ESAIM: PS 24 (2020) 21–38
https://doi.org/10.1051/ps/2019017

ESAIM: Probability and Statistics
www.esaim-ps.org

THE BERRY-ESSEEN BOUND OF A WAVELET ESTIMATOR IN NON-RANDOMLY DESIGNED NONPARAMETRIC REGRESSION MODEL BASED ON ANA ERRORS*

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Abstract. Consider the nonparametric regression model $Y_{ni} = g(t_{ni}) + \varepsilon_i,\ i = 1, 2, \ldots, n,\ n \geq 1$, where $\varepsilon_i,\ 1 \leq i \leq n$, are asymptotically negatively associated (ANA, for short) random variables. Under some appropriate conditions, the Berry-Esseen bound of the wavelet estimator of $g(\cdot)$ is established. In addition, some numerical simulations are provided in this paper. The results obtained in this paper generalize some corresponding ones in the literature.

Mathematics Subject Classification. 62E20, 62G05.

Received September 21, 2018. Accepted July 13, 2019.

1. Introduction

Berry-Esseen bound is known as the upper bound on the rate of convergence in the central limit theorem. In the past, many people were interested in the Berry-Esseen bounds theorem. For example, Xue [25] discussed the Berry-Esseen bound of an estimator for the variance in a semi-parametric regression model under some mild conditions; Liang and Baek [12] gave the Berry-Esseen bound for density estimator under negatively associated sequences; Wang and Zhang [23] provided the Berry-Esseen bound for linear negatively quadrant-dependent sequences; Cai and Roussas [3] established Berry-Esseen bound for a smooth estimator of the distribution function; Liang and Li [14] investigated the Berry-Esseen bound of the weighted estimator in a nonparametric regression model with linear process errors.

Consider the following nonparametric regression model:

$$Y_{ni} = g(t_{ni}) + \varepsilon_i, \quad i = 1, \ldots, n, \tag{1.1}$$

where the regression function $g(\cdot)$ is an unknown Borel measurable function defined on [0,1], $\{t_{ni},\ 1 \leq i \leq n\}$ are non-random design points and $\{\varepsilon_i\}$ are random errors.

*Supported by the National Natural Science Foundation of China (11671012, 11871072, 11701004, 11701005), the Natural Science Foundation of Anhui Province (1808085QA03, 1908085QA01, 1908085QA07), the Research Project of Chaohu University (XLZ-201903, XLY-201905) and the Project on Reserve Candidates for Academic and Technical Leaders of Anhui Province (2017H123).

Keywords and phrases: Nonparametric regression model, asymptotically negatively associated random variables, wavelet estimator, Berry-Esseen bound.

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Article published by EDP Sciences

It is well known that regression model has a wide range of applications about prediction in communications and control systems, and is an important tool of data analysis. In recent years, the weighting function estimation of $g(\cdot)$ has got extensive investigation. For instance, Georgiev [6] proposed a general weighted regression estimator of $g(\cdot)$ and subsequently it has been studied by many scholars. One can refer to Georgiev [7] under independent random errors, Roussas *et al.* [18] under strong mixing random errors, Yang [26] under negatively associated errors, Wang and Si [22] under negatively orthant dependent errors, Chen *et al.* [4] under martingale difference errors, Shen [19] under extended negatively dependent errors, and so on.

It is well known that the wavelet method is very good to adapt to local features of unknown curves. Hence, many authors have applied wavelet procedures to estimate the general nonparametric model. For instance, Antoniadis *et al.* [1] proposed the wavelet estimator of $g(\cdot)$ as follows:

$$\hat{g}_n(t) = \sum_{i=1}^{n} Y_{ni} \int_{A_i} E_m(t,s)\mathrm{d}s, \quad A_i = [s_{i-1}, s_i), \tag{1.2}$$

where $A_1, A_2, \ldots, A_n$ is a partition of interval $[0,1]$ with $t_{ni} \in A_i$. The wavelet kernel $E_m(t,s)$ can be defined as follows:

$$E_0(t,s) = \sum_{j\in Z} \varphi(t-j)\varphi(s-j), \qquad E_m(t,s) = 2^m E_0(2^m t, 2^m s), \tag{1.3}$$

where $m = m(n) > 0$ is an integer depending only on n, and φ is a scaling function.

Li *et al.* [16] derived the Berry-Esseen bound of the wavelet estimator for a nonparametric regression model with linear process errors generated by φ-mixing sequences. Ding and Li [5] obtained the Berry-Esseen bound of the wavelet estimator of $g(\cdot)$, and the rate of the normal approximation was shown as $O(n^{-\frac{1}{6}})$ under errors form a linear process based on ρ-mixing random variables. Zhou and Lin [29] studied the asymptotic properties of wavelet estimator in a semiparametric regression model with the mixing dependent errors structure. The aim of this work is to further investigate the Berry-Esseen bound of the wavelet estimator $\hat{g}_n(t)$ in nonparametric regression model (1.1) under asymptotically negatively associated random errors.

Now, let us recall some dependent structures. Joag-Dev and Proschan [9] introduced the following concept of negatively associated random variables which have wide applications in multivariate statistical analysis and reliability.

Definition 1.1. A finite family of random variables $\{X_i, 1 \le i \le n\}$ is said to be negatively associated (NA, for short) if for any disjoint subsets A and B of $\{1, 2, \ldots, n\}$,

$$Cov\left(f_1(X_i, i \in A), f_2(X_j, j \in B)\right) \le 0, \tag{1.4}$$

whenever f_1 and f_2 are any real coordinatewise nondecreasing functions such that this covariance exists. An infinite family of random variables $\{X_n, n \ge 1\}$ is NA if every finite subfamily is NA.

Another important concept of dependent random variables is ρ^*-mixing, which was introduced by Bradley [2] as follows.

Definition 1.2. A sequence of random variables $\{X_n, n \ge 1\}$ is called ρ^*-mixing, if the mixing coefficient

$$\rho^*(s) = \sup\{\rho(S,T) : S, T \subset \mathrm{N}, dist(S,T) \ge s\} \to 0, \quad as \quad s \to \infty, \tag{1.5}$$

where

$$\rho(S,T) = \sup\left\{ \frac{|EXY - EXEY|}{\sqrt{Var(X)\cdot Var(Y)}} : X \in L_2\left(\sigma(S)\right), Y \in L_2\left(\sigma(T)\right) \right\}, \tag{1.6}$$

N is the set of positive integers, and $\sigma(S)$ and $\sigma(T)$ are the σ-fields generated by $\{X_i, i \in S\}$ and $\{X_i, i \in T\}$, respectively.

Zhang and Wang [30] introduced the following concept of asymptotically negatively associated random variables.

Definition 1.3. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be asymptotically negatively associated (ANA or ρ^--mixing, for short) if

$$\rho^-(s) = \sup\{\rho^-(S,T) : S,T \subset \mathrm{N}, dist(S,T) \geq s\} \to 0, \quad as \quad s \to \infty, \tag{1.7}$$

where

$$\rho^-(S,T) = 0 \vee \left\{ \frac{Cov\left(f_1(X_i, i \in S), f_2(X_j, j \in T)\right)}{\sqrt{Var(f_1(X_i, i \in S)) \cdot Var(f_2(X_j, j \in T))}} : f_1, f_2 \in \mathcal{C} \right\}, \tag{1.8}$$

and $\mathcal{C}$ is the set of nondecreasing functions.

An array of random variables $\{X_{ni},\ 1 \leq i \leq n,\ n \geq 1\}$ is said to be rowwise ANA if for every $n \geq 1$, $\{X_{ni}, 1 \leq i \leq n\}$ are ANA random variables.

It is obvious to see that $\rho^-(s) \leq \rho^*(s)$ and that a sequence of ANA random variables is NA if and only if $\rho^-(1) = 0$. On the other hand, Zhang and Wang [30] provided an example of ANA random variables, which is neither NA or ρ^*-mixing. So, ANA random variables include ρ^*-mixing and NA random variables as special cases. Consequently, the study of the limit properties for ANA random variables is of much interest.

Remark 1.4. We point out that ρ^*-mixing and NA are both ANA. However, the converse is not always true. The following gives an example of an ANA sequence which is neither NA nor ρ^*-mixing.

Example 1.5. Let $\{\xi_n, n \geq 1\}$, $\{\eta_n, n \geq 1\}$ and $\{\tau_n, n \geq 1\}$ be three independent sequences of independent and identically distributed standard normal random variables. Let

$$X_n = \begin{cases} \xi_m, & if\ n = 2m-1 \\ -\xi_m, & if\ n = 2m \end{cases}, \quad Y_n = \begin{cases} \eta_m, & if\ n = 2^{2m-1} \\ -\eta_m, & if\ n = 2^{2m} \\ \tau_n, & \text{otherwise} \end{cases},$$

and $Z_n = X_n^2 + Y_n$. Then $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ are two independent sequences of identically distributed NA normal random variables. Also, $\{X_n, n \geq 1\}$ is a ρ^*-mixing sequence with $\rho^*(2) = 0$. From Property P3 of Zhang and Wang [30], we can see that $\{Z_n, n \geq 1\}$ is ANA with $\rho^-(2) = 0$. But $\{Z_n, n \geq 1\}$ is neither NA nor ρ^*-mixing, since

$$Cov\left(Z_{2m-1}, Z_{2m}\right) = Cov\left(X_{2m-1}^2, X_{2m}^2\right) = E\xi_m^4 - (E\xi_m^2)^2 = 2 > 0$$

and

$$\frac{Cov\left(Z_{2^{2m-1}}, Z_{2^{2m}}\right)}{Var(Z_{2^{2m-1}})Var(Z_{2^{2m}})} = -\frac{1}{3} \nrightarrow 0 \ \text{ as } \ \mathrm{dist}(2^{2m-1}, 2^{2m}) = 2^{2m-1} \to \infty.$$

This example can be found in Zhang and Wang [30].

Since the concept of ANA random variables was introduced by Zhang and Wang [30], many interesting results have been found. For example, Zhang and Wang [30] established some Rosenthal type inequalities, complete convergence and almost sure summability on the convergence rate with respect to the strong law of large numbers; Zhang [27] obtained moment inequalities, characteristic functions and the central limit theorems; Zhang [28] investigated the central limit theorem for ANA random fields under lower moment conditions or the Lindeberg condition; Wang and Lu [21] established some inequalities for the maximum of partial sums and weak convergence; Wang and Zhang [24] obtained the law of the iterated logarithm; Liu and Liu [15] showed moments of the maximum of normed partial sums; Wu and Jiang [20] studied the almost sure convergence for sequences of ANA random variables; Huang *et al.* [8] investigated the complete convergence and complete moment convergence for weighted sums of arrays of rowwise ANA random variables, and so on.

In this paper, we further investigate model (1.1) and derive a Berry-Esseen type bound for the estimator $\hat{g}_n(\cdot)$ in (1.2) under the ANA errors $\{\varepsilon_i\}$ satisfying some mild regularity conditions. Under appropriate conditions, the Berry-Esseen type bound can attain $O(n^{-1/4})$.

To prove the main results, we recall some definitions used in this paper.

Definition 1.6. A father wavelet φ is said to be q-regular $(S_q; q \in \mathrm{N})$ if for any $l \leq q$, and for any integer k, one has $|\frac{d^l\varphi}{dx^l}| \leq C_k(1+|x|)^{-1}$, where C_k is a generic constant depending only on k.

Definition 1.7. A function space $H^\gamma(\gamma \in \mathrm{R})$ is said to be Sobolev space with order γ, i.e., if $h \in H^\gamma$ then $\int |\hat{h}(\omega)|^2(1+\omega^2)^\gamma \mathrm{d}\omega < \infty$, where $\hat{h}$ is the Fourier transform of h.

The paper is organized as follows. In next section, we list some assumptions and give the main results. Some numerical simulations are provided in Section 3. Some lemmas and proofs of the main results are presented in Section 4, and the proofs of the lemmas are given in Appendix A.

2. Main results

In what follows, we will use C, C_p and C_i to denote positive constants whose values are unimportant and may vary. Set $\|X\|_r = (E|X|^r)^{1/r}$, $\|X\|_{2,1} = \int_0^\infty P^{1/2}(|X| \geq x)dx$, $x^+ = xI(x>0)$, $x^- = -xI(x<0)$, and $\lfloor x \rfloor$ denotes the integer part of x, $\Phi(u)$ is defined as a standard normal distribution.

Now, we shall give some assumptions:

(A1) $\{\varepsilon_i,\ i \geq 1\}$ are ANA random variables with $E\varepsilon_i = 0$, $\sup_{i\geq 1} E|\varepsilon_i|^{2+\delta} < \infty$ for some $\delta > 0$, $u(n) =: \sup\limits_{k\geq 1} \sum\limits_{j:|j-k|\geq n} (Cov(\varepsilon_j, \varepsilon_k))^- \to 0$ and $v(n) =: \sum_{j=n}^\infty \rho^-(j) \to 0$ as $n \to \infty$.

(A2) The spectral density function $f(\omega)$ of $\{\varepsilon_i\}$ satisfies $0 < C_1 \leq f(\omega) \leq C_2 < \infty$ for all $\omega \in (-\pi, \pi)$.

(A3) (i) $\varphi(\cdot)$ is q-regular with order $q > u$, where u is defined in **(A4)** (ii);

(ii) $\varphi(\cdot)$ satisfies the Lipschitz condition with order 1, has a compact support and $|\hat{\varphi}(\xi) - 1| = O(\xi)$ as $\xi \to 0$, where $\hat{\varphi}$ is the Fourier transform of φ.

(A4) (i) $g(\cdot)$ satisfies the Lipschitz condition with order 1;

(ii) $g(\cdot) \in H^u, u > \frac{3}{2}$, where $H^u(u \in \mathbb{R})$ is the Sobolev space with order u.

(A5) $\max\limits_{1\leq i\leq n} \left|s_i - s_{i-1} - \frac{1}{n}\right| = o(\frac{1}{n})$.

(A6) Set $p = p(n) \to \infty$ and $q = q(n) \to \infty$ as $n \to \infty$, write $k = \lfloor \frac{n}{p+q} \rfloor$ such that $p + q \leq n, \frac{q}{p} \to 0$, and $\zeta_{in} \to 0, i = 1, 2$, where $\zeta_{1n} =: \frac{q}{p}2^m, \zeta_{2n} =: \frac{p}{n}2^m$.

Remark 2.1. We point out that $u(q) \to 0$ and $v(q) \to 0$ are easily satisfied in Assumption **(A1)**. One can refer to Li *et al.* [16], Cai and Roussas [3], Liang and Li [14] for details. Assumptions **(A2)**-**(A5)** are mild regular conditions for the wavelet estimate in recent literature, such as Li *et al.* [16] and Liang and Li [14] among others. In Assumption **(A6)**, $p,\ q$ and 2^m can be defined as increasing sequences, and $\zeta_{in} \to 0,\ i = 1, 2$ are easily satisfied, if $p,\ q$ and 2^m are chosen as follows: $p \sim n^{\delta_1},\ q \sim n^{\delta_2}$ and $2^m \sim n^{\delta_3}$ $(\delta_1 > \delta_2 > 0,\ \delta_3 > 0,\ \delta_2 + \delta_3 < \delta_1 < 1 - \delta_3)$. For example, taking $\delta_1 = 0.6,\ \delta_2 = 0.32$ and $\delta_3 = 0.21$, we get $\zeta_{in} \to 0,\ i = 1, 2$.

Consider the nonparametric regression model (1.1). Let $\sigma_n^2(t) = Var(\hat{g}_n(t))$ and $S_n(t) = \sigma_n^{-1}(t)\{\hat{g}_n(t) - E\hat{g}_n(t)\}$ for each $n \geq 1$. Our main results on the uniform Berry-Esseen bounds for the estimator $\hat{g}_n(\cdot)$ in (1.2) are presented as follows.

Theorem 2.2. *Suppose that Assumptions **(A1)**–**(A6)** hold. Then for any $t \in [0,1]$,*

$$\sup_u |P(S_n(t) \le u) - \Phi(u)| \le C \left[\zeta_{1n}^{\frac{2+\delta}{6+2\delta}} + \zeta_{2n}^{\frac{2+\delta}{6+2\delta}} + \zeta_{2n}^{\frac{\delta}{2}} + (u(q) + v(q))^{\frac{1}{3}} \right].$$

If we take $p = \lfloor n^\tau \rfloor$ and $q = \lfloor n^{2\tau - 1} \rfloor$ for some $\frac{1}{2} < \tau < 1$ in Theorem 2.2, we can obtain the following result.

Corollary 2.3. *Suppose that Assumptions **(A1)**–**(A6)** hold. If $\sup_{i \ge 1} E|\varepsilon_i|^{2+\delta} < \infty$ for some $\delta \ge \sqrt{3} - 1$, $\frac{2^m}{n} = O(n^{-\theta})$, $u(n) + v(n) = O\left(n^{-\frac{3(\theta-\tau)(2+\delta)}{(4\tau-2)(3+\delta)}}\right)$, where $\frac{1}{2} < \tau < \theta \le 1$, then for any $t \in [0,1]$,*

$$\sup_u |P(S_n(t) \le u) - \Phi(u)| = O\left(n^{-\frac{(\theta-\tau)(2+\delta)}{6+2\delta}}\right).$$

If we take $\lfloor p = n^{1/2} \rfloor$ and $q = \lfloor \log n \rfloor$ in Theorem 2.2, we can also obtain the following result.

Corollary 2.4. *Suppose that Assumptions **(A1)**–**(A6)** hold. If $\sup_{i \ge 1} E|\varepsilon_i|^{2+\delta} < \infty$ for some $\delta \ge \sqrt{3} - 1$, $\frac{2^m}{n} = O(n^{-\theta})$, $u(n) + v(n) = O\left(\exp\{\frac{-3(2\theta-1)(2+\delta)n}{12+4\delta}\}\right)$, where $\frac{1}{2} < \theta \le 1$, then for any $t \in [0,1]$,*

$$\sup_u |P(S_n(t) \le u) - \Phi(u)| = O\left((n^{-(\theta - \frac{1}{2})} \log n)^{\frac{2+\delta}{6+2\delta}}\right).$$

Remark 2.5. It is easy to check that if $\delta = 1$, the Berry-Esseen bound for $\hat{g}_n(\cdot)$ approximates to $n^{-3/16}$ for $\tau \approx 1/2$ and $\theta \approx 1$, and if δ can be sufficiently large, the Berry-Esseen bound is nearly $n^{-1/4}$. Hence, the results above generalize the corresponding ones of Ding and Li [5] from ρ-mixing setting to ρ^--mixing setting.

Remark 2.6. Since ρ^*-mixing implies ANA, so our results are also available for ρ^*-mixing random errors. It is worth to mention that as far as we know, there is no literature investigating the Berry-Esseen bound for the estimator (1.2) under the assumption of ANA random errors.

3. Simulation study

To evaluate the finite sample performance of the wavelet estimator, we simulate data from the following model:

$$Y_{ni} = g(t_{ni}) + \varepsilon_i, \quad i = 1, \ldots, n, \; n \ge 1, \tag{3.1}$$

where $t_{ni} = \frac{i-0.5}{n}$, $i = 1, 2, 3, \ldots, n$.
Case 1. Let random vector $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \sim N(\mathbf{0}, \Sigma)$, where $\mathbf{0}$ represents zero vector and

$$\Sigma = \begin{pmatrix} \frac{1}{2}+v^2 & -v & 0 & \cdots & 0 & 0 & 0 \\ -v & \frac{1}{2}+v^2 & -v & \cdots & 0 & 0 & 0 \\ 0 & -v & \frac{1}{2}+v^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2}+v^2 & -v & 0 \\ 0 & 0 & 0 & \cdots & -v & \frac{1}{2}+v^2 & -v \\ 0 & 0 & 0 & \cdots & 0 & -v & \frac{1}{2}+v^2 \end{pmatrix}_{n \times n}, \quad v = 0.1.$$

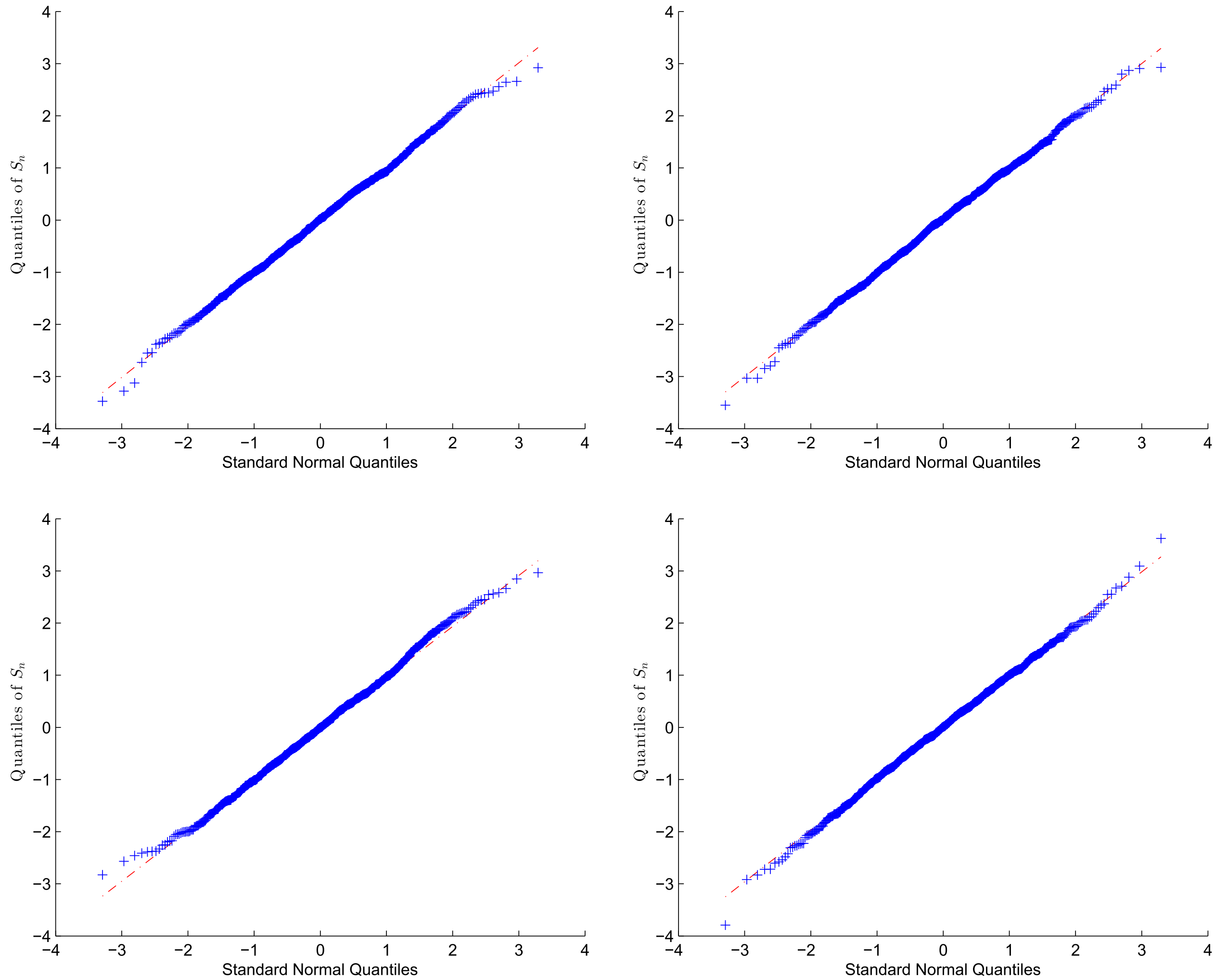

FIGURE 1. Q-Q plots of $S_n(t)$ $g_1(t)$.

By Joag-Dev and Proschan [9], it can be seen that $\varepsilon_1, \ \varepsilon_2, \ \cdots, \ \varepsilon_n$ are NA random variables, and thus ANA random variables. We take two functions for $g(\cdot)$:

$$g_1(t) = \cos(t) \qquad \text{and} \qquad g_2(t) = \begin{cases} -2t^3 - \frac{1}{32}, & if \ t < \frac{1}{4} \\ \frac{1}{2}t - \frac{3}{16}, & if \ \frac{1}{4} \leq t \leq \frac{3}{4} \\ t^2 - \frac{3}{8}, & if \ t > \frac{3}{4} \end{cases}.$$

For a given sample size n, take $2^m = \lfloor n^{1/3} \rfloor$ and $s_i = \frac{i}{n}$. We choose the scale function $\varphi(x) = I(0 \leq x \leq 1)$. One can easily check that the Assumptions **(A1)**–**(A6)** are satisfied. From the model above, taking $n = 500, 1000, 1500$ and 2000, we obtain estimator $\hat{g}_n(t)$ as the simulation results. Repeat calculating 1000 times, we get estimators $\hat{g}_{n1}(t), \ \hat{g}_{n2}(t), \ldots, \hat{g}_{n,1000}(t)$ of $g(t)$. We depict the Q-Q plots of $S_n(t)$ statistics versus the quantiles of standard normal distribution for $g_1(t)$ and $g_2(t)$.

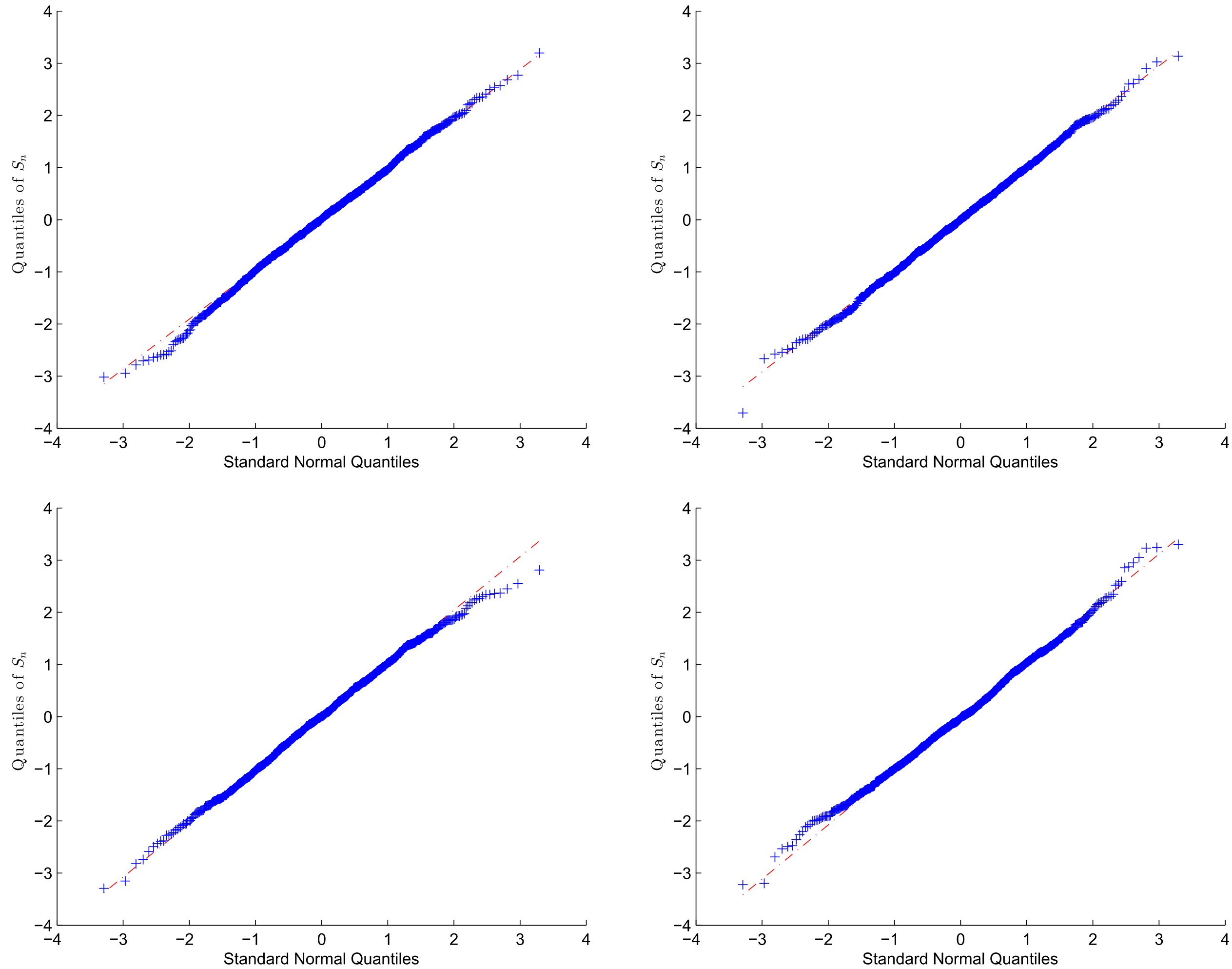

FIGURE 2. Q-Q plots of $S_n(t)$ $g_2(t)$.

According to Theorem 2.2, the distribution of $S_n(t)$ should be asymptotically normal. Figures 1 and 2 depict the Q-Q plot for $g_1(t)$ and $g_2(t)$ with $n = 500$, 1000, 1500 and 2000, respectively. The Q-Q plots show a good fit of the $S_n(t)$ to normal distribution.

Case 2. For fixed integer m, let $e_i \overset{iid}{\backsim} N(0, \sigma_0^2)$, where $\sigma_0^2 = 1/(m+1)$. Let $\varepsilon_i = \Sigma_{k=0}^{m} e_{i+k}$ for each $i \geq 1$. Then $\{\varepsilon_i,\ i \geq 1\}$ is a sequence of ρ^*-mixing random variables and thus a sequence of ρ^--mixing random variables with $E\varepsilon_i = 0$ and $Var(\varepsilon_i) = 1$. For simplicity, take $m = 2$. The other settings are the same as those in **Case 1**. We get similar conclusion as in **Case 1**. Consider the length of the paper, the Q-Q plots of $S_n(t)$ are omitted here.

Case 3. Suppose that $\{\varepsilon_n, n \geq 1\}$ has the same distribution as that of $\{Z_n, n \geq 1\}$ in Example 1.5. Then $\{\varepsilon_n, n \geq 1\}$ is a sequence of ρ^--mixing random variables which is neither NA nor ρ^*-mixing. The other settings are the same as those in **Case 1**. We get some similar conclusion as in **Case 1** and **Case 2**, so the details are also omitted here.

To compute the uniform Berry-Esseen bounds for the estimator (1.2), we first compute the empirical distribution functions of S_n to estimate $F_n(u)$, and then estimate the maximum values of $|F_n(u) - \Phi(u)|$ for $u = -3, -2.99, -2.98, \ldots, 2.98, 2.99, 3$. The results are shown in Table 1.

It can be easily seen in Table 1 that for the three cases and the two functions, the uniform Berry-Esseen bounds decrease markedly as the sample size n increases. These simulation results basically agree with our theoretical results.

TABLE 1. The uniform Berry-Esseen bounds.

Case	n $n^{-1/4}$	500 0.2115	1000 0.1778	1500 0.1607	2000 0.1495
1	$g_1(t)$	0.0276	0.0202	0.0190	0.0172
	$g_2(t)$	0.0220	0.0195	0.0185	0.0178
2	$g_1(t)$	0.0219	0.0209	0.0196	0.0141
	$g_2(t)$	0.0264	0.0199	0.0187	0.0163
3	$g_1(t)$	0.0231	0.0194	0.0183	0.0169
	$g_2(t)$	0.0235	0.0200	0.0181	0.0138

4. Proofs of main results

From (1.2), we can see that

$$S_n =: S_n(t) = \sigma_n^{-1}(t)\sum_{i=1}^{n}\varepsilon_i\int_{A_i}E_m(t,s)\mathrm{d}s =: \sum_{i=1}^{n}W_{ni}.$$

Set

$$y_{nv} = \sum_{i=k_v}^{k_v+p-1}W_{ni},\ \xi_{nv} = \sum_{i=l_v}^{l_v+q-1}W_{ni},\ \gamma_{nk+1} = \sum_{i=k(p+q)-n+1}^{n}W_{ni},$$

$$k_v = (v-1)(p+q)+1,\ l_v = (v-1)(p+q)+p+1,\ v = 1,\ldots,k,$$

$$S'_n = \sum_{v=1}^{k}y_{nv},\ S''_n = \sum_{v=1}^{k}\xi_{nv},\ S'''_n = \gamma_{nk+1},\ s_n^2 = \sum_{v=1}^{k}Var(y_{nv}).$$

Then

$$S_n = S'_n + S''_n + S'''_n.$$

Let $\eta_{n1},\eta_{n2},\ldots,\eta_{nk}$ be independent random variables and $\eta_{nv} \overset{d}{=} y_{nv}$ for each $v = 1,2,\ldots,k$, Denote $T_n = \sum_{v=1}^{k}\eta_{nv}$.

To prove the main results of the paper, we need the following important lemmas, whose proofs will be given in Appendix.

Lemma 4.1. *Assume that **(A1)**–**(A6)** hold. Then for any $\delta \geq 0$,*

$$(i)\ E(S''_n)^{2+\delta} \leq C\zeta_{1n}^{1+\frac{\delta}{2}}, \qquad E(S'''_n)^{2+\delta} \leq C\zeta_{2n}^{1+\frac{\delta}{2}};$$

$$(ii)\ P(|S''_n| \geq \zeta_{1n}^{\frac{2+\delta}{6+2\delta}}) \leq C\zeta_{1n}^{\frac{2+\delta}{6+2\delta}}, \qquad P(|S'''_n| \geq \zeta_{2n}^{\frac{2+\delta}{6+2\delta}}) \leq C\zeta_{2n}^{\frac{2+\delta}{6+2\delta}}.$$

Lemma 4.2. *Assume that **(A1)**–**(A6)** hold. Then*

$$|s_n^2 - 1| \leq C\left[\zeta_{1n}^{\frac{1}{2}} + \zeta_{2n}^{\frac{1}{2}} + u(q) + v(q)\right].$$

Lemma 4.3. *Assume that **(A1)**–**(A6)** hold. Then*

$$\sup_u \left| P\left(\frac{T_n}{s_n} \leq u\right) - \Phi(u)\right| \leq C\zeta_{2n}^{\frac{\delta}{2}}.$$

Lemma 4.4. *Assume that **(A1)**–**(A6)** hold. Then*

$$\sup_u |P(S'_n \leq u) - P(T_n \leq u)| \leq C\left[\zeta_{2n}^{\frac{\delta}{2}} + (u(q) + v(q))^{\frac{1}{3}}\right]. \tag{4.1}$$

Lemma 4.5. *(cf. Liang and Fan [13]) Let X and $Y_1, Y_2, \ldots, Y_m$ be random variables. Then for positive numbers $\omega_1, \omega_2, \ldots, \omega_m$,*

$$\begin{aligned} &\sup_u \left| P\left(X + \sum_{i=1}^m Y_i \leq u\right) - \Phi(u)\right| \\ &\leq \sup_u |P(X \leq u) - \Phi(u)| + \sum_{i=1}^m \frac{\omega_i}{\sqrt{2\pi}} + \sum_{i=1}^m P(|Y_i| > \omega_i). \end{aligned} \tag{4.2}$$

Now, we turn to prove the main results of the paper.

Proof of Theorem 2.2. It can be easily checked that

$$\begin{aligned} &\sup_u |P(S'_n \leq u) - \Phi(u)| \\ &\leq \sup_u |P(S'_n \leq u) - P(T_n \leq u)| + \sup_u \left| P(T_n \leq u) - \Phi(\frac{u}{s_n})\right| + \sup_u \left|\Phi(\frac{u}{s_n}) - \Phi(u)\right| \\ &=: J_{1n} + J_{2n} + J_{3n}. \end{aligned} \tag{4.3}$$

According to Lemma 4.4 and Lemma 4.3, we can get that

$$J_{1n} \leq C\left[\zeta_{2n}^{\frac{\delta}{2}} + (u(q) + v(q))^{\frac{1}{3}}\right] \tag{4.4}$$

and

$$J_{2n} = \sup_u \left| P\left(\frac{T_n}{s_n} \leq \frac{u}{s_n}\right) - \Phi\left(\frac{u}{s_n}\right)\right| = \sup_u \left| P\left(\frac{T_n}{s_n} \leq u\right) - \Phi(u)\right| \leq C\zeta_{2n}^{\frac{\delta}{2}}. \tag{4.5}$$

It is well known that (*cf.* Lemma 5.2 in Petrov [17]),

$$\sup_{-\infty<u<\infty} |\Phi(pu)-\Phi(u)| \le \frac{(p-1)I(p\ge 1)}{\sqrt{2\pi e}} + \frac{(p^{-1}-1)I(0<p<1)}{\sqrt{2\pi e}}.$$

Thus, by Lemma 4.2, we have

$$\begin{aligned} J_{3n} &\le \frac{(s_n-1)I(s_n\ge 1)}{\sqrt{2\pi e}} + \frac{(s_n^{-1}-1)I(0<s_n<1)}{\sqrt{2\pi e}} \\ &\le \frac{\max\{|s_n-1|,|s_n-1|/s_n\}}{\sqrt{2\pi e}} \\ &\le C\max\{|s_n-1|,|s_n-1|/s_n\}(s_n+1) \\ &\le C|s_n^2-1| \le C\left[\zeta_{1n}^{\frac12}+\zeta_{2n}^{\frac12}+u(q)+v(q)\right], \end{aligned} \tag{4.6}$$

which, together with (4.3)–(4.5), yields that

$$\sup_u |P(S_n'\le u)-\Phi(u)| \le C\left[\zeta_{1n}^{\frac12}+\zeta_{2n}^{\frac12}+\zeta_{2n}^{\frac{\delta}{2}}+(u(q)+v(q))^{\frac13}\right]. \tag{4.7}$$

Thus, by Lemma 4.5, Lemma 4.1(ii) and (4.7), we have

$$\begin{aligned} &\sup_u |P(S_n\le u)-\Phi(u)| \\ &\le C\sup_u |P(S_n'\le u)-\Phi(u)| + C\zeta_{1n}^{\frac{2+\delta}{6+2\delta}} + C\zeta_{2n}^{\frac{2+\delta}{6+2\delta}} + CP(|S_n''|\ge \zeta_{1n}^{\frac{2+\delta}{6+2\delta}}) + CP(|S_n'''|\ge \zeta_{2n}^{\frac{2+\delta}{6+2\delta}}) \\ &\le C\left[\zeta_{1n}^{\frac{2+\delta}{6+2\delta}}+\zeta_{2n}^{\frac{2+\delta}{6+2\delta}}+\zeta_{2n}^{\frac{\delta}{2}}+(u(q)+v(q))^{\frac13}\right]. \end{aligned} \tag{4.8}$$

This completes the proof of Theorem 2.2. ☐

Proof of Corollary 2.3. Let $p=\lfloor n^{\tau}\rfloor$ and $q=\lfloor n^{2\tau-1}\rfloor$. Noting that $\delta\ge\sqrt{3}-1$, we have by Theorem 2.2 that

$$\sup_u |P(S_n\le u)-\Phi(u)| \le C\left[\zeta_{1n}^{\frac{2+\delta}{6+2\delta}}+\zeta_{2n}^{\frac{2+\delta}{6+2\delta}}+(u(q)+v(q))^{\frac13}\right]. \tag{4.9}$$

Since $\frac{2^m}{n}=O(n^{-\theta})$ and $u(n)+v(n)=O\left(n^{-\frac{3(\theta-\tau)(2+\delta)}{(4\tau-2)(3+\delta)}}\right)$, we have

$$\zeta_{1n}^{\frac{2+\delta}{6+2\delta}} = O\left(n^{-\frac{(\theta-\tau+1)(2+\delta)}{6+2\delta}}\right) = O\left(n^{-\frac{(\theta-\tau)(2+\delta)}{6+2\delta}}\right), \qquad \zeta_{2n}^{\frac{2+\delta}{6+2\delta}} = O\left(n^{-\frac{(\theta-\tau)(2+\delta)}{6+2\delta}}\right), \tag{4.10}$$

and

$$(u(q)+v(q))^{\frac13} = O\left(q^{-\frac{(\theta-\tau)(2+\delta)}{(4\tau-2)(3+\delta)}}\right) = O\left(n^{-\frac{(\theta-\tau)(2+\delta)}{6+2\delta}}\right). \tag{4.11}$$

Therefore, the desired result follows from (4.9)–(4.11) immediately. The proof is completed. ☐

Proof of Corollary 2.4. Set $p = \lfloor n^{\frac{1}{2}} \rfloor$ and $q = \lfloor \log n \rfloor$. Noting that $\frac{2^m}{n} = O(n^{-\theta})$ and $u(n) + v(n) = O\left(\exp\{\frac{-3(2\theta-1)(2+\delta)n}{12+4\delta}\}\right)$, we can get that

$$\zeta_{1n}^{\frac{2+\delta}{6+2\delta}} = O\left((n^{-(\theta-\frac{1}{2})}\log n)^{\frac{2+\delta}{6+2\delta}}\right), \tag{4.12}$$

$$\zeta_{2n}^{\frac{2+\delta}{6+2\delta}} = O(n^{-\frac{(\theta-\frac{1}{2})(2+\delta)}{6+2\delta}}) = O\left((n^{-(\theta-\frac{1}{3})}\log n)^{\frac{2+\delta}{6+2\delta}}\right) \tag{4.13}$$

and

$$(u(q)+v(q))^{\frac{1}{3}} = O\left(\exp\left\{\frac{-3(2\theta-1)(2+\delta)q}{(12+4\delta)}\right\}\right)^{\frac{1}{3}} = O\left((n^{-(\theta-\frac{1}{2})}\log n)^{\frac{2+\delta}{6+2\delta}}\right), \tag{4.14}$$

which, together with (4.9), yield the desired result. This completes the proof of the corollary. □

Appendix A.

Lemma A.1. *(cf. Li and Guo [11]) Under Assumptions **(A3)**–**(A5)**, we have*

$$(i)\left|\int_{A_i} E_m(t,s)\mathrm{d}s\right| = O(\frac{2^m}{n}),\ i=1,\ldots,n; \quad (ii)\sum_{i=1}^n\left(\int_{A_i} E_m(t,s)\mathrm{d}s\right)^2 = O(\frac{2^m}{n});$$

$$(iii)\sup_m\int_0^1 |E_m(t,s)|\mathrm{d}s \le C; \qquad (iv)\sum_{i=1}^n\left|\int_{A_i} E_m(t,s)\mathrm{d}s\right| \le C.$$

Lemma A.2. *Assume that Assumptions **(A1)**–**(A5)** are satisfied. Then*

$$C_1\frac{2^m}{n} \le \sigma_n^2(t) \le C_2\frac{2^m}{n}. \tag{A.1}$$

Proof. Similar to the proof of Lemma 2.2 in Li and Guo [17], it follows that

$$\sigma_n^2(t) = Var\left(\sum_{k=1}^n \varepsilon_{nk}\int_{A_k} E_m(t,s)\mathrm{d}s\right) = \int_{-\pi}^{\pi} f(\omega)\left|\sum_{k=1}^n\int_{A_k} E_m(t,s)\mathrm{d}s\cdot e^{-ik\omega}\right|^2 \mathrm{d}\omega, \tag{A.2}$$

which together with Assumption **(A2)** yields that

$$C_1\sum_{k=1}^n\left(\int_{A_k} E_m(t,s)\mathrm{d}s\right)^2 \le \sigma_n^2(t) \le C_2\sum_{k=1}^n\left(\int_{A_k} E_m(t,s)\mathrm{d}s\right)^2. \tag{A.3}$$

In addition, it is easy to see that

$$
\begin{aligned}
&\left|\sum_{i=1}^{n}\left(\int_{A_i} E_m(t,s)\mathrm{d}s\right)^2 - \frac{1}{n}\sum_{i=1}^{n}\int_{A_i} E_m^2(t,s)\mathrm{d}s\right| \\
&\le \sum_{i=1}^{n}\left|(s_i-s_{i-1})^2 E_m^2(t,\xi_i^{(1)}) - \frac{s_i-s_{i-1}}{n}E_m^2(t,\xi_i^{(2)})\right| \\
&= O(n^{-1})\sum_{i=1}^{n}\left|\left(s_i-s_{i-1}-\frac{1}{n}\right)E_m^2(t,\xi_i^{(1)}) - \frac{1}{n}\left(E_m^2(t,\xi_i^{(2)})-E_m^2(t,\xi_i^{(1)})\right)\right|, \qquad \text{(A.4)}
\end{aligned}
$$

where both $\xi_i^{(1)}$ and $\xi_i^{(2)}$ belong to A_i. Note that the number of terms contributing to the above sum is of order $O(\frac{n}{2^m})$, $\sup_{t,s} E_m^2(t,s) = O(2^{2m})$, and

$$
\left|E_m^2(t,\xi_i^{(2)})-E_m^2(t,\xi_i^{(1)})\right| = 2^{2m}\left|E_0^2(2^m t, 2^m\xi_i^{(2)}) - E_0^2(2^m t, 2^m \xi_i^{(1)})\right| = O\left(\frac{2^{3m}}{n}\right),
$$

which together with (A.4) and Assumption **(A5)** yields that

$$
\begin{aligned}
&\left|\sum_{i=1}^{n}\left(\int_{A_i} E_m(t,s)\mathrm{d}s\right)^2 - \frac{1}{n}\sum_{i=1}^{n}\int_{A_i} E_m^2(t,s)\mathrm{d}s\right| \\
&= O\left(\frac{1}{n}\frac{n}{2^m}\right)\left(o(\frac{2^{2m}}{n}) + O(\frac{2^{3m}}{n^2})\right) = o\left(\frac{2^m}{n}\right). \qquad \text{(A.5)}
\end{aligned}
$$

From (A.3) and (A.5), we obtain

$$
\begin{aligned}
\sigma_n^2(t) &\ge C_1\frac{1}{n}\int_0^1 E_m^2(t,s)\mathrm{d}s + o(\frac{2^m}{n}) \\
&\ge C_1\frac{2^{2m}}{n}\int_0^1 E_0^2(2^m t, 2^m s)\mathrm{d}s + o(\frac{2^m}{n}) = C_1\frac{2^m}{n}. \qquad \text{(A.6)}
\end{aligned}
$$

The desired result (A.1) now follows from (A.3), (A.6) and Lemma A.1(ii) immediately. □

Lemma A.3. *(cf. Zhang and Wang [21]) Suppose that $\{X_k, k\in\mathbb{N}^d\}$ is an ANA random field with $EX_k = 0$ and $\|X_k\|_p < \infty$ for some $p \ge 2$ and all k. Then there exists a positive constant C_p depending only on p and $\rho^-(\cdot)$ such that for any finite set $S\subset\mathbb{N}^d$,*

$$
E\left|\sum_{k\in S}X_k\right|^p \le C_p\left\{\sum_{k\in S}E|X_k|^p + \left(\sum_{k\in S}E|X_k|^2\right)^{p/2}\right\}.
$$

When $d=1$, the Rosenthal-type inequality remains true for the maximal partial sums, if some conditions on $\rho^-(\cdot)$ are added.

Lemma A.4. *(cf. Zhang [27]) Suppose that $f_1(x)$ and $f_2(y)$ are real, bounded and absolutely continuous functions on $\mathbb{R}$ with $|f_1'(x)|\le C_1$ and $|f_2'(y)|\le C_2$. Then for any random variables X and Y,*

$$
|Cov\{f_1(X), f_2(Y)\}| \le C_1C_2\left\{-Cov(X,Y) + 8\rho^-(X,Y)\|X\|_{2,1}\|Y\|_{2,1}\right\}.
$$

Lemma A.5. *Let $\{X_i, i \geq 1\}$ be a sequence of ANA random variables with finite variances and $\{a_i, i \geq 1\}$ be a sequence of nonnegative (or nonpositive) constants. Denote by $\eta_l = \sum_{i=(l-1)(p+q)+1}^{(l-1)(p+q)+p} a_i X_i$ for $1 \leq l \leq m$, where p and q are two integers. Then for any $t_l \in \mathbb{R}$, $l = 1, 2, \cdots, m$,*

$$\begin{aligned}
&\left| E\exp\left(i\sum_{l=1}^{m} t_l\eta_l\right) - \prod_{l=1}^{m} E\exp\left(it_l\eta_l\right)\right| \\
&\leq 8\sum_{1\leq l<k\leq m} |t_l||t_k| \left\{ -Cov\left(\sum_{i=(l-1)(p+q)+1}^{(l-1)(p+q)+p} a_iX_i, \sum_{j=(k-1)(p+q)+1}^{(k-1)(p+q)+p} a_jX_j\right)\right. \\
&\quad \left. + 8\left\|\sum_{i=(l-1)(p+q)+1}^{(l-1)(p+q)+p} a_iX_i\right\|_{2,1}\left\|\sum_{j=(k-1)(p+q)+1}^{(k-1)(p+q)+p} a_jX_j\right\|_{2,1} \rho^-(j-i)\right\}. \qquad (A.7)
\end{aligned}$$

Proof. According to Property P2 in Zhang and Wang [30], we can get that $\eta_l = \sum_{i=(l-1)(p+q)+1}^{(l-1)(p+q)+p} a_i X_i$, $l = 1, 2, \ldots, m$ are also ANA random variables. In view of Lemma A.4, and similar to the proof of Theorem 3.3 in Zhang [27], we can obtain (A.7) immediately. The proof is completed. □

Proof of Lemma 4.1. By Lemma A.3, Assumption **(A1)**, Lemma A.2 and Lemma A.1(i), we obtain that

$$\begin{aligned}
E|S_n''|^{2+\delta} &= E\left|\sum_{m=1}^{k}\sum_{i=l_m}^{l_m+q-1} \sigma_n^{-1}\varepsilon_i \int_{A_i} E_m(t,s)\mathrm{d}s\right|^{2+\delta} \\
&\leq C\sum_{m=1}^{k}\sum_{i=l_m}^{l_m+q-1}\left|\sigma_n^{-1}\int_{A_i} E_m(t,s)\mathrm{d}s\right|^{2+\delta} + C\left(\sum_{m=1}^{k}\sum_{i=l_m}^{l_m+q-1}\sigma_n^{-2}\left|\int_{A_i} E_m(t,s)\mathrm{d}s\right|^2\right)^{1+\frac{\delta}{2}} \\
&\leq C\sum_{m=1}^{k}\sum_{i=l_m}^{l_m+q-1}\left(\frac{2^m}{n}\right)^{1+\frac{\delta}{2}} + C\left(kq\frac{2^m}{n}\right)^{1+\frac{\delta}{2}} \\
&\leq C\left(kq\frac{2^m}{n}\right)^{1+\frac{\delta}{2}} \leq C\zeta_{1n}^{1+\frac{\delta}{2}}
\end{aligned}$$

and

$$\begin{aligned}
E|S_n'''|^{2+\delta} &= E\left|\sum_{i=k(p+q)+1}^{n} \sigma_n^{-1}\varepsilon_i\int_{A_i} E_m(t,s)\mathrm{d}s\right|^{2+\delta} \\
&\leq C\sum_{i=k(p+q)+1}^{n}\left|\sigma_n^{-1}\int_{A_i} E_m(t,s)\mathrm{d}s\right|^{2+\delta} + \left(\sum_{i=k(p+q)+1}^{n}\left|\sigma_n^{-1}\int_{A_i} E_m(t,s)\mathrm{d}s\right|^2\right)^{1+\delta/2} \\
&\leq C\sum_{i=k(p+q)+1}^{n}\left(\frac{2^m}{n}\right)^{1+\frac{\delta}{2}} + C\left(\sum_{i=k(p+q)+1}^{n}\frac{2^m}{n}\right)^{1+\frac{\delta}{2}}
\end{aligned}$$

$$\leq C\left\{\lfloor n-k(p+q)\rfloor \frac{2^m}{n}\right\}^{1+\frac{\delta}{2}} \leq C\left(p\frac{2^m}{n}\right)^{1+\frac{\delta}{2}} = C\zeta_{2n}^{1+\frac{\delta}{2}}.$$

By the arguments above and according to Markov's inequality, we have

$$P\left(|S_n''| \geq \zeta_{1n}^{(2+\delta)/(6+2\delta)}\right) \leq \frac{E|S_n''|^{2+\delta}}{\zeta_{1n}^{(2+\delta)^2/(6+2\delta)}} \leq C\frac{\zeta_{1n}^{1+\delta/2}}{\zeta_{1n}^{(2+\delta)^2/(6+2\delta)}} = C\zeta_{1n}^{\frac{2+\delta}{6+2\delta}}$$

and

$$P\left(|S_n'''| \geq \zeta_{2n}^{(2+\delta)/(6+2\delta)}\right) \leq \frac{E|S_n'''|^{2+\delta}}{\zeta_{2n}^{(2+\delta)^2/(6+2\delta)}} \leq C\frac{\zeta_{2n}^{1+\delta/2}}{\zeta_{2n}^{(2+\delta)^2/(6+2\delta)}} = C\zeta_{2n}^{\frac{2+\delta}{6+2\delta}}.$$

The proof is completed. □

Proof of Lemma 4.2. Let $\Gamma_n = \sum_{1\leq i<j\leq k} Cov(y_{ni}, y_{nj})$. Then $s_n^2 = E(S_n')^2 - 2\Gamma_n$ and $ES_n^2 = 1$,

$$E(S_n')^2 = E[S_n - (S_n'' + S_n''')]^2 = 1 + E(S_n'' + S_n''')^2 - 2E[S_n(S_n'' + S_n''')].$$

By the C_r-inequality, Lemma 4.1 (i) and the Cauchy-Schwarz inequality, we have

$$E(S_n'' + S_n''')^2 \leq 2[E(S_n'')^2 + E(S_n''')^2] \leq C(\zeta_{1n} + \zeta_{2n}),$$

$$E[S_n(S_n'' + S_n''')] \leq E^{1/2}(S_n^2)E^{1/2}(S_n'')^2 + E^{\frac{1}{2}}(S_n^2)E^{\frac{1}{2}}(S_n''')^2 \leq C(\zeta_{1n}^{\frac{1}{2}} + \zeta_{2n}^{\frac{1}{2}})$$

and thus,

$$|E(S_n')^2 - 1| = |E(S_n'')^2 + E(S_n''')^2 - 2E\{S_n(S_n'' + S_n''')\}| \leq C\left(\zeta_{1n}^{\frac{1}{2}} + \zeta_{2n}^{\frac{1}{2}}\right). \tag{A.8}$$

On the other hand, note that $\|X\|_{2,1} \leq \frac{r}{r-2}\|X\|_r$ $(r > 2)$ (*cf.* Ledoux and Talagrand [10], p. 251). From the definition of ANA, Lemma A.2, Lemma A.1(i), Lemma A.4, we have

$$\begin{aligned}
|\Gamma_n| &\leq \sum_{1\leq i<j\leq k} |Cov(y_{ni}, y_{nj})| \\
&\leq \sum_{1\leq i<j\leq k} \sum_{u=k_i}^{k_i+p-1} \sum_{v=k_j}^{k_j+p-1} |Cov(W_u, W_v)| \\
&\leq \sum_{1\leq i<j\leq k} \sum_{u=k_i}^{k_i+p-1} \sum_{v=k_j}^{k_j+p-1} \sigma_n^{-2} \left|\int_{A_u} E_m(t,s)\mathrm{d}s \int_{A_v} E_m(t,s)\mathrm{d}s\right| |Cov(\varepsilon_u, \varepsilon_v)| \\
&\leq \sum_{1\leq i<j\leq k} \sum_{u=k_i}^{k_i+p-1} \sum_{v=k_j}^{k_j+p-1} \left|\int_{A_u} E_m(t,s)\mathrm{d}s\right| \{-Cov(\varepsilon_u, \varepsilon_v) + 8\rho^-(v-u)\|\varepsilon_u\|_{2,1}\|\varepsilon_v\|_{2,1}\}
\end{aligned}$$

$$\begin{aligned}&\leq C\sum_{1\leq i<j\leq k}\sum_{u=k_i}^{k_i+p-1}\sum_{v=k_j}^{k_j+p-1}\left|\int_{A_u}E_m(t,s)\mathrm{d}s\right|(Cov(\varepsilon_u,\varepsilon_v))^-\\&\quad+C\sum_{1\leq i<j\leq k}\sum_{u=k_i}^{k_i+p-1}\sum_{v=k_j}^{k_j+p-1}\left|\int_{A_u}E_m(t,s)\mathrm{d}s\right|\rho^-(v-u)\|\varepsilon_u\|_{2+\delta}\|\varepsilon_v\|_{2+\delta}\\&=: I_2+I_3.\end{aligned}\tag{A.9}$$

According to Lemma A.1(iv) and Assumption **(A1)**, we have

$$\begin{aligned}I_2&\leq C\sum_{i=1}^{k-1}\sum_{u=k_i}^{k_i+p-1}\left|\int_{A_u}E_m(t,s)\mathrm{d}s\right|\sum_{j=i+1}^{k}\sum_{v=k_j}^{k_j+p-1}(Cov(\varepsilon_u,\varepsilon_v))^-\\&\leq C\sum_{i=1}^{k-1}\sum_{u=k_i}^{k_i+p-1}\left|\int_{A_u}E_m(t,s)\mathrm{d}s\right|\sup_{k\geq1}\sum_{j:|j-k|\geq q}(Cov(\varepsilon_j,\varepsilon_k))^-\\&\leq C\sum_{u=1}^{n}\left|\int_{A_u}E_m(t,s)\mathrm{d}s\right|u(q)\leq Cu(q)\end{aligned}\tag{A.10}$$

and

$$\begin{aligned}I_3&\leq C\sum_{1\leq i<j\leq k}\sum_{u=k_i}^{k_i+p-1}\sum_{v=k_j}^{k_j+p-1}\left|\int_{A_u}E_m(t,s)\mathrm{d}s\right|\rho^-(v-u)\\&\leq C\sum_{i=1}^{k-1}\sum_{u=k_i}^{k_i+p-1}\left|\int_{A_u}E_m(t,s)\mathrm{d}s\right|\sum_{j=i+1}^{k}\sum_{v=k_j}^{k_j+p-1}\rho^-(v-u)\\&\leq C\sum_{u=1}^{n}\left|\int_{A_u}E_m(t,s)\mathrm{d}s\right|\sum_{j=q}^{\infty}\rho^-(j)\leq Cv(q).\end{aligned}\tag{A.11}$$

Combining (A.8) and (A.9)–(A.11), we can get that

$$|s_n^2-1|\leq|E(S_n')^2-1|+2|\Gamma_n|\leq C\left[\zeta_{1n}^{1/2}+\zeta_{2n}^{1/2}+u(q)+v(q)\right].$$

The proof is completed. □

Proof of Lemma 4.3. The proof is similar to that of Lemma 3.3 in Li *et al.* [16]. So we omit the details. □

Proof of Lemma 4.4. Assume that $\Phi(t)$ and $\Psi(t)$ are the characteristic functions of S_n' and T_n, respectively. For any $T>0$, it follows by Esseen inequality that

$$\begin{aligned}&\sup_u|P(S_n'\leq u)-P(T_n\leq u)|\\&\leq\int_{-T}^{T}\left|\frac{\Phi(t)-\Psi(t)}{t}\right|\mathrm{d}t+T\sup_u\int_{|y|\leq c/T}|P(T_n\leq u+y)-P(T_n\leq u)|\mathrm{d}y\\&=: I_4+I_5.\end{aligned}\tag{A.12}$$

By Lemma A.5, we can get that

$$
\begin{aligned}
&|\Phi(t)-\Psi(t)| = \left| E\exp\left(it\sum_{m=1}^{k} y_{nm}\right) - \prod_{m=1}^{k} E\exp\left(ity_{nm}\right)\right| \\
&\leq Ct^2\left\{\sum_{1\leq l<m\leq k} (Cov(y_{nl},y_{nm}))^{-}\right. \\
&\quad \left. + \sum_{1\leq l<m\leq k} \left\|\sum_{u=k_l}^{k_l+p-1} \varepsilon_u\sigma_n^{-1}\int_{A_u} E_m(t,s)\mathrm{d}s\right\|_{2,1} \left\|\sum_{v=k_m}^{k_m+p-1} \varepsilon_v\sigma_n^{-1}\int_{A_v} E_m(t,s)\mathrm{d}s\right\|_{2,1} \rho^{-}(v-u)\right\} \\
&=: Ct^2\{I_6+I_7\}.
\end{aligned}
\tag{A.13}
$$

For I_6, by (A.9)–(A.11), we have

$$
I_6 \leq \sum_{1\leq l<m\leq k} |Cov(y_{nl},y_{nm})| \leq C\left(u(q)+v(q)\right).
$$

Note that $\|X\|_{2,1} \leq \frac{2+\delta}{\delta}\|X\|_{2+\delta}$ (*cf.* Ledoux and Talagrand [10], p.251). By C_r-inequality, Lemma A.3, Assumption **(A1)**, Lemma A.2, and Lemma A.1(iv), we conclude that

$$
\begin{aligned}
I_7 &\leq C\sum_{1\leq l<m\leq k} \left\|\sum_{u=k_l}^{k_l+p-1} \varepsilon_u\sigma_n^{-1}\int_{A_u} E_m(t,s)\mathrm{d}s\right\|_{2+\delta} \left\|\sum_{v=k_m}^{k_m+p-1} \varepsilon_v\sigma_n^{-1}\int_{A_v} E_m(t,s)\mathrm{d}s\right\|_{2+\delta} \rho^{-}(v-u) \\
&\leq C\sum_{1\leq l<m\leq k} \left\{\sum_{u=k_l}^{k_l+p-1}\left|\sigma_n^{-1}\int_{A_u} E_m(t,s)\mathrm{d}s\right|^{2+\delta} + \left(\sum_{u=k_l}^{k_l+p-1}\left|\sigma_n^{-1}\int_{A_u} E_m(t,s)\mathrm{d}s\right|^{2}\right)^{1+\delta/2}\right\}^{1/(2+\delta)} \\
&\quad \cdot \left\{\sum_{v=k_m}^{k_m+p-1}\left|\sigma_n^{-1}\int_{A_v} E_m(t,s)\mathrm{d}s\right|^{2+\delta} + \left(\sum_{v=k_m}^{k_m+p-1}\left|\sigma_n^{-1}\int_{A_v} E_m(t,s)\mathrm{d}s\right|^{2}\right)^{1+\delta/2}\right\}^{1/(2+\delta)} \rho^{-}(v-u) \\
&\leq C\sum_{1\leq l<m\leq k}\sum_{u=k_l}^{k_l+p-1}\left|\sigma_n^{-1}\int_{A_u} E_m(t,s)\mathrm{d}s\right| \sum_{v=k_m}^{k_m+p-1}\left|\sigma_n^{-1}\int_{A_v} E_m(t,s)\mathrm{d}s\right| \rho^{-}(v-u) \\
&\leq C\sum_{l=1}^{k-1}\sum_{u=k_l}^{k_l+p-1}\left|\int_{A_u} E_m(t,s)\mathrm{d}s\right| \sum_{m=l+1}^{k}\sum_{v=k_m}^{k_m+p-1}\rho^{-}(v-u) \\
&\leq C\sum_{u=1}^{n}\left|\int_{A_u} E_m(t,s)\mathrm{d}s\right| v(q) \leq Cv(q).
\end{aligned}
$$

Hence,

$$
I_4 = \int_{-T}^{T}\left|\frac{\Phi(t)-\Psi(t)}{t}\right|\mathrm{d}t \leq C(u(q)+v(q))T^2. \tag{A.14}
$$

Noting that $s_n \to 1$, we have by Lemma 4.3 that

$$\begin{aligned}
&\sup_u |P(T_n \le u+y) - P(T_n \le u)| \\
= &\sup_u |P(\frac{T_n}{s_n} \le \frac{u+y}{s_n}) - P(\frac{T_n}{s_n} \le \frac{u}{s_n})| \\
\le &\sup_u |P(\frac{T_n}{s_n} \le \frac{u+y}{s_n}) - \Phi(\frac{u+y}{s_n})| + \sup_u |\Phi(\frac{u+y}{s_n}) - \Phi(\frac{u}{s_n})| \\
&+ \sup_u |P(\frac{T_n}{s_n} \le \frac{u}{s_n}) - \Phi(\frac{u}{s_n})| \\
\le &\, 2\sup_u |P(\frac{T_n}{s_n} \le \frac{u}{s_n}) - \Phi(u)| + \sup_u |\Phi(\frac{u+y}{s_n}) - \Phi(\frac{u}{s_n})| \\
\le &\, C\{\zeta_{2n}^{\delta/2} + \frac{|y|}{s_n}\} \le C\{\zeta_{2n}^{\delta/2} + |y|\}. \qquad (A.15)
\end{aligned}$$

Therefore

$$I_5 = \sup_u \int_{|y|\le c/T} |P(T_n \le u+y) - P(T_n \le u)| \mathrm{d}y \le C\{\zeta_{2n}^{\delta/2} + 1/T\} \qquad (A.16)$$

Thus, combining (A.12), (A.14) with (A.17) and taking $T = (u(q)+v(q))^{-1/3}$, we have that

$$\begin{aligned}
&\sup_u |P(S'_n \le u) - P(T_n \le u)| \\
\le &\int_{-T}^{T} \left| \frac{\Phi_1(t) - \Psi_1(t)}{t} \right| \mathrm{d}t + T \sup_u \int_{|y|\le c/T} |P(T_n \le u+y) - P(T_n \le u)| \mathrm{d}y \\
\le &\, C\left(\zeta_{2n}^{\delta/2} + (u(q)+v(q))T^2 + \frac{1}{T}\right) \\
\le &\, C\left(\zeta_{2n}^{\delta/2} + (u(q)+v(q))^{1/3}\right). \qquad (A.17)
\end{aligned}$$

The proof is completed. □

Acknowledgements. The authors are most grateful to the Editor and anonymous referee for carefully reading the manuscript and valuable suggestions which helped in improving an earlier version of this paper.

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