

## LARGE DEVIATION ESTIMATES FOR BRANCHING RANDOM WALKS\*

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**Abstract.** For the branching random walk drifting to  $-\infty$  we study large deviations-type estimates for the first passage time. We prove the corresponding law of large numbers and the central limit theorem.

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### 1. INTRODUCTION

#### 1.1. Branching random walk

We consider a discrete-time one-dimensional branching random walk. An initial particle is located at the origin. At time 1 it gives birth to  $N$  new particles, and then dies. Each of the particles is positioned randomly on the real line. Their displacements with respect to the parent are given by independent copies of some given random variable  $X$ . Next, at time 2 all the individuals produce independently their own children and die. All the particles in the second generation are located according to the same rule, with respect to the positions of their parents. This procedure perpetuates itself. The resulting system is called a branching random walk. It can be represented as an infinite tree  $\mathcal{T} = \bigcup_{k \geq 0} \{1, 2, \dots, N\}^k$ , where  $o = \emptyset$  denotes the initial ancestor and word  $\gamma$  of length  $n$  corresponds to individuals in the  $n$ th generation. With every node  $\gamma$  one can associate a real random variable  $X_\gamma$  representing its displacement according to the parent. Then, the position of  $\gamma$  is given by  $S_\gamma$ , the sum of all weights on the path from  $o$  to  $\gamma$ . The collection of positions  $\{S_\gamma\}_{\gamma \in \mathcal{T}}$  forms the branching random walk.

In this paper we are interested in the behavior of

$$M_n = \max_{|\gamma|=n} S_\gamma,$$

the maximal position of the branching random walk after  $n$  steps. (Usually one considers the minimal position, however it is sufficient to replace the displacement of every particle  $X_\gamma$ , by  $-X_\gamma$ .)

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The starting point of our considerations is the law of large numbers proved in a sequence of papers by Hammersley [15], Kingman [19] and Biggins [6],

$$\lim_{n \rightarrow \infty} \frac{M_n}{n} = \rho^* \quad \text{a.s.} \quad (1.1)$$

(see (1.4) below for the definition of  $\rho^*$ ). Last years, in a number of papers, the problem of fluctuations, *i.e.* behavior of  $M_n - n\rho^*$ , was considered. Addario-Berry and Reed [1] and Hu and Shi [17] exhibited a logarithmic correction and next Aïdékon [4] proved that the normalized fluctuations converge in law to some random shift of a Gumbel variable. Precise large deviations were proved by Rouault [22], see also [12, 14] for recent results. We refer the reader to a recent book by Shi [23] for an overview of the related results.

In this paper, we consider the case when the branching random walk drifts to  $-\infty$ , *i.e.*  $\rho^* < 0$  however  $X > 0$  with some positive probability. Some of the particles can still exceed a large level  $u$  and apart from knowing the probability of large deviations, we point out the moment when it arises. More precisely we consider the first passage time

$$\tau_u := \inf\{n : M_n \geq u\}. \quad (1.2)$$

Then, conditioning on the event  $\{\tau_u < \infty\}$  we prove limit theorems related to  $\tau_u$ : the law of large numbers, the central limit theorem and large deviations. All these results are formulated as Theorems 1.1 and 1.2.

## 1.2. Main results

Let  $\mathcal{T} = \bigcup_{k \geq 0} \{1, 2, \dots, N\}^k$  be an infinite tree, where  $\{1, 2, \dots, N\}^0 = \{\emptyset\}$ . For  $\gamma = (i_1, \dots, i_n) \in \mathcal{T}$  we denote the length of  $\gamma$  by  $|\gamma| = n$ , by  $\gamma i$  we denote the vertex  $(i_1, i_2, \dots, i_n, i)$  and we put  $\gamma|_k = (i_1, \dots, i_k)$  for  $k \leq |\gamma|$ . Thus, every node  $\gamma \in \mathcal{T}$  has  $N$  children of the form  $\gamma i$ . With the tree  $\mathcal{T}$  we associate an i.i.d sequence  $\{X_\gamma\}_{\gamma \in \mathcal{T}}$  with the law of some generic random variable  $X$ . We assume that the distribution of  $X$  is non-lattice, *i.e.* it is not supported on any of the sets  $a\mathbb{Z} + b$ ,  $a > 0$ . For any  $\gamma = (i_1, \dots, i_n) \in \mathcal{T}$  define

$$S_\gamma = X_{(i_1)} + X_{(i_1, i_2)} + \dots + X_{(i_1, i_2, \dots, i_n)} = X_{\gamma|_1} + \dots + X_{\gamma|_n},$$

then  $S_\gamma$  denotes the position of the particle represented by  $\gamma$ .

In this paper we are interested in the behavior of  $M_n = \max_{|\gamma|=n} S_\gamma$ , *i.e.* the maximal position of the branching random walk after  $n$  steps. Its properties are determined by the moment and the cumulant generating functions

$$\begin{aligned} \psi(s) &= \mathbb{E} \left[ \sum_{i=1}^N e^{sX_i} \right] = N \mathbb{E} [e^{sX_1}] =: N\lambda(s), \\ \Psi(s) &= \log \psi(s), \Lambda(s) = \log \lambda(s). \end{aligned}$$

We define two parameters

$$\alpha_\infty = \sup\{\alpha : \Psi(\alpha) < \infty\}, \quad \rho_\infty = \sup_{\alpha < \alpha_\infty} \{\Psi'(\alpha)\}.$$

Our rate function is just the convex conjugate (the Fenchel-Legendre transform) of  $\Psi$  and is defined by

$$\Psi^*(x) = \sup_{s \in \mathbb{R}} \{sx - \Psi(s)\}, \quad x \in \mathbb{R},$$

similarly one can define the rate function of displacements  $\Lambda^*$ . Its various properties can be found in Dembo, Zeitouni [13]. We will often use that given  $\alpha < \alpha_\infty$  and  $\rho = \Psi'(\alpha)$

$$\Psi^*(\rho) = \rho\alpha - \Psi(\alpha). \tag{1.3}$$

Now we are able to explain the parameter  $\rho^*$  in (1.1). It can be expressed both in terms of  $\Psi$  and the rate function of displacements  $\Lambda^*$ :

$$\rho^* = \inf_{s>0} \frac{\Psi(s)}{s} = \inf\{s : \Lambda^*(s) \geq \log N\}. \tag{1.4}$$

We assume that  $\rho^* < 0$  and we study the behavior of the first passage time  $\tau_u$  defined in (1.2). It is known (see Jelenkovic and Olvera-Cravioto [18]) that if  $\psi(\alpha_0) = 1$  for some  $\alpha_0 \in (0, \alpha_\infty)$ , then

$$\mathbb{P}[\tau_u < \infty] = \mathbb{P}\left[\sup_{\gamma \in \mathcal{T}} S_\gamma > u\right] \sim c_+ e^{-\alpha_0 u}. \tag{1.5}$$

Here we obtain limit theorems for  $\tau_u$  conditioned on  $\{\tau_u < \infty\}$ .

**Theorem 1.1 (Law of large numbers and Central limit theorem).** *If  $\psi(\alpha_0) = 1$  for some  $\alpha_0 \in (0, \alpha_\infty)$  and the distribution of  $X$  is non-lattice, then*

$$\frac{\tau_u}{u} \Big|_{\tau_u < \infty} \xrightarrow{\mathbb{P}} \frac{1}{\rho_0}$$

and

$$\frac{\tau_u - u/\rho_0}{\sigma_0 \rho_0^{-3/2} \sqrt{u}} \Big|_{\tau_u < \infty} \xrightarrow{d} N(0, 1),$$

where  $\rho_0 = \Psi'(\alpha_0)$ ,  $\sigma_0 = \Psi''(\alpha_0)$ .<sup>1</sup>

**Theorem 1.2 (Large deviations).** *Assume that the distribution of  $X$  is non-lattice,  $\rho \in (0, \rho_\infty)$  and let  $\alpha$  satisfies  $\Psi'(\alpha) = \rho$ .*

*If  $\psi(\alpha) > 1$  (in particular  $\rho > \rho_0$ ), then*

$$\mathbb{P}\left[\frac{\tau_u}{u} < \frac{1}{\rho}\right] \sim \frac{C_1 \psi(\alpha)^{-\Theta(u)}}{\sqrt{u}} e^{-\frac{\Psi^*(\rho)}{\rho} u}, \tag{1.6}$$

for  $\Theta(u) = u/\rho - \lfloor u/\rho \rfloor$ .

*If  $\psi(\alpha) < 1$  (in particular  $\rho < \rho_0$  if  $\rho_0$  is well defined), then*

$$\mathbb{P}\left[\frac{\tau_u}{u} > \frac{1}{\rho}\right] \sim \frac{C_2 \psi(\alpha)^{-\Theta(u)}}{\sqrt{u}} e^{-\frac{\Psi^*(\rho)}{\rho} u}. \tag{1.7}$$

Moreover

$$\mathbb{P}\left[\tau_u = \left\lfloor \frac{u}{\rho} \right\rfloor\right] \sim \frac{C_3 \psi(\alpha)^{-\Theta(u)}}{\sqrt{u}} e^{-\frac{\Psi^*(\rho)}{\rho} u}. \tag{1.8}$$

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<sup>1</sup>The symbol  $\xrightarrow{\mathbb{P}}$  (resp.  $\xrightarrow{d}$ ) denotes convergence in probability (resp. in distribution).

The above results generalize classical estimates for random walks (von Bahr [24], Lalley [20], Höglund [16], see also Buraczewski and Maślanka [11]). Similar results were proved recently for analogous first passage times in the context of random difference equation (see [7, 8]) and branching process in random environment (see Afanasyev [2, 3] and Buraczewski and Dyszewski [10]).

## 2. AUXILIARY DEFINITIONS AND RESULTS

### 2.1. Some further definitions

Here, we collect some additional definitions that will be needed in the proof. If  $\gamma_1 = i_1^1 i_2^1 \dots i_{n_1}^1 \in \mathcal{T}$  and  $\gamma_2 = i_1^2 i_2^2 \dots i_{n_2}^2 \in \mathcal{T}$  then we write  $\gamma_1 \gamma_2 = i_1^1 i_2^1 \dots i_{n_1}^1 i_1^2 i_2^2 \dots i_{n_2}^2$  for the element of  $\mathcal{T}$  obtained by juxtaposition. In particular  $\gamma \emptyset = \emptyset \gamma = \gamma$ . We write  $\gamma' < \gamma$  if  $\gamma'$  is a proper prefix of  $\gamma$ , *i.e.*  $\gamma' = (i_1, \dots, i_k)$  for some  $k < n$ . For two vertices  $\gamma_1$  and  $\gamma_2$  we denote by  $\gamma_0 = \gamma_1 \wedge \gamma_2$  the longest common subsequence of  $\gamma_1$  and  $\gamma_2$ , *i.e.* the maximal  $\gamma_0$  such that both  $\gamma_0 \leq \gamma_1$  and  $\gamma_0 \leq \gamma_2$ . Moreover, we write  $\gamma' \leq \gamma$  if  $\gamma' < \gamma$  or  $\gamma' = \gamma$ .

We will consider different subsets of the tree and maximums over these subsets, namely we define

$$T_{\gamma,n} = \{\gamma\gamma' : |\gamma'| = n\},$$

$$\mathbb{T}_{\gamma,n} = \bigcup_{k=1}^n T_{\gamma,k} = \{\gamma\gamma' : 0 < |\gamma'| \leq n\}.$$

Thus  $T_{\gamma,n}$  is the set of progeny of  $\gamma$  in the generation  $|\gamma| + n$  and  $\mathbb{T}_{\gamma,n}$  consists of all descendants of  $\gamma$  up to time  $|\gamma| + n$ . If  $\gamma = o$  we omit the vertex in the subscript and just write

$$T_n = T_{o,n} = \{\gamma' : |\gamma'| = n\},$$

that is  $T_n$  is exactly the  $n$ th generation. Moreover we define

$$M_\gamma = \max_{n \leq |\gamma|} S_{\gamma|n},$$

$$M_A = \max_{\gamma \in A} S_\gamma, \quad \text{for any } A \subset \mathcal{T},$$

$$\bar{M}_n = \max_{|\gamma| \leq n} S_\gamma = \max_{k \leq n} M_k.$$

To simplify our notation we consider a generic random walk  $\mathbb{S} = \mathbb{X}_1 + \dots + \mathbb{X}_n$  where  $\mathbb{X}_i$  are i.i.d. copies of  $X$ . Then for any  $\gamma \in T_n$ ,  $S_\gamma$  and  $\mathbb{S}_n$  have the same law. Define  $\mathbb{M}_n = \max_{1 \leq k \leq n} \mathbb{S}_k$  and  $\bar{\tau}_u = \inf\{k : \mathbb{S}_k > u\}$ . We define also  $\mathbb{S}_i^n = \mathbb{S}_n - \mathbb{S}_{n-i} = \mathbb{X}_{n-i+1} + \dots + \mathbb{X}_n$ .

### 2.2. Large deviations for classical random walks

The proofs of our results strongly base on large deviation estimates of the usual random walk. We recall here Petrov’s Theorem [21]. Similar result was proved independently by Bahadur and Rao [5], but since we need uniform estimates we follow here [21]. We state here slightly more general version that will be needed in subsequent sections.

**Lemma 2.1.** *Assume that the law of  $X$  is non-lattice. Choose  $\rho$  such that  $\mathbb{E}X < \rho < \rho_\infty$ , let  $\alpha$  be the parameter satisfying  $\Lambda'(\alpha) = \rho$  and let  $\sigma = \Lambda''(\alpha)$ . Then*

$$\mathbb{P}[\mathbb{S}_{n+j_n} > n(\rho + \varepsilon_n)] \sim \frac{1}{\alpha\sigma\sqrt{2\pi n}} e^{-n\Lambda^*(\rho)} \lambda(\alpha)^{j_n} e^{-\alpha n \varepsilon_n} e^{-\frac{(-\rho j_n + n \varepsilon_n)^2}{2\sigma^2 n}}$$

as  $n \rightarrow \infty$  uniformly with respect to

$$\sqrt{n}|\varepsilon_n| \leq \delta_n \rightarrow 0 \quad \text{and} \quad |j_n| \leq b\sqrt{n \log n}.$$

*Proof.* First let us recall Theorem 2 in [21] saying that

$$\mathbb{P}[\mathbb{S}_n > n(\rho + \varepsilon_n)] \sim \frac{1}{\alpha\sigma\sqrt{2\pi n}} e^{n(-\Lambda^*(\rho) - \alpha\varepsilon_n - \frac{\varepsilon_n^2}{2\sigma^2}(1+|\varepsilon_n|))}$$

as  $n \rightarrow \infty$ , uniformly for  $|\varepsilon_n| < \delta_n \rightarrow 0$ . Replacing in the formula above  $(n, \varepsilon_n)$  by  $(n + j_n, -\frac{\rho j_n}{n+j_n} + \frac{n\varepsilon_n}{n+j_n})$  we obtain

$$\begin{aligned} \mathbb{P}[\mathbb{S}_{n+j_n} > n(\rho + \varepsilon_n)] &= \mathbb{P}\left[\mathbb{S}_{n+j_n} > (n + j_n)\left(\rho - \frac{\rho j_n}{n + j_n} + \frac{n\varepsilon_n}{n + j_n}\right)\right] \\ &\sim \frac{1}{\alpha\sigma\sqrt{2\pi(n + j_n)}} e^{-(n+j_n)\Lambda^*(\rho)} e^{-\alpha(-\rho j_n + n\varepsilon_n)} e^{-\frac{(-\rho j_n + n\varepsilon_n)^2}{2\sigma^2(n+j_n)}} \\ &\sim \frac{1}{\alpha\sigma\sqrt{2\pi n}} e^{-n\Lambda^*(\rho)} \lambda(\alpha)^{j_n} e^{-\alpha n\varepsilon_n} e^{-\frac{(-\rho j_n + n\varepsilon_n)^2}{2\sigma^2 n}}. \end{aligned}$$

The last line follows from the fact, that in the considered region  $e^{\frac{(-\rho j_n + n\varepsilon_n)^2}{2\sigma^2(n+j_n)}} e^{-\frac{(-\rho j_n + n\varepsilon_n)^2}{2\sigma^2 n}}$  converges uniformly to 1 as  $n$  tends to  $\infty$ . □

### 2.3. Large deviations for first passage times for random walks

The following Lemma was proved in [11, 16]

**Lemma 2.2.** *If the distribution of  $X$  is non-lattice,  $\mathbb{E}X < 0$  and  $\rho = \Lambda'(\alpha) > 0$  for some  $\rho, \alpha$ , then*

$$\mathbb{P}[\bar{\tau}_{n\rho} = n] = \mathbb{P}[\mathbb{M}_{n-1} \leq n\rho, \mathbb{S}_n > n\rho] \sim \frac{c(\rho)e^{-\Lambda^*(\rho)n}}{\sqrt{n}}.$$

*Proof.* The Lemma is just Theorem 2.3 in [11]. □

However, for our purpose a uniform version of this result is needed.

**Lemma 2.3.** *If the distribution of  $X$  is non-lattice,  $\mathbb{E}X < 0$  and  $\rho = \Lambda'(\alpha) > 0$  for some  $\rho, \alpha$ , then*

$$\mathbb{P}[\bar{\tau}_{n\rho+a_n} = n] \sim \frac{c(\rho)e^{-\alpha a_n} e^{-\Lambda^*(\rho)n}}{\sqrt{n}} \tag{2.1}$$

uniformly as  $n \rightarrow \infty$  for any sequence  $a_n$  such that  $-K \log n < a_n < K \log n$  for some constant  $K$ .

If  $\rho = \rho_0$  and  $\alpha = \alpha_0$ , then

$$\mathbb{P}[\bar{\tau}_{n\rho_0} = n + j_n] \sim \frac{c(\rho_0)}{\sqrt{n}N^{j_n}} e^{-\Lambda^*(\rho_0)n} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}} \tag{2.2}$$

uniformly as  $n \rightarrow \infty$  for any sequence  $j_n$  such that  $|j_n| \leq b\sqrt{n \log n}$  for some constant  $b$ .

We omit the proof since it is very close to the arguments presented in [11]. Only a few places require some minor corrections. Below in the proof we use similar techniques and we will explain in details the role of uniformity.

### 3. UNIFORM LARGE DEVIATIONS FOR $\tau_u$

The aim of this section is to prove a stronger version of (1.8) with perturbed parameters. In the next section we will use this result to deduce Theorems 1.1 and 1.2. To simplify our presentation we state both results separately.

**Proposition 3.1.** *For any  $\rho \in (0, \rho_\infty)$  and  $\alpha$  such that  $\Psi'(\alpha) = \rho$ ,*

$$\mathbb{P}[\tau_{n\rho+a_n} = n] = \mathbb{P}[\overline{M}_{n-1} \leq n\rho + a_n, M_n > n\rho + a_n] \sim C(\rho) \frac{e^{-\alpha a_n} e^{-\Psi^*(\rho)n}}{\sqrt{n}} \tag{3.1}$$

*uniformly as  $n \rightarrow \infty$  for any sequence  $a_n$  such that  $-K \log n < a_n < K \log n$  for some large constant  $K$ .  
If  $\rho = \rho_0$  and  $\alpha = \alpha_0$ , then*

$$\mathbb{P}[\tau_{n\rho_0} = n + j_n] = \mathbb{P}[\overline{M}_{n+j_n-1} \leq n\rho_0, M_{n+j_n} > n\rho_0] \sim \frac{C(\rho_0)}{\sqrt{n}} e^{-\alpha_0 n \rho_0} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}} \tag{3.2}$$

*uniformly as  $n \rightarrow \infty$  for any  $|j_n| < b\sqrt{n \log n}$ .*

Before passing to the proof let us explain in more details the meaning of uniformicity in the statement of the Proposition. Denote by  $L(n, a_n)$  the left side expression in (3.1) and by  $R(n, a_n)$  the right one. Then we are going to prove for fixed constant  $K$  the following: for every  $\varepsilon > 0$  there exists  $N$  such that for all  $n > N$  and all sequences  $\{a_n\}_n$  satisfying  $-K \log n < a_n < K \log n$  we have

$$1 - \varepsilon \leq \frac{L(n, a_n)}{R(n, a_n)} \leq 1 + \varepsilon.$$

#### 3.1. Lower and upper estimates

**Lemma 3.2.** *There are constants  $c_1, c_2 > 0$  such that*

$$c_1 \leq \sqrt{n} e^{\alpha a_n} e^{\Psi^*(\rho)n} \mathbb{P}[\tau_{n\rho+a_n} = n] \leq c_2 \tag{3.3}$$

*uniformly as  $n \rightarrow \infty$  for any  $a_n$  such that  $-K \log n < a_n < K \log n$  for some large constant  $K$ .  
Moreover, if  $\rho = \rho_0$  and  $\alpha = \alpha_0$ , then for  $c_3, c_4 > 0$*

$$c_3 \leq \sqrt{n} e^{\alpha_0 \rho_0 n} e^{\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}} \mathbb{P}[\tau_{n\rho_0} = n + j_n] \leq c_4 \tag{3.4}$$

*uniformly as  $n \rightarrow \infty$  for any  $|j_n| < b\sqrt{n \log n}$ .*

*Proof.* We prove here only the second part of the Lemma, i.e. (3.4). The first part (3.3) can be proved exactly in the same way with obvious changes.

**Step 1. Upper estimates.** For  $\gamma \in T_{n+j_n}$  define

$$C_\gamma = \{M_\gamma|_{n+j_n-1} \leq n\rho_0, S_\gamma > n\rho_0\}$$

to be the event that the path from  $o$  to  $\gamma$  exceeds  $n\rho_0$  for the first time exactly at  $\gamma$ . Then

$$\{\tau_{n\rho_0} = n + j_n\} \subset \bigcup_{\gamma \in T_{n+j_n}} C_\gamma$$

and Lemma 2.3 implies immediately

$$\begin{aligned} \mathbb{P}[\tau_{n\rho_0} = n + j_n] &\leq \mathbb{P}\left[\bigcup_{\gamma \in T_{n+j_n}} C_\gamma\right] \leq \sum_{\gamma \in T_{n+j_n}} \mathbb{P}[C_\gamma] \\ &\leq \frac{c|T_{n+j_n}|}{\sqrt{n}N^{n+j_n}} ce^{-\alpha_0\rho_0 n} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}} \\ &= \frac{c}{\sqrt{n}} e^{-\alpha_0\rho_0 n} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}}, \end{aligned}$$

which gives the upper bound (3.4).

**Step 2. Lower estimates.** Unfortunately, to obtain the lower bound the way we need to pass is quite long and tedious. The idea is quite simple and bases on arguments presented in [9]. We choose a sparse subset  $U$  of  $T_{n+j_n}$ , prove that the corresponding trajectories are almost independent and we apply Lemma 2.3. For a large constant  $L$  (its value will be specified below) we define

$$U = \{\gamma \in T_{n+j_n} : \gamma = \gamma|_{n+j_n-L} \underbrace{(1, \dots, 1)}_{L \text{ times}}\}.$$

This is the set of elements of  $T_{n+j_n}$ , whose last  $L$  indices are 1's. Note that  $|U| = N^{n+j_n-L}$ .

For  $\gamma \in U$  denote

$$W_\gamma = \{\gamma' \in \mathbb{T}_{\gamma|_{n+j_n-L}, L-1} \text{ and } \gamma' \text{ is not an ancestor of } \gamma\}.$$

Thus  $W_\gamma$  is the set of all elements of  $\mathbb{T}_{\gamma|_{n+j_n-L}, L-1}$  except those lying on the path between  $\gamma|_{n+j_n-L}$  and  $\gamma$ . We define

$$\tilde{C}_\gamma = C_\gamma \cap \{X_{\gamma'} < 0 \text{ for all } \gamma' \in W_\gamma\}.$$

Observe

$$\{\tau_{n\rho_0} = n + j_n\} \supset \left(\bigcup_{\gamma \in U} C_\gamma\right) \cap \{M_{T_{n+j_n-1}} \leq n\rho_0\} \supset \bigcup_{\gamma \in U} \left(\tilde{C}_\gamma \cap \{M_{T_{n+j_n-1}} \leq n\rho_0\}\right)$$

Our proof consists of two steps. First we prove that (see step 2A below)

$$\mathbb{P}\left[\bigcup_{\gamma \in U} \tilde{C}_\gamma\right] > \frac{\varepsilon_1(L)}{\sqrt{n}} e^{-\alpha_0\rho_0 n} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}}, \tag{3.5}$$

for  $\varepsilon_1(L) = cp_L N^{-L}$  and next (see step 2B below)

$$\mathbb{P}\left[\left(\bigcup_{\gamma \in U} \tilde{C}_\gamma\right) \cap \{M_{T_{n+j_n-1}} > n\rho_0\}\right] < \frac{\varepsilon_1(L)}{2\sqrt{n}} \cdot e^{-\alpha_0\rho_0 n} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}}. \tag{3.6}$$

Then combining (3.5) and (3.6) we obtain

$$\begin{aligned} \mathbb{P}[\tau_{n\rho_0} = n + j_n] &\geq \mathbb{P}\left[\left(\bigcup_{\gamma \in U} \tilde{C}_\gamma\right) \cap \{M_{T_{n+j_n-1}} \leq n\rho_0\}\right] \\ &= \mathbb{P}\left[\bigcup_{\gamma \in U} \tilde{C}_\gamma\right] - \mathbb{P}\left[\left(\bigcup_{\gamma \in U} \tilde{C}_\gamma\right) \cap \{M_{T_{n+j_n-1}} > n\rho_0\}\right] \\ &> \frac{\varepsilon_1(L)}{2\sqrt{n}} \cdot e^{-\alpha_0\rho_0 n} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}} \end{aligned}$$

**Step 2A. Proof of (3.5).** We observe that by the inclusion-exclusion formula

$$\mathbb{P}\left[\bigcup_{\gamma \in U} \tilde{C}_\gamma\right] \geq \sum_{\gamma \in U} \mathbb{P}(\tilde{C}_\gamma) - \sum_{\gamma, \gamma' \in U, \gamma \neq \gamma'} \mathbb{P}(\tilde{C}_\gamma \cap \tilde{C}_{\gamma'}). \tag{3.7}$$

Since  $S_\gamma$  and  $X_{\gamma'}$  for  $\gamma' \in W_\gamma$  are independent

$$\mathbb{P}(\tilde{C}_\gamma) = p_L \mathbb{P}(C_\gamma),$$

for

$$p_L = \mathbb{P}[X_{\gamma'} < 0 \text{ for all } \gamma' \in W_\gamma] = \mathbb{P}[X_1 < 0]^{|W_\gamma|} > 0.$$

We prove that the sum of intersections in (3.7) is of smaller order. For this purpose we group all  $\gamma' \in U$  depending on the level of the first common ancestor with  $\gamma$ . Given  $k \leq n + j_n - L + 1$  there are  $N^{n+j_n-L-k}$  vertices  $\gamma' \in U$  such that  $|\gamma \wedge \gamma'| = k$ . Therefore, applying the Markov inequality (with some  $\beta$  such that  $\psi(\beta) < 1$ ), we have

$$\begin{aligned} \sum_{\gamma, \gamma' \in U, \gamma \neq \gamma'} \mathbb{P}(\tilde{C}_\gamma \cap \tilde{C}_{\gamma'}) &= p_L \sum_{\gamma, \gamma' \in U, \gamma \neq \gamma'} \mathbb{P}(\tilde{C}_\gamma \cap C_{\gamma'}) \\ &\leq p_L \sum_{\gamma \in U} \sum_{k=0}^{n+j_n-L-1} \mathbb{P}\left[\tilde{C}_\gamma \cap \{S_{\gamma|k} \leq n\rho_0 \text{ and } S_{\gamma'} > n\rho_0 \text{ for some } \gamma' \in U \text{ with } |\gamma \wedge \gamma'| = k\}\right] \\ &\leq p_L \sum_{\gamma \in U} \sum_{k=0}^{n+j_n-L-1} N^{n+j_n-L-k} \mathbb{P}[\tilde{C}_\gamma] \mathbb{P}[S_{n+j_n-k} > 0] \\ &\leq p_L \sum_{\gamma \in U} \mathbb{P}[\tilde{C}_\gamma] N^{-L} \sum_{k=0}^{n+j_n-L-1} \psi(\beta)^{n+j_n-k} \\ &\leq p_L C \lambda(\beta)^L \sum_{\gamma \in U} \mathbb{P}[\tilde{C}_\gamma]. \end{aligned}$$

Therefore by Lemma 2.3, for large  $L$  such that  $\varepsilon_L = Cp_L \lambda(\beta)^L < 1/2$ ,

$$\begin{aligned} \mathbb{P}\left[\bigcup_{\gamma \in U} \tilde{C}_\gamma\right] &\geq (1 - \varepsilon_L) \sum_{\gamma \in U} \mathbb{P}[\tilde{C}_\gamma] \\ &\geq \frac{p_L |U|}{2\sqrt{n} N^{n+j_n}} e^{-\alpha_0\rho_0 n} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}} \\ &\geq \frac{\varepsilon_1(L)}{\sqrt{n}} \cdot e^{-\alpha_0\rho_0 n} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}}, \end{aligned}$$

which proves (3.5).

**Step 2B. Proof of (3.6).** Pick any  $\alpha_{\min} < \beta < \alpha_0$ . Define  $\eta_1 = \frac{\lambda(\beta)}{\lambda(\alpha_0)} = \psi(\beta) < 1$  and  $\eta_2 = \eta_1 e^{\delta(\alpha_0 - \beta)}$ . Take  $\delta$  such that  $\eta_2 < 1$ . We define

$$V_{\gamma,L} = \{S_{\gamma|k} \leq n\rho_0 - \delta(n + j_n - k) \text{ for all } k \leq n + j_n - L\} \tag{3.8}$$

for  $\gamma \in U$  and estimate

$$\begin{aligned} \mathbb{P}\left[\left(\bigcup_{\gamma \in U} \tilde{C}_\gamma\right) \cap \{M_{T_{n+j_n-1}} > n\rho_0\}\right] &\leq N^{n+j_n-L} \mathbb{P}[\tilde{C}_\gamma \cap \{M_{T_{n+j_n-1}} > n\rho_0\}] \\ &\leq N^{n+j_n-L} \left(\mathbb{P}[\tilde{C}_\gamma \cap V_{\gamma,L}^c] + \mathbb{P}[\tilde{C}_\gamma \cap V_{\gamma,L} \cap \{M_{T_{n+j_n-1}} > n\rho_0\}]\right). \end{aligned} \tag{3.9}$$

In the next steps we estimate separately the probabilities above.

**Step 2B1.** We prove that for any  $L > 0$

$$\mathbb{P}[\tilde{C}_\gamma \cap V_{\gamma,L}^c] = p_L \mathbb{P}[C_\gamma \cap V_{\gamma,L}^c] \leq \frac{C p_L \kappa_1^L}{\sqrt{n} N^n} e^{-\alpha_0 \rho_0 n} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}} \tag{3.10}$$

for some  $\kappa_1 < 1$  and  $C > 0$ .

We write

$$\begin{aligned} \mathbb{P}[C_\gamma \cap V_{\gamma,L}^c] &\leq \mathbb{P}[S_{n+j_n} > u, S_k > n\rho_0 - \delta(n + j_n - k) \text{ for some } k \leq n + j_n - L] \\ &\leq \sum_{k=1}^{n+j_n-L} \mathbb{P}[S_{n+j_n} > n\rho_0, S_k > n\rho_0 - \delta(n + j_n - k)] \\ &\leq \sum_{k=1}^{n+j_n-L} \sum_{m=0}^{\infty} \mathbb{P}[S_k + S_{n+j_n-k}^{n+j_n} > n\rho_0, m < S_k - n\rho_0 + \delta(n + j_n - k) \leq m + 1] \\ &\leq \sum_{k=1}^{n+j_n-L} \sum_{m=0}^{\infty} \mathbb{P}[S_k > n\rho_0 - \delta(n + j_n - k) + m] \mathbb{P}[S_{n+j_n-k}^{n+j_n} > \delta(n + j_n - k) - (m + 1)] \\ &= \sum_{k=1}^{n+j_n-L} \sum_{m=0}^{\infty} \mathbb{P}[S_k > n\rho_0 - \delta(n + j_n - k) + m] \mathbb{P}[S_{n+j_n-k} > \delta(n + j_n - k) - (m + 1)]. \end{aligned}$$

Fix  $K > 0$  such that  $\eta_2^{K \log n} < \frac{1}{n^{1+d}}$  for  $d = \frac{\rho_0^2 b}{2\sigma_0^2}$ . We divide the sum into two terms and study both of them independently.

**Case 1:** We first consider indices  $k \leq n + j_n - K \log n$ . By the Markov inequality

$$\begin{aligned} \sum_{k=1}^{n+j_n-K \log n} \sum_{m=0}^{\infty} \mathbb{P}[S_k > n\rho_0 - \delta(n + j_n - k) + m] \mathbb{P}[S_{n+j_n-k} > \delta(n + j_n - k) - (m + 1)] \\ \leq \sum_{k=1}^{n+j_n-K \log n} \lambda(\alpha_0)^k e^{-\alpha_0 \rho_0 n} e^{\alpha_0 \delta(n+j_n-k)} \lambda(\beta)^{n+j_n-k} e^{-\beta \delta(n+j_n-k)} \sum_{m=0}^{\infty} e^{-\alpha_0 m} e^{\beta(m+1)} \end{aligned}$$

$$\begin{aligned} &\leq \frac{C e^{-\alpha_0 \rho_0 n}}{N^{n+j_n}} \sum_{k=1}^{n+j_n-K \log n} \eta_2^{n+j_n-k} \leq \frac{C e^{-\alpha_0 \rho_0 n}}{N^{n+j_n}} \eta_2^{K \log n} \\ &= o\left(\frac{e^{-\alpha_0 \rho_0 n}}{\sqrt{n} N^{n+j_n}} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}}\right). \end{aligned}$$

**Case 2:** Consider  $n + j_n - K \log n < k \leq n + j_n - L$ . By Lemma 2.1 and the Markov inequality

$$\begin{aligned} &\sum_{k=n+j_n-K \log n+1}^{n+j_n-L} \sum_{m=0}^{\infty} \mathbb{P}[\mathbb{S}_k > \rho n_0 - \delta(n + j_n - k) + m] \mathbb{P}[\mathbb{S}_{n+j_n-k} > \delta(n + j_n - k) - (m + 1)] \\ &\leq C \sum_{k=n+j_n-K \log n+1}^{n+j_n-L} \sum_{m < n^{1/4}} \frac{1}{\sqrt{n} N^k} e^{-\alpha_0 \rho_0 n} e^{\alpha_0 \delta(n+j_n-k)} e^{-\alpha_0 m} e^{-\frac{(\rho_0(k-n)+\delta(n+j_n-k)-m)^2}{2\sigma_0^2 n}} e^{-\beta \delta(n+j_n-k)} e^{\beta m} \lambda(\beta)^{n+j_n-k} \\ &\quad + C \sum_{k=n+j_n-K \log n+1}^{n+j_n-L} \sum_{m \geq n^{1/4}} \frac{1}{N^k} e^{-\alpha_0 \rho_0 n} e^{\alpha_0 \delta(n+j_n-k)} e^{-\alpha_0 m} \cdot e^{-\beta \delta(n+j_n-k)} e^{\beta m} \lambda(\beta)^{n+j_n-k} \\ &\leq \frac{C e^{-\alpha_0 \rho_0 n}}{N^{n+j_n}} \sum_{k=n+j_n-K \log n+1}^{n+j_n-L} \eta_2^{n+j_n-k} \left( \frac{e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}}}{\sqrt{n}} \sum_{m < n^{1/4}} e^{-(\alpha_0-\beta)m} + \sum_{m \geq n^{1/4}} e^{-(\alpha_0-\beta)m} \right) \\ &\leq \frac{C e^{-\alpha_0 \rho_0 n}}{N^{n+j_n}} \cdot \eta_2^L \left( \frac{e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}}}{\sqrt{n}} + e^{-(\alpha_0-\beta)n^{1/4}} \right) \leq \frac{C \eta_2^L}{\sqrt{n} N^{n+j_n}} e^{-\alpha_0 \rho_0 n} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}}. \end{aligned}$$

This proves (3.10).

**Step 2B<sub>2</sub>.** Now we are going to prove

$$\mathbb{P}[\tilde{C}_\gamma \cap V_{\gamma,L} \cap \{M_{T_{n+j_n-1}} > n\rho_0\}] \leq \frac{C p_L \kappa_2^L}{\sqrt{n} N^{n+j_n}} e^{-\alpha_0 \rho_0 n} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}} \tag{3.11}$$

We have

$$\begin{aligned} &\mathbb{P}[\tilde{C}_\gamma \cap V_{\gamma,L} \cap \{M_{T_{n+j_n-1}} > n\rho_0\}] \leq \mathbb{P}[\tilde{C}_\gamma \cap V_{\gamma,L} \cap \{S_{\gamma'} > n\rho_0 \text{ for some } |\gamma' \wedge \gamma| < n + j_n - L\}] \\ &\leq p_L \sum_{i=0}^{n+j_n-L} \sum_{k=1}^{\infty} \mathbb{P}[C_\gamma \cap \{C_{\gamma|i} < u - \delta(n+j_n-i) \text{ and } S_{\gamma|i\gamma'} > \rho_0 n \text{ for some } |\gamma'| = k \text{ and } |\gamma \wedge (\gamma|i)\gamma'| = i\}] \\ &\leq p_L \sum_{i=0}^{n+j_n-L} \sum_{k=1}^{\infty} N^k \mathbb{P}[C_\gamma] \mathbb{P}[\mathbb{S}_k > \delta(n + j_n - i)] \\ &\leq p_L \mathbb{P}[C_\gamma] \sum_{i=0}^{n+j_n-L} \sum_{k=1}^{\infty} \psi(\beta)^k e^{-\beta \delta(n+j_n-i)} \\ &\leq C p_L \mathbb{P}[C_\gamma] \sum_{i=0}^{n+j_n-L} e^{-\beta \delta(n+j_n-i)} \\ &\leq C p_L (e^{-\beta \delta})^L \mathbb{P}[C_\gamma]. \end{aligned}$$

**Step 2B<sub>3</sub>.** Estimates (3.9), (3.10) and (3.11) imply that for some  $\kappa_3 < 1$

$$\begin{aligned} \mathbb{P}\left[\bigcup_{\gamma \in U} \tilde{C}_\gamma \cap \{M_{T_{n+j_n-1}} > n\rho_0\}\right] &\leq \frac{C\rho_L \kappa_3^L}{N^L \sqrt{n}} e^{-\alpha_0 \rho_0 n} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}} \\ &\leq \frac{C\kappa_3^L \varepsilon_1(L)}{\sqrt{n}} e^{-\alpha_0 \rho_0 n} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}} \end{aligned}$$

and choosing large  $L$  such that  $C\kappa_3^L < 1/2$  we conclude (3.6) and complete proof of the Lemma. □

**3.2. Proof of Proposition 3.1**

The proof of the asymptotic behavior relies again on the block decomposition. We focus here on the proof of (3.2). Fix large  $L$  and consider the family of  $N^{n+j_n-L}$  disjoint sets  $\mathbb{T}_{\gamma,L}$  for  $\gamma \in T_{n+j_n-L}$ . Given such a  $\gamma$  define

$$\begin{aligned} B_{\gamma,L} &= \{M_\gamma \leq n\rho_0, M_{\mathbb{T}_{\gamma,L-1}} \leq n\rho_0, M_{T_{\gamma,L}} > n\rho_0\} \\ &= \left\{ \max_{k \leq n+j_n-L} S_{\gamma|k} \leq n\rho_0, \max_{|\gamma'| \leq L-1} S_{\gamma\gamma'} \leq n\rho_0, \max_{|\gamma'|=L} S_{\gamma\gamma'} > n\rho_0 \right\} \end{aligned} \tag{3.12}$$

This is the event that on the path from  $o$  to  $\gamma$  and in the subtree  $\mathbb{T}_{\gamma,L-1}$  the value  $n\rho_0$  is not exceeded, but  $S_{\gamma'} > n\rho_0$  for some  $\gamma' \in T_{\gamma,L} \subset T_{n+j_n}$ . The set  $T_{\gamma,L}$  consists of  $N^L$  elements and the corresponding paths from the root are dependent. The following Lemma is the main step of the proof.

**Lemma 3.3.** *There exists  $L_0$  such that for any fixed  $L > L_0$  and any  $\gamma \in T_{n+j_n-L}$*

$$\mathbb{P}(B_{\gamma,L}) \sim \frac{cC_L}{\sqrt{n}N^{n+j_n-L}} e^{-\alpha_0 \rho_0 n} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}}$$

as  $n \rightarrow \infty$  uniformly for  $|j_n| \leq b\sqrt{n \log n}$ , where  $C_L = \mathbb{E}\left[(e^{\alpha \bar{M}_L} - e^{\alpha M_{L-1}})_+\right]$

*Proof.* Observe first that for fixed  $\gamma \in T_{n+j_n-L}$  we have

$$\begin{aligned} \mathbb{P}[M_{\mathbb{T}_{\gamma,L-1}} \leq n\rho_0, M_{T_{\gamma,L}} > n\rho_0] &= \mathbb{P}[M_\gamma > n\rho_0, M_{\mathbb{T}_{\gamma,L-1}} \leq n\rho_0, M_{T_{\gamma,L}} > n\rho_0] \\ &\quad + \mathbb{P}[M_\gamma \leq n\rho_0, M_{\mathbb{T}_{\gamma,L-1}} \leq n\rho_0, M_{T_{\gamma,L}} > n\rho_0] \\ &= \mathbb{P}[M_\gamma > n\rho_0, M_{\mathbb{T}_{\gamma,L-1}} \leq n\rho_0, M_{T_{\gamma,L}} > n\rho_0] + \mathbb{P}[B_{\gamma,L}]. \end{aligned}$$

**Step 1.** Notice that

$$\mathbb{P}[M_\gamma > n\rho_0, M_{T_{\gamma,L}} > n\rho_0] \leq \frac{C(\alpha, \beta)\delta^L}{\sqrt{n}N^{n+j_n-L}} e^{-\alpha_0 \rho_0 n} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}}, \tag{3.13}$$

where  $\delta = \psi(\beta) < 1$  provided  $\beta < \alpha_0$ . Indeed, by Lemma 3.3 in [11] (appropriately modified), for some fixed  $|\gamma'_0| = L$  we have

$$\begin{aligned} \mathbb{P}[M_\gamma > n\rho_0, M_{T_{\gamma,L}} > n\rho_0] &= \mathbb{P}\left[\max_{k \leq n+j_n-L} S_{\gamma|k} > n\rho_0, \max_{|\gamma'|=L} S_{\gamma\gamma'} > n\rho_0\right] \\ &\leq N^L \mathbb{P}\left[\max_{k \leq n+j_n-L} S_{\gamma|k} > n\rho_0, S_{\gamma\gamma'_0} > n\rho_0\right] \end{aligned}$$

$$= \frac{C\delta^L}{\sqrt{n}N^{n-L}} e^{-\alpha_0\rho_0 n} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}}.$$

which proves (3.13).

**Step 2.** Thus, in order to prove the Proposition, it is sufficient to show that for some large fixed  $L$

$$\mathbb{P} [M_{T_{\gamma,L-1}} \leq n\rho_0, M_{T_{\gamma,L}} > n\rho_0] \sim \frac{CC_L}{\sqrt{n}N^{n+j_n-L}} e^{-\alpha_0\rho_0 n} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}}$$

We write

$$\begin{aligned} \mathbb{P} [M_{T_{\gamma,L-1}} \leq n\rho_0, M_{T_{\gamma,L}} > n\rho_0] &= \mathbb{P} \left[ S_\gamma \leq n\rho_0 - n^{\frac{1}{4}}, M_{T_{\gamma,L-1}} \leq n\rho_0, M_{T_{\gamma,L}} > n\rho_0 \right] \\ &+ \mathbb{P} \left[ n\rho_0 - n^{\frac{1}{4}} < S_\gamma < n\rho_0, M_{T_{\gamma,L-1}} \leq n\rho_0, M_{T_{\gamma,L}} > n\rho_0, \max_{|\gamma'|=L} S_{\gamma'} \geq n^{\frac{1}{4}} \right] \\ &+ \mathbb{P} \left[ n\rho_0 - n^{\frac{1}{4}} < S_\gamma < n\rho_0, M_{T_{\gamma,L-1}} \leq n\rho_0, M_{T_{\gamma,L}} > n\rho_0, \max_{|\gamma'|=L} S_{\gamma'} < n^{\frac{1}{4}} \right] \end{aligned} \tag{3.14}$$

We prove that the first two terms are negligible and one needs to consider the asymptotic behavior of the third one.

To estimate the first summand fix  $\beta > \alpha$  and observe that by Markov’s inequality with functions  $e^{\alpha_0 x}$  and  $e^{\beta x}$  for fixed  $|\gamma'| = L$  we have

$$\begin{aligned} &\mathbb{P}[S_\gamma \leq n\rho_0 - n^{\frac{1}{4}}, M_{T_{\gamma,L}} > n\rho_0] \\ &\leq N^L \sum_{m \geq 0} \mathbb{P} \left[ -(m+1) < \mathbb{S}_{n+j_n-L} - (n\rho_0 - n^{\frac{1}{4}}) \leq -m, \mathbb{S}_{n+j_n} > n\rho_0 \right] \\ &\leq N^L \sum_{m \geq 0} \mathbb{P} \left[ \mathbb{S}_{n+j_n-L} > n\rho_0 - n^{\frac{1}{4}} - (m+1) \right] \mathbb{P} \left[ \mathbb{S}_L > n^{\frac{1}{4}} + m \right] \\ &\leq N^L \sum_{m \geq 0} \lambda(\alpha_0)^{n+j_n-L} e^{-\alpha_0\rho_0 n} e^{\alpha_0 n^{\frac{1}{4}}} e^{\alpha_0(m+1)} \lambda(\beta)^L e^{-\beta n^{\frac{1}{4}}} e^{-\beta m} \\ &= e^{-\alpha_0\rho_0 n} N^{L-n-j_n} e^{(\alpha_0-\beta)n^{\frac{1}{4}}} \lambda(\alpha_0)^{-L} \lambda(\beta)^L \sum_{m \geq 0} e^{\alpha_0(m+1)} e^{-\beta m} \\ &= o\left(\frac{1}{\sqrt{n}N^{n+j_n-L}} e^{-\alpha_0\rho_0 n} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}}\right). \end{aligned} \tag{3.15}$$

The same argument proves

$$\mathbb{P} [S_\gamma > n\rho_0 - n^{\frac{1}{4}}, \max_{|\gamma'|=L} S_{\gamma'} > n^{\frac{1}{4}}] = o\left(\frac{1}{\sqrt{n}N^{n+j_n-L}} e^{-\alpha_0\rho_0 n} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}}\right). \tag{3.16}$$

Finally we write

$$\begin{aligned} &\mathbb{P} \left[ n\rho_0 - n^{\frac{1}{4}} < S_\gamma < n\rho_0, M_{T_{\gamma,L-1}} \leq n\rho_0, M_{T_{\gamma,L}} > n\rho_0, \max_{|\gamma'|=L} S_{\gamma'} < n^{\frac{1}{4}} \right] \\ &= \int_{0 \leq y \leq x < n^{\frac{1}{4}}} \mathbb{P} [n\rho_0 - x < \mathbb{S}_{n+j_n-L} < n\rho_0 - y] \mathbb{P} \left[ \max_{|\gamma'| \leq L-1} S_{\gamma'} \in dy, \max_{|\gamma'|=L} S_{\gamma'} \in dx \right]. \end{aligned} \tag{3.17}$$

Now we apply Lemma 2.1

$$\mathbb{P}[\mathbb{S}_{n+j_n-L} \geq n\rho_0 - y] \sim C(\rho_0)e^{-\alpha_0\rho_0 n} N^{-(n+j_n-L)} e^{\alpha_0 y} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}}.$$

Analogously

$$\mathbb{P}[\mathbb{S}_{n+j_n-L} \geq n\rho_0 - x] \sim C(\rho_0)e^{-\alpha_0\rho_0 n} N^{-(n+j_n-L)} e^{\alpha_0 x} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}}.$$

Back to (3.17) we end up with

$$\begin{aligned} &\mathbb{P}\left[n\rho_0 - n^{\frac{1}{4}} < S_\gamma < n\rho_0, S_\gamma + \max_{|\gamma'| \leq L-1} S_{\gamma'} \leq n\rho_0, \max_{|\gamma'|=L} S_{\gamma\gamma'} > n\rho_0, \max_{|\gamma'|=L} S_{\gamma'} < n^{\frac{1}{4}}\right] \\ &\sim \frac{C(\rho_0)}{\sqrt{n}N^{n+j_n-L}} e^{-\alpha_0\rho_0 n} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}} \mathbb{E}\left[\left(e^{\alpha\overline{M}_L} - e^{\alpha M_{L-1}}\right)_+\right]. \end{aligned}$$

Note that by the moment assumptions the expectation above is finite, hence we conclude the Lemma.  $\square$

*Proof of Proposition 3.1 – formula (3.2).* We apply Lemma 3.3 and proceed similarly as in the proof of Lemma 3.2 i.e. we divide all the elements of  $T_{n+j_n}$  into disjoint sets and use strongly that

$$\{\tau_{n\rho_0} = n + j_n\} \subset \bigcup_{\gamma \in T_{n+j_n-L}} B_{\gamma,L},$$

for  $B_{\gamma,L}$  defined in (3.12).

We will prove that for any  $\varepsilon > 0$  there is a constant  $C_L$  depending on the parameter  $L$ , that will be specified below, and such that

$$\begin{aligned} C_L - \varepsilon &\leq \liminf_{n \rightarrow \infty} \mathbb{P}[\tau_{n\rho_0} = n + j_n] \cdot \sqrt{n} e^{\alpha_0\rho_0 n} e^{\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}} \\ &\leq \limsup_{n \rightarrow \infty} \mathbb{P}[\tau_{n\rho_0} = n + j_n] \cdot \sqrt{n} e^{\alpha_0\rho_0 n} e^{\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}} \leq C_L + \varepsilon. \end{aligned} \tag{3.18}$$

Then, passing with  $\varepsilon$  to 0 and applying Lemma 3.2, we obtain

$$\lim_{n \rightarrow \infty} \mathbb{P}[\tau_{n\rho_0} = n + j_n] \cdot \sqrt{n} e^{\alpha_0\rho_0 n} e^{\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}} = C,$$

for some  $C \in (0, \infty)$ .

Our proof consists of two steps. First we prove that

$$\left| \mathbb{P}\left[\bigcup_{\gamma \in T_{n+j_n-L}} B_{\gamma,L}\right] - \sum_{\gamma \in T_{n+j_n-L}} \mathbb{P}(B_{\gamma,L}) \right| \leq \frac{\varepsilon}{2\sqrt{n}} e^{-\alpha_0\rho_0 n} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}} \tag{3.19}$$

which in view of Lemma 3.3 entails

$$\begin{aligned}
 c_L - \varepsilon/2 &\leq \liminf_{n \rightarrow \infty} \mathbb{P} \left[ \bigcup_{\gamma \in T_{n+j_n-1}} B_{\gamma,L} \right] \cdot \sqrt{n} e^{\alpha_0 \rho_0 n} e^{\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}} \\
 &\leq \limsup_{n \rightarrow \infty} \mathbb{P} \left[ \bigcup_{\gamma \in T_{n+j_n-1}} B_{\gamma,L} \right] \cdot \sqrt{n} e^{\alpha_0 \rho_0 n} e^{\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}} \leq C_L + \varepsilon/2.
 \end{aligned}$$

Next we show

$$\mathbb{P} \left[ \bigcup_{\gamma \in T_{n+j_n-1}} B_{\gamma,L} \setminus \{\tau_{n\rho_0} = n + j_n\} \right] \leq \frac{\varepsilon}{2\sqrt{n}} e^{-\alpha_0 \rho_0 n} e^{-\frac{\rho_0^2 j_n^2}{2\sigma_0^2 n}}. \tag{3.20}$$

Thus, (3.19) and (3.20) imply (3.18) and the main result.

**Step 1. Proof of (3.19).** By the inclusion-exclusion formula

$$\left| \mathbb{P} \left[ \bigcup_{\gamma \in T_{n+j_n-L}} B_{\gamma,L} \right] - \sum_{\gamma \in T_{n+j_n-L}} \mathbb{P}[B_{\gamma,L}] \right| \leq \sum_{\gamma, \gamma' \in T_{n+j_n-L}, \gamma \neq \gamma'} \mathbb{P}[B_{\gamma,L} \cap B_{\gamma',L}].$$

Choose  $\beta$  such that  $\psi(\beta) < 1$ , then the Markov inequality entails

$$\begin{aligned}
 \sum_{\gamma, \gamma' \in T_{n+j_n-L}, \gamma \neq \gamma'} \mathbb{P}[B_{\gamma,L} \cap B_{\gamma',L}] &= \sum_{\gamma \in T_{n+j_n-L}} \sum_{k=0}^{n+j_n-L-1} \sum_{\{\gamma' \in T_{n+j_n-L} : |\gamma \wedge \gamma'| = k\}} \mathbb{P}[B_{\gamma,L} \cap B_{\gamma',L}] \\
 &\leq \sum_{\gamma \in T_{n+j_n-L}} \sum_{k=0}^{n+j_n-L-1} \sum_{\{\gamma' \in T_{n+j_n-L} : |\gamma \wedge \gamma'| = k\}} \mathbb{P} \left[ B_{\gamma,L} \cap \{S_{\gamma'|_k} \leq n\rho_0 \text{ and } M_{T_{\gamma',L}} > n\rho_0\} \right] \\
 &\leq \sum_{\gamma \in T_{n+j_n-L}} \sum_{k=0}^{n+j_n-L-1} N^{n+j_n-L-k} N^L \mathbb{P}[B_{\gamma,L}] \mathbb{P}[S_{n+j_n-k} > 0] \\
 &\leq \sum_{\gamma \in T_{n+j_n-L}} \mathbb{P}[B_{\gamma,L}] \cdot \sum_{k=0}^{n+j_n-L-1} \psi(\beta)^{n+j_n-k} \\
 &\leq C\psi(\beta)^L \cdot \sum_{\gamma \in T_{n+j_n-L}} \mathbb{P}[B_{\gamma,L}]
 \end{aligned}$$

which proves (3.19).

**Step 2. Proof of (3.20).** We proceed as in the proof of Lemma 3.2. Recall the definition of  $V_{\gamma,L}$  given in (3.8). Then for some  $\gamma \in T_{n+j_n-L}$

$$\begin{aligned}
 \mathbb{P} \left[ \bigcup_{\gamma \in T_{n+j_n-L}} B_{\gamma,L} \setminus \{\tau_{n\rho_0} = n + j_n\} \right] &= \mathbb{P} \left[ \bigcup_{\gamma \in T_{n+j_n-L}} B_{\gamma,L} \cap \{M_{n+j_n-1} > n\rho_0\} \right] \\
 &= N^{n+j_n-L} \mathbb{P} \left[ B_{\gamma,L} \cap \{M_{n+j_n-1} > n\rho_0\} \right] \\
 &\leq N^{n+j_n-L} \left( \mathbb{P}[B_{\gamma,L} \cap V_{\gamma,L}^c] + \mathbb{P}[B_{\gamma,L} \cap V_{\gamma,L} \cap \{M_{n+j_n-1} > n\rho_0\}] \right).
 \end{aligned}$$

Reasoning as in the proof of Lemma 3.2 (see the proof of (3.10) in step 2B<sub>1</sub>) one can prove

$$\mathbb{P}[B_{\gamma,L} \cap V_{\gamma,L}^c] \leq C\kappa_1^L \mathbb{P}[B_{\gamma,L}] \tag{3.21}$$

and (see the proof of (3.11) in step 2B<sub>2</sub>)

$$\mathbb{P}[B_{\gamma,L} \cap V_{\gamma,L} \cap \{M_{n+j_n-1} > u\}] \leq C\kappa_2^L \mathbb{P}[B_{\gamma,L}]. \tag{3.22}$$

Combining (3.21) with (3.22) we obtain (3.20). □

#### 4. PROOFS OF THEOREMS 1.1 AND 1.2

##### 4.1. Law of large numbers for $\tau_u$

The law of large numbers follows essentially from the following Lemma

**Lemma 4.1.** *Assume that  $\Psi(\alpha_0 + \varepsilon) < \infty$  for some  $\varepsilon > 0$ . Let  $n_1 = n_u - b\sqrt{n_u \log n_u}$  and  $n_2 = n_u + b\sqrt{n_u \log n_u}$  for  $n_u = u/\rho_0$ . Then for any  $\delta > 0$  one can choose large  $b > 0$  such that*

$$\mathbb{P}[\tau_u \leq n_1] \leq Ce^{-\alpha_0 u} u^{-\delta} \tag{4.1}$$

and

$$\mathbb{P}[\tau_u \geq n_2] \leq Ce^{-\alpha_0 u} u^{-\delta} \tag{4.2}$$

*Proof.* Taking  $\gamma$  some fixed elements of  $T_{n_1}$  and applying the Taylor expansion

$$\Psi(\alpha_0 + \varepsilon) + \rho\varepsilon + \frac{\varepsilon^2}{2}\psi''(s)$$

for  $s \in [\alpha_0, \alpha_0 + \varepsilon]$ , we obtain

$$\begin{aligned} \mathbb{P}[M_\gamma > u] &\leq \sum_{k=1}^{|\gamma|} \mathbb{P}[S_k > u] \leq \sum_{k=1}^{|\gamma|} \lambda(\alpha_0 + \varepsilon)^k e^{-(\alpha_0 + \varepsilon)u} \\ &\leq C(\varepsilon)N^{-|\gamma|} \psi(\alpha_0 + \varepsilon)^{|\gamma|} e^{-(\alpha_0 + \varepsilon)u} \\ &\leq C(\varepsilon)N^{-|\gamma|} e^{|\gamma|(\varepsilon\rho_0 + c\varepsilon^2)} e^{-(\alpha_0 + \varepsilon)u}. \end{aligned}$$

Therefore choosing  $\varepsilon = \sqrt{\frac{\log n_u}{n_u}}$  we have

$$\begin{aligned} \mathbb{P}[\tau_u \leq n_1] &\leq N^{n_1} \mathbb{P}[M_\gamma > u] \\ &\leq Ce^{n_1(\varepsilon\rho_0 + c\varepsilon^2)} e^{-(\alpha_0 + \varepsilon)u} \\ &\leq Ce^{(n_u - b\sqrt{n_u \log n_u})(\varepsilon\rho_0 + c\varepsilon^2)} e^{-(\alpha_0 + \varepsilon)u} \\ &\leq Ce^{(\varepsilon^2 - b\varepsilon\rho_0) \log n_u} e^{-\alpha_0 u} \\ &\leq Cu^{-\delta} e^{-\alpha_0 u} \end{aligned}$$

for appropriate large  $b$ .

The proof of (4.2) is similar, but instead of  $M_\gamma$  one has to consider  $\{S_{\gamma|k} > u \text{ for some } k > n_2\}$  for some infinite path  $\gamma$ . □

*Proof of Theorem 1.1 – LLN.* The result is a direct consequence of even stronger result, namely using (1.5) and Lemma 4.1 we see that for any  $\varepsilon > 0$  and large  $u$

$$\begin{aligned} \mathbb{P}\left[\left|\frac{\tau_u}{u} - \rho_0\right| > \varepsilon \mid \tau_u < \infty\right] &\leq \mathbb{P}[\tau_u < u(\rho_0 - \varepsilon) \mid \tau_u < \infty] + \mathbb{P}[\tau_u > u(\rho_0 + \varepsilon) \mid \tau_u < \infty] \\ &\leq \mathbb{P}[\tau_u < n_1 \mid \tau_u < \infty] + \mathbb{P}[\tau_u > n_2 \mid \tau_u < \infty] \\ &\leq Cu^{-\delta}. \end{aligned}$$

□

**4.2. Central limit theorem for  $\tau_u$**

*Proof of Theorem 1.1 – CLT.* Fix  $y \in \mathbb{R}$  and choose  $b$  as in Lemma (4.1). Let  $n = u/\rho_0$ . Then applying Lemma 4.1

$$\begin{aligned} \mathbb{P}\left[\frac{\tau_u - u/\rho_0}{\sigma_0\rho_0^{-3/2}\sqrt{u}} \leq y\right] &\sim \mathbb{P}[n - b\sqrt{n \log n} < \tau_{\rho_0 n} \leq n + \sigma_0\rho_0^{-1}\sqrt{ny}] \\ &= \sum_{-b\sqrt{n \log n} < j \leq \sigma_0\rho_0^{-1}\sqrt{ny}} \mathbb{P}[\tau_{\rho_0 n} = n + j] \\ &\sim Ce^{-\alpha_0\rho_0 n} \sum_{-b\sqrt{n \log n} < j \leq \sigma_0\rho_0^{-1}\sqrt{ny}} \frac{1}{\sqrt{n}} e^{-\frac{\rho_0^2}{2\sigma_0^2}\left(\frac{j}{\sqrt{n}}\right)^2} \end{aligned}$$

Observe that the last expression is just the Riemann sum, thus

$$\mathbb{P}\left[\frac{\tau_u - u/\rho_0}{\sigma_0\rho_0^{-3/2}\sqrt{u}} \leq y\right] \sim Ce^{-\alpha_0\rho_0 n} \int_{-\infty}^{\frac{\sigma_0 y}{\rho_0}} e^{-\frac{1}{2}\left(\frac{\rho_0 s}{\sigma_0}\right)^2} ds = C'e^{-\alpha_0 u} \Phi(y).$$

□

**4.3. Large deviations of  $\tau_u$**

**Lemma 4.2.** *If  $\rho > \rho_0$ , then*

$$\mathbb{P}[\overline{M}_{n_u - D \log n_u} > u] = \mathbb{P}[M_k > u \text{ for some } k \leq n_u - D \log n_u] = o\left(\frac{1}{\sqrt{u}} e^{-\frac{\Psi^*(\rho)}{\rho} u}\right). \tag{4.3}$$

for appropriately large constant  $D$ .

*If  $\rho < \rho_0$ , then*

$$\mathbb{P}[M_k > u \text{ for some } k > n_u + D \log n_u] = o\left(\frac{1}{\sqrt{u}} e^{-\frac{\Psi^*(\rho)}{\rho} u}\right). \tag{4.4}$$

*Proof.* We prove only the first part of the Lemma *i.e.* (4.3), the second one requires similar arguments. By the Markov inequality for  $\varepsilon > 0$  we have

$$\begin{aligned} \mathbb{P}[\overline{M}_{n_u - D \log n_u} > u] &\leq \sum_{k=1}^{n_u - D \log n_u} \mathbb{P}[M_k > u] \leq \sum_{k=1}^{n_u - D \log n_u} \sum_{\gamma \in T_k} \mathbb{P}[S_\gamma > u] \\ &\leq \sum_{k=1}^{n_u - D \log n_u} N^k \mathbb{P}[S_k > u] \leq \sum_{k=1}^{n_u - D \log n_u} N^k \lambda(\alpha + \varepsilon)^k e^{-(\alpha + \varepsilon)u} \\ &\leq \sum_{k=1}^{n_u - D \log n_u} e^{\Psi(\alpha + \varepsilon)k} k e^{-(\alpha + \varepsilon)u}. \end{aligned}$$

Now we expand  $\Psi$  into a Taylor series

$$\Psi(\alpha + \varepsilon) = \Psi(\alpha) + \rho\varepsilon + \frac{\varepsilon^2}{2} \Psi''(s)$$

for some  $s \in [\alpha, \alpha + \varepsilon]$  and choose  $\varepsilon = 1/\sqrt{u}$ . Then

$$\begin{aligned} \mathbb{P}[\overline{M}_{n_u - D \log n_u} > u] &\leq n_u e^{(\Psi(\alpha) + \rho\varepsilon + c\varepsilon^2)(n_u - D \log n_u)} e^{-(\alpha + \varepsilon)u} \\ &\leq C e^{-\alpha u} \cdot n_u^{1 - D\Psi(\alpha)} = o\left(\frac{1}{\sqrt{u}} e^{-\frac{\Psi^*(\rho)}{\rho} u}\right) \end{aligned}$$

for appropriately large  $D$ . □

*Proof of Theorem 1.2.* First we prove (1.8). Take  $\Theta(u) = n_u - \lfloor n_u \rfloor$ . Then by Proposition 3.1

$$\begin{aligned} \mathbb{P}[\tau_u = \lfloor n_u \rfloor] &= \mathbb{P}[\tau_{\lfloor n_u \rfloor \rho + \Theta(u)\rho} = \lfloor n_u \rfloor] \\ &\sim \frac{C e^{-\alpha \Theta(u)\rho} e^{\Psi^*(\rho)\lfloor n_u \rfloor}}{\sqrt{\lfloor n_u \rfloor \rho}} \\ &\sim \frac{C \psi(\alpha)^{-\Theta(u)}}{\sqrt{u}} e^{-\frac{\Psi^*(\rho)}{\rho} u} \end{aligned}$$

Now, Lemma 4.2 and Proposition 3.1 entail

$$\begin{aligned} \mathbb{P}[\tau_u < n_u] &\sim \mathbb{P}[n_u - D \log n_u < \tau_u < n_u] \\ &= \sum_{j=0}^{D \log n_u} \mathbb{P}[\tau_u = \lfloor n_u \rfloor - j] \\ &= \sum_{j=0}^{D \log n_u} \mathbb{P}[\tau_{(\lfloor n_u \rfloor - j)\rho + (\Theta(u) + j)\rho} = \lfloor n_u \rfloor - j] \\ &\sim \sum_{j=0}^{D \log n_u} C(\rho) e^{-\alpha(\Theta(u) + j)\rho} \frac{e^{-\Psi^*(\rho)(\lfloor n_u \rfloor - j)}}{\sqrt{\rho(\lfloor n_u \rfloor - j)}} \end{aligned}$$

$$\begin{aligned} &\sim \frac{C(\rho)\psi(\alpha)^{-\Theta(u)}}{\sqrt{u}} e^{-\frac{\Psi^*(\rho)}{\rho}u} \sum_{j=0}^{D \log n_u} \psi(\alpha)^{-j} \\ &\sim \frac{\psi(\alpha)C(\rho)}{\psi(\alpha) - 1} \psi(\alpha)^{-\Theta(u)} \frac{1}{\sqrt{u}} e^{-\frac{\Psi^*(\rho)}{\rho}u}, \end{aligned}$$

which proves (1.6). Analogous arguments can be used to prove (1.7). We omit details  $\square$

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