{E}\left[\mathcal{AS}_{2,2}(U, \mathbf{t}, \omega)\right]^{1/2} \int_\omega \mathbb{E}\left[(P_\omega(V_{\mathbf{t}})(\mathbf{x}) - \sum_{k=0}^{n} \sqrt{\lambda_k(\mathbf{t}, \omega)} e_k(\mathbf{x}) X_k)^2\right] \mathrm{d}\mathbf{x}\,,
\end{aligned} \tag{4.4}$$

where we used the Fubini theorem in the last inequality. Using the dominated convergence theorem in (4.4) with integral domination given by $\sup_{n\in\mathbb{N}}\sup_{\mathbf{x}\in\omega}\mathbb{E}\left[(P_\omega(V_{\mathbf{t}})(\mathbf{x})-\sum_{k=0}^n\sqrt{\lambda_k(\mathbf{t},\omega)}e_k(\mathbf{x})X_k)^2\right]$ we conclude that $(I_n)_{n\in\mathbb{N}}$ converges to $\mathcal{AS}_{2,2}(U,\mathbf{t},\omega)$ in $L^1(\mathcal{A})$. Thus there exists a subsequence of $(I_n)_{n\in\mathbb{N}}$ which converges almost surely to $\mathcal{AS}_{2,2}(U,\mathbf{t},\omega)$. We also have $I_n=\int_\omega\left(\sum_{k=0}^n\sqrt{\lambda_k(\mathbf{t},\omega)}e_k(\mathbf{x})X_k\right)^2\mathrm{d}\mathbf{x}=\sum_{k=0}^n\lambda_k(\omega,k)X_k^2$ by orthonormality and thus the sequence $(I_n)_{n\in\mathbb{N}}$ is almost surely non-decreasing. We get that $(I_n)_{n\in\mathbb{N}}$ converges almost surely to $\mathcal{AS}_{2,2}(U,\mathbf{t},\omega)$ which can be rewritten as

$$\mathcal{AS}_{2,2}(U,\mathbf{t},\omega)=\sum_{k\in\mathbb{Z}}\lambda_k(\mathbf{t},\omega)X_k^2\quad\text{almost surely.}$$

The characterization of $(\lambda_k(\mathbf{t},\omega),e_k(\mathbf{x}))$ is given by the Karhunen-Loeve theorem and $e_k(\mathbf{x})$ is solution of the following Fredholm equation for all $\mathbf{x}\in\omega$

$$\int_\omega\Delta(\mathbf{t},\mathbf{x}-\mathbf{y})e_k(\mathbf{y})\,\mathrm{d}\mathbf{y}=\lambda_k(\mathbf{t},\omega)e_k(\mathbf{x}).$$

Setting $Z_k=X_k^2$ concludes the proof. □

Note that if $\omega=\mathbb{T}^2$ then the solution of the Fredholm equation is given by the Fourier series of Γ.

Appendix A. Multidimensional central limit theorems

In this section we provide an extension of ([35], Thm. 2) to the multidimensional case.

We recall the notion of dependency graph as introduced in [35]. Let $(X_i)_{i\in\mathbb{N}}$ be $\mathbb{R}^d$-valued random variables. A graph is a dependency graph for $(X_i)_{i\in\mathbb{N}}$ if the two following conditions are satisfied.

1. There is a one-to-one correspondence between $(X_i)_{i\in\mathbb{N}}$ and the vertices of the graph.
2. If two sets of vertices are not connected then the corresponding random variables are independent.

Theorem A.1. *Let $(X_{i,j})_{(i,j)\in\mathbb{N}^2}$ be a sequence of $\mathbb{R}^d$-valued random variables and $(N_n)_{n\in\mathbb{N}}\in\mathbb{N}^{\mathbb{N}}$. For any $n\in\mathbb{N}$, assume that there exists $A_n, M_n\geqslant 0$ such that for any $j\in\mathbb{N}$, $\|X_{n,j}\|\leqslant A_n$ and that the dependency graph of $(X_{n,j})_{j\in\mathbb{N}}$ is of degree M_n at most. For any $n\in\mathbb{N}$ let $S_n=\sum_{j=1}^{N_n}X_{n,j}$ and $C_n=\mathrm{Cov}\left[S_n\right]$. Assume that there exists $m_0\in\mathbb{N}$ and $C\in\mathcal{M}_d(\mathbb{R})$ such that for any $n\in\mathbb{N}$, $\lim_{n\to+\infty}(N_n/M_n)^{1/m_0}M_nA_n=0$ and $\lim_{n\to+\infty}C_n=C$. Then, $S_n-\mathbb{E}\left[S_n\right]$ converges (in the weak sense) towards $\mathcal{N}(0,C)$.*

Proof. Let $a\in\mathbb{R}^d$ and consider $(X_{i,j}^a)_{(i,j)\in\mathbb{N}^2}$ such that for any $i,j\in\mathbb{N}$, $X_{i,j}^a=\langle X_{i,j},a\rangle$. We also introduce for any $n\in\mathbb{N}$, $S_n^a=\sum_{j=1}^{N_n}x_{n,j}^a$. Assume that $a^\top Ca=0$. Then, using the Bienaymé -Tchebychev inequality, we have for any $\varepsilon>0$

$$\lim_{n\to+\infty}\mathbb{P}\left[|S_n^a-\mathbb{E}\left[S_n^a\right]|>\varepsilon\right]\leqslant\lim_{n\to+\infty}\varepsilon^{-2}a^\top C_na=0. \tag{A.1}$$

Hence, $\langle a,S_n-\mathbb{E}\left[S_n\right]\rangle$ converges (in the weak sense) towards $\langle a,Z\rangle$ with Z a d-dimensional Gaussian random variable with zero mean and covariance matrix C. If $a^\top C_na\neq 0$ then using [35] we have that $\langle a,S_n-\mathbb{E}\left[S_n\right]\rangle$ converges (in the weak sense) towards $\langle a,Z\rangle$. We conclude using the Cramér-Wold theorem, [15], Theorem 1. □

Similarly to ([35], Thm. 2), we can replace the condition $\|X_{n,i}\|\leqslant A_n$ by the following condition: for any $a\in\mathbb{R}^d$ with $a\neq 0$

$$\lim_{n\to+\infty}M_n\sum_{j=1}^{N_n}\mathbb{E}\left[\|X_{n,j}\|^2\mathbb{1}_{\|X_{n,j}\|>A_n\|a\|}\right]=0. \tag{A.2}$$

Indeed, this implies that for any $a \in \mathbb{R}^d$, $\lim_{n\to+\infty} M_n \sum_{j=1}^{N_n} \mathbb{E}\left[\langle a, X_{n,j}\rangle^2 \mathbb{1}_{|\langle a, X_{n,j}\rangle| > A_n}\right] = 0$, which is the Lindeberg type condition identified in ([35], Thm. 2).

Appendix B. Asymptotic theorems – discrete case

We start by introducing two notions which will be crucial in order to derive a law of large numbers and a central limit theorem in broad settings. The R-independence, see Definition B.1, ensures long-range independence whereas stochastic domination will replace integrability conditions in the standard law of large numbers or central limit theorem.

The notion of R-independence generalizes to $\mathbb{R}^2$ and $\mathbb{Z}^2$ the associated one-dimensional concept, see [5] and its extension to $\mathbb{N}^2$ [44, 53].

Definition B.1 (R-independence)**.** Let $d \in \mathbb{N}$, $\Omega = \mathbb{R}^2$ or $\Omega = \mathbb{Z}^2$ and V be a d-dimensional random field over Ω. Let $K_1, K_2 \subset \Omega$ be two compact sets, and $V|_{K_i}$ be the restriction of V to K_i, $i \in \{1, 2\}$. We say that V is R-independent, with $R \geqslant 0$, if $V|_{K_1}$ is independent from $V|_{K_2}$ as soon as $d_\infty(K_1, K_2) = \min_{\mathbf{x}\in K_1, \mathbf{y}\in K_2} \|\mathbf{x} - \mathbf{y}\|_\infty > R$.

Note that in the case of $\Omega = \mathbb{Z}^2$, compacts sets K_1 and K_2 are finite sets of indices. This notion of R-independence will replace the traditional assumption of independence in asymptotic theorems.

Definition B.2 (Uniform domination)**.** Let $\Omega = \mathbb{R}^2$ or $\Omega = \mathbb{Z}^2$ and let $V, \widetilde{V}$ be real random fields over Ω. We say that:

(a) $\widetilde{V}$ uniformly stochastically dominates V if for any $\alpha \geqslant 0$ and $\mathbf{x} \in \Omega$, $\mathbb{P}\left[V(\mathbf{x}) \geqslant \alpha\right] \leqslant \mathbb{P}\left[\widetilde{V}(\mathbf{x}) \geqslant \alpha\right]$;

(b) $\widetilde{V}$ uniformly almost surely dominates V if for any $\mathbf{x} \in \Omega$, $V(\mathbf{x}) \leqslant \widetilde{V}(\mathbf{x})$ almost surely.

Note that if $\widetilde{V}$ uniformly almost surely dominates V then $\widetilde{V}$ uniformly stochastically dominates $\widetilde{V}$.

The following theorem is a two-dimensional law of large numbers with weak dependence assumptions. It is a slight modification of Corollary 4.1 (ii) in [53].

Theorem B.3. *Let $d \in \mathbb{N}$. Let $(m_k)_{k\in\mathbb{N}}, (n_k)_{k\in\mathbb{N}}$ be two positive increasing integer sequences and $(\omega_k)_{k\in\mathbb{N}}$ be the sequence of subsets such that for any $k \in \mathbb{N}$, $\omega_k = [\![0, m_k]\!] \times [\![0, n_k]\!]$. Let V be a d-dimensional R-independent random field over $\mathbb{Z}^2$, with $R \geqslant 0$, such that $\|V(\mathbf{x}) - \mathbb{E}\left[V(\mathbf{x})\right]\|$ is uniformly stochastically dominated by $\widetilde{V}$, a real second-order stationary random field over $\mathbb{Z}^2$. Then V is a second-order random field. In addition, assume that there exists $\mu \in \mathbb{R}^d$ such that $\lim_{k\to+\infty} |\omega_k|^{-1} \sum_{\mathbf{x}\in\omega_k} \mathbb{E}\left[V(\mathbf{x})\right] = \mu$. Then it holds that*

$$\lim_{k\to+\infty} |\omega_k|^{-1} \sum_{\mathbf{x}\in\omega_k} V(\mathbf{x}) \underset{\text{a.s}}{=} \mu. \tag{B.1}$$

Proof. Without loss of generality we can suppose that $d = 1$ and that for any $\mathbf{x} \in \mathbb{Z}^2$, $\mathbb{E}\left[V(\mathbf{x})\right] = 0$. In order to apply Corollary 4.1 (ii) in [53] we must check that:

(a) V is $R-$independent;

(b) $|V|$ is uniformly stochastically dominated by a random field $\widetilde{V}$ and there exists $r \in [1, 2[$ such that for any $\mathbf{x} \in \mathbb{Z}^2$, $\mathbb{E}\left[\widetilde{V}^r(\mathbf{x}) \log^+(\widetilde{V}(\mathbf{x}))\right]$ is finite.

Item (a) is given in the statement of Theorem B.3 and $|V|$ is uniformly stochastically dominated by the random field $\widetilde{V}_0$ defined for any $\mathbf{x} \in \mathbb{Z}^2$ by $\widetilde{V}_0(\mathbf{x}) = \widetilde{V}(\mathbf{0})$. Since $\mathbb{E}\left[\widetilde{V}(\mathbf{0})^2\right]$ is finite so is $\mathbb{E}\left[\widetilde{V}(\mathbf{0}) \log^+(\widetilde{V}(\mathbf{0}))\right]$ which implies (b). Then it holds that

$$\lim_{k\to+\infty} \sum_{\mathbf{x}\in\omega_k} \left(V(\mathbf{x}) - \mathbb{E}\left[V(\mathbf{x})\right]\right) \underset{a.s}{=} 0.$$

Using that $\lim_{k\to+\infty} |\omega_k|^{-1} \sum_{\mathbf{x}\in\omega_k} \mathbb{E}[U(\mathbf{x})] = \mu$, we conclude the proof. □

We now turn to an extension of the central limit theorem to two-dimensional random fields with weak dependence assumptions. This result is a consequence of Theorem A.1.

Theorem B.4. *Under the hypotheses of Theorem B.3 and assuming that there exist* $\mu \in \mathbb{R}^d$ *and* $C \in \mathcal{M}_d(\mathbb{R})$ *such that*

(a) $\lim_{k\to+\infty} |\omega_k|^{-1/2} \sum_{\mathbf{x}\in\omega_k} (\mathbb{E}[V](\mathbf{x}) - \mu) = 0$;
(b) $\lim_{k\to+\infty} |\omega_k|^{-1} \sum_{\mathbf{x},\mathbf{y}\in\omega_k} \mathrm{Cov}[V(\mathbf{x}), V(\mathbf{y})] = C$.

Then it holds that

$$\lim_{k\to+\infty} |\omega_k|^{-1/2} \sum_{\mathbf{x}\in\omega_k} (V(\mathbf{x}) - \mu) \underset{\mathcal{L}}{=} \mathcal{N}(0, C). \tag{B.2}$$

Proof. For any $i, j \in \mathbb{N}$, let $X_{i,j} = (V(\mathbf{x}_j) - \mathbb{E}[V(\mathbf{x}_j)])|\omega_i|^{-1/2}$ with $(\mathbf{x}_j)_{j\in\mathbb{N}}$ such that for any $k \in \mathbb{N}$, $\{V(\mathbf{x}_j),\ j \in [\![1, |\omega_k|]\!]\} = \{V(\mathbf{x}), \mathbf{x} \in \omega_k\}$. For any $n \in \mathbb{N}$, let $N_n = |\omega_n|$. Then, we have that for any $n \in \mathbb{N}$, $\sum_{j=1}^{N_n} X_{n,j} = |\omega_k|^{-1/2} \sum_{\mathbf{x}\in\omega_k} (V(\mathbf{x}) - \mathbb{E}[V(\mathbf{x})])$. Since V is R-independent each vertex of the dependency graph of $(X_{i,j})_{i,j\in\mathbb{N}^2}$ has its degree bounded by $(2R+1)^2$ and therefore for any $n \in \mathbb{N}$, $M_n = (2R+1)^2$. For any $n \in \mathbb{N}$, let $A_n = |\omega_n|^\alpha$ with $\alpha \in (1/3, 1/2)$. Using that $\tilde{V}$ uniformly stochastically dominates $(\|V(\mathbf{x}) - \mathbb{E}[V(\mathbf{x})]\|)_{\mathbf{x}\in\mathbb{Z}^2}$ we obtain that for any $a \in \mathbb{R}^d$

$$\begin{aligned}
\sum_{j=1}^{N_n} \mathbb{E}\left[\|X_{n,j}\|^2 \mathbb{1}_{\|X_{n,j}\|^2 > A_n^2 \|a\|^{-2}}\right] &= |\omega_n|^{-1} \sum_{\mathbf{x}\in\omega_n} \mathbb{E}\left[\|V(\mathbf{x}) - \mathbb{E}[V(\mathbf{x})]\|^2 \mathbb{1}_{\|V(\mathbf{x}) - \mathbb{E}[V(\mathbf{x})]\|^2 > A_n^2\|a\|^{-2}|\omega_n|}\right] \\
&= |\omega_n|^{-1} \sum_{\mathbf{x}\in\omega_n} \int_0^{+\infty} \mathbb{P}\left[\|V(\mathbf{x}) - \mathbb{E}[V(\mathbf{x})]\|^2 \geqslant \max(A_n^2\|a\|^{-2}|\omega_n|, t)\right] \mathrm{d}t \\
&\leqslant |\omega_n|^{-1} \sum_{\mathbf{x}\in\omega_n} \int_0^{+\infty} \mathbb{P}\left[\tilde{V}(\mathbf{x}) \geqslant \max(A_n^2\|a\|^{-2}|\omega_n|, t)\right] \mathrm{d}t \\
&\leqslant \mathbb{E}\left[\tilde{V}(\mathbf{0}) \mathbb{1}_{\tilde{V}(\mathbf{0}) > A_n^2\|a\|^{-2}|\omega_n|}\right].
\end{aligned}$$

Hence, since $\lim_{n\to+\infty} A_n^2|\omega_n| = \lim_{n\to+\infty} |\omega_n|^{1-2\alpha} = +\infty$ we get that

$$\lim_{n\to+\infty} \sum_{j=1}^{N_n} \mathbb{E}\left[\|X_{n,j}\|^2 \mathbb{1}_{\|X_{n,j}\|^2 > A_n^2\|a\|^{-2}}\right] = 0. \tag{B.3}$$

Letting $m_0 = 3$ we get that

$$\lim_{n\to+\infty} (N_n/M_n)^{1/m_0} M_n A_n = (2R+1)^{2(1+1/3)} |\omega_n|^{1/3-\alpha} = 0. \tag{B.4}$$

In addition, we have that for any $n \in \mathbb{N}$

$$C_n = \mathrm{Cov}\left[\sum_{j=1}^{N_n} X_{n,j}\right] = |\omega_n|^{-1} \mathrm{Cov}\left[\sum_{\mathbf{x}\in\omega_n} V(\mathbf{x})\right] = |\omega_n|^{-1} \sum_{\mathbf{x},\mathbf{y}\in\omega_n} \mathrm{Cov}[V(\mathbf{x}), V(\mathbf{y})]. \tag{B.5}$$

Hence, combining (B.3), (B.4), (B.5), (b) and Theorem A.1, we get that

$$\lim_{k\to+\infty} |\omega_k|^{-1/2} \sum_{\mathbf{x}\in\omega_k} (V(\mathbf{x}) - \mathbb{E}[V(\mathbf{x})]) \underset{\mathcal{L}}{=} \mathcal{N}(0, C). \tag{B.6}$$

Combining (B.6) and (a) concludes the proof. □

The following lemma explicits a class of Gaussian random fields over $\mathbb{Z}^2$ such that the R-independence property holds for some $R \geqslant 0$.

Lemma B.5. *Let $f \in \mathbb{R}^{\mathbb{Z}^2}$ with finite support $\operatorname{Supp}(f) \subset [\![-r, r]\!]^2$, where $r \in \mathbb{N}$. Let W be a Gaussian white noise over $\mathbb{Z}^2$ and $V = f * W$ then V is a R-independent second-order random field with $R = 2r$.*

Proof. V is a Gaussian random field such that for any $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^2$

$$\mathbb{E}[V(\mathbf{x})] = 0\,, \qquad \operatorname{Cov}[V(\mathbf{x}), V(\mathbf{y})] = \sum_{\mathbf{x}',\mathbf{y}'\in\mathbb{Z}^2} f(\mathbf{x}-\mathbf{x}')f(\mathbf{y}-\mathbf{y}')\operatorname{Cov}\left[W(\mathbf{x}'), W(\mathbf{y}')\right] = \Gamma_f(\mathbf{x}-\mathbf{y}). \tag{B.7}$$

Note that since $\operatorname{Supp}(f) \subset [\![-r, r]\!]$ we have $\operatorname{Supp}(\Gamma_f) \subset [\![-R, R]\!]$ with $R = 2r$. For any $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^2$ such that $\|\mathbf{x}-\mathbf{y}\|_\infty > R$, using (B.7), we obtain

$$\operatorname{Cov}[V(\mathbf{x}), V(\mathbf{y})] = \Gamma_f(\mathbf{x}-\mathbf{y}) = 0. \tag{B.8}$$

Let $K_1, K_2 \subset \mathbb{Z}^2$ two finite sets with $\sup_{\mathbf{x}\in K_1, \mathbf{y}\in K_2} \|\mathbf{x}-\mathbf{y}\|_\infty > R$ and consider $V|_{K_i}$ the restriction of V to K_i for $i = \{1, 2\}$. Using (B.8), we get that for any $\mathbf{x} \in K_1$, $\mathbf{y} \in K_2$ we have

$$\operatorname{Cov}[V|_{K_1}(\mathbf{x}), V|_{K_2}(\mathbf{y})] = 0.$$

As a consequence, $\operatorname{Cov}[V|_{K_1}, V|_{K_2}] = 0$ and $V|_{K_1}$ and $V|_{K_2}$ are uncorrelated. Since $V|_{K_1}, V|_{K_2}$ are Gaussian random fields we get that $V|_{K_1}, V|_{K_2}$ are R-independent. □

The following lemma gives specific conditions on random fields in order for Theorems B.3 and B.4 to hold.

Lemma B.6. *Let $d \in \mathbb{N}$. Let $(m_k)_{k\in\mathbb{N}}, (n_k)_{k\in\mathbb{N}}$ be two positive increasing integer sequences and $(\omega_k)_{k\in\mathbb{N}}$ be the sequence of subsets given for any $k \in \mathbb{N}$ by, $\omega_k = [\![0, m_k]\!] \times [\![0, n_k]\!]$. Let V be a d-dimensional R-independent second-order stationary random field over $\mathbb{Z}^2$, with $R \geqslant 0$. Then for all $k \in \mathbb{N}$*

(a) $|\omega_k|^{-1} \sum_{\mathbf{x}\in\omega_k} \mathbb{E}[V(\mathbf{x})] = \mathbb{E}[V(\mathbf{0})]$;
(b) $\lim_{k\to+\infty} |\omega_k|^{-1} \sum_{\mathbf{x},\mathbf{y}\in\omega_k} \operatorname{Cov}[V(\mathbf{x}), V(\mathbf{y})] = \sum_{\mathbf{x}\in\mathbb{Z}^2} \operatorname{Cov}[V(\mathbf{x}), V(\mathbf{0})]$.

In addition, equations (B.1) and (B.2) hold with $\mu = \mathbb{E}[V(\mathbf{0})]$ and $C = \sum_{\mathbf{x}\in\mathbb{Z}^2} \operatorname{Cov}[V(\mathbf{x}), V(\mathbf{0})]$ which is finite.

Proof. Item (a) is immediate by stationarity. Concerning (b), for any $k \in \mathbb{N}$ we have by stationarity

$$|\omega_k|^{-1} \sum_{\mathbf{x},\mathbf{y}\in\omega_k} \operatorname{Cov}[V(\mathbf{x}), V(\mathbf{y})] = |\omega_k|^{-1} \sum_{\mathbf{x},\mathbf{y}\in\omega_k} \operatorname{Cov}[V(\mathbf{x}-\mathbf{y}), V(\mathbf{0})] = \sum_{\mathbf{x}\in\mathbb{Z}^2} \operatorname{Cov}[V(\mathbf{x}), V(\mathbf{0})]\, g_k(\mathbf{x})\,,$$

where $g_k \in \mathbb{R}^{\mathbb{Z}^2}$ satisfies for any $\mathbf{x} \in \mathbb{Z}^2$, $g_k(\mathbf{x}) = |\omega_k|^{-1} \mathbb{1}_{\omega_k} * \check{\mathbb{1}}_{\omega_k}(\mathbf{x})$. For any $k \in \mathbb{N}$, $\mathbf{x} \in \mathbb{Z}^2$ we have $0 \leqslant g_k(\mathbf{x}) \leqslant 1$ and $\lim_{k\to+\infty} g_k(\mathbf{x}) = 1$. For any $\mathbf{x} \in \mathbb{Z}^2$ such that $\|\mathbf{x}\|_\infty > R$, $\operatorname{Cov}[V(\mathbf{x}), V(\mathbf{0})] = 0$ and then $\sum_{\mathbf{x}\in\mathbb{Z}^2} |\operatorname{Cov}[V(\mathbf{x}), V(\mathbf{0})]| < +\infty$. Using the dominated convergence theorem we get that

$$\lim_{k\to+\infty} |\omega_k|^{-1} \sum_{\mathbf{x},\mathbf{y}\in\omega_k} \operatorname{Cov}[V(\mathbf{x}), V(\mathbf{y})] = \sum_{\mathbf{x}\in\mathbb{Z}^2} \operatorname{Cov}[V(\mathbf{x}), V(\mathbf{0})].$$

We obtain equations (B.1) and (B.2) by applying Theorems B.3 and B.4. □

Lemma B.7. *Let $f \in \mathbb{R}^{\mathbb{Z}^2}$, $f \neq 0$, a function with finite support. Then it holds for any $\mathbf{t} \in \mathbb{Z}^2$, $\Gamma_f(\mathbf{t}) \leqslant \Gamma_f(\mathbf{0})$, with equality if and only if $\mathbf{t} = \mathbf{0}$.*

Proof. For any $\mathbf{t} \in \mathbb{Z}^2$, let $\tau_{\mathbf{t}} f = f(\cdot + \mathbf{t})$. By the definition of the autocorrelation Γ_f and using the Cauchy-Schwarz inequality we get that for any $\mathbf{t} \in \mathbb{Z}^2$

$$\Gamma_f(\mathbf{t}) = \langle \tau_{\mathbf{t}} f, f \rangle \leqslant \|f\|_2^2 \leqslant \Gamma_f(\mathbf{0}) \ ,$$

with equality if and only if $f = \alpha \tau_{\mathbf{t}} f$, with $\alpha \neq 0$ since $f \neq 0$. This implies that $\mathrm{Supp}\,(\tau_{\mathbf{t}}(f)) = \mathrm{Supp}\,(f)$. As a consequence $\mathbf{t} = \mathbf{0}$, which concludes the proof. □

The following lemma ensures that items (a) and (b) in Theorem B.4 are satisfied in the template similarity case when imposing summability conditions over v.

Lemma B.8. *Under the hypotheses of Theorem 3.4, assuming (A3.2) with $\ell \in \mathbb{N}$ and $p = 2\ell$. There exist $\mu_{p,p} > 0$ and $\sigma_{p,p} \geqslant 0$ such that for any $\mathbf{t} \in \Omega$*

(a) $\lim_{k \to +\infty} |\omega_k|^{\frac{1}{2}} \left(\frac{1}{|\omega_k|} \mathbb{E}\left[\mathcal{TS}_{p,p}(U, v, \mathbf{t}, \omega_k)\right] - \mu_{p,p}(\mathbf{t}) \right) = 0$;

(b) $\lim_{k \to +\infty} \frac{1}{|\omega_k|} \mathrm{Var}\left[\mathcal{TS}_{p,p}(U, v, \mathbf{t}, \omega_k)\right] = \sigma_{p,p}^2(\mathbf{t})$.

Proof. (a) For any $k \in \mathbb{N}$ we have that

$$\begin{aligned}
\mathbb{E}\left[\mathcal{TS}_{p,p}(U, v, \mathbf{t}, \omega_k)\right] &= \sum_{\mathbf{x} \in \omega_k} \mathbb{E}\left[(v(\mathbf{x}) - U(\mathbf{x} + \mathbf{t}))^{2\ell}\right] \\
&= \sum_{j=0}^{2\ell} \binom{2\ell}{j} \sum_{\mathbf{x} \in \omega_k} (-1)^j v(\mathbf{x})^j \mathbb{E}\left[U(\mathbf{x})^{2\ell - j}\right] \\
&= \sum_{j=0}^{\ell} \binom{2\ell}{2j} \sum_{\mathbf{x} \in \omega_k} v(\mathbf{x})^{2j} \mathbb{E}\left[U(\mathbf{x})^{2(\ell - j)}\right] = \sum_{j=0}^{\ell} \binom{2\ell}{2j} \mathbb{E}\left[U(\mathbf{0})\right]^{2(\ell - j)} \sum_{\mathbf{x} \in \omega_k} v(\mathbf{x})^{2j}.
\end{aligned}$$

Let $\mu_{p,p} = \sum_{j=0}^{\ell} \binom{2\ell}{2j} \mathbb{E}\left[U(\mathbf{0})\right]^{2(\ell - j)} \beta_j$ and using (a) of (A3.2) we get that

$$\lim_{k \to +\infty} |\omega_k|^{\frac{1}{2}} \left(\frac{1}{|\omega_k|} \mathbb{E}\left[\mathcal{TS}_{p,p}(U, v, \mathbf{t}, \omega_k)\right] - \mu_{p,p}(\mathbf{t}) \right) = 0.$$

Now since $\mu_{p,p} \geqslant \mathbb{E}\left[U(\mathbf{0})^{2\ell}\right] \geqslant \mathbb{E}\left[U(\mathbf{0})^2\right]^{\ell} \geqslant \Gamma_f(\mathbf{0}) > 0$ we have that $\mu_{p,p} > 0$.
(b) For any $k \in \mathbb{N}$ we have that

$$\begin{aligned}
\mathrm{Var}\left[\mathcal{TS}_{p,p}(U, v, \mathbf{t}, \omega_k)\right] &= \sum_{\mathbf{x}, \mathbf{y} \in \omega_k} \mathrm{Cov}\left[(U(\mathbf{x}) - v(\mathbf{x}))^{2\ell}, (U(\mathbf{y}) - v(\mathbf{y}))^{2\ell}\right] \\
&= \sum_{\mathbf{x}, \mathbf{y} \in \omega_k} \sum_{i,j=0}^{\ell} \binom{2\ell}{2i} \binom{2\ell}{2j} v(\mathbf{x})^{2i} v(\mathbf{y})^{2j} \, \mathrm{Cov}\left[U(\mathbf{x})^{2(\ell - i)}, U(\mathbf{y})^{2(\ell - j)}\right] \\
&= \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{Z}^2} \sum_{i,j=0}^{\ell} \binom{2\ell}{2i} \binom{2\ell}{2j} v_k(\mathbf{x})^{2i} v_k(\mathbf{x} + \mathbf{y})^{2j} \, \mathrm{Cov}\left[U(\mathbf{y})^{2(\ell - i)}, U(\mathbf{0})^{2(\ell - j)}\right] \\
&= \sum_{i,j=0}^{\ell} \binom{2\ell}{2i} \binom{2\ell}{2j} \left\langle v_k^{2i} * \check{v}_k^{2j}, \mathrm{Cov}\left[U(\cdot)^{2(\ell - i)}, U(\mathbf{0})^{2(\ell - j)}\right] \right\rangle.
\end{aligned}$$

Let $\sigma_{p,p} = \sum_{i,j=0}^{\ell} \binom{2\ell}{2i}\binom{2\ell}{2j} \left\langle \gamma_{i,j}, \mathrm{Cov}\left[U(\cdot)^{2(\ell-i)}, U(\mathbf{0})^{2(\ell-j)}\right]\right\rangle$. Using (b) in (A3.2) we can conclude. □

Note that this lemma is also valid in the continuous case.

Appendix C. Asymptotic theorems – continuous case

We now turn to the continuous setting. We start by stating the continuous counterparts of Theorems B.3 and B.4. The following theorem, given here for completeness, can be found with different assumptions (in the one-dimensional case) in [42].

Theorem C.1. *Let $d \in \mathbb{N}$. Let $(m_k)_{k\in\mathbb{N}}, (n_k)_{k\in\mathbb{N}}$ be two positive increasing integer sequences and $(\omega_k)_{k\in\mathbb{N}}$ be the sequence of subsets given for any $k \in \mathbb{N}$ by, $\omega_k = [0, m_k] \times [0, n_k]$. Let V be a d-dimensional R-independent random field over $\mathbb{R}^2$, with $R \geqslant 0$, such that $\|V(\mathbf{x}) - \mathbb{E}[V(\mathbf{x})]\|$ is uniformly stochastically dominated by $\widetilde{V}$, a stationary random field of order $r > 2$ over $\mathbb{R}^2$. Then V is a second-order random field. In addition, assume V is sample path continuous and that there exists $\mu \in \mathbb{R}^d$ given by $\lim_{k\to+\infty} |\omega_k|^{-1} \int_{\mathbf{x}\in\omega_k} \mathbb{E}[V(\mathbf{x})]\,\mathrm{d}\mathbf{x} = \mu$. Then it holds that*

$$\lim_{k\to+\infty} |\omega_k|^{-1} \int_{\mathbf{x}\in\omega_k} V(\mathbf{x})\mathrm{d}\mathbf{x} \underset{a.s.}{=} \mu. \tag{C.1}$$

Proof. Without loss of generality we can suppose that $d = 1$ and that for any $\mathbf{x} \in \Omega$, $\mathbb{E}[V(\mathbf{x})] = 0$. Let $(\sigma_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ given for any $k \in \mathbb{N}$ by

$$\sigma_k^2 = \mathbb{E}\left[\left(k^{-2}\int_{\Omega_k} V(\mathbf{x})\mathrm{d}\mathbf{x}\right)^2\right] , \tag{C.2}$$

with $\Omega_k = [0, k]^2$. Since V is R-independent, for any $\mathbf{x}, \mathbf{y} \in \Omega$ such that $\|\mathbf{x} - \mathbf{y}\|_\infty > R$, we have $C(\mathbf{x}, \mathbf{y}) = 0$. Hence for k large enough we obtain

$$\int_{\Omega_k}\int_{\Omega_k} C(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \leqslant \int_{\mathbf{x}\in\Omega_k}\int_{\|\mathbf{y}\|_\infty\leqslant R} |C(\mathbf{x},\mathbf{x}+\mathbf{y})|\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} \leqslant k^2|\bar{\mathrm{B}}_\infty(0,R)| \sup_{\Omega_k\times\bar{\mathrm{B}}_\infty(0,R)} |C(\mathbf{x},\mathbf{x}+\mathbf{y})|. \tag{C.3}$$

Using that $\widetilde{V}$ uniformly stochastically dominates $|V|$, the stationarity of $\widetilde{V}$, and the Cauchy-Schwarz inequality, we obtain for any $\mathbf{x}, \mathbf{y} \in \Omega$,

$$|C(\mathbf{x},\mathbf{x}+\mathbf{y})| = |\mathbb{E}[V(\mathbf{x})V(\mathbf{x}+\mathbf{y})]| \leqslant \mathbb{E}\left[\widetilde{V}^2(\mathbf{x})\right]^{1/2}\mathbb{E}\left[\widetilde{V}^2(\mathbf{x}+\mathbf{y})\right]^{1/2} \leqslant \mathbb{E}\left[\widetilde{V}^2(\mathbf{0})\right]. \tag{C.4}$$

Combining (C.2), (C.3) and (C.4) we get that for any $k \in \mathbb{N}$

$$\sigma_k^2 \leqslant Mk^{-2} ,$$

with $M = |\bar{\mathrm{B}}_\infty(0,R)|\mathbb{E}\left[\widetilde{V}^2(\mathbf{0})\right]$. Thus the series $\sum_{k\in\mathbb{N}} \sigma_k^2$ converges and $\sum_{k\in\mathbb{N}}\left(k^{-2}\int_{\Omega_k} V(\mathbf{x})\mathrm{d}\mathbf{x}\right)^2$ is finite almost surely. This proves that $\lim_{k\to+\infty} k^{-2}\int_{\Omega_k} V(\mathbf{x})\mathrm{d}\mathbf{x} = 0$ almost surely. Using [16, p. 95] we get that

$$\lim_{k\to+\infty} \sup_{\Omega_k\subset\omega\subset\Omega_{k+1}} \left| |\omega|^{-1}\int_\omega V(\mathbf{x})\mathrm{d}\mathbf{x} - k^{-2}\int_{\Omega_k} V(\mathbf{x})\mathrm{d}\mathbf{x}\right| \underset{a.s.}{=} 0.$$

Combining this result with $\lim_{k\to+\infty} k^{-2}\int_{\Omega_k} V(\mathbf{x})\mathrm{d}\mathbf{x} = 0$ implies that $\lim_{k\to+\infty} |\omega_k|^{-1}\int_{\mathbf{x}\in\omega_k} V(\mathbf{x})\mathrm{d}\mathbf{x} \underset{a.s.}{=} 0$. □

The following theorem is an application of ([33], Thm. 1.7.1).

Theorem C.2. *Under the hypotheses of Theorem C.1 and assuming that there exist $\mu \in \mathbb{R}^d$ and $C \in \mathcal{M}_d(\mathbb{R})$ such that*

(a) $\lim_{k\to+\infty} |\omega_k|^{-1/2}\int_{\mathbf{x}\in\omega_k} (\mathbb{E}[V](\mathbf{x}) - \mu)\,\mathrm{d}\mathbf{x} = 0$;
(b) $\lim_{k\to+\infty} |\omega_k|^{-1}\int_{\mathbf{x},\mathbf{y}\in\omega_k} \mathrm{Cov}[V(\mathbf{x}), V(\mathbf{y})]\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} = C$.

Then it holds that

$$\lim_{k\to+\infty} |\omega_k|^{-1/2}\int_{\mathbf{x}\in\omega_k} (V(\mathbf{x}) - \mu)\,\mathrm{d}\mathbf{x} \underset{\mathcal{L}}{=} \mathcal{N}(0, C). \tag{C.5}$$

Proof. Let $a \in \mathbb{R}^d$. We consider the d-dimensional random field ξ over $\mathbb{R}^2$ defined for any $\mathbf{x} \in \mathbb{R}^2$ by $\xi(\mathbf{x}) = V(\mathbf{x}) - \mathbb{E}[V(\mathbf{x})]$. We define also the weight functions $(g_n)_{n\in\mathbb{N}}$ given for any $n \in \mathbb{N}$ by $g_n(\mathbf{x}) = |\omega_n|^{-1/2}\mathbb{1}_{\mathbf{x}\in\omega_n}$. For any $n \in \mathbb{N}$, let $S_n = \int_{\mathbb{R}^2} g_n(\mathbf{x})\xi(\mathbf{x})\mathrm{d}\mathbf{x}$. We have for any $n \in \mathbb{N}$,

$$S_n = |\omega_n|^{-1/2}\int_{\mathbf{x}\in\omega_n} (V(\mathbf{x}) - \mathbb{E}[V(\mathbf{x})])\mathrm{d}\mathbf{x}.$$

Let ξ^a be the one-dimensional random field over $\mathbb{R}^2$ such that for any $\mathbf{x}\tilde{A} \in \mathbb{R}^2$, $\xi^a(\mathbf{x}) = \langle a, \xi(\mathbf{x})\rangle$ and $(S_n^a)_{n\in\mathbb{N}}$ be the sequence of real-valued random variables such that for any $n \in \mathbb{N}$, $S_n^a = \langle a, S_n\rangle$. Then for any $n \in \mathbb{N}$, $S_n^a = \int_{\mathbb{R}^2} g_n(\mathbf{x})\xi^a(\mathbf{x})\mathrm{d}\mathbf{x}$. Using (b) we have that

$$\lim_{n\to+\infty} \mathbb{E}\left[(S_n^a)^2\right] = \lim_{n\to+\infty} |\omega_n|^{-1}\int_{\mathbf{x},\mathbf{y}\in\omega_n} a^\top \mathrm{Cov}[V(\mathbf{x}), V(\mathbf{y})]\,a = a^\top C a. \tag{C.6}$$

By assumption, ξ^a is stochastically dominated by $\|a\|\widetilde{V}$ and therefore for any $\mathbf{x} \in \mathbb{R}^2$ we have

$$\mathbb{E}[|\xi^a|^r] < +\infty. \tag{C.7}$$

Combining (C.6), (C.7), the fact that V is R-independent ([33], Thm. 1.7.1) and (a) we obtain that

$$\lim_{k\to+\infty} \left\langle a, |\omega_k|^{-1/2}\int_{\mathbf{x}\in\omega_k} (V(\mathbf{x}) - \mu)\,\mathrm{d}\mathbf{x}\right\rangle \underset{\mathcal{L}}{=} \mathcal{N}(0, a^\top C a). \tag{C.8}$$

We conclude the proof upon using the Cramér-Wold theorem ([15], Thm. 1).

□

The following lemmas are the continuous versions of Lemma B.6 and B.7.

Lemma C.3. *Let $d \in \mathbb{N}$. Let $(m_k)_{k\in\mathbb{N}}, (n_k)_{k\in\mathbb{N}}$ be two positive increasing integer sequences and $(\omega_k)_{k\in\mathbb{N}}$ be the sequence of subsets such that for any $k \in \mathbb{N}$, $\omega_k = [0, m_k] \times [0, n_k]$. Let V be a d-dimensional, R-independent random field of order $r > 2$ over $\mathbb{R}^2$, with $R \geqslant 0$. Assume that V is sample path continuous, then for all $k \in \mathbb{N}$*

(a) $|\omega_k|^{-1}\int_{\mathbf{x}\in\omega_k} \mathbb{E}[V(\mathbf{x})]\,\mathrm{d}\mathbf{x} = \mathbb{E}[V(\mathbf{0})]$;
(b) $\lim_{k\to+\infty} |\omega_k|^{-1}\int_{\mathbf{x},\mathbf{y}\in\omega_k} \mathrm{Cov}[V(\mathbf{x}), V(\mathbf{y})]\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} = \int_{\mathbf{x}\in\mathbb{R}^2} \mathrm{Cov}[V(\mathbf{x}), V(\mathbf{0})]\,\mathrm{d}\mathbf{x}$.

In addition, (C.1) *and* (C.5) *hold with $\mu = \mathbb{E}[V(\mathbf{0})]$ and $C = \int_{\mathbf{x}\in\mathbb{R}^2} \mathrm{Cov}[V(\mathbf{x}), V(\mathbf{0})]\,\mathrm{d}\mathbf{x}$.*

Proof. (a) The proof is immediate since for any $\mathbf{x} \in \mathbb{R}^2$, $\mathbb{E}[V(\mathbf{x})] = \mathbb{E}[V(\mathbf{0})]$.

(b) For any $k \in \mathbb{N}$ we have by stationarity

$$|\omega_k|^{-1} \int_{\mathbf{x},\mathbf{y} \in \omega_k} \mathrm{Cov}\left[V(\mathbf{x}), V(\mathbf{y})\right] \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} = |\omega_k|^{-1} \int_{\mathbf{x},\mathbf{y} \in \omega_k} \mathrm{Cov}\left[V(\mathbf{x}-\mathbf{y}), V(\mathbf{0})\right] \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}.$$

By the Fubini-Lebesgue theorem we obtain that for any $k \in \mathbb{N}$,

$$|\omega_k|^{-1} \int_{\mathbf{x},\mathbf{y} \in \omega_k} \mathrm{Cov}\left[V(\mathbf{x}), V(\mathbf{y})\right] \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} = \int_{\mathbf{x} \in \mathbb{R}^2} \mathrm{Cov}\left[V(\mathbf{x}), V(\mathbf{0})\right] g_k(\mathbf{x}) \mathrm{d}\mathbf{x} \ ,$$

where $g_k \in L^\infty(\mathbb{R}^2)$ satisfies for any $\mathbf{x} \in \mathbb{R}^2$, $g_k(\mathbf{x}) = |\omega_k|^{-1} \mathbb{1}_{\omega_k} * \check{\mathbb{1}}_{\omega_k}(\mathbf{x})$. For any $k \in \mathbb{N}$, $\mathbf{x} \in \mathbb{R}^2$ we have $0 \leqslant g_k(\mathbf{x}) \leqslant 1$ and $\lim_{k \to +\infty} g_k(\mathbf{x}) = 1$. For any $\mathbf{x} \in \mathbb{R}^2$ such that $\|\mathbf{x}\|_\infty > R_{\mathbf{t}}$, $\mathrm{Cov}\left[V(\mathbf{x}), V(\mathbf{0})\right] = 0$ and then

$$\int_{\mathbf{x} \in \mathbb{R}^2} |\,\mathrm{Cov}\left[V(\mathbf{x}), V(\mathbf{0})\right]| \mathrm{d}\mathbf{x} < +\infty.$$

Using the dominated convergence theorem we get that

$$|\omega_k|^{-1} \int_{\mathbf{x},\mathbf{y} \in \omega_k} \mathrm{Cov}\left[V(\mathbf{x}), V(\mathbf{y})\right] \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} = \int_{\mathbf{x} \in \mathbb{R}^2} \mathrm{Cov}\left[V(\mathbf{x}), V(\mathbf{0})\right] \mathrm{d}\mathbf{x} \ ,$$

Since V is R-independent we conclude the proof by applying Theorem C.1 and C.2.

□

Lemma C.4. *Let Γ be a function over $\mathbb{R}^2$, $\Gamma \neq 0$, such that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, $C(\mathbf{x}, \mathbf{y}) = \Gamma(\mathbf{x} - \mathbf{y})$ with C the covariance function of V a second-order random field over $\mathbb{R}^2$. Assume that Γ has finite support. Then it holds for any $\mathbf{t} \in \mathbb{R}^2$, $\Gamma(\mathbf{t}) \leqslant \Gamma(\mathbf{0})$, with equality if and only if $\mathbf{t} = \mathbf{0}$.*

Proof. Upon replacing for any $\mathbf{x} \in \mathbb{R}^2$, $V(\mathbf{x})$ by $V(\mathbf{x}) - \mathbb{E}\left[V(\mathbf{x})\right]$ we suppose that $\mathbb{E}\left[V(\mathbf{x})\right] = 0$. Using the Cauchy-Schwarz inequality and the stationarity of V we get for any $\mathbf{t} \in \mathbb{R}^2$ and $\mathbf{x} \in \mathbb{R}^2$

$$\Gamma(\mathbf{t}) = \mathbb{E}\left[V(\mathbf{x}+\mathbf{t})V(\mathbf{x})\right] \leqslant \mathbb{E}\left[V(\mathbf{x}+\mathbf{t})^2\right]^{1/2} \mathbb{E}\left[V(\mathbf{x})^2\right]^{1/2} \leqslant \mathbb{E}\left[V(\mathbf{x})^2\right] \leqslant \Gamma(\mathbf{0}).$$

with equality if and only if $V(\mathbf{x}+\mathbf{t}) = \alpha(\mathbf{x})V(\mathbf{x})$ with $\alpha(\mathbf{x}) \in \mathbb{R}$. Since V is stationary and $V \neq 0$ we get that for any $\mathbf{x}, \mathbf{t} \in \mathbb{R}^2$, $\mathbb{E}\left[V(\mathbf{x}+\mathbf{t})^2\right] = \mathbb{E}\left[V(\mathbf{x})^2\right] > 0$. Thus $\alpha(\mathbf{x})^2 = 1$ and for all $n \in \mathbb{N}$, $V(n\mathbf{t}) = \pm V(\mathbf{0})$. If $\mathbf{t} \neq 0$ then there exists $n \in \mathbb{N}$ such that $n\mathbf{t} \notin \mathrm{Supp}\,(\Gamma)$ and then we have

$$0 = \Gamma(n\mathbf{t}) = \mathbb{E}\left[V(n\mathbf{t})V(\mathbf{0})\right] = \pm \mathbb{E}\left[V(\mathbf{0}^2)\right] \neq 0 \ ,$$

which is absurd. Thus the equality in the inequality holds if and only if $\mathbf{t} = 0$. □

Appendix D. Explicit constants

In order to derive precise constants in Theorems 3.4 and 3.6 we use the following lemma which is a consequence of the Isserlis formula [32].

Lemma D.1. *Let U and V be two zero-mean, real-valued Gaussian random variable and $k, \ell \in \mathbb{N}$. We have*

$$\mathbb{E}\left[U^{2k}V^{2\ell}\right] = \sum_{j=0}^{k\wedge\ell} r_{j,k,\ell}\mathbb{E}\left[U^2\right]^{k-j}\mathbb{E}\left[V^2\right]^{\ell-j}\mathbb{E}\left[UV\right]^{2j} \quad \textit{and} \quad \mathrm{Cov}\left[U^{2k}, V^{2\ell}\right] = \sum_{j=1}^{k\wedge\ell} r_{j,k,\ell}\mathbb{E}\left[U^2\right]^{k-j}\mathbb{E}\left[V^2\right]^{\ell-j}\mathbb{E}\left[UV\right]^{2j} \ ,$$

with $r_{j,k,\ell}$ defined by (3.4).

Proof. Let $k, \ell \in \mathbb{N}$. Using Isserlis formula [32] we obtain that $\mathbb{E}\left[U^{2k}V^{2\ell}\right]$ is the sum over all the partitions in pairs of $\{\underbrace{U, \ldots, U}_{2k \text{ times}}, \underbrace{V, \ldots, V}_{2\ell \text{ times}}\}$ of the product of the expectations given by a pair partition. Given a pair partition we identify three different cases, $\{U, U\}$, $\{V, V\}$ and $\{U, V\}$. We only need to count the number of times each case appears in the sum. We denote the number of $\{U, U\}$ couples in a given pair partition p by $n_{U,U}(p)$. In the same fashion we define $n_{U,V}(p)$ and $n_{V,V}(p)$. We have $2k = 2n_{U,U}(p) + n_{U,V}(p)$ which proves that $n_{U,V}(p)$ is even. We denote by $\mathscr{P}_j$ the number of pair partitions p such that $n_{U,V}(p) = 2j$, with $j \in [\![0, k\wedge\ell]\!]$.

The cardinality of $\mathscr{P}_j$ is given by $r_{j,k,\ell}$. Indeed, in order to select $2j$ pair $\{U, V\}$ we select $2j$ elements among $2k$ (selection of replicates of U), same for V which gives $\binom{2k}{2j}\binom{2\ell}{2j}$ possibilities. Considering all the bijections between these elements we construct all the possible $2j$ pairs $\{U, V\}$. Given $2j$ pairs $\{U, V\}$ we must construct $k - j$ pairs $\{U, U\}$ and $\ell - j$ pairs $\{V, V\}$ in order to obtain a pair partition of $\mathscr{P}_j$. The number of pairs partition of a set with $\ell - j$ elements is given $q_{\ell-j}$. As a consequence we obtain for all $j \in [\![0, k\wedge\ell]\!]$

$$|\mathscr{P}_j| = q_{k-j}q_{\ell-j}\binom{2k}{2j}\binom{2\ell}{2j}(2j)! = r_{j,k,\ell}.$$

Summing over $j \in [\![0, k\wedge\ell]\!]$ we obtain all the possible pair partition and we get

$$\mathbb{E}\left[U^{2k}V^{2\ell}\right] = \sum_{j=0}^{k\wedge\ell} r_{j,k,\ell}\mathbb{E}\left[U^2\right]^{k-j}\mathbb{E}\left[V^2\right]^{\ell-j}\mathbb{E}\left[UV\right]^{2j}. \tag{D.1}$$

Using that $r_{0,k} = q_k^2$, respectively $r_{0,\ell} = q_\ell^2$ and $\mathbb{E}\left[U^{2k}\right] = q_k\mathbb{E}\left[U^2\right]^k$, respectively $\mathbb{E}\left[V^{2\ell}\right] = q_\ell\mathbb{E}\left[V^2\right]^\ell$, we obtain that the first term in the right-hand side sum of (D.1) is equal to $\mathbb{E}\left[U^{2k}\right]\mathbb{E}\left[V^{2\ell}\right]$. Hence by removing this term we obtain the covariance and conclude the proof. □

Proof of Proposition 3.5. The proof is divided into three parts. First we consider the case $i = p$ then the case $i = sc$ and finally the case $i = \cos$.

1. Let $i = p$ with $p = 2\ell$ and $\ell \in \mathbb{N}$, $\mathbf{t} \in \mathbb{Z}^2\backslash\{\mathbf{0}\}$ and $V_\mathbf{t}$ the Gaussian random field given for any $\mathbf{x} \in \mathbb{Z}^2$ by $V_\mathbf{t}(\mathbf{x}) = U(\mathbf{x}) - U(\mathbf{x} + \mathbf{t})$. Note that for all $\mathbf{x} \in \mathbb{Z}^2$ we have $V_\mathbf{t}(\mathbf{x})^{2\ell} = V_{p,\mathbf{t}}(\mathbf{x})$. For any $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^2$ we have

$$\mathbb{E}\left[V_\mathbf{t}(\mathbf{x})\right] = 0 \ , \qquad \mathrm{Cov}\left[V_\mathbf{t}(\mathbf{x}), V_\mathbf{t}(\mathbf{y})\right] = 2\Gamma_f(\mathbf{x} - \mathbf{y}) - \Gamma_f(\mathbf{x} - \mathbf{y} - \mathbf{t}) - \Gamma_f(\mathbf{x} - \mathbf{y} + \mathbf{t}) = \Delta_f(\mathbf{t}, \mathbf{x} - \mathbf{y}) \ , \tag{D.2}$$

with Δ_f given by (3.5). We show in proof of Theorem 3.4, see equation (3.1), that for any $\mathbf{t} \in \mathbb{Z}^2\backslash\{\mathbf{0}\}$

$$\mu_p(\mathbf{t}) = \mathbb{E}\left[V_\mathbf{t}^{2\ell}(\mathbf{0})\right]^{1/2\ell} \ , \qquad \sigma_p(\mathbf{t})^2 = \sum_{\mathbf{x}\in\mathbb{Z}^2} \mathrm{Cov}\left[V_\mathbf{t}^{2\ell}(\mathbf{x}), V_\mathbf{t}^{2\ell}(\mathbf{0})\right]\mathbb{E}\left[V_\mathbf{t}^{2\ell}(\mathbf{0})\right]^{1/\ell-2}/(2\ell)^2. \tag{D.3}$$

Combining (D.2), (D.3) and Lemma D.1 we get that

(a) $\mu_p(\mathbf{t}) = q_{2\ell}^{1/(2\ell)} \Delta_f(\mathbf{t}, \mathbf{0})^{1/2}$;
(b) $\sigma_p(\mathbf{t})^2 = \sum_{\mathbf{x}\in\mathbb{Z}^2} \left(\sum_{j=1}^{\ell} r_{j,\ell} \Delta_f(\mathbf{t}, \mathbf{0})^{2(\ell-j)} \Delta_f(\mathbf{t}, \mathbf{x})^{2j}\right) q_\ell^{1/\ell-2} \Delta_f(\mathbf{t}, \mathbf{0})^{1-2\ell}/(2\ell)^2$.

Exchanging the sums in (b) we get $\sigma_p(\mathbf{t})^2 = \frac{q_\ell^{1/\ell-2}}{(2\ell)^2} \sum_{j=1}^{\ell} r_{j,\ell} \left(\frac{\|\Delta_f(\mathbf{t},\cdot)\|_{2j}}{\Delta_f(\mathbf{t},\mathbf{0})}\right)^{2j} \Delta_f(\mathbf{t}, \mathbf{0})$.

2. Let $i = sc$, $\mathbf{t} \in \mathbb{Z}^2 \backslash \{\mathbf{0}\}$ and $V_{sc,\mathbf{t}}$ be a Gaussian random field given for any $\mathbf{x} \in \mathbb{Z}^2$, by $V_{\mathbf{t}}(\mathbf{x}) = U(\mathbf{x})U(\mathbf{x}+\mathbf{t})$. For any $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^2$ we have

$$\mathbb{E}\left[V_{sc,\mathbf{t}}(\mathbf{x})\right] = \Gamma_f(\mathbf{t}) , \qquad \mathrm{Cov}\left[V_{sc,\mathbf{t}}(\mathbf{x}), V_{sc,\mathbf{t}}(\mathbf{y})\right] = \Gamma_f(\mathbf{x}-\mathbf{y}) - \Gamma_f(\mathbf{x}-\mathbf{y}-\mathbf{t})\Gamma_f(\mathbf{x}-\mathbf{y}+\mathbf{t}) = \widetilde{\Delta}_f(\mathbf{t}, \mathbf{x}-\mathbf{y}) , \tag{D.4}$$

with $\widetilde{\Delta}_f$ given by (3.5). We show in the proof of Theorem 3.4, see (3.1), that for any $\mathbf{t} \in \mathbb{Z}^2 \backslash \{\mathbf{0}\}$

$$\mu_{sc}(\mathbf{t}) = \mathbb{E}\left[V_{sc,\mathbf{t}}(\mathbf{0})\right] , \qquad \sigma_{sc}(\mathbf{t})^2 = \sum_{\mathbf{x}\in\mathbb{Z}^2} \mathrm{Cov}\left[V_{sc,\mathbf{t}}(\mathbf{x}), V_{sc,\mathbf{t}}(\mathbf{0})\right]. \tag{D.5}$$

Combining (D.4) and (D.5) we get that
(a) $\mu_{sc}(\mathbf{t}) = \Gamma_f(\mathbf{t})$;
(b) $\sigma_{sc}(\mathbf{t})^2 = \sum_{\mathbf{x}\in\mathbb{Z}^2} \widetilde{\Delta}_f(\mathbf{t}, \mathbf{x})$,
which concludes the proof in the case $i = sc$.
3. We now consider the case $i = \cos$. Recall that in the proof of Theorem 3.4 we show that

$$\mathcal{AS}_{s_{\cos}}(U, \mathbf{t}, \omega_k) = h\left(|\omega_k|^{-1} \sum_{\mathbf{x}\in\omega_k} V_{\cos,\mathbf{t}}(\mathbf{x})\right) ,$$

where for any $x \in \mathbb{R}$, $y, z > 0$

$$h(x, y, z) = xy^{-1/2}z^{-1/2} , \qquad V_{\cos,\mathbf{t}}(\mathbf{x}) = \begin{pmatrix} -U(\mathbf{x})U(\mathbf{x}+\mathbf{t}) \\ U(\mathbf{x})^2 \\ U(\mathbf{x}+\mathbf{t})^2 \end{pmatrix}.$$

Applying Lemma B.6 there exist $\tilde{\mu}_{\cos}(\mathbf{t})$ and $\tilde{C}_{\cos}(\mathbf{t})$ such that

(a) $\lim_{k\to+\infty} \frac{1}{|\omega_k|} V_{\cos,\mathbf{t}} \underset{a.s.}{=} \tilde{\mu}_{\cos}(\mathbf{t})$;
(b) $\lim_{k\to+\infty} |\omega_k|^{\frac{1}{2}} \left(\frac{1}{|\omega_k|} V_{\cos,\mathbf{t}} - \tilde{\mu}_{\cos}(\mathbf{t})\right) \underset{\mathcal{L}}{=} \mathcal{N}\left(0, \tilde{C}_{\cos}(\mathbf{t})\right)$,

with

$$\begin{aligned} \tilde{\mu}_{\cos}(\mathbf{t}) &= \begin{pmatrix} \Gamma_f(\mathbf{t}) \\ \Gamma_f(\mathbf{0}) \\ \Gamma_f(\mathbf{0}) \end{pmatrix} , \\ \tilde{C}_{\cos}(\mathbf{t}) &= \begin{pmatrix} \|\Gamma_f\|^2 + \Gamma_f * \check{\Gamma}_f(2\mathbf{t}) & 2\Gamma_f * \check{\Gamma}_f(\mathbf{t}) & 2\Gamma_f * \check{\Gamma}_f(\mathbf{t}) \\ 2\Gamma_f * \check{\Gamma}_f(\mathbf{t}) & 2\|\Gamma_f\|^2 & 2\|\Gamma_f\|^2 \\ 2\Gamma_f * \check{\Gamma}_f(\mathbf{t}) & 2\|\Gamma_f\|^2 & 2\|\Gamma_f\|^2 \end{pmatrix}. \end{aligned} \tag{D.6}$$

In addition, for any $x \in \mathbb{R}$, $y, z > 0$

$$\nabla h(x, y, z) = \begin{pmatrix} y^{-1/2}z^{-1/2} \\ -(1/2)xy^{-3/2}z^{-1/2} \\ -(1/2)xy^{-1/2}z^{-3/2} \end{pmatrix}.$$

Combining this result, (D.6) and the multivariate Delta method we get that

$$\mu_{\cos}(\mathbf{t}) = h(\Gamma_f(\mathbf{t}), \Gamma_f(\mathbf{0}), \Gamma_f(\mathbf{0})) = \Gamma_f(\mathbf{t})/\Gamma_f(\mathbf{0}) \ ,$$

$$\sigma_{\cos}(\mathbf{t})^2 = \nabla h(\Gamma_f(\mathbf{t}), \Gamma_f(\mathbf{0}), \Gamma_f(\mathbf{0}))^\top \begin{pmatrix} \|\Gamma_f\|^2 + \Gamma_f * \check{\Gamma}_f(2\mathbf{t}) & 2\Gamma_f * \check{\Gamma}_f(\mathbf{t}) & 2\Gamma_f * \check{\Gamma}_f(\mathbf{t}) \\ 2\Gamma_f * \check{\Gamma}_f(\mathbf{t}) & 2\|\Gamma_f\|^2 & 2\|\Gamma_f\|^2 \\ 2\Gamma_f * \check{\Gamma}_f(\mathbf{t}) & 2\|\Gamma_f\|^2 & 2\|\Gamma_f\|^2 \end{pmatrix} \nabla h(\Gamma_f(\mathbf{t}), \Gamma_f(\mathbf{0}), \Gamma_f(\mathbf{0}))$$

$$= \Gamma_f(\mathbf{0})^{-2} \left\{ \|\Gamma_f\|_2^2 \left(1 + 2\frac{\Gamma_f(\mathbf{t})^2}{\Gamma_f(\mathbf{0})^2}\right) - 4\frac{\Gamma_f(\mathbf{t})}{\Gamma_f(\mathbf{0})}\Gamma_f * \check{\Gamma}_f + \Gamma_f * \check{\Gamma}_f(2\mathbf{t}) \right\} \ ,$$

which concludes the proof. □

Proof of Proposition 3.7. The proof is divided in two parts. First we treat the case $i = p$ then the case $i = sc$ and $i = \cos$.

Let $p = 2\ell$ with $\ell \in \mathbb{N}$. Lemma B.8 gives us that

$$\mu_{p,p} = \sum_{j=0}^{\ell} \binom{2\ell}{2j} \mathbb{E}\left[U(\mathbf{0})\right]^{2(\ell-j)} \beta_j \quad \text{and} \quad \sigma_{p,p} = \sum_{i,j=0}^{\ell} \binom{2\ell}{2i}\binom{2\ell}{2j} \left\langle \gamma_{i,j}, \operatorname{Cov}\left[U(\cdot)^{2(\ell-i)}, U(\mathbf{0})^{2(\ell-j)}\right]\right\rangle.$$

Using Lemma D.1 we obtain that

$$\mu_{p,p} = \Gamma_f(\mathbf{0})^\ell \sum_{j=0}^{\ell} \binom{2\ell}{2j} q_{\ell-j} \Gamma_f(\mathbf{0})^{-j}\beta_j \quad \text{and} \quad \sigma_{p,p} = \sum_{i,j=0}^{\ell} \binom{2\ell}{2i}\binom{2\ell}{2j} \sum_{m=1}^{\ell-i\wedge\ell-j} r_{m,k,\ell} \left\langle \gamma_{i,j}, \Gamma_f^{2m}\right\rangle \Gamma_f(\mathbf{0})^{2\ell-i-j-2m}.$$

We conclude using (3.7).

For $i = sc$ and $i = \cos$, the result is given in the proof of Theorem 3.6.

□

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