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A PROBABILISTIC APPROACH TO QUASILINEAR PARABOLIC PDES WITH OBSTACLE AND NEUMANN PROBLEMS*

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Abstract. In this paper, by a probabilistic approach we prove that there exists a unique viscosity solution to obstacle problems of quasilinear parabolic PDEs combined with Neumann boundary conditions and algebra equations. The existence and uniqueness for adapted solutions of fully coupled forward-backward stochastic differential equations with reflections play a crucial role. Compared with existing works, in our result the spatial variable of solutions of PDEs lives in a region without convexity constraints, the second order coefficient of PDEs depends on the gradient of the solution, and the required conditions for the coefficients are weaker.

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1. Introduction

We will adopt a probabilistic approach to study the following obstacle problems of second order quasilinear parabolic PDEs combined with Neumann boundary conditions and algebra equations, with some slight abuse of notions,

$$\begin{cases} \big(u(t,x)-L(t,x)\big)\wedge\{(u(t,x)-U(t,x))\vee[-\partial_t u(t,x)-(\mathcal{L}_t u+g)(t,x,u,v)]\}=0,\\ v(t,x)=\sigma^*(t,x,u(t,x),v(t,x))\nabla u(t,x),\\ u(T,x)=\Phi(x),\quad x\in\mathcal{O}_1;\quad \dfrac{\partial u}{\partial n}(t,x)=0,\quad x\in\partial\mathcal{O}_1, \end{cases} \tag{1.1}$$

where we write $(\mathcal{L}_t u+g)(t,x,u,v):=\frac{1}{2}\operatorname{tr}\{(\sigma\sigma^*)(t,x,u(t,x),v(t,x))\mathrm{D}^2u(t,x)\}+\langle b(t,x,u(t,x),v(t,x)),\nabla u(t,x)\rangle + g(t,x,u(t,x),v(t,x))$. From the viewpoint of nonlinear Feynman-Kac formula, proposed by Peng [16], the above PDE system (1.1) should be related with the following fully coupled forward-backward stochastic differential

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equation with reflections (FBSDER for short):

$$\begin{cases} \mathrm{d}X_s^{t,x} = b(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})\,\mathrm{d}s + \sigma(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})\,\mathrm{d}B_s + \mathrm{d}\eta_s^{t,x}, \\ \mathrm{d}Y_s^{t,x} = g(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})\,\mathrm{d}s - \langle Z_s^{t,x},\, \mathrm{d}B_s\rangle + \mathrm{d}\zeta_s^{t,x}, \quad s \in [t, T], \\ X_t^{t,x} = x,\ Y_T^{t,x} = \Phi(X_T^{t,x}), \end{cases} \tag{1.2}$$

where $\eta_\cdot^{t,x}$ and $\zeta_\cdot^{t,x}$ are two adapted processes with bounded variations, which are the reflection processes that keep $X_\cdot^{t,x}$ living in a bounded connected region $\overline{\mathcal{O}}_1$ without convexity constraints and $Y_\cdot^{t,x}$ living in $\mathcal{O}_2 := [L, U]$, respectively. The probabilistic interpretation for PDE (1.1) can be expressed as $u(t, x) = Y_t^{t,x}$, in viscosity sense. Probabilistic approaches to obstacle problems and Neumann problems have been studied by many researchers. El Karoui, Kapoudjian, Pardoux, Peng and Quenez [5] first presented the existence and uniqueness for viscosity solutions of semilinear parabolic PDEs with one obstacle by backward stochastic differential equations (BSDEs for short) with reflections. Recently, Li, Peng and Liu [10] showed the quasilinear case by reflected BSDEs coupled with stochastic differential equations (SDEs for short) when $b(t, x, y, z) = b(t, x, y)$ and $\sigma(t, x, y, z) = \sigma(t, x)$, *i.e.*, b is independent of z and σ is independent of (y, z).

Pardoux and Tang [15] first proposed the probabilistic interpretation for viscosity solutions of quasilinear PDEs with Cauchy problems by forward-backward stochastic differential equations (FBSDEs for short), which is extended from the semilinear case of Peng [16] and Pardoux and Peng [13]. For technical reasons, however, their result requires that σ does not depend on z, then the second order coefficient σ of PDEs is independent of the gradient of solutions, $\nabla u(t, x)$. Abraham and Riviere [1] attempted to deal with the case of σ being dependent of z by imposing an identity of $\sigma\sigma^*$ involved in z. Nevertheless, they need that the coefficients b and g are independent of z. Recently, Wu and Yu [18] addressed this issue by solving an algebra equation (see the second identity in (1.1)) when $\sigma(t, x, y, z)$ satisfies Lipschitz continuity and a so called G-monotonicity conditions with respect to z. When showing the probabilistic interpretation for Hamilton-Jacobi-Bellman equation by a controlled system of fully coupled FBSDE, Li and Wei [9] also solved this algebra equation by a contraction method, which is different from the method of Wu and Yu [18]. Yet they impose a monotonicity condition on the test function in the definition of viscosity solutions for PDEs.

Earlier works on Neumann problems by probabilistic approaches are referred to Hsu [6], Hu [7] and the references therein. Ma and Cvitanić [11] first considered the obstacle problems of quasilinear PDEs with Neumann boundary conditions by FBSDER (1.2). They obtained both existence and uniqueness for viscosity solutions. However, the main weakness of their study is also the independence of the second order coefficient σ in the gradient of solutions $u(t, x)$. They still required that the spatial variable x of $u(t, x)$ lives in a closed convex region, and the coefficients $b(t, x, y, z)$ and $g(t, x, y, z)$ are differentiable in z when proving the existence result. For more precise results on probabilistic approach to Neumann problems we refer to Pardoux and Zhang [14], Ren and Xia [17], Biswas, Ishii, Saha and Wang [3], etc.

The main purpose of this paper is to generalize the results of Wu and Yu [18] and Ma and Cvitanić [11] to the quasilinear parabolic PDEs with obstacle and Neumann problems. We will prove that there exists a unique viscosity solution for quasilinear parabolic PDE (1.1) in a general setting. Our existence result will enjoy the following features: (i) the region $\overline{\mathcal{O}}_1$ of spatial variable x is not necessarily convex; (ii) the second order coefficient σ of PDEs may depend on the gradient of solution, $\nabla u(t, x)$; (iii) the algebra equation is solved without the G-monotonicity condition; (iv) the differentiability for coefficients b and g in z is eliminated. Our conclusions seem to be new because anyone of these four issues has not been addressed in the literature as far as we know. For the previous four issues to make sense, the conditions for coefficients of FBSDER (1.2) should be weakened accordingly. FBSDER (1.2) is solved under some monotonicity and Lipschitz conditions by the method of contraction mapping, since other approaches to FBSDEs are invalid for the case of FBSDERs (see Ma and Cvitanić [11] for details). The regularity of solutions with respect to the initial conditions is also proved in our framework. The uniqueness result for viscosity solutions is proved by the same methods of Barles, Buckdahn and Pardoux [2] and Ma and Cvitanić [11] with necessary modifications, under additional restrictions that $\sigma(t, x, y, z) = \sigma(t, x)$ and $g(t, x, y, z)$ is nondecreasing in y.

The rest of this paper is organized as follows: Section 2 gives all necessary preliminaries and the assumptions on coefficients of FBSDERs; Section 3 proves the existence and uniqueness for solutions of FBSDERs by a contraction mapping method; Section 4 shows the procedures of solving the algebra equation and the quasilinear parabolic PDEs with obstacle and Neumann problems; Appendix A illustrates some standard *a priori* estimates for FBSDERs.

2. Preliminaries and assumptions

Let $T > 0$ be a given real number, $(\Omega, \mathcal{F}, \mathbf{P})$ a probability space carrying a standard d-dimensional Brownian motion $(B_t)_{t\geq 0}$ and $(\mathcal{F}_t)_{t\geq 0}$ the natural σ-algebra filtration generated by $(B_t)_{t\geq 0}$. We assume that $\mathcal{F}_T = \mathcal{F}$ and $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous and complete. The Euclidean norms of a vector $x \in \mathbf{R}^n$ and a matrix $z \in \mathbf{R}^{n\times d}$ will be denoted by $|x|$ and $|z| := \sqrt{\mathrm{tr}\,(zz^*)}$, where and hereafter z^* represents the transpose of z. Let $\mathbf{H}$ be an Euclidean space. We denote by $\mathcal{S}^2(0,T;\mathbf{H})$ the set of $\mathbf{H}$-valued, $(\mathcal{F}_t)$-progressively measurable and continuous processes $(u_t)_{t\in[0,T]}$ such that $\mathbf{E}\big[\sup_{t\in[0,T]} |u_t|^2\big] < \infty$. Let $\mathcal{H}^2(0,T;\mathbf{H})$ denote the set of all $\mathbf{H}$-valued and $(\mathcal{F}_t)$-progressively measurable processes $(v_t)_{t\in[0,T]}$ satisfying that $\|v\|^2_{\mathcal{H}^2} := \mathbf{E}\big[\int_0^T |v_s|^2\,\mathrm{d}s\big] < \infty$. Moreover, let $\mathcal{A}(0,T;\mathbf{H})$ represent the set of all $\mathbf{H}$-valued, continuous and $(\mathcal{F}_t)$-progressively measurable processes whose paths are of bounded variation and vanish at initial time $t = 0$.

Let $\mathcal{O}_1$ be an open connected bounded subset of $\mathbf{R}^n$ given by $\mathcal{O}_1 = \{x \in \mathbf{R}^n : \phi(x) > 0\}$ with $\phi \in \mathrm{C}^2(\mathbf{R}^n;\mathbf{R})$, and such that $\partial\mathcal{O}_1 = \{x \in \mathbf{R}^n : \phi(x) = 0\}$, with $\nabla\phi(x) = 1$ for all $x \in \partial\mathcal{O}_1$. Observe that $\nabla\phi(x)$ coincides with the unit normal pointing toward the interior of $\mathcal{O}_1$ at $x \in \partial\mathcal{O}_1$. Another observation is that ϕ, $\nabla\phi$ and $\mathrm{D}^2\phi$ are bounded in $\overline{\mathcal{O}}_1$. Then there exists a constant $C_0 > 0$ such that

$$2\langle x' - x, \nabla\phi(x)\rangle + C_0|x - x'|^2 \geq 0, \quad \forall\, x \in \partial\mathcal{O}_1, \quad \forall\, x' \in \overline{\mathcal{O}}_1. \tag{2.1}$$

We suppose that $0 \in \overline{\mathcal{O}}_1$. Next we consider a moving interval $\mathcal{O}_2(\omega,t,x) := [L(\omega,t,x), U(\omega,t,x)]$, where both L and U are continuous random fields such that $L(\omega,t,x) < 0 < U(\omega,t,x)$ for all $(\omega,t,x) \in \Omega\times[0,T]\times\mathbf{R}^n$. We always suppose that L and U are Itô's processes with parameter x of class C^2, *i.e.*, $\big(L(\omega,t,\cdot), U(\omega,t,\cdot)\big) \in \mathrm{C}^2(\mathbf{R}^n;\mathbf{R}\times\mathbf{R})$, and

$$L(\omega,t,x)\ (\text{or } U(\omega,t,x)) \ = \int_0^t J(\omega,s,x)\,\mathrm{d}s + \int_0^t \langle K(\omega,s,x),\, \mathrm{d}B_s\rangle,$$

where J and K are measurable random fields with $\big(J(\cdot,\cdot,x), K(\cdot,\cdot,x)\big) \in \mathcal{H}^2(0,T;\mathbf{R}\times\mathbf{R}^d)$ for each $x \in \mathbf{R}^n$. It is obvious by the Itô-Wentzell formula that $L(\omega,t,u_t)$ and $U(\omega,t,u_t)$, $t \in [0,T]$ are also Itô's processes for any Itô's process $(u_t)_{t\in[0,T]}$. For simplicity of notations, we will denote $M := \overline{\mathcal{O}}_1 \times \mathbf{R}\times\mathbf{R}^d$; an element in M is denoted by $\theta := (x,y,z)$ with $x \in \overline{\mathcal{O}}_1$, $y \in \mathbf{R}$ and $z \in \mathbf{R}^d$. Similarly, we use $\Theta := (X,Y,Z)$, and so on. We will solve a fully coupled FBSDE with reflections of the following type:

$$\begin{cases} X_t = x + \displaystyle\int_0^t b(\omega,s,\Theta_s)\,\mathrm{d}s + \int_0^t \sigma(\omega,s,\Theta_s)\,\mathrm{d}B_s + \eta_t, \\ Y_t = \Phi(\omega,X_T) + \displaystyle\int_t^T g(\omega,s,\Theta_s)\,\mathrm{d}s - \int_t^T \langle Z_s,\,\mathrm{d}B_s\rangle + \zeta_T - \zeta_t, \quad t \in [0,T]. \end{cases} \tag{2.2}$$

Here, the functions $b(\omega,t,\theta) : \Omega\times[0,T]\times M \mapsto \mathbf{R}^n$, $\sigma(\omega,t,\theta) : \Omega\times[0,T]\times M \mapsto \mathbf{R}^{n\times d}$, $g(\omega,t,\theta) : \Omega\times[0,T]\times M \mapsto \mathbf{R}$ and $\Phi(\omega,x) : \Omega\times\mathbf{R}^n \mapsto \mathbf{R}$ are continuous with respect to $\theta \in M$. And for each θ, the processes $b(\cdot,\cdot,\theta)$, $\sigma(\cdot,\cdot,\theta)$ and $g(\cdot,\cdot,\theta)$ are $(\mathcal{F}_t)$-progressively measurable and $\Phi(\cdot,x)$ is $\mathcal{F}_T$-measurable. Henceforth, we will suppress ω in the coefficients of FBSDERs. Actually, in FBSDER (2.2), the first equation is a SDE with reflection and the second equation is a BSDE with two reflecting barriers. The adapted solutions of FBSDER (2.2) is defined analogously in Definition 2.1 of Ma and Cvitanić [11], which is rewritten as follows.

Definition 2.1. A quintuple process $(\Theta_t, \eta_t, \zeta_t)_{t\in[0,T]}$ is called an adapted solution of FBSDER (2.2) if it satisfies FBSDER (2.2) almost surely and

- (i) $(X_t, Y_t)_{t\in[0,T]} \in \mathcal{S}^2(0,T;\mathbf{R}^n\times\mathbf{R})$, $(Z_t)_{t\in[0,T]}\in\mathcal{H}^2(0,T;\mathbf{R}^d)$ and $(\eta_t,\zeta_t)_{t\in[0,T]}\in\mathcal{A}(0,T;\mathbf{R}^n\times\mathbf{R})$;
- (ii) $\mathbf{P}-$ a.s., $X_t\in\overline{\mathcal{O}}_1$ and $Y_t\in\mathcal{O}_2(t,X_t)$ for each $t\in[0,T]$;
- (iii) $\mathbf{P}-$ a.s., for each $t\in[0,T]$, $|\eta|_t=\int_0^t \mathbf{1}_{\partial\mathcal{O}_1}(X_s)\,\mathrm{d}|\eta|_s$ and $\eta_t=\int_0^t\nabla\phi(X_s)\,\mathrm{d}|\eta|_s$;
- (iv) for any continuous and $(\mathcal{F}_t)$-progressively measurable process $(Y_t')_{t\in[0,T]}$ with $\mathbf{P}-$ a.s., $Y_t'\in\mathcal{O}_2(t,X_t)$ for each $t\in[0,T]$, we have $\mathbf{P}-$ a.s., $(Y_t-Y_t')\,\mathrm{d}\zeta_t\le 0$ for each $t\in[0,T]$.

Remark 2.2. It should be noted that, thanks to assumptions on L and U, *i.e.*, $0\in(L,U)$, we can choose $Y_t'\equiv 0$ in (iv) of Definition 2.1, whence $\mathbf{P}-$ a.s., $Y_t\,\mathrm{d}\zeta_t\le 0$ for each $t\in[0,T]$.

Throughout this paper we shall make use of the following assumptions, which are similar as that in Pardoux and Tang [15] and weaker than those in Ma and Cvitanić [11]. In stating them, we use a common constant $K\ge 0$.

- (H1) $\mathbf{E}\big[\int_0^T|g(t,0,0,0)|^2\,\mathrm{d}t\big]+\mathbf{E}|\Phi(0)|^2<\infty$;
- (H2) The coefficients b, σ, ϕ, $\nabla\phi$ and $\mathrm{D}^2\phi$ are uniformly bounded by K;
- (H3) The coefficients b, σ and g are monotone, of linear growth and Lipschitz continuous, *i.e.*, there exist constants K_{bx}, $K_{gy}\in\mathbf{R}$ and $L_{\sigma z}\ge 0$ such that $\mathrm{d}\mathbf{P}\times\mathrm{d}t-$ *a.e.*, for each θ, θ_1 and $\theta_2\in M$,
  - (a) $\langle x_1-x_2, b(t,x_1,y,z)-b(t,x_2,y,z)\rangle\le K_{bx}|x_1-x_2|^2$,
    $|b(t,x,y_1,z_1)-b(t,x,y_2,z_2)|\le K|y_1-y_2|+K|z_1-z_2|$;
  - (b) $|\sigma(t,\theta_1)-\sigma(t,\theta_2)|^2\le K^2|x_1-x_2|^2+K^2|y_1-y_2|^2+{L_{\sigma z}}^2|z_1-z_2|^2$;
  - (c) $(y_1-y_2)(g(t,x,y_1,z)-g(t,x,y_2,z))\le K_{gy}|y_1-y_2|^2$,
    $|g(t,x_1,y,z_1)-g(t,x_2,y,z_2)|\le K|x_1-x_2|+K|z_1-z_2|$,
    $|g(t,\theta)|\le|g(t,x,0,z)|+K|y|$;
- (H4) The function Φ satisfies $\mathbf{P}-$ a.s., $L(T,x)\le\Phi(x)\le U(T,x)$ for each $x\in\overline{\mathcal{O}}_1$, and there exists a constant $L_\Phi\ge 0$ such that $\mathbf{P}-$ a.s., for each x_1 and $x_2\in\overline{\mathcal{O}}_1$, $|\Phi(x_1)-\Phi(x_2)|\le L_\Phi|x_1-x_2|$.

3. Existence and uniqueness for solutions of FBSDERs

For any $\lambda\ge 0$, we define an equivalent norm on $\mathcal{H}^2(0,T;\mathbf{R}^d)$ by $\|\xi\|_\lambda^2:=\mathbf{E}\big[\int_0^T \mathrm{e}^{-\lambda t}|\xi_t|^2\,\mathrm{d}t\big]$. Next we present *a priori* estimates for solutions of FBSDER (2.2), which are useful for constructing the contraction mapping. Some general estimates are referred to Appendix A. It is emphasized that the lack of convexity for the domain $\overline{\mathcal{O}}_1$ makes the steps of *a priori* estimates for the SDE with reflection more complicated than those in Ma and Cvitanić [11].

Lemma 3.1. *Assume that (H2)–(H4) hold with K, $L_{\sigma z}$, $L_\Phi\ge 0$ and K_{bx}, $K_{gy}\in\mathbf{R}$, $(\Theta_t,\eta_t,\zeta_t)_{t\in[0,T]}$ and $(\Theta_t',\eta_t',\zeta_t')_{t\in[0,T]}$ are two solutions of FBSDER (2.2). Set $\hat{\xi}_\cdot:=\xi_\cdot-\xi_\cdot'$, where $\xi=X$, Y, Z, η, ζ, respectively. Let $\lambda\ge 0$ and $C_i>0$ $(i=1,2,\ldots,6)$ with $C_6<1$. Then we have*

$$\mathrm{e}^{-\lambda T}\mathbf{E}|\hat{X}_T|^2+\lambda_1\|\hat{X}\|_\lambda^2\le k_1\|\hat{Y}\|_\lambda^2+k_2\|\hat{Z}\|_\lambda^2, \tag{3.1}$$

where $\lambda_1:=\lambda-\mathrm{e}^{2C_0K}\big(2K_{bx}+{C_1}^{-1}+{C_2}^{-1}+K^2+C_0K^2(2+5K+2C_0K^2+2{C_3}^{-1}+2{C_4}^{-1})\big)$, $k_1:=\mathrm{e}^{2C_0K}(K^2C_1+K^2+2C_0C_3K^4)$ *and* $k_2:=\mathrm{e}^{2C_0K}(K^2C_2+{L_{\sigma z}}^2+2C_0C_4K^2{L_{\sigma z}}^2)$*; and*

$$\lambda_2\|\hat{Y}\|_\lambda^2+(1-C_6)\|\hat{Z}\|_\lambda^2\le{L_\Phi}^2\mathrm{e}^{-\lambda T}\mathbf{E}|\hat{X}_T|^2+C_5\|\hat{X}\|_\lambda^2, \tag{3.2}$$

where $\lambda_2:=-\lambda-2K_{gy}-K^2{C_5}^{-1}-K^2{C_6}^{-1}$.

Proof. Denote $\hat{\phi}_r := \mathrm{e}^{-C_0[\phi(X_r)+\phi(X'_r)]}$ for each $r \in [0,T]$. Itô's formula to $\mathrm{e}^{-\lambda r}\hat{\phi}_r|\hat{X}_r|^2$ on the interval $[s,t]$ gives that, for each $0 \le s \le t \le T$,

$$
\begin{aligned}
&\mathrm{e}^{-\lambda t}\hat{\phi}_t|\hat{X}_t|^2 + \lambda \int_s^t \mathrm{e}^{-\lambda r}\hat{\phi}_r|\hat{X}_r|^2\,\mathrm{d}r \\
&= \mathrm{e}^{-\lambda s}\hat{\phi}_s|\hat{X}_s|^2 + \int_s^t \mathrm{e}^{-\lambda r}\hat{\phi}_r\left[\langle 2\hat{X}_r, b(r,\Theta_r)-b(r,\Theta'_r)\rangle\,\mathrm{d}r + \langle 2\hat{X}_r, (\sigma(r,\Theta_r)-\sigma(r,\Theta'_r))\,\mathrm{d}B_r\rangle\right] \\
&\quad + \int_s^t \mathrm{e}^{-\lambda r}\hat{\phi}_r\left[\langle 2\hat{X}_r, \nabla\phi(X_r)\,\mathrm{d}|\eta|_r - \nabla\phi(X'_r)\,\mathrm{d}|\eta'|_r\rangle + |\sigma(r,\Theta_r)-\sigma(r,\Theta'_r)|^2\,\mathrm{d}r\right] \\
&\quad - C_0\int_s^t \mathrm{e}^{-\lambda r}\hat{\phi}_r|\hat{X}_r|^2\Big[\big(\langle\nabla\phi(X_r), b(r,\Theta_r)\rangle + \langle\nabla\phi(X'_r), b(r,\Theta'_r)\rangle\big)\,\mathrm{d}r \\
&\quad + \langle\nabla\phi(X_r), \sigma(r,\Theta_r)\,\mathrm{d}B_r\rangle + \langle\nabla\phi(X'_r), \sigma(r,\Theta'_r)\,\mathrm{d}B_r\rangle + |\nabla\phi(X_r)|^2\,\mathrm{d}|\eta|_r + |\nabla\phi(X'_r)|^2\,\mathrm{d}|\eta'|_r \\
&\quad + \frac{1}{2}\operatorname{tr}\left\{\mathrm{D}^2\phi(X_r)(\sigma\sigma^*)(r,\Theta_r) + \mathrm{D}^2\phi(X'_r)(\sigma\sigma^*)(r,\Theta'_r)\right\}\mathrm{d}r\Big] \\
&\quad + \frac{{C_0}^2}{2}\int_s^t \mathrm{e}^{-\lambda r}\hat{\phi}_r|\hat{X}_r|^2|\sigma^*(r,\Theta_r)\nabla\phi(X_r) + \sigma^*(r,\Theta'_r)\nabla\phi(X'_r)|^2\,\mathrm{d}r \\
&\quad - 2C_0\int_s^t \mathrm{e}^{-\lambda r}\hat{\phi}_r\hat{X}_r^*(\sigma(r,\Theta_r)-\sigma(r,\Theta'_r))(\sigma^*(r,\Theta_r)\nabla\phi(X_r) + \sigma^*(r,\Theta'_r)\nabla\phi(X'_r))\,\mathrm{d}r.
\end{aligned}
\tag{3.3}
$$

Note that $|\nabla\phi(x)| = 1$ for each $x \in \partial\mathcal{O}_1$. It follows from (2.1) and (iii) in Definition 2.1 that $\mathbf{P}-$ a.s.,

$$
\begin{aligned}
&\int_s^t \mathrm{e}^{-\lambda r}\hat{\phi}_r\left[2\langle\hat{X}_r, \nabla\phi(X_r)\rangle - C_0|\hat{X}_r|^2|\nabla\phi(X_r)|^2\right]\,\mathrm{d}|\eta|_r \le 0, \\
&\int_s^t \mathrm{e}^{-\lambda r}\hat{\phi}_r\left[-2\langle\hat{X}_r, \nabla\phi(X'_r)\rangle - C_0|\hat{X}_r|^2|\nabla\phi(X'_r)|^2\right]\,\mathrm{d}|\eta'|_r \le 0.
\end{aligned}
$$

Plugging the previous two inequalities into (3.3), then using (H2) and the inequality $\operatorname{tr}\{AB\} \le |A||B|$, we deduce that

$$
\begin{aligned}
&\mathrm{e}^{-\lambda t}\mathrm{e}^{-2C_0K}\mathbf{E}|\hat{X}_t|^2 + \lambda\mathrm{e}^{-2C_0K}\int_s^t \mathrm{e}^{-\lambda r}\mathbf{E}|\hat{X}_r|^2\,\mathrm{d}r \\
&\le \mathrm{e}^{-\lambda s}\mathbf{E}|\hat{X}_s|^2 + \int_s^t \mathrm{e}^{-\lambda r}\mathbf{E}\left[\hat{\phi}_r\langle 2\hat{X}_r, b(r,\Theta_r)-b(r,\Theta'_r)\rangle + |\sigma(r,\Theta_r)-\sigma(r,\Theta'_r)|^2\right]\,\mathrm{d}r \\
&\quad + C_0K^2(2+K+2C_0K^2)\int_s^t \mathrm{e}^{-\lambda r}\mathbf{E}|\hat{X}_r|^2\,\mathrm{d}r + 2C_0K^2\int_s^t \mathrm{e}^{-\lambda r}\mathbf{E}\left[2|\hat{X}_r||\sigma(r,\Theta_r)-\sigma(r,\Theta'_r)|\right]\,\mathrm{d}r.
\end{aligned}
$$

Therefore, after taking $t = T$ and $s = 0$, it is not hard to verify (3.1) by (H3)(a), (H3)(b) and the basic inequality $2ab \le ca^2 + b^2/c$ for each $c > 0$.

Now applying Itô's formula to $\mathrm{e}^{-\lambda s}|\hat{Y}_s|^2$ from t to T, we derive that for each $t \in [0,T]$,

$$
\begin{aligned}
&\mathrm{e}^{-\lambda t}\mathbf{E}|\hat{Y}_t|^2 - \lambda\int_t^T \mathrm{e}^{-\lambda s}\mathbf{E}|\hat{Y}_s|^2\,\mathrm{d}s + \int_t^T \mathrm{e}^{-\lambda s}\mathbf{E}|\hat{Z}_s|^2\,\mathrm{d}s \\
&= \mathrm{e}^{-\lambda T}\mathbf{E}|\Phi(X_T)-\Phi(X'_T)|^2 + \int_t^T \mathrm{e}^{-\lambda s}\mathbf{E}\left[2\hat{Y}_s\big(g(s,\Theta_s)-g(s,\Theta'_s)\big)\,\mathrm{d}s + \langle 2\hat{Y}_s,\,\mathrm{d}\hat{\zeta}_s\rangle\right].
\end{aligned}
$$

Then, by taking $t = 0$, (iv) in Definition 2.1, (H3)(c), (H4) and the basic inequality mentioned above we can derive (3.2). Then the proof of Lemma 3.1 is finished. $\square$

Now we introduce a mapping Γ from $\mathcal{H}^2(0,T;\mathbf{R}^n)$ to itself defined as follows: for each $x \in \overline{\mathcal{O}}_1$, let $\overline{X}_\cdot := \Gamma(X_\cdot)$ be a solution of the SDE with reflection,

$$\overline{X}_t = x + \int_0^t b(s, \overline{X}_s, Y_s, Z_s)\,\mathrm{d}s + \int_0^t \sigma(s, \overline{X}_s, Y_s, Z_s)\,\mathrm{d}B_s + \overline{\eta}_t, \quad t \in [0,T], \tag{3.4}$$

where $(Y_\cdot, Z_\cdot)$ is the solution of the following BSDE with reflections,

$$Y_t = \Phi(X_T) + \int_t^T g(s, X_s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T \langle Z_s,\, \mathrm{d}B_s\rangle + \zeta_T - \zeta_t, \quad t \in [0,T]. \tag{3.5}$$

We note that Theorem 1 in Marín-Rubio and Real [12] and assumptions (H2), (H3)(a)–(H3)(b) ensure the existence and uniqueness for the adapted solution of SDE with reflection; and Theorem 2.2 in Xu [19] and assumptions (H1), (H3)(c) and (H4) guarantee that BSDE with reflections admits a unique solution. Next we need the following assumption (H5) to ensure the mapping Γ is a contraction, leading to the existence and uniqueness for solutions of FBSDER (2.2). We use the same notations λ, C_i, λ_1, λ_2, k_1 and k_2 defined in Lemma 3.1.

(H5) The constants $\lambda \geq 0$, $C_i > 0$ $(i = 1, \ldots, 6)$ such that $C_6 < 1$, $\lambda_1 > 0$, $\lambda_2 > 0$ and

$$0 < \mu(\lambda_1, \lambda_2) := \left(L_\Phi{}^2 \vee \frac{C_5}{\lambda_1}\right)\left(\frac{k_1}{\lambda_2} \vee \frac{k_2}{1 - C_6}\right) < 1.$$

Theorem 3.2. *Let assumptions (H1)–(H5) hold. Then FBSDER (2.2) admits a unique adapted solution.*

Proof. We choose any $(X_\cdot^i, \eta_\cdot^i)$ $(i = 1, 2)$ in $\mathcal{H}^2(0,T;\mathbf{R}^n) \times \mathcal{A}(0,T;\mathbf{R}^n)$, and let $(Y_\cdot^i, Z_\cdot^i, \zeta_\cdot^i)$ and $(\overline{X}_\cdot^i, \overline{\eta}_\cdot^i)$ be the corresponding solutions of (3.5) and (3.4), respectively. Set $\Delta\xi := \xi^1 - \xi^2$, where $\xi = X$, Y, Z, $\overline{X}$. Since λ_1, $\lambda_2 > 0$ and $C_6 < 1$, it is not hard to deduce from (3.1) and (3.2) in Lemma 3.1 that

$$\mathrm{e}^{-\lambda T}\mathbf{E}|\Delta\overline{X}_T|^2 + \lambda_1\|\Delta\overline{X}\|_\lambda^2 \leq \mu(\lambda_1, \lambda_2)\left\{\mathrm{e}^{-\lambda T}\mathbf{E}|\Delta X_T|^2 + \lambda_1\|\Delta X\|_\lambda^2\right\}.$$

The assumption (H5) leads to $0 < \mu(\lambda_1, \lambda_2) < 1$. Then Γ is a contraction mapping. Therefore, we know that FBSDER (2.2) admits a unique solution $(\Theta_t)_{t\in[0,T]} \in \mathcal{H}^2(0,T;\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^d)$, $(\eta_t, \zeta_t)_{t\in[0,T]} \in \mathcal{A}(0,T;\mathbf{R}^n \times \mathbf{R})$ satisfying (ii)–(iv) in Definition 2.1. It reduces to prove that $(X_t, Y_t)_{t\in[0,T]} \in \mathcal{S}^2(0,T;\mathbf{R}^n \times \mathbf{R})$. Itô's formula to $\phi(X_r)$ on the interval $[t,T]$ yields that, for each $t \in [0,T]$,

$$|\eta|_t = \phi(X_T) - \phi(X_t) - \int_t^T \left[\frac{1}{2}\operatorname{tr}\left\{\mathrm{D}^2\phi(X_r)\sigma\sigma^*(r,\Theta_r)\right\} + \langle\nabla\phi(X_r), b(r,\Theta_r)\rangle\right]\mathrm{d}r - \int_t^T \langle\nabla\phi(X_r), \sigma(r,\Theta_r)\,\mathrm{d}B_r\rangle.$$

It follows from (H2) and the Burkholder-Davis-Gundy (BDG for short) inequality that $\mathbf{E}[\sup_{t\in[0,T]}|\eta|_t^2] < \infty$, whence $\mathbf{E}[\sup_{t\in[0,T]}|\eta_t|^2] < \infty$. Thanks to (H2), BDG's inequality and the SDE with reflection in (2.2), we obtain that $(X_t)_{t\in[0,T]} \in \mathcal{S}^2(0,T;\mathbf{R}^n)$. Then by Itô's formula to $\mathrm{e}^{-\lambda s}|Y_s|^2$ on the interval $[t,T]$, the linear growth of g, Remark 2.2 and BDG's inequality we can deduce that $(Y_t)_{t\in[0,T]} \in \mathcal{S}^2(0,T;\mathbf{R})$, which completes the proof of Theorem 3.2. $\square$

Remark 3.3. From the quantities of parameters λ_1, λ_2, k_1 and k_2 and assumption (H5), we can find some further restrictions on the constants appearing in (H2)–(H4), such as K_{gy} is negative with a large enough absolute value and K_{bx} may be negative ensuring λ_1 and λ_2 are positive, $L_{\sigma z}L_\Phi$ may take value in $[0,1]$

ensuring $\mu(\lambda_1, \lambda_2)$ small enough. More analyses on the parameters are referred to Pardoux and Tang [15] and Ma and Cvitanić [11].

Example 3.4. In order to show that the assumption (H5) is not void, we give a simple example. If parameters appearing in (H2)–(H4) are all prescribed as $K = L_{\sigma z} = 1$, $K_{bx} = -25$, $K_{gy} = -50$ and $L_\Phi = 1/8$, we can take $C_0 = C_1 = C_3 = C_5 = 1$, $C_2 = 1/8$, $C_4 = 1/16$, $C_6 = 11/16$ and suitable $\lambda \in [52, 97]$ to satisfy (H5).

4. Obstacle problems of quasilinear parabolic PDEs with Neumann boundary conditions

In this section we investigate obstacle problems of second order quasilinear parabolic PDEs with Neumann boundary conditions by solutions of FBSDERs, including the probabilistic interpretation and the uniqueness for solutions in viscosity sense. For that purpose, we need the following assumption to put the FBSDER in a Markovian framework.

(H6) The coefficients b, σ, g, Φ and the moving boundaries L and U are all deterministic, *i.e.*, independent of ω; and $b(t,\theta)$, $\sigma(t,\theta)$, $g(t,\theta)$ are continuous with respect to (t,θ).

We consider the FBSDER over a subinterval $[t,T] \subset [0,T]$, and the initial data is $(t,x) \in [0,T] \times \overline{\mathcal{O}}_1$,

$$\begin{cases} X_s^{t,x} = x + \displaystyle\int_t^s b(r, \Theta_r^{t,x})\,\mathrm{d}r + \int_t^s \sigma(r, \Theta_r^{t,x})\,\mathrm{d}B_r + \int_t^s \nabla\phi(X_r^{t,x})\,\mathrm{d}|\eta^{t,x}|_r, \\ Y_s^{t,x} = \Phi(X_T^{t,x}) + \displaystyle\int_s^T g(r, \Theta_r^{t,x})\,\mathrm{d}r - \int_s^T \langle Z_r^{t,x}, \mathrm{d}B_r\rangle + \zeta_T^{t,x} - \zeta_s^{t,x}, \quad s \in [t,T]. \end{cases} \tag{4.1}$$

Note that (H1) is trivially satisfied if (H6) holds. By Theorem 3.2 we know that FBSDER (4.1) admits a unique solution $(\Theta_s^{t,x}, \eta_s^{t,x}, \zeta_s^{t,x})_{s\in[t,T]}$ provided that assumptions (H2)–(H6) hold. Here, the process $(Y_s^{t,x})_{s\in[t,T]}$ is $\mathcal{F}_s^t$-adapted with standard techniques of time shifting, where $\mathcal{F}_s^t$ is the natural filtration generated by $(B_s - B_t)_{s\in[t,T]}$ and augmented by $\mathbf{P}$-null sets. Consequently, $Y_t^{t,x}$ is deterministic. We define the function $u : [0,T] \times \overline{\mathcal{O}}_1 \mapsto \mathbf{R}$ as $u(t,x) := Y_t^{t,x}$. From the uniqueness for solutions, we obtain the Markov property of u, *i.e.*, $u(t+\varepsilon, X_{t+\varepsilon}^{t,x}) = Y_{t+\varepsilon}^{t+\varepsilon, X_{t+\varepsilon}^{t,x}} = Y_{t+\varepsilon}^{t,x}$ holds for each $\varepsilon > 0$.

We now relate the function $u(t,x) = Y_t^{t,x}$ to the following obstacle problem for second order quasilinear parabolic PDEs combined with a Neumann boundary condition and an algebra equation:

$$\begin{cases} \big(u(t,x) - L(t,x)\big) \wedge \{(u(t,x) - U(t,x)) \vee [-\partial_t u(t,x) - (\mathcal{L}_t u + g)(t,x,u(t,x),v(t,x))]\} = 0, \\ v(t,x) = \sigma^*(t,x,u(t,x),v(t,x))\nabla u(t,x), \quad (t,x) \in [0,T) \times \mathcal{O}_1, \\ u(T,x) = \Phi(x), \quad x \in \mathcal{O}_1, \\ \dfrac{\partial u}{\partial n}(t,x) = 0, \quad (t,x) \in [0,T) \times \partial\mathcal{O}_1, \end{cases} \tag{4.2}$$

where we have written, for each $t \in [0,T]$ and $\theta \in M$,

$$(\mathcal{L}_t u)(t,\theta) := \frac{1}{2}\sum_{i,j=1}^n (\sigma\sigma^*)(t,\theta)\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x) + \sum_{i=1}^n b_i(t,\theta)\frac{\partial u}{\partial x_i}(t,x),$$

$$\frac{\partial u}{\partial n}(t,x) := \sum_{i=1}^n \frac{\partial\phi}{\partial x_i}(x)\frac{\partial u}{\partial x_i}(t,x).$$

The reason of the presence of the algebra equation lies in that $\sigma(t,x,y,z)$ may depend on z. We now give a notion of viscosity solutions for PDE (4.2), which is adapted from Crandall, Ishii and Lions [4], Ma and Cvitanić [11] and Wu and Yu [18].

Definition 4.1. A function $u \in \mathrm{C}([0,T]\times\overline{\mathcal{O}}_1;\mathbf{R})$ is called a viscosity sub- (resp., super-) solution of PDE (4.2) if $u(T,x) \le \Phi(x)$ (resp., $u(T,x)\ge\Phi(x)$) for all $x\in\overline{\mathcal{O}}_1$, and for any $\varphi\in\mathrm{C}^{1,2}([0,T]\times\overline{\mathcal{O}}_1;\mathbf{R})$ whenever $(t,x)\in[0,T)\times\overline{\mathcal{O}}_1$ is a global maximum (resp., minimum) of $u-\varphi$, then there must exist a $v(t,x)\in\mathbf{R}^d$ satisfying,

$$\begin{cases}(u-L)\wedge\{(u-U)\vee[-\partial_t\varphi-(\mathcal{L}_t\varphi+g)(t,x,u(t,x),v(t,x))]\}\le 0,\\ v(t,x)=\sigma^*(t,x,u(t,x),v(t,x))\nabla\varphi(t,x),\quad x\in\mathcal{O}_1;\\ \dfrac{\partial\varphi}{\partial n}\wedge\big\{(u-L)\wedge\{(u-U)\vee[-\partial_t\varphi-(\mathcal{L}_t\varphi+g)(t,x,u(t,x),v(t,x))]\}\big\}\le 0,\quad x\in\partial\mathcal{O}_1.\end{cases}\tag{4.3}$$

$$\left(\text{resp., }\begin{cases}(u-L)\vee\{(u-U)\wedge[-\partial_t\varphi-(\mathcal{L}_t\varphi+g)(t,x,u(t,x),v(t,x))]\}\ge 0,\\ v(t,x)=\sigma^*(t,x,u(t,x),v(t,x))\nabla\varphi(t,x),\quad x\in\mathcal{O}_1;\\ \dfrac{\partial\varphi}{\partial n}\vee\big\{(u-L)\vee\{(u-U)\wedge[-\partial_t\varphi-(\mathcal{L}_t\varphi+g)(t,x,u(t,x),v(t,x))]\}\big\}\ge 0,\quad x\in\partial\mathcal{O}_1.\end{cases}\right)$$

Moreover, a function $u\in\mathrm{C}([0,T]\times\overline{\mathcal{O}}_1;\mathbf{R})$ is called a viscosity solution of PDE (4.2) if it is both a viscosity subsolution and a viscosity supersolution.

We also need the following notations of parabolic second order semijets of function u at (t,x).

$$\begin{aligned}\mathcal{P}^{2,+}u(t,x) &:= \{(\partial_t\varphi(t,x),\nabla\varphi(t,x),\mathrm{D}^2\varphi(t,x)) : \varphi\in\mathrm{C}^{1,2}([0,T]\times\overline{\mathcal{O}}_1;\mathbf{R})\\ &\qquad \text{and } u-\varphi \text{ has a global maximum at } (t,x)\},\\ \mathcal{P}^{2,-}u(t,x) &:= \{(\partial_t\varphi(t,x),\nabla\varphi(t,x),\mathrm{D}^2\varphi(t,x)) : \varphi\in\mathrm{C}^{1,2}([0,T]\times\overline{\mathcal{O}}_1;\mathbf{R})\\ &\qquad \text{and } u-\varphi \text{ has a global minimum at } (t,x)\}.\end{aligned}$$

With the same notations, we record the closures of the superjets $\mathcal{P}^{2,+}$ as follows, and the closures of the subjets $\mathcal{P}^{2,-}$ can be defined similarly.

$$\begin{aligned}\overline{\mathcal{P}}^{2,+}u(t,x) := \{(p,q,X)\in\mathbf{R}\times\mathbf{R}^n\times S^n :\ &\text{there exists a sequence } (t_n,x_n,p_n,q_n,X_n)\in\\ &[0,T)\times\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n\times S^n \text{ such that } (p_n,q_n,X_n)\in\mathcal{P}^{2,+}u(t_n,x_n)\\ &\text{and } (t_n,x_n,p_n,q_n,X_n)\to(t,x,p,q,X)\},\end{aligned}$$

where and hereafter S^n denotes the collection of all $n\times n$ symmetric matrices.

4.1. Regularity of solutions with respect to initial data

This subsection aims to interpret the regularity of the function $u(t,x)=Y_t^{t,x}$, $(t,x)\in[0,T]\times\overline{\mathcal{O}}_1$, *i.e.*, $u(t,x)$ is Hölder continuous in t and Lipschitz continuous in x. The following proposition shows the accurate Lipschitz constant, which will be used later. A general version of regularity of solutions of FBSDERs is referred to Appendix A.

Proposition 4.2. *Assume that (H2)–(H6) hold. Then the function* $u(t,x)=Y_t^{t,x}$, $(t,x)\in[0,T]\times\overline{\mathcal{O}}_1$ *enjoys the following regular property,*

$$|u(t,x)-u(t',x')|\le C|t-t'|^{1/2}+L_{ux}^{t,t'}|x-x'|,$$

where $C \geq 0$ depends on ϕ, b, σ, g, Φ, C_0, K and T, and

$$L_{ux}^{t,t'} := \sqrt{\frac{\mathrm{e}^{\lambda|t'-t|+C_0K}}{1-\mu(\lambda_1,\lambda_2)}\left({L_\Phi}^2 \vee \frac{C_5}{\lambda_1}\right)}.$$

Particularly, $|u(t,x)-u(t,x')| \leq L_{ux}|x-x'|$, *where* $L_{ux} := L_{ux}^{t,t'}|_{t=t'}$.

Proof. Let all assumptions hold and (t,x), $(t',x') \in [0,T]\times\overline{\mathcal{O}}_1$ are given. Without loss of generality, we assume that $t' \geq t$. Since $Y_t^{t,x}$ is deterministic, we have that

$$|u(t,x)-u(t',x')| = \mathbf{E}|Y_t^{t,x} - Y_{t'}^{t',x'}| \leq \mathbf{E}|Y_t^{t,x} - Y_{t'}^{t,x}| + \mathbf{E}|Y_{t'}^{t,x} - Y_{t'}^{t',x'}|. \tag{4.4}$$

We now estimate the first term on the right hand side of the previous inequality. Let us set $\overline{Y}_r^{t,x} := Y_r^{t,x} - Y_{t'}^{t,x}$, and $\overline{\xi}(r,\theta) := \xi(r,x,y+Y_{t'}^{t,x},z)$ for each $r\in[t,T]$ and $\theta \in M$, where $\xi = b$, σ, g. Note that $(\Theta_s^{t,x},\eta_s^{t,x},\zeta_s^{t,x})_{s\in[t,T]}$ is the unique solution of FBSDER (4.1). Then $(X_s^{t,x},\overline{Y}_s^{t,x},Z_s^{t,x},\eta_s^{t,x},\zeta_s^{t,x})_{s\in[t,t']}$ is a solution of the following FBSDER,

$$\begin{cases} X_s^{t,x} = x + \displaystyle\int_t^s \overline{b}(r,X_r^{t,x},\overline{Y}_r^{t,x},Z_r^{t,x})\,\mathrm{d}r + \int_t^s \overline{\sigma}(r,X_r^{t,x},\overline{Y}_r^{t,x},Z_r^{t,x})\,\mathrm{d}B_r + \int_t^s \nabla\phi(X_r^{t,x})\,\mathrm{d}|\eta^{t,x}|_r, \\ \overline{Y}_s^{t,x} = \displaystyle\int_s^{t'} \overline{g}(r,X_r^{t,x},\overline{Y}_r^{t,x},Z_r^{t,x})\,\mathrm{d}r - \int_s^{t'} \langle Z_r^{t,x},\,\mathrm{d}B_r\rangle + \zeta_{t'}^{t,x} - \zeta_s^{t,x}, \quad s\in[t,t']. \end{cases}$$

Thus, we know that $\overline{Y}_s^{t,x} \in \mathcal{O}_2(s,X_s^{t,x})$ for each $s\in[t,t']$. Itô's formula to $|\overline{Y}_r^{t,x}|^2$ on the interval $[t,t']$ and Remark 2.2 yield that

$$\mathbf{E}|\overline{Y}_t^{t,x}|^2 + \mathbf{E}\left[\int_t^{t'} |Z_r^{t,x}|^2\,\mathrm{d}r\right] \leq \mathbf{E}\left[\int_t^{t'} 2\overline{Y}_r^{t,x} g(r,\Theta_r^{t,x})\,\mathrm{d}r\right]. \tag{4.5}$$

Note that (H6) and (H3)(c) imply that $|g(t,\theta)| \leq C(1+|x|+|y|+|z|)$ for each $t\in[0,T]$ and $\theta\in M$. In view of $X_\cdot^{t,x} \in \overline{\mathcal{O}}_1$ and (H6), we know that $\mathcal{O}_2(s,X_s^{t,x})$ is bounded for each $s\in[t,T]$ since L and U are continuous. Thus we get that $Y_\cdot^{t,x}$ is bounded. Hence, there exists a constant $C\geq 0$, whose value could vary from line to line, such that

$$\mathbf{E}\left[\int_t^{t'} 2\overline{Y}_r^{t,x} g(r,\Theta_r^{t,x})\,\mathrm{d}r\right] \leq C|t'-t| + C\mathbf{E}\left[\int_t^{t'} 2|\overline{Y}_r^{t,x}||Z_r^{t,x}|\,\mathrm{d}r\right] \leq C|t'-t| + \frac{1}{2}\mathbf{E}\left[\int_t^{t'} |Z_r^{t,x}|^2\,\mathrm{d}r\right].$$

Plugging the previous inequality into (4.5) yields that

$$\mathbf{E}|Y_t^{t,x} - Y_{t'}^{t,x}|^2 = \mathbf{E}|\overline{Y}_t^{t,x}|^2 \leq C|t'-t|. \tag{4.6}$$

We proceed to estimate the second term on the right hand side of (4.4). For simplicity, we denote $\hat{\Theta}_\cdot := \Theta_\cdot^{t,x} - \Theta_\cdot^{t',x'}$. Analogous arguments to Lemma 3.1, we can deduce that for each $\lambda \geq 0$,

$$\mathrm{e}^{-\lambda T}\mathbf{E}|\hat{X}_T|^2 + \lambda_1\int_{t'}^T \mathrm{e}^{-\lambda r}\mathbf{E}|\hat{X}_r|^2\,\mathrm{d}r \leq \mathrm{e}^{-\lambda t'}\mathbf{E}|X_{t'}^{t,x} - x'|^2 + k_1\int_{t'}^T \mathrm{e}^{-\lambda r}\mathbf{E}|\hat{Y}_r|^2\,\mathrm{d}r + k_2\int_{t'}^T \mathrm{e}^{-\lambda r}\mathbf{E}|\hat{Z}_r|^2\,\mathrm{d}r,$$

and

$$
\begin{aligned}
&\mathrm{e}^{-\lambda t'}\mathbf{E}|\hat{Y}_{t'}|^2 + \lambda_2 \int_{t'}^{T} \mathrm{e}^{-\lambda r}\mathbf{E}|\hat{Y}_r|^2\,\mathrm{d}r + (1 - C_6)\int_{t'}^{T} \mathrm{e}^{-\lambda r}\mathbf{E}|\hat{Z}_r|^2\,\mathrm{d}r \\
&\leq {L_\Phi}^2 \mathrm{e}^{-\lambda T}\mathbf{E}|\hat{X}_T|^2 + C_5 \int_{t'}^{T} \mathrm{e}^{-\lambda r}\mathbf{E}|\hat{X}_r|^2\,\mathrm{d}r,
\end{aligned} \tag{4.7}
$$

where we use the same quantities of λ_i and k_i $(i = 1, 2)$ defined in Lemma 3.1. Applying similar computations to construct the contraction mapping in the proof of Theorem 3.2, we have that

$$
\mathrm{e}^{-\lambda T}\mathbf{E}|\hat{X}_T|^2 + \lambda_1 \int_{t'}^{T} \mathrm{e}^{-\lambda r}\mathbf{E}|\hat{X}_r|^2\,\mathrm{d}r \leq \frac{\mathrm{e}^{-\lambda t'}}{1-\mu(\lambda_1,\lambda_2)}\mathbf{E}|X_{t'}^{t,x} - x'|^2.
$$

Combining the previous inequality with (4.7) yields that

$$
\mathbf{E}|\hat{Y}_{t'}|^2 \leq \frac{1}{1-\mu(\lambda_1,\lambda_2)}\left({L_\Phi}^2 \vee \frac{C_5}{\lambda_1}\right)\mathbf{E}|X_{t'}^{t,x} - x'|^2. \tag{4.8}
$$

Next we estimate $\mathbf{E}|X_{t'}^{t,x} - x'|^2$. Set $\phi_s := \mathrm{e}^{-C_0\phi(X_s^{t,x})}$ for each $s \in [t, T]$. For each $\lambda \geq 0$, Itô's formula to $\mathrm{e}^{-\lambda s}\phi_s|X_s^{t,x} - x'|^2$ from t to t' arrives at

$$
\begin{aligned}
\mathrm{e}^{-\lambda t'}\mathbf{E}\phi_{t'}|X_{t'}^{t,x} - x'|^2 &\leq \mathrm{e}^{-\lambda t}\phi_t|x - x'|^2 + \mathbf{E}\int_t^{t'} \mathrm{e}^{-\lambda r}\phi_r[\langle 2X_r^{t,x} - x', b(r,\Theta_r^{t,x})\rangle + |\sigma(r,\Theta_r^{t,x})|^2]\,\mathrm{d}r \\
&\quad - C_0\mathbf{E}\int_t^{t'} \mathrm{e}^{-\lambda r}\phi_r|X_r^{t,x} - x'|^2[\langle\nabla\phi(X_r^{t,x}), b(r,\Theta_r^{t,x})\rangle + \frac{1}{2}\operatorname{tr}\{\mathrm{D}^2\phi(X_r^{t,x})\sigma\sigma^*(r,\Theta_r^{t,x})\}]\,\mathrm{d}r \\
&\quad + \frac{{C_0}^2}{2}\mathbf{E}\int_t^{t'} \mathrm{e}^{-\lambda r}\phi_r|X_r^{t,x} - x'|^2|\sigma^*(r,\Theta_r^{t,x})\nabla\phi(X_r^{t,x})|^2\,\mathrm{d}r \\
&\quad - 2C_0\mathbf{E}\int_t^{t'} \mathrm{e}^{-\lambda r}\phi_r(X_r^{t,x} - x')^*\sigma(r,\Theta_r^{t,x})\sigma^*(r,\Theta_r^{t,x})\nabla\phi(X_r^{t,x})\,\mathrm{d}r.
\end{aligned}
$$

By the aid of (H2) and $X_\cdot^{t,x}$, $x' \in \overline{\mathcal{O}}_1$, we can derive that there exists a constant $C \geq 0$ such that

$$
\mathbf{E}|X_{t'}^{t,x} - x'|^2 \leq \mathrm{e}^{\lambda|t'-t| + C_0K}|x - x'|^2 + C|t - t'|.
$$

Combining the previous inequality with (4.4), (4.6) and (4.8) and then applying Hölder's inequality yield the desired result. □

4.2. Solving the algebra equation

A significant procedure to handle the dependence of σ in z is solving the algebra equation. To this end, we need the following auxiliary conclusion whose proof is omitted here since it is classical in the theory of viscosity solutions.

Lemma 4.3. *Assume that $u \in \mathrm{C}([0,T]\times\mathbf{R}^n;\mathbf{R})$ is Lipschitz continuous in x. For any $\varphi \in \mathrm{C}^{1,2}([0,T]\times\overline{\mathcal{O}}_1;\mathbf{R})$ whenever $(t,x) \in [0,T)\times\overline{\mathcal{O}}_1$ is a global maximum (resp., minimum) of $u - \varphi$ and $u(t,x) - \varphi(t,x) = 0$, then $|\nabla\varphi(t,x)| \leq L$, where L is the Lipschitz constant of u.*

We would like to solve the algebra equation in the case where $\sigma(t,x,y,z)$ is Lipschitz continuous in z, without the G-monotonicity condition which Wu and Yu [18] and Li and Wei [9] relied on heavily. As a compensation, we need a compatibility condition for the constants appearing in (H2)–(H5). We adopt the contraction mapping method which is similar to Li and Wei [9]. But compared with their Definition 4.9, we impose the restriction (H7) to the parameters instead of the test function in the definition of the viscosity solution, which seems to be more reasonable. Example 3.4 is still applicable to show the validity of the following assumption, where the same notations in Lemma 3.1 are adopted.

(H7) The constants $\lambda \geq 0$, $C_i > 0$ $(i = 1, \dots, 6)$ such that $C_6 < 1$, $\lambda_1 > 0$, $\lambda_2 > 0$ and $L_{\sigma z} L_{ux} < 1$, where L_{ux} is defined in Proposition 4.2.

Proposition 4.4. *Assume that (H2)–(H7) hold,* $u(t,x) = Y_t^{t,x}$, $(t,x) \in [0,T] \times \overline{\mathcal{O}}_1$, *and given some* $\varphi \in \mathrm{C}^{1,2}([0,T] \times \overline{\mathcal{O}}_1; \mathbf{R})$ *such that* $u - \varphi$ *attains a global maximum (or minimum) at* $(t,x) \in [0,T) \times \overline{\mathcal{O}}_1$ *and* $u(t,x) - \varphi(t,x) = 0$. *Then there exists a domain*

$$U^{t,x}(\delta) := \{(s,q) \in [0,T] \times \overline{\mathcal{O}}_1 : |s-t| < \delta, |q-x| < \delta\},$$

where $\delta > 0$ *is a constant depending on* t, x *and* φ, *such that the following algebra equation*

$$z = \sigma^*(s,q,u(s,q),z)\nabla\varphi(s,q)$$

admits a unique solution $z = v(s,q)$ *for any* $(s,q) \in U^{t,x}(\delta)$*; and* $z = v(s,q)$ *is continuous in* $U^{t,x}(\delta)$.

Proof. Let the assumptions hold. By Proposition 4.2 we know that $u(t,x)$ is Lipschitz continuous with respect to x. Then Lemma 4.3 implies that $|\nabla\varphi(t,x)| \leq L_{ux}$. Moreover, there exists a $\alpha > 0$ such that $L_{\sigma z}|\nabla\varphi(t,x)| \leq L_{\sigma z} L_{ux} < \alpha < 1$. It follows from (H7) and the continuity of $\nabla\varphi$ that there exists a small enough constant $\delta > 0$ such that $L_{\sigma z}|\nabla\varphi(s,q)| < \alpha$ for all $(s,q) \in U^{t,x}(\delta)$.

We first treat the existence of unique solution. If $|\nabla\varphi(s,q)| = 0$, then $z = 0$ is the solution. Now we consider the case $|\nabla\varphi(s,q)| > 0$. For fixed $(s,q) \in U^{t,x}(\delta)$, take any $z \in \mathbf{R}^d$ and define an operator $\Psi : z \mapsto Z$ as $Z := \sigma^*(s,q,u(s,q),z)\nabla\varphi(s,q)$, which is actually a mapping from $\mathbf{R}^d$ to $\mathbf{R}^d$. Take another $z' \in \mathbf{R}^d$ and set $Z' := \Psi(z')$, $\hat{Z} := Z - Z'$ and $\hat{z} := z - z'$. Then it follows from (H3)(b) that $|\hat{Z}| \leq L_{\sigma z}|\nabla\varphi(s,q)||\hat{z}|$, which means that Ψ is a contraction mapping. The fixed point theorem indicates that the algebra equation admits a unique solution $z \in \mathbf{R}^d$. Since z is determined by (s,q), we can put $z = v(s,q)$.

We proceed to handle with the continuity of v. Since $\nabla\varphi$ is continuous, for the α given above, there exists a $\delta > 0$ satisfying

$$L_{\sigma z}|\nabla\varphi(s,q) - \nabla\varphi(t,x)| \leq 1 - \alpha, \quad \forall\ (s,q) \in U^{t,x}(\delta).$$

For fixed $(s_0,q_0) \in U^{t,x}(\delta)$, we have that, for each $(s,q) \in U^{t,x}(\delta)$,

$$\begin{aligned} v(s,q) - v(s_0,q_0) = {} & \sigma^*(s,q,u(s,q),v(s_0,q_0))\nabla\varphi(s,q) - \sigma^*(s_0,q_0,u(s_0,q_0),v(s_0,q_0))\nabla\varphi(s_0,q_0) \\ & + [\sigma^*(s,q,u(s,q),v(s,q)) - \sigma^*(s,q,u(s,q),v(s_0,q_0))]\nabla\varphi(t,x) \\ & + [\sigma^*(s,q,u(s,q),v(s,q)) - \sigma^*(s,q,u(s,q),v(s_0,q_0))][\nabla\varphi(s,q) - \nabla\varphi(t,x)]. \end{aligned}$$

We denote by I^σ the first line on the right hand side of the above equality. Hence, making inner product with $v(s,q) - v(s_0,q_0)$ in the above equality yields that,

$$|v(s,q) - v(s_0,q_0)|^2 \leq |I^\sigma||v(s,q) - v(s_0,q_0)| + L_{\sigma z}L_{ux}|v(s,q) - v(s_0,q_0)|^2 + (1-\alpha)|v(s,q) - v(s_0,q_0)|^2,$$

from which it follows that $\big(\alpha - L_{\sigma z}L_{ux}\big)|v(s,q) - v(s_0,q_0)| \leq |I^\sigma|$. Then the continuity of u, $\nabla\varphi$ and σ implies the continuity of v in $U^{t,x}(\delta)$. ∎

4.3. Viscosity solution: existence result

Now we can give the probabilistic interpretation for viscosity solutions of PDE (4.2) and its proof.

Theorem 4.5. *Let assumptions (H2)–(H7) hold. The function $u(t,x) = Y_t^{t,x}$, $(t,x) \in [0,T] \times \overline{\mathcal{O}}_1$ is a viscosity solution of PDE (4.2).*

Proof. The continuity of u follows from Proposition 4.2. It is sufficient to show that u is a viscosity subsolution of PDE (4.2) since the proof of the other assertion is symmetric. Noticing that $u(T,x) = \Phi(x)$ for all $x \in \overline{\mathcal{O}}_1$, we only need to prove (4.3).

Let $\varphi \in \mathrm{C}^{1,2}([0,T] \times \overline{\mathcal{O}}_1; \mathbf{R})$ such that $u - \varphi$ attains the global maximum at $(t,x) \in [0,T) \times \overline{\mathcal{O}}_1$, and by slight modifications we can assume without loss of generality that $u(t,x) - \varphi(t,x) = 0$. Accordingly, Proposition 4.4 yields that there exists a domain $U^{t,x}(\delta)$ with $\delta > 0$ depending on t, x and φ, such that the following algebra equation admits a unique solution v,

$$v(s,q) = \sigma^*(s,q,u(s,q),v(s,q))\nabla\varphi(s,q), \quad \forall\ (s,q) \in U^{t,x}(\delta).$$

It reduces to prove the two inequalities in (4.3). From (ii) in Definition 2.1 we know that $L(t,x) \leq u(t,x) = Y_t^{t,x} \leq U(t,x)$, and $u(t,x) = L(t,x)$ would imply the two inequalities of (4.3) no matter $(t,x) \in [0,T) \times \mathcal{O}_1$ or $(t,x) \in [0,T) \times \partial\mathcal{O}_1$. So we assume without loss of generality that $u(t,x) - L(t,x) > 2\delta > 0$. Here the δ can be chosen as different symbols and we choose the same one just for simplicity. Let us set

$$\tau_\delta := \inf\{s > t : |X_s^{t,x} - x| \wedge L_{\sigma z}|\nabla\varphi(s,X_s^{t,x}) - \nabla\varphi(t,x)| \geq \delta,\ u(s,X_s^{t,x}) - L(s,X_s^{t,x}) \leq \delta\}.$$

From the continuity of $X_\cdot^{t,x}$, $\nabla\varphi$, u and L, we can derive that $\mathbf{P}-$a.s., $\tau_\delta > t$ holds. Setting $\tau_{\delta,\varepsilon} := \tau_\delta \wedge (t+\varepsilon)$ with $0 < \varepsilon \leq (T-t) \wedge \delta$, we have that for each $r \in [t,T]$, $\mathbf{P}-$a.s.,

$$\mathbf{1}_{r \leq \tau_{\delta,\varepsilon}} v(r,X_r^{t,x}) = \mathbf{1}_{r \leq \tau_{\delta,\varepsilon}} \sigma^*(r,X_r^{t,x},u(r,X_r^{t,x}),v(r,X_r^{t,x}))\nabla\varphi(r,X_r^{t,x}). \tag{4.9}$$

Since $u(t,x)$ is continuous in (t,x), analogous arguments to Proposition 2.6 in Wu and Yu [18] yields that $u(\tau,\xi) = Y_\tau^{\tau,\xi}$ for any stopping time τ valued in $[0,T]$ and random variable $\xi \in \mathrm{L}^2(\Omega,\mathcal{F}_\tau,\mathbf{P};\mathbf{R}^n)$. Hence, for any $0 \leq t < t+\varepsilon \leq T$ with ε small enough, we have

$$u(t,x) = Y_t^{t,x} = u(\tau_{\delta,\varepsilon},X_{\tau_{\delta,\varepsilon}}^{t,x}) + \int_t^{\tau_{\delta,\varepsilon}} g(r,\Theta_r^{t,x})\,\mathrm{d}r - \int_t^{\tau_{\delta,\varepsilon}} \langle Z_r^{t,x},\,\mathrm{d}B_r\rangle + \zeta_{\tau_{\delta,\varepsilon}}^{t,x}.$$

Note that $u(r,X_r^{t,x}) = Y_r^{t,x}$ and $u(\tau_{\delta,\varepsilon},X_{\tau_{\delta,\varepsilon}}^{t,x}) \leq \varphi(\tau_{\delta,\varepsilon},X_{\tau_{\delta,\varepsilon}}^{t,x})$. Itô's formula applied to $\varphi(r,X_r^{t,x})$ implies

$$\begin{aligned}\varphi(t,x) = \varphi(\tau_{\delta,\varepsilon},X_{\tau_{\delta,\varepsilon}}^{t,x}) &- \int_t^{\tau_{\delta,\varepsilon}} \partial_r\varphi(r,X_r^{t,x})\,\mathrm{d}r - \int_t^{\tau_{\delta,\varepsilon}} \langle\nabla\varphi(r,X_r^{t,x}), b(r,X_r^{t,x},u(r,X_r^{t,x}),Z_r^{t,x})\rangle\,\mathrm{d}r \\ &- \int_t^{\tau_{\delta,\varepsilon}} \frac{1}{2}\,\mathrm{tr}\,\{\sigma\sigma^*(r,X_r^{t,x},u(r,X_r^{t,x}),Z_r^{t,x})\mathrm{D}^2\varphi(r,X_r^{t,x})\}\,\mathrm{d}r \\ &- \int_t^{\tau_{\delta,\varepsilon}} \langle\sigma^*(r,X_r^{t,x},u(r,X_r^{t,x}),Z_r^{t,x})\nabla\varphi(r,X_r^{t,x}),\,\mathrm{d}B_r\rangle + \int_t^{\tau_{\delta,\varepsilon}} \frac{\partial\varphi}{\partial n}(r,X_r^{t,x})\mathbf{1}_{\partial\mathcal{O}_1}(X_r^{t,x})\,\mathrm{d}|\eta^{t,x}|_r.\end{aligned}$$

Thereby, setting $\widetilde{Z}_\cdot^{t,x} := \sigma^*(\,\cdot\,,X_\cdot^{t,x},u(\,\cdot\,,X_\cdot^{t,x}),Z_\cdot^{t,x})\nabla\varphi(\,\cdot\,,X_\cdot^{t,x})$ and $\hat{Z}_\cdot^{t,x} := Z_\cdot^{t,x} - \widetilde{Z}_\cdot^{t,x}$,

$$\int_t^{\tau_{\delta,\varepsilon}} G(r,X_r^{t,x},u(r,X_r^{t,x}),Z_r^{t,x})\,\mathrm{d}r - \int_t^{\tau_{\delta,\varepsilon}} \langle\hat{Z}_r^{t,x},\,\mathrm{d}B_r\rangle + \zeta_{\tau_{\delta,\varepsilon}}^{t,x} - \int_t^{\tau_{\delta,\varepsilon}} \frac{\partial\varphi}{\partial n}(r,X_r^{t,x})\mathbf{1}_{\partial\mathcal{O}_1}(X_r^{t,x})\,\mathrm{d}|\eta^{t,x}|_r \geq 0, \tag{4.10}$$

where G is defined as $G(r,\theta') := \partial_r\varphi(r,x') + (\mathcal{L}_r\varphi)(r,\theta') + g\big(r,\theta'\big)$ for each $\theta' \in M$. By the aid of (H2) and (H3) we can deduce that there exists a bounded and adapted process $C(r)$ such that $\mathbf{P}-$ a.s., for each $r \in [t,\tau_{\delta,\varepsilon}]$,

$$|G(r,X_r^{t,x},u(r,X_r^{t,x}),Z_r^{t,x}) - G(r,X_r^{t,x},u(r,X_r^{t,x}),v(r,X_r^{t,x}))| \leq C(r)|v(r,X_r^{t,x}) - Z_r^{t,x}|. \tag{4.11}$$

In the sequel, we will estimate the right hand side term of the previous inequality. In view of (4.9) and the definition of $\widetilde{Z}^{t,x}$, we have that $\mathbf{P}-$ a.s., for each $r \in [t,\tau_{\delta,\varepsilon}]$,

$$\begin{aligned} v(r,X_r^{t,x}) - \widetilde{Z}_r^{t,x} &= [\sigma^*(r,X_r^{t,x},u(r,X_r^{t,x}),v(r,X_r^{t,x})) - \sigma^*(r,X_r^{t,x},u(r,X_r^{t,x}),Z_r^{t,x})]\nabla\varphi(t,x) \\ &\quad + [\sigma^*(r,X_r^{t,x},u(r,X_r^{t,x}),v(r,X_r^{t,x})) - \sigma^*(r,X_r^{t,x},u(r,X_r^{t,x}),Z_r^{t,x})]\big(\nabla\varphi(r,X_r^{t,x}) - \nabla\varphi(t,x)\big). \end{aligned}$$

Lemma 4.3 and the assumption (H7) indicate that there exists a α with $0 < \alpha < 1$ such that $L_{\sigma z}|\nabla\varphi(t,x)| \leq L_{\sigma z}L_{ux} < \alpha$. Hence, by the definition of $\tau_{\delta,\varepsilon}$ and choosing $\delta \leq (1-\alpha)/2$ we have that $\mathbf{P}-$ a.s., for each $r \in [t,\tau_{\delta,\varepsilon}]$,

$$L_{\sigma z}|\nabla\varphi(r,X_r^{t,x}) - \nabla\varphi(t,x)| \leq \frac{1-\alpha}{2}.$$

Thus, we can deduce from (H3)(b) that

$$\langle v(r,X_r^{t,x}) - \widetilde{Z}_r^{t,x}, v(r,X_r^{t,x}) - Z_r^{t,x}\rangle \leq \frac{1+\alpha}{2}|v(r,X_r^{t,x}) - Z_r^{t,x}|^2.$$

Then by adding and subtracting $Z_r^{t,x}$ in the first term of the inner product we have that

$$\begin{aligned} &\langle (v(r,X_r^{t,x}) - Z_r^{t,x}) - (\widetilde{Z}_r^{t,x} - Z_r^{t,x}), v(r,X_r^{t,x}) - Z_r^{t,x}\rangle \\ &= |v(r,X_r^{t,x}) - Z_r^{t,x}|^2 - \langle \widetilde{Z}_r^{t,x} - Z_r^{t,x}, v(r,X_r^{t,x}) - Z_r^{t,x}\rangle \leq \frac{1+\alpha}{2}|v(r,X_r^{t,x}) - Z_r^{t,x}|^2, \end{aligned}$$

from which it follows that

$$|v(r,X_r^{t,x}) - Z_r^{t,x}|^2 \leq \frac{2}{1-\alpha}|v(r,X_r^{t,x}) - Z_r^{t,x}||\widetilde{Z}_r^{t,x} - Z_r^{t,x}|.$$

Hence, the previous inequality together with (4.11) indicates that

$$|G(r,X_r^{t,x},u(r,X_r^{t,x}),Z_r^{t,x}) - G(r,X_r^{t,x},u(r,X_r^{t,x}),v(r,X_r^{t,x}))| \leq \frac{2}{1-\alpha}C(r)|\hat{Z}_r^{t,x}|.$$

Thus, there exists a bounded process $c(r)$ with $|c(r)| \leq 2C(r)/(1-\alpha)$ such that

$$G(r,X_r^{t,x},u(r,X_r^{t,x}),Z_r^{t,x}) - G(r,X_r^{t,x},u(r,X_r^{t,x}),v(r,X_r^{t,x})) = \langle c(r),\hat{Z}_r^{t,x}\rangle.$$

Then (4.10) becomes

$$\begin{aligned} &\int_t^{\tau_{\delta,\varepsilon}} [G(r,X_r^{t,x},u(r,X_r^{t,x}),v(r,X_r^{t,x})) + \langle c(r),\hat{Z}_r^{t,x}\rangle]\,\mathrm{d}r - \int_t^{\tau_{\delta,\varepsilon}} \frac{\partial\varphi}{\partial n}(r,X_r^{t,x})\mathbf{1}_{\partial\mathcal{O}_1}(X_r^{t,x})\,\mathrm{d}|\eta^{t,x}|_r \\ &\quad - \int_t^{\tau_{\delta,\varepsilon}} \langle \hat{Z}_r^{t,x},\,\mathrm{d}B_r\rangle + \zeta_{\tau_{\delta,\varepsilon}}^{t,x} \geq 0. \end{aligned}$$

The boundedness of $c(\cdot)$ renders the process $e_r^t := \exp\{\int_t^r \langle c(s), \mathrm{d}B_s\rangle - \frac{1}{2}\int_t^r |c(s)|^2 \,\mathrm{d}s\}$, $r \in [t,T]$, is a martingale. Thus, by Girsanov's theorem we can define a new probability measure $\widetilde{\mathbf{P}}$ *via* $\mathrm{d}\widetilde{\mathbf{P}}/\mathrm{d}\mathbf{P} = e_T^t$, so that $\widetilde{B}_r := B_r - B_t - \int_t^r c(s)\,\mathrm{d}s$ is a $\widetilde{\mathbf{P}}$-Brownian motion on $[t,T]$. Hence, we see that

$$\widetilde{\mathbf{E}}\int_t^{\tau_{\delta,\varepsilon}} G(r, X_r^{t,x}, u(r,X_r^{t,x}), v(r,X_r^{t,x}))\,\mathrm{d}r - \widetilde{\mathbf{E}}\int_t^{\tau_{\delta,\varepsilon}} \frac{\partial\varphi}{\partial n}(r,X_r^{t,x})\mathbf{1}_{\partial\mathcal{O}_1}(X_r^{t,x})\,\mathrm{d}|\eta^{t,x}|_r + \widetilde{\mathbf{E}}[\zeta_{\tau_{\delta,\varepsilon}}^{t,x}] \geq 0,$$

where $\widetilde{\mathbf{E}}$ represents the expectation with respect to $\widetilde{\mathbf{P}}$. Note that $u(r,X_r^{t,x}) \in \mathcal{O}_2(r,X_r^{t,x})$. It follows from the definition of $\tau_{\delta,\varepsilon}$ that $\mathbf{P}-$ a.s., for each $r \in [t,\tau_{\delta,\varepsilon}]$,

$$L(r,X_r^{t,x}) < u(r,X_r^{t,x}) - \delta < u(r,X_r^{t,x}) \leq U(r,X_r^{t,x}),$$

which together with (iv) in Definition 2.1 implies that

$$0 \geq \int_t^{\tau_{\delta,\varepsilon}} \left[Y_r^{t,x} - \left(u(r,X_r^{t,x}) - \delta\right)\right]\mathrm{d}\zeta_r^{t,x} = \delta\zeta_{\tau_{\delta,\varepsilon}}^{t,x}.$$

Thus we get that $\widetilde{\mathbf{E}}[\zeta_{\tau_{\delta,\varepsilon}}^{t,x}] \leq 0$, whence

$$\widetilde{\mathbf{E}}\int_t^{\tau_{\delta,\varepsilon}} G(r, X_r^{t,x}, u(r,X_r^{t,x}), v(r,X_r^{t,x}))\,\mathrm{d}r - \widetilde{\mathbf{E}}\int_t^{\tau_{\delta,\varepsilon}} \frac{\partial\varphi}{\partial n}(r,X_r^{t,x})\mathbf{1}_{\partial\mathcal{O}_1}(X_r^{t,x})\,\mathrm{d}|\eta^{t,x}|_r \geq 0. \tag{4.12}$$

We first consider the case $(t,x) \in [0,T) \times \mathcal{O}_1$. It reduces to show that

$$0 \leq \partial_t\varphi(t,x) + (\mathcal{L}_t\varphi)(t,x,u(t,x),v(t,x)) + g\big(t,x,u(t,x),v(t,x)\big) = G(t,x,u(t,x),v(t,x)).$$

Because $X_r^{t,x} \notin \partial\mathcal{O}_1$ for $r \in [t,\tau_{\delta,\varepsilon}]$ with δ and ε sufficiently small, the second term on the left hand side of (4.12) vanishes in this case. Dividing both sides by ε leads to

$$\frac{1}{\varepsilon}\widetilde{\mathbf{E}}\int_t^{t+\varepsilon} \mathbf{1}_{r\leq\tau_\delta} G(r, X_r^{t,x}, u(r,X_r^{t,x}), v(r,X_r^{t,x}))\,\mathrm{d}r \geq 0.$$

Then Lebesgue lemma (see Proposition 2.2 in Jiang [8]) after taking $\varepsilon \to 0$ yields that $G(t,x,u(t,x),v(t,x)) \geq 0$.

We now consider the case $(t,x) \in [0,T) \times \partial\mathcal{O}_1$. We aim at showing the third inequality in (4.3). It suffices to show that either $\partial\varphi/\partial n \leq 0$ or $G(t,x,u(t,x),v(t,x)) \geq 0$ hold. We suppose that both $-\partial\varphi/\partial n < -\varepsilon_0 < 0$ and $G(t,x,u(t,x),v(t,x)) < -\varepsilon_0 < 0$ hold, then we will obtain a contradiction. Let us set

$$\tau_0 := \inf\left\{s > t : \left(-\frac{\partial\varphi}{\partial n}(s,X_s^{t,x})\right) \wedge G(s,X_s^{t,x},u(s,X_s^{t,x}),v(s,X_s^{t,x})) \geq -\frac{\varepsilon_0}{2}\right\}.$$

We still have that $\mathbf{P}-$ a.s., $\tau_0 > t$, thanks to the continuity of $\partial\varphi/\partial n$ and G. Analogous arguments as previous to obtain (4.12) yield that (4.12) still holds true when the stopping time $\tau_{\delta,\varepsilon}$ is substituted by $\tau_{\delta,\varepsilon}^0 := \tau_{\delta,\varepsilon} \wedge \tau_0$. Noticing that $|\eta^{t,x}|_\cdot$ is nondecreasing, we deduce that

$$-\widetilde{\mathbf{E}}\int_t^{\tau_{\delta,\varepsilon}^0} \frac{\partial\varphi}{\partial n}(r,X_r^{t,x})\mathbf{1}_{\partial\mathcal{O}_1}(X_r^{t,x})\,\mathrm{d}|\eta^{t,x}|_r \leq -\frac{\varepsilon_0}{2}\widetilde{\mathbf{E}}[|\eta^{t,x}|_{\tau_{\delta,\varepsilon}^0} - |\eta^{t,x}|_t] \leq 0.$$

Moreover, it follows from the definition of $\tau^0_{\delta,\varepsilon}$ that $\mathbf{P}(\tau^0_{\delta,\varepsilon} > t) = 1$. Thus, we have that $\widetilde{\mathbf{P}}(\tau^0_{\delta,\varepsilon} > t) = 1$, whence $\widetilde{\mathbf{E}}[\tau^0_{\delta,\varepsilon} - t] > 0$. Therefore, we can derive that

$$\widetilde{\mathbf{E}} \int_t^{\tau^0_{\delta,\varepsilon}} G(r, X_r^{t,x}, u(r, X_r^{t,x}), v(r, X_r^{t,x}))\, \mathrm{d}r \le -\frac{\varepsilon_0}{2} \widetilde{\mathbf{E}}[\tau^0_{\delta,\varepsilon} - t] < 0,$$

which contradicts the inequality (4.12) with $\tau_{\delta,\varepsilon}$ replaced by $\tau^0_{\delta,\varepsilon}$. The proof of Theorem 4.5 is completed. $\square$

4.4. Viscosity solution: uniqueness result

In this subsection we proceed to prove the uniqueness for the viscosity solutions of PDE (4.2). The uniqueness result is an immediate consequence of the following comparison theorem, whose proof is an variation of Barles, Buckdahn and Pardoux [2] and Ma and Cvitanić [11].

Theorem 4.6. *Assume that (H2)–(H7) hold, $\sigma(t,\theta) = \sigma(t,x)$, and $g(t,\theta)$ is nondecreasing in y for each $(t,x,z) \in [0,T] \times \overline{\mathcal{O}}_1 \times \mathbf{R}^d$. Let u_1 and u_2 be a viscosity subsolution and supersolution of PDE (4.2), respectively, such that $u_1(t,x) \ge L(t,x)$ and $u_2(t,x) \le U(t,x)$ for all $(t,x) \in [0,T] \times \overline{\mathcal{O}}_1$, and both u_1 and u_2 are continuous in t and Lipschitz continuous in x. Then $u_1(t,x) \le u_2(t,x)$ holds for all $(t,x) \in [0,T] \times \overline{\mathcal{O}}_1$.*

Proof. Analogous to Ma and Cvitanić [11], we only need to prove $u_1 \le u_2$ in $[0,T] \times \mathcal{O}_1$ since both u_1 and u_2 are continuous. We define a subset of $\mathcal{O}_1$ as $\mathcal{O}_1^\alpha := \{x \in \overline{\mathcal{O}}_1 : d(x, \partial\mathcal{O}_1) \ge \alpha\}$ for each $\alpha > 0$. We choose a $\alpha_0 > 0$ such that $\mathcal{O}_1^\alpha \neq \varnothing$ for each $0 < \alpha \le \alpha_0$. Hence, we only need to prove that for each $0 < \alpha \le \alpha_0$, $u_1(t,x) \le u_2(t,x)$ for all $(t,x) \in [0,T] \times \mathcal{O}_1^\alpha$. We first show the following auxiliary lemma. Since our conditions are weaker than the exiting works, we give the detailed proof for completeness.

Lemma 4.7. *Suppose that the assumptions of Theorem 4.6 are in force. Then for each $0 < \alpha \le \alpha_0$, the function $w(t,x) := u_1(t,x) - u_2(t,x)$, $(t,x) \in [0,T] \times \mathcal{O}_1^\alpha$ is a viscosity subsolution of the following PDE:*

$$\begin{cases} w(t,x) \wedge F_w(t,x,w,\partial_t w, \nabla w, \mathrm{D}^2 w) = 0, \quad (t,x) \in [0,T) \times \mathcal{O}_1^\alpha; \\ w(T,x) = 0, \quad x \in \mathcal{O}_1^\alpha, \end{cases} \tag{4.13}$$

where $F_w(t,x,w,p,q,X) := -p - \frac{1}{2}\operatorname{tr}\{\sigma\sigma^*(t,x)X\} - \langle b(t,x,u_1(t,x),0), q\rangle - \widetilde{K}(|w| + |q||\sigma(t,x)|)$, and $\widetilde{K}$ is a positive constant depending on K, g and the Lipschitz constants of u_1 and u_2.

Proof of Lemma 4.7. Let us fix arbitrarily a $\alpha \in (0,\alpha_0]$, $\varphi \in \mathrm{C}^{1,2}([0,T] \times \mathcal{O}_1^\alpha; \mathbf{R})$ such that $(t_0,x_0) \in [0,T) \times \mathcal{O}_1^\alpha$ is a global maximum point of $w - \varphi$. Without loss of generality, we can assume that $w(t_0,x_0) = \varphi(t_0,x_0)$. Then Lemma 4.3 implies that $\nabla\varphi$ is uniformly bounded by $L_{u_1} + L_{u_2}$, where L_{u_1} and L_{u_2} denote the Lipschitz constants of u_1 and u_2, respectively. Note that $u_1(T,x) \le \Phi(x)$ and $u_2(T,x) \ge \Phi(x)$. Thereby, $w(T,x) \le 0$ holds. By the definition of viscosity subsolution, it reduces to prove the following inequality,

$$w(t_0,x_0) \wedge F_w(t_0,x_0,w,\partial_t\varphi, \nabla\varphi, \mathrm{D}^2\varphi)|_{(t,x)=(t_0,x_0)} \le 0. \tag{4.14}$$

Since u_1 is a viscosity subsolution and u_2 is a viscosity supersolution, Definition 4.1 indicates that $u_1(t,x) \le U(t,x)$ and $u_2(t,x) \ge L(t,x)$. In conjunction with assumptions $u_1(t,x) \ge L(t,x)$ and $u_2(t,x) \le U(t,x)$, we have that $u_1(t,x)$, $u_2(t,x) \in [L(t,x), U(t,x)]$ for all $(t,x) \in [0,T] \times \mathcal{O}_1$. Thus, if (t_0,x_0) satisfies either $u_1(t_0,x_0) = L(t_0,x_0)$ or $u_2(t_0,x_0) = U(t_0,x_0)$, then $w(t_0,x_0) \le 0$, whence (4.14) holds. Hence, we assume without loss of generality that

$$[u_1(t_0,x_0) - L(t_0,x_0)][u_2(t_0,x_0) - U(t_0,x_0)] \neq 0. \tag{4.15}$$

Next we define a function as follows, for each $\varepsilon > 0$, $t \in [0,T]$ and x, $y \in \mathcal{O}_1^\alpha$,

$$\psi_\varepsilon(t,x,y) := u_1(t,x) - u_2(t,y) - \frac{|x-y|^2}{\varepsilon} - \varphi(t,x).$$

For each $\varepsilon > 0$, we suppose that $(t^\varepsilon, x^\varepsilon, y^\varepsilon)$ is the maximum point of ψ_ε. Then Proposition 3.7 of Crandall, Ishii and Lions [4] implies that $(t^\varepsilon, x^\varepsilon, y^\varepsilon)$ tends to (t_0, x_0, x_0) and $|x^\varepsilon - y^\varepsilon|^2/\varepsilon$ tends to 0 as $\varepsilon \to 0$. Moreover, it follows from Theorem 8.3 in Crandall, Ishii and Lions [4] that there exist $(X,Y) \in S^n \times S^n$ and $a \in \mathbf{R}^n$ such that, denoting $p^\varepsilon := 2(x^\varepsilon - y^\varepsilon)/\varepsilon$ and the identity matrix by I,

$$(a + \partial_t\varphi(t^\varepsilon, x^\varepsilon), p^\varepsilon + \nabla\varphi(t^\varepsilon, x^\varepsilon), X) \in \overline{\mathcal{P}}^{2,+} u_1(t^\varepsilon, x^\varepsilon), \quad (a, p^\varepsilon, Y) \in \overline{\mathcal{P}}^{2,-} u_2(t^\varepsilon, y^\varepsilon),$$

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \frac{2}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \begin{pmatrix} \mathrm{D}^2\varphi(t^\varepsilon, x^\varepsilon) & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.16}$$

Then Definition 4.1 and the definitions of semijets yield that

$$\begin{cases} (u_1 - L) \wedge \{(u_1 - U) \vee [-(a + \partial_t\varphi) - F(t,x,u_1,p^\varepsilon + \nabla\varphi, X)]\}|_{(t,x)=(t^\varepsilon,x^\varepsilon)} \le 0, \\ (u_2 - U) \vee \{(u_2 - L) \wedge [-a - F(t,y,u_2,p^\varepsilon, Y)]\}|_{(t,y)=(t^\varepsilon,y^\varepsilon)} \ge 0, \end{cases}$$

where for each $(t,x,u,q,X) \in [0,T] \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times S^n$,

$$F(t,x,u,q,X) := \frac{1}{2}\,\mathrm{tr}\,\{\sigma\sigma^*(t,x)X\} + \langle b(t,x,u,\sigma^*(t,x)q), q\rangle + g(t,x,u,\sigma^*(t,x)q).$$

The continuity of u_1, u_2, L and U, and the fact that $(t^\varepsilon, x^\varepsilon, y^\varepsilon) \to (t_0, x_0, x_0)$ indicate that $(t^\varepsilon, x^\varepsilon)$ and $(t^\varepsilon, y^\varepsilon)$ can also satisfy (4.15) when ε is small enough. Thus, we get $u_1(t^\varepsilon, x^\varepsilon) > L(t^\varepsilon, x^\varepsilon)$ and $u_2(t^\varepsilon, y^\varepsilon) < U(t^\varepsilon, y^\varepsilon)$, which results in

$$\begin{cases} -a - \partial_t\varphi(t^\varepsilon, x^\varepsilon) - F(t^\varepsilon, x^\varepsilon, u_1(t^\varepsilon, x^\varepsilon), p^\varepsilon + \nabla\varphi(t^\varepsilon, x^\varepsilon), X) \le 0, \\ -a - F(t^\varepsilon, y^\varepsilon, u_2(t^\varepsilon, y^\varepsilon), p^\varepsilon, Y) \ge 0. \end{cases} \tag{4.17}$$

Subtracting the second inequality from the first one in (4.17) yields that

$$\begin{aligned} &\partial_t\varphi(t^\varepsilon, x^\varepsilon) + \frac{1}{2}\big[\,\mathrm{tr}\,\{\sigma\sigma^*(t^\varepsilon, x^\varepsilon)X\} - \mathrm{tr}\,\{\sigma\sigma^*(t^\varepsilon, y^\varepsilon)Y\}\big] \\ &+ \big[\langle b(t^\varepsilon, x^\varepsilon, u_1(t^\varepsilon, x^\varepsilon), \sigma^*(t^\varepsilon, x^\varepsilon)(p^\varepsilon + \nabla\varphi(t^\varepsilon, x^\varepsilon))), p^\varepsilon + \nabla\varphi(t^\varepsilon, x^\varepsilon)\rangle - \langle b(t^\varepsilon, y^\varepsilon, u_2(t^\varepsilon, y^\varepsilon), \sigma^*(t^\varepsilon, y^\varepsilon)p^\varepsilon), p^\varepsilon\rangle\big] \\ &+ \big[g(t^\varepsilon, x^\varepsilon, u_1(t^\varepsilon, x^\varepsilon), \sigma^*(t^\varepsilon, x^\varepsilon)(p^\varepsilon + \nabla\varphi(t^\varepsilon, x^\varepsilon))) - g(t^\varepsilon, y^\varepsilon, u_2(t^\varepsilon, y^\varepsilon), \sigma^*(t^\varepsilon, y^\varepsilon)p^\varepsilon)\big] \ge 0. \end{aligned} \tag{4.18}$$

The second term of the first line, the second line and the left hand side term of the last line in (4.18) are denoted by I_1^ε, I_2^ε and I_3^ε, respectively. Then it follows from the inequality (4.16) and (H3)(b) that

$$I_1^\varepsilon \le \frac{1}{2}\,\mathrm{tr}\,\{\sigma\sigma^*(t^\varepsilon, x^\varepsilon)\mathrm{D}^2\varphi(t^\varepsilon, x^\varepsilon)\} + \frac{|x^\varepsilon - y^\varepsilon|^2}{\varepsilon}.$$

Using (H3)(a) and (H3)(b) we can deduce that

$$I_2^\varepsilon = \langle b(t^\varepsilon, x^\varepsilon, u_1(t^\varepsilon, x^\varepsilon), \sigma^*(t^\varepsilon, x^\varepsilon)(p^\varepsilon + \nabla\varphi(t^\varepsilon, x^\varepsilon))) - b(t^\varepsilon, y^\varepsilon, u_2(t^\varepsilon, y^\varepsilon), \sigma^*(t^\varepsilon, y^\varepsilon)p^\varepsilon), p^\varepsilon\rangle$$

$$
\begin{aligned}
&+\langle b(t^\varepsilon,x^\varepsilon,u_1(t^\varepsilon,x^\varepsilon),\sigma^*(t^\varepsilon,x^\varepsilon)(p^\varepsilon+\nabla\varphi(t^\varepsilon,x^\varepsilon))),\nabla\varphi(t^\varepsilon,x^\varepsilon)\rangle\\
\le{}& 2K_{bx}\frac{|x^\varepsilon-y^\varepsilon|^2}{\varepsilon}+K|p^\varepsilon|\big(|u_1(t^\varepsilon,x^\varepsilon)-u_2(t^\varepsilon,y^\varepsilon)|+2K\frac{|x^\varepsilon-y^\varepsilon|^2}{\varepsilon}+|\sigma(t^\varepsilon,x^\varepsilon)||\nabla\varphi(t^\varepsilon,x^\varepsilon)|\big)\\
&+\langle b(t^\varepsilon,x^\varepsilon,u_1(t^\varepsilon,x^\varepsilon),0),\nabla\varphi(t^\varepsilon,x^\varepsilon)\rangle+K|\nabla\varphi(t^\varepsilon,x^\varepsilon)||\sigma(t^\varepsilon,x^\varepsilon)||p^\varepsilon+\nabla\varphi(t^\varepsilon,x^\varepsilon)|.
\end{aligned}
$$

Since g satisfies the monotonicity condition in (H3)(c) and $g(t,x,y,z)$ is nondecreasing in y, we have

$$
g(t,x,y_1,z)-g(t,x,y_2,z)\le |K_{gy}||y_1-y_2|.
$$

Hence, it follows from (H3)(b) and (H3)(c) that

$$
I_3^\varepsilon\le K|x^\varepsilon-y^\varepsilon|+|K_{gy}||u_1(t^\varepsilon,x^\varepsilon)-u_2(t^\varepsilon,y^\varepsilon)|+2K^2\frac{|x^\varepsilon-y^\varepsilon|^2}{\varepsilon}+K|\sigma(t^\varepsilon,x^\varepsilon)||\nabla\varphi(t^\varepsilon,x^\varepsilon)|.
$$

In view of $w(t^\varepsilon,x^\varepsilon)-\varphi(t^\varepsilon,x^\varepsilon)\le w(t_0,x_0)-\varphi(t_0,x_0)=0$, we can deduce by the definition of ψ_ε that

$$
\begin{aligned}
0=\psi_\varepsilon(t_0,x_0,x_0)\le\psi_\varepsilon(t^\varepsilon,x^\varepsilon,y^\varepsilon)&=(w(t^\varepsilon,x^\varepsilon)-\varphi(t^\varepsilon,x^\varepsilon))+(u_2(t^\varepsilon,x^\varepsilon)-u_2(t^\varepsilon,y^\varepsilon))-\frac{|x^\varepsilon-y^\varepsilon|^2}{\varepsilon}\\
&\le(u_2(t^\varepsilon,x^\varepsilon)-u_2(t^\varepsilon,y^\varepsilon))-\frac{|x^\varepsilon-y^\varepsilon|^2}{\varepsilon}.
\end{aligned}
$$

Then we have that $|p^\varepsilon|\le 2|u_2(t^\varepsilon,x^\varepsilon)-u_2(t^\varepsilon,y^\varepsilon)|/|x^\varepsilon-y^\varepsilon|\le 2L_{u_2}$. Thereby, plugging the estimates of I_i^ε $(i=1,2,3)$ into (4.18) and then sending $\varepsilon\to0$, we compute by the boundedness of $|p^\varepsilon|$ and $|\nabla\varphi|$ that,

$$
\begin{aligned}
&-\partial_t\varphi(t_0,x_0)-\frac12\operatorname{tr}\{\sigma\sigma^*(t_0,x_0)\mathrm{D}^2\varphi(t_0,x_0)\}-\langle b(t_0,x_0,u_1(t_0,x_0),0),\nabla\varphi(t_0,x_0)\rangle\\
&-(K+KL_{u_1}+5KL_{u_2})|\sigma(t_0,x_0)||\nabla\varphi(t_0,x_0)|-(|K_{gy}|+2KL_{u_2})|w(t_0,x_0)|\le0.
\end{aligned}
$$

After choosing $\widetilde K:=K+|K_{gy}|+KL_{u_1}+7KL_{u_2}$, we obtain the desired result. □

Coming back to the proof of Theorem 4.6. It suffices to prove that for any $0<\alpha\le\alpha_0$, $w(t,x)\le0$ holds for each $(t,x)\in[0,T]\times\mathcal{O}_1^\alpha$. Let us fix arbitrarily a $\alpha\in(0,\alpha_0]$. For each $(t,x)\in[0,T]\times\mathcal{O}_1^\alpha$, we define the function $\Psi(t,x):=\exp\{(\widetilde C(T-t)+A)\psi(x)\}$, where $\psi(x):=(1+|x|^2)^{1/2}$ and constants A, $\widetilde C>0$ will be chosen later. By some easy computations we have the following estimates of the first and second derivatives of $\psi(x)$,

$$
|\nabla\psi(x)|=\frac{|x|}{\psi(x)}\le1,\qquad |\mathrm{D}^2\psi(x)|=\left|\frac{I}{\psi(x)}-\frac{xx^*}{\psi^3(x)}\right|\le\frac{|I|}{\psi(x)}+\frac{|x|^2}{\psi^2(x)}\frac{1}{\psi(x)}\le n+1.
$$

In what follows, we denote $t_1:=T-A/\widetilde C$. Then the previous estimates yield that, for each $t\in[t_1,T]$,

$$
\begin{aligned}
\partial_t\Psi(t,x)=-\widetilde C\psi(x)\Psi(t,x),\quad |\nabla\Psi(t,x)|&=(\widetilde C(T-t)+A)\psi(x)|\Psi(t,x)|\le2A\Psi(t,x),\\
|\mathrm{D}^2\Psi(t,x)|&=|(\widetilde C(T-t)+A)^2\Psi(t,x)\nabla\psi(x)(\nabla\psi(x))^*+(\widetilde C(T-t)+A)\Psi(t,x)\mathrm{D}^2\psi(x)|\\
&\le2A(2A+n+1)\Psi(t,x).
\end{aligned}
$$

Thus, it follows from (H2) and $\psi(x)$, $\Psi(t,x)\ge1$ that, for each $(t,x)\in[t_1,T]\times\mathcal{O}_1^\alpha$,

$$
F_w(t,x,\Psi,\partial_t\Psi,\nabla\Psi,\mathrm{D}^2\Psi)
$$

$$\geq \widetilde{C}\psi(x)\Psi(t,x) - \frac{1}{2}K^2|\mathrm{D}^2\Psi(t,x)| - K|\nabla\Psi(t,x)| - \widetilde{K}\Psi(t,x) - \widetilde{K}K|\nabla\Psi(t,x)|$$
$$\geq \Psi(t,x)\{\widetilde{C} - \widetilde{K}A(2A+n+1) - 2AK - \widetilde{K} - 2AK\widetilde{K}\} > 0,$$

provided that $\widetilde{C}$ is large enough.

Next we choose suitable A and $\widetilde{C}$ such that $\lim_{|x|\to\infty}|w(t,x)|\exp\{-A|x|\} = 0$. For each $\varepsilon > 0$, define

$$M^\alpha(\varepsilon) := \max_{[t_1,T]\times\mathcal{O}_1^\alpha}\left\{\big(w(t,x) - \varepsilon\Psi(t,x)\big)\exp\{-\widetilde{K}(T-t)\}\right\}.$$

In the sequel, we will prove that $M^\alpha(\varepsilon) \leq 0$. Recall that (t,x) belongs to the compact set $[t_1,T]\times\mathcal{O}_1^\alpha$, $w(t,x)$ and $\Psi(t,x)$ are continuous. Then for each $\varepsilon > 0$, there exists $(t^\varepsilon, x^\varepsilon) \in [t_1,T]\times\mathcal{O}_1^\alpha$ where the maximum value $M^\alpha(\varepsilon)$ is achieved. Even if $\mathcal{O}_1^\alpha$ is unbounded, the maximum point can also be taken since $w(t,x) - \varepsilon\Psi(t,x) < 0$ when $|x| \to \infty$. For each $(t,x) \in [t_1,T]\times\mathcal{O}_1^\alpha$, define $\varphi(t,x) := \varepsilon\Psi(t,x) + M^\alpha(\varepsilon)\exp\{\widetilde{K}(T-t)\}$. Then $\varphi \in \mathrm{C}^{1,2}([t_1,T]\times\mathcal{O}_1^\alpha;\mathbf{R})$, $\varphi(t^\varepsilon,x^\varepsilon) = w(t^\varepsilon,x^\varepsilon)$ and $w(t,x) - \varphi(t,x) \leq 0$ for all $(t,x) \in [t_1,T]\times\mathcal{O}_1^\alpha$. From Lemma 4.7 we know that $w(t,x)$ is a viscosity subsolution of PDE (4.13). We suppose that $M^\alpha(\varepsilon) > 0$ for some $\varepsilon > 0$, from which it follows that $w(t^\varepsilon,x^\varepsilon) > 0$. Then by the definition of viscosity subsolution we have $F_w(t^\varepsilon,x^\varepsilon,w,\partial_t\varphi,\nabla\varphi,\mathrm{D}^2\varphi)|_{(t,x)=(t^\varepsilon,x^\varepsilon)} \leq 0$. On the other hand, by some direct calculations we derive that

$$F_w(t^\varepsilon,x^\varepsilon,w,\partial_t\varphi,\nabla\varphi,\mathrm{D}^2\varphi)|_{(t,x)=(t^\varepsilon,x^\varepsilon)} = \varepsilon F_w(t^\varepsilon,x^\varepsilon,\Psi,\partial_t\Psi,\nabla\Psi,\mathrm{D}^2\Psi)|_{(t,x)=(t^\varepsilon,x^\varepsilon)} > 0.$$

This contradiction indicates that $M^\alpha(\varepsilon) \leq 0$ holds for each $\varepsilon > 0$. Thus, we have that $w(t,x) \leq \varepsilon\Psi(t,x)$ for all $(t,x) \in [t_1,T]\times\mathcal{O}_1^\alpha$. Sending $\varepsilon \to 0$ yields that $w(t,x) \leq 0$ for all $(t,x) \in [t_1,T]\times\mathcal{O}_1^\alpha$. Finally, applying the same arguments on the intervals $[t_{i+1},t_i]$ $(i = 1,2,\cdots)$, where $t_{i+1} := (t_i - A/\widetilde{C})^+$, we obtain that for any $0 < \alpha \leq \alpha_0$, $w(t,x) \leq 0$ holds for all $(t,x) \in [0,T]\times\mathcal{O}_1^\alpha$. □

Appendix A. Standard estimates for solutions of FBSDERs

In this section we illustrate some *a priori* estimates for FBSDERs, which are useful for investigating the theory of FBSDERs. With analogous arguments to Lemma 3.1 and recalling that $0 \in \overline{\mathcal{O}}_1$ and $0 \in (L,U)$, we can conclude the following *a priori* estimates for adapted solutions of FBSDER (2.2), in which we set $\xi_t^0 := \xi(t,0,0,0)$, $\xi = b$, σ, g.

Lemma A.1. *Assume that (H1)–(H4) hold with K, $L_{\sigma z}$, $L_\Phi \geq 0$ and K_{bx}, $K_{gy} \in \mathbf{R}$, $(\Theta_t,\eta_t,\zeta_t)_{t\in[0,T]}$ is an adapted solution of FBSDER (2.2). Let $\lambda \geq 0$, $\varepsilon > 0$, $C_i > 0$ $(i = 1,\ldots,6)$ with $C_6 < 1$, we have*

$$\mathrm{e}^{-\lambda T}\mathbf{E}|X_T|^2 + \lambda_1^\varepsilon\|X\|_\lambda^2 \leq \mathrm{e}^{2C_0K}|x|^2 + k_1^\varepsilon\|Y\|_\lambda^2 + k_2^\varepsilon\|Z\|_\lambda^2 + \frac{\mathrm{e}^{2C_0K}}{\varepsilon}\left[\|b^0\|_\lambda^2 + (\varepsilon + 1 + 2C_0K^2)\|\sigma^0\|_\lambda^2\right], \tag{A.1}$$

where $\lambda_1^\varepsilon := \lambda_1 - \varepsilon\mathrm{e}^{2C_0K}(1 + K^2 + 2C_0K^2)$, $k_1^\varepsilon := k_1 + \varepsilon\mathrm{e}^{2C_0K}K^2$, $k_2^\varepsilon := k_2 + \varepsilon\mathrm{e}^{2C_0K}{L_{\sigma z}}^2$ and λ_1, k_1, k_2 are defined in Lemma 3.1; and

$$\lambda_2^\varepsilon\|Y\|_\lambda^2 + (1 - C_6)\|Z\|_\lambda^2 \leq \mathrm{e}^{-\lambda T}\left[(1+\varepsilon){L_\Phi}^2\mathbf{E}|X_T|^2 + \left(1 + \frac{1}{\varepsilon}\right)\mathbf{E}|\Phi(0)|^2\right] + \frac{1}{\varepsilon}\|g^0\|_\lambda^2 + C_5\|X\|_\lambda^2, \tag{A.2}$$

where $\lambda_2^\varepsilon := \lambda_2 - \varepsilon$ and λ_2 is defined in Lemma 3.1.

By (H5) and the procedure of establishing the contraction mapping in the proof of Theorem 3.2, we can prove the following estimate for solutions of FBSDER (2.2) with the help of Lemma A.1.

Proposition A.2. *Assume that (H1)–(H5) hold, $(\Theta_t, \eta_t, \zeta_t)_{t\in[0,T]}$ is an adapted solution of FBSDER (2.2). Then there exists a positive constant C such that*

$$\mathbf{E}\bigg[\int_0^T (|X_s|^2 + |Y_s|^2 + |Z_s|^2)\,\mathrm{d}s\bigg] \le C\mathbf{E}\bigg[|x|^2 + |\Phi(0)|^2 + \int_0^T \left(|b_s^0|^2 + |\sigma_s^0|^2 + |g_s^0|^2\right)\mathrm{d}s\bigg].$$

Proof. Let all assumptions hold. Then by Lemma A.1 we have that

$$\mathrm{e}^{-\lambda T}\mathbf{E}|X_T|^2 + \lambda_1^\varepsilon \|X\|_\lambda^2 \le A(\varepsilon) + k_1^\varepsilon \|Y\|_\lambda^2 + k_2^\varepsilon \|Z\|_\lambda^2,$$
$$\lambda_2^\varepsilon \|Y\|_\lambda^2 + (1 - C_6)\|Z\|_\lambda^2 \le B(\varepsilon) + (1+\varepsilon){L_\Phi}^2 \mathrm{e}^{-\lambda T}\mathbf{E}|X_T|^2 + C_5\|X\|_\lambda^2,$$

where we write

$$A(\varepsilon) := \mathrm{e}^{2C_0K}|x|^2 + \frac{\mathrm{e}^{2C_0K}}{\varepsilon}\left[\|b^0\|_\lambda^2 + (\varepsilon + 1 + 2C_0K^2)\|\sigma^0\|_\lambda^2\right],\ B(\varepsilon) := \Big(1 + \frac{1}{\varepsilon}\Big)\mathrm{e}^{-\lambda T}\mathbf{E}|\Phi(0)|^2 + \frac{1}{\varepsilon}\|g^0\|_\lambda^2.$$

Note that $\lambda_i^\varepsilon \to \lambda_i$, $k_i^\varepsilon \to k_i$, $(i = 1, 2)$, as $\varepsilon \to 0$. So (H5) implies that there exists a sufficiently small $\varepsilon > 0$ such that λ_1^ε, $\lambda_2^\varepsilon > 0$. Then it is easily follows from the previous inequalities that

$$\mathrm{e}^{-\lambda T}\mathbf{E}|X_T|^2 + \lambda_1^\varepsilon\|X\|_\lambda^2 \le A(\varepsilon) + \frac{B(\varepsilon)}{\lambda_2^\varepsilon} + \mu(\lambda_1^\varepsilon, \lambda_2^\varepsilon)\left[\mathrm{e}^{-\lambda T}\mathbf{E}|X_T|^2 + \lambda_1^\varepsilon\|X\|_\lambda^2\right],$$

where we denote

$$\mu(\lambda_1^\varepsilon, \lambda_2^\varepsilon) := \left[\frac{k_1^\varepsilon}{\lambda_2^\varepsilon} \vee \frac{k_2^\varepsilon}{1 - C_6}\right]\left((1+\varepsilon){L_\Phi}^2 \vee \frac{C_5}{\lambda_1^\varepsilon}\right).$$

Since $\mu(\lambda_1^\varepsilon, \lambda_2^\varepsilon)$ tends to $\mu(\lambda_1, \lambda_2)$ as $\varepsilon \to 0$, we choose a small enough ε such that $0 < \mu(\lambda_1^\varepsilon, \lambda_2^\varepsilon) < 1$. Then

$$\mathrm{e}^{-\lambda T}\mathbf{E}|X_T|^2 \le \frac{A(\varepsilon)}{1 - \mu(\lambda_1^\varepsilon, \lambda_2^\varepsilon)} + \frac{B(\varepsilon)}{\lambda_2^\varepsilon\big(1 - \mu(\lambda_1^\varepsilon, \lambda_2^\varepsilon)\big)}, \quad \|X\|_\lambda^2 \le \frac{A(\varepsilon)}{\lambda_1^\varepsilon\big(1 - \mu(\lambda_1^\varepsilon, \lambda_2^\varepsilon)\big)} + \frac{B(\varepsilon)}{\lambda_1^\varepsilon\lambda_2^\varepsilon\big(1 - \mu(\lambda_1^\varepsilon, \lambda_2^\varepsilon)\big)},$$

from which we can deduce the desired result. □

Analogous arguments to Lemma 3.1, the following standard estimates for solutions of (4.1) can be obtained with the help of Proposition A.2 and BDG's inequality.

Proposition A.3. *Assume that (H1)–(H5) hold, and $(\Theta_\cdot^{t,x}, \eta_\cdot^{t,x}, \zeta_\cdot^{t,x})$ and $(\Theta_\cdot^{t',x'}, \eta_\cdot^{t',x'}, \zeta_\cdot^{t',x'})$ are the adapted solutions of (4.1) corresponding to the initial points (t, x) and (t', x'), respectively. Then there exists a positive constant C such that*

$$\mathbf{E}\bigg[\sup_{s\in[t,T]} |X_s^{t,x}|^4\bigg] \le C(1 + |x|^4), \quad \mathbf{E}\bigg[\sup_{s\in[t,T]} |Y_s^{t,x}|^2 + \int_t^T |Z_s^{t,x}|^2\,\mathrm{d}s\bigg] \le C(1 + |x|^2);$$
$$\mathbf{E}\bigg[\sup_{s\in[t\vee t',T]} |X_s^{t,x} - X_s^{t',x'}|^4\bigg] \le C(|x - x'|^4 + |t - t'|^2),$$
$$\mathbf{E}\bigg[\sup_{s\in[t\vee t',T]} |Y_s^{t,x} - Y_s^{t',x'}|^2 + \int_{t\vee t'}^T |Z_s^{t,x} - Z_s^{t',x'}|^2\,\mathrm{d}s\bigg] \le C(|x - x'|^2 + |t - t'|).$$

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