BECKNER INEQUALITIES FOR MOEBIUS MEASURES ON SPHERES[☆]

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Abstract. In this paper, we consider the Moebius measures μ_x^n indexed by dimension n and $|x| < 1$ on the unit sphere S^{n-1} in $\mathbb{R}^n$ ($n \geq 3$), and provide a precise two-sided estimate on the order of the Beckner inequality constant with exponent $p \in [1, 2)$ in the three parameters. As special cases for $p = 1$ and p tending to 2, our results cover those in Barthe *et al.* [*Forum Math.* (submitted for publication)] for $n \geq 3$ and explore an interesting phenomenon.

1. Introduction

Let S^{n-1} be the unit sphere in Euclidean space $\mathbb{R}^n$ ($n \geq 3$), equipped with the geodesic distance d and the uniform probability measure μ^n. More precisely, let σ_{n-1} be the uniform measure on S^{n-1} and $s_{n-1} := n\pi^{n/2}/\Gamma(1 + n/2)$ be its total mass, then $\mu^n := \sigma_{n-1}/s_{n-1}$. For $x \in \mathbb{R}^n$ with $|x| < 1$, we consider the probability measure μ_x^n on S^{n-1}, which is given by

$$\mathrm{d}\mu_x^n(y) = \frac{(1 - |x|^2)^{\frac{n-1}{2}}}{(1 - \langle x, y \rangle)^{n-1}} \mathrm{d}\mu^n(y), \quad y \in S^{n-1}, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ stands for the scalar product, $|\cdot|$ the modulus on $\mathbb{R}^n$. It is the so-called Moebius measure we are working on. In fact, this probability is the image of μ^n under the Moebius transformation ϕ_x on $\mathbb{R}^n$, defined by

$$\phi_x(y) := \frac{x - P_x y - (1 - |x|^2)^{1/2} Q_x y}{1 - \langle x, y \rangle},$$

where P_x is the orthogonal projection on the subspace generated by x, and Q_x is the orthogonal projection on the subspace orthogonal to x. Notice that ϕ_x maps S^{n-1} onto S^{n-1}, $\frac{\partial |\phi_x|}{\partial r}$ is the radial derivative of the radial

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component of ϕ_x, and the factor

$$\frac{|detD\phi_x(y)|}{|\frac{\partial|\phi_x|}{\partial r}(y)|} := \frac{(1-|x|^2)^{\frac{n-1}{2}}}{(1-\langle x,y\rangle)^{n-1}}$$

is known as the invariant Poisson kernel $P(x,y)$: as a function of x, it is not harmonic but satisfies the equation $\tilde{\triangle}P(\cdot,y) = 0$, where $\tilde{\triangle}$ denotes the invariant Laplacian (see [11] for further information about this measure).

We say a probability measure μ on a connected complete Riemannian manifold M satisfies a *Beckner inequality* (see [6]) with exponent $p \in [1,2)$, if there exists one positive constant C such that for any $f: M \to \mathbb{R}$ smooth enough,

$$\frac{p}{2-p}\left(\int f^2 \mathrm{d}\mu - \left(\int |f|^p \mathrm{d}\mu\right)^{\frac{2}{p}}\right) \leq C\int |\nabla f|^2 \mathrm{d}\mu, \tag{1.2}$$

where ∇ is the gradient. The best constant is denoted by $C_p(\mu)$. Notice that $C_p(\mu)$ is nondecreasing in $p \geq 1$ since the left-hand side of (1.2) is also nondecreasing (see [12], p. 301).

The classical *Poincaré inequality* corresponds to the case $p = 1$:

$$\mathrm{Var}_\mu(f) := \int f^2 \mathrm{d}\mu - \left(\int f d\mu\right)^2 \leq C\int |\nabla f|^2 \mathrm{d}\mu,$$

while the best constant is denoted by $C_P(\mu)$. Of course, $C_P(\mu) \leq C_p(\mu)$.

The *logarithmic Sobolev inequality* amounts to the limit case when p tends to 2:

$$\mathrm{Ent}_\mu(f^2) \leq 2C\int |\nabla f|^2 \mathrm{d}\mu,$$

since

$$\lim_{p\uparrow 2}\frac{p(\int f^2 \mathrm{d}\mu - (\int |f|^p \mathrm{d}\mu)^{\frac{2}{p}})}{2-p} = \mathrm{Ent}_\mu(f^2),$$

where

$$\mathrm{Ent}_\mu(f^2) := \mu(f^2 \log f^2) - \mu(f^2)\log(\mu(f^2))$$

is the entropy of f^2 w.r.t. μ. The optimal constant for this inequality is denoted by $C_{LS}(\mu)(\geq C_p(\mu)/2)$.

It's well-known that Poincaré and logarithmic Sobolev inequalities are essential tools in the study of concentration of measure and in the estimation of all relaxation time of various ergodic systems [8]. Moreover, if a probability measure μ satisfies logarithmic Sobolev inequality with constant $C > 0$, then μ also satisfies Poincaré inequality with the same constant C. Beckner inequality (1.2) with exponent $1 \leq p < 2$ interpolates between Poincaré and logarithmic Sobolev. Moreover, if μ satisfies a logarithmic Soboelv inequality with constant C, then it satisfies a Beckner inequality with $2C$, while a Poincaré inequality with constant C implies a Beckner inequality with $\frac{p}{2-p}C$ (see [2], p. 384).

In [11], Schechtman and Schmuckenschläger proved that harmonic measures with density $\frac{1-|x|^2}{|y-x|^n}$ and Moebius measures on spheres have a uniform Gaussian concentration property for any $|x| < 1$, which is similar to the

one of μ_0^n. More specifically, they showed that for every 1-Lipschitz function $f : S^{n-1} \to \mathbb{R}$, it holds

$$\mu_x^n \left(\left| f - \int f \, \mathrm{d}\mu_x^n \right| > t \right) \le 3e^{-\delta n t^2}, \quad \forall t > 0,$$

where δ is a universal constant, independent of x and n. Motivated by this work, in [4, 5], Barthe, Ma and the second author investigated whether this property could be recovered from a stronger logarithmic Sobolev inequality, with constant c/n, independent of x. Taking advantage of the fact that the density of both harmonic and Moebius measures only depend on one coordinate, they transferred the problem of estimating the corresponding constant for μ_x^n to the one for only one dimensional probability $\nu_{|x|,n}$ on $[0, \pi]$, where $\nu_{|x|,n}$ is the image probability of μ_x^n under the map $y \to d(y, e_1)$, with e_1 the first component of the canonical basis in $\mathbb{R}^n$.

In [5], the authors showed that for Moebius measures, for any dimension $n \ge 3$ and $|x| < 1$, the logarithmic Sobolev constant $C_{LS}(\mu_x^n)$ has the order $\frac{1}{n} \log(1 + \frac{1}{1-|x|})$, while the Poincaré constant $C_P(\mu_x^n)$ essentially depends on the dimension: $C_P(\mu_x^n) \le \frac{2}{n+1}$. Similar results are obtained in [4] for harmonic measures. Furthermore, Du, Lei and Ma extended the research on harmonic measures to the Beckner inequality in [7].

So the aim of this paper is to generalize the results in [5] and investigate $C_p(\mu_x^n)$ for Moebius measures. After all, Poincaré inequality and logarithmic Sobolev inequality correspond respectively to the special cases $p = 1$ and $p \to 2$. Following the idea of [4], we transfer the estimate on $C_p(\mu_x^n)$ to that on the optimal Beckner constant of $\nu_{|x|,n}$ and then Theorem 11 in [3] helps us to finally offer a two-sided estimate with precise order in n, x and p on $C_p(\mu_x^n)$. As special cases $p = 1$ and $p \to 2$, not only do our results recover those obtained in [5], but also an interesting phenomenon on spectral gap is explored.

The organization of this paper is as follows: in Section 2, we make some preparations, including an important lemma allowing us to reduce the problem for the Moebius measure μ_x^n to that for one dimensional probability $\nu_{|x|,n}$ on $[0, \pi]$; the third section is the main results and their proofs.

2. Preliminaries

In this section, we state the comparison between the constants for μ_x^n and those for $\nu_{|x|,n}$, and we also collect some general results on the one dimensional probabilities on $\mathbb{R}$.

2.1. Key lemma and proposition

The following comparison lemma and proposition are crucial, which show the idea of our proof. The lemma reduces the high dimension problem to one dimension case, and the proposition gives the characterization on Beckner inequality in one dimension case.

Lemma 2.1. *([4]) Let M be a probability measure on S^{n-1} with*

$$dM(x) = \varphi(d(x, e_1))\mathrm{d}\sigma_{n-1}(x), \quad x \in S^{n-1},$$

where φ is measurable. Let ν be the image of M by the map $x \mapsto d(x, e_1)$. Then we have

$$C_p(\nu) \le C_p(M) \le \max \left\{ C_p(\nu), \frac{1}{n-2} \right\}. \tag{2.1}$$

Proposition 2.2. *([3]) Let $p \in (1, 2)$, and let μ, ν be Borel probabilities on $\mathbb{R}$, $\mathrm{d}\nu(x) = h(x)\mathrm{d}x$ and m be the median of μ. Let C be the optimal constant such that for every smooth $f : \mathbb{R} \to \mathbb{R}$ one has*

$$\int f^2 \mathrm{d}\mu - \left(\int |f|^p \mathrm{d}\mu \right)^{\frac{2}{p}} \le C \int f'^2 \mathrm{d}\nu.$$

Then $\max(b_-(p), b_+(p)) \le C \le 4\max(B_-(p), B_+(p))$, *where*

$$b_+(p) = \sup_{x>m} \mu([x,\infty)) \left(1 - \left(1 + \frac{1}{2\mu([x,\infty))} \right)^{\frac{p-2}{p}} \right) \int_m^x \frac{1}{h},$$

$$B_+(p) = \sup_{x>m} \mu([x,\infty)) \left(1 - \left(1 + \frac{(p-1)^{\frac{p}{p-2}}}{\mu([x,\infty))} \right)^{\frac{p-2}{p}} \right) \int_m^x \frac{1}{h},$$

$$b_-(p) = \sup_{x<m} \mu((-\infty,x]) \left(1 - \left(1 + \frac{1}{2\mu((-\infty,x])} \right)^{\frac{p-2}{p}} \right) \int_x^m \frac{1}{h},$$

$$B_-(p) = \sup_{x<m} \mu((-\infty,x]) \left(1 - \left(1 + \frac{(p-1)^{\frac{p}{p-2}}}{\mu((-\infty,x])} \right)^{\frac{p-2}{p}} \right) \int_x^m \frac{1}{h}.$$

Remark 2.3. Specially, if $\nu = \mu$, then $C_p(\mu) = \frac{p}{2-p}C$, that is,

$$\frac{p}{2-p} \max(b_-(p), b_+(p)) \le C_p(\mu) \le \frac{4p}{2-p} \max(B_-(p), B_+(p)).$$

Remark 2.4. For any $y \in (0, 1/2]$,

$$\frac{1 - \left(1 + \frac{(p-1)^{p/(p-2)}}{y} \right)^{(p-2)/p}}{1 - \left(1 + \frac{1}{2y} \right)^{(p-2)/p}} \le \frac{1 - (1 + 2(p-1)^{p/(p-2)})^{(p-2)/p}}{1 - 2^{(p-2)/p}} \le 5.$$

From the inequality above, we have $B_+(p) \le 5b_+(p)$, $B_-(p) \le 5b_-(p)$, then for the optimal constant C, $\max(b_-(p), b_+(p)) \le C \le 20\max(b_-(p), b_+(p))$.

2.2. Auxiliary lemmas

Lemma 2.5.

$$\sqrt{2(n-1)\pi} < \frac{(n-1)s_{n-1}}{s_{n-2}} < \sqrt{2(n-1)\pi \frac{n-1}{n-2}}.$$

Proof. Let $I_{n-2} := \int_0^{\pi/2} (\sin\phi)^{n-2} \mathrm{d}\phi$, then $\frac{(n-1)s_{n-1}}{s_{n-2}} = 2(n-1)I_{n-2}$.

Moreover,

$$\begin{cases} I_{2m} = \dfrac{(2m-1)!!}{(2m)!!} \dfrac{\pi}{2} \\ I_{2m-1} = \dfrac{(2m-2)!!}{(2m-1)!!} \end{cases}$$

Obviously, $I_{2m+1} < I_{2m} < I_{2m-1}$, that is,

$$\frac{(2m)!!}{(2m+1)!!} < \frac{(2m-1)!!}{(2m)!!}\frac{\pi}{2} < \frac{(2m-2)!!}{(2m-1)!!}.$$

Furthermore, we have

$$\frac{1}{\sqrt{\pi(m+1/2)}} < \frac{(2m-1)!!}{2m!!} < \frac{1}{\sqrt{\pi m}},$$

$$\sqrt{\frac{\pi}{4m}} < \frac{(2m-2)!!}{(2m-1)!!} < \sqrt{\frac{\pi}{4m}\left(1+\frac{1}{2m}\right)}.$$

According to the estimates above, we can obtain the desired estimates on $\dfrac{(n-1)s_{n-1}}{s_{n-2}}$. □

Lemma 2.6. *For any $a, b > 0$, and $p \in (1, 2)$,*

$$1 - (1 + ab)^{\frac{p-2}{p}} \leq \left(1 - (1+a)^{\frac{p-2}{p}}\right) + \left(1 - (1+b)^{\frac{p-2}{p}}\right).$$

Proof.

$$\left(1 - (1+a)^{\frac{p-2}{p}} + 1 - (1+b)^{\frac{p-2}{p}}\right) - \left(1 - (1+ab)^{\frac{p-2}{p}}\right)$$

$$= \left(1 + (1+ab)^{\frac{p-2}{p}} - (1+a)^{\frac{p-2}{p}} - (1+b)^{\frac{p-2}{p}}\right)$$

$$\geq \left(1 + (1+a+b+ab)^{\frac{p-2}{p}} - (1+a)^{\frac{p-2}{p}} - (1+b)^{\frac{p-2}{p}}\right)$$

$$= \left(1 + (1+a)^{\frac{p-2}{p}}(1+b)^{\frac{p-2}{p}} - (1+a)^{\frac{p-2}{p}} - (1+b)^{\frac{p-2}{p}}\right)$$

$$= \left(1 - (1+a)^{\frac{p-2}{p}}\right)\left(1 - (1+b)^{\frac{p-2}{p}}\right) \geq 0$$

Lemma 2.7. *For any $C_1, C_2 > 0$ fixed, the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ given by* □

$$f(r) := r\left(1 - \left(1 + \frac{C_1}{r}\right)^{-C_2}\right)$$

is increasing.

Proof.

$$f'(r) = 1 - \left(1 + \frac{C_1}{r}\right)^{-C_2} - \frac{C_1 C_2}{r}\left(1 + \frac{C_1}{r}\right)^{-C_2-1}$$

$$f''(r) = -\frac{C_1^2 C_2(C_2+1)}{r^3}\left(1 + \frac{C_1}{r}\right)^{-C_2-2} < 0$$

Therefore f' is decreasing. From $\lim_{r\to\infty} f'(r) = 0$, we have $f' > 0$, and then the desired result is obtained. □

Lemma 2.8. *For $n \geq 3$, $1 \leq p \leq 2$, define*

$$b_-(n,p) = \sup_{\alpha \in (0,\pi/2)} \int_0^\alpha \sin^{n-2}\theta d\theta \left(1 - \left(1 + \frac{s_{n-1}/s_{n-2}}{2\int_0^\alpha \sin^{n-2}\theta d\theta}\right)^{\frac{p-2}{p}}\right) \int_\alpha^{\pi/2} \frac{d\theta}{\sin^{n-2}\theta},$$

$$b_+(n,p) = \sup_{\alpha \in (\pi/2,\pi)} \int_\alpha^\pi \sin^{n-2}\theta d\theta \left(1 - \left(1 + \frac{s_{n-1}/s_{n-2}}{2\int_\alpha^\pi \sin^{n-2}\theta d\theta}\right)^{\frac{p-2}{p}}\right) \int_{\pi/2}^\alpha \frac{d\theta}{\sin^{n-2}\theta}.$$

Let $b(n,p) := b_-(n,p) = b_+(n,p)$, then $\dfrac{\log 2}{8(n-1)}\dfrac{2-p}{p} \leq b(n,p) \leq \dfrac{1}{n-1}\dfrac{2-p}{p}$.

Proof. On one hand, from $\frac{s_{n-2}}{s_{n-1}}\int_0^\alpha \sin^{n-2}\theta d\theta \leq \frac{1}{2}$ for any $\alpha < \pi/2$, we have

$$b_-(n,p) \geq \left(1 - 2^{\frac{p-2}{p}}\right) \sup_{\alpha \in (0,\pi/2)} \int_0^\alpha \sin^{n-2}\theta d\theta \int_\alpha^{\pi/2} \frac{d\theta}{\sin^{n-2}\theta}$$

$$\geq \frac{1}{4(n-1)}\left(1 - 2^{\frac{p-2}{p}}\right).$$

The lower bound is obtained by $1 - 2^{\frac{p-2}{p}} \geq \dfrac{\log 2}{2}\dfrac{2-p}{p}$ for any $p \in [1,2]$.

On the other hand, by $C_p(\mu^n) = \frac{1}{n-1}$ (see [7]) and $\frac{2-p}{p}C_p(\mu^n) \geq b_-(n,p)$, we get

$$b_-(n,p) \leq \frac{1}{n-1}\frac{2-p}{p}.$$

□

Lemma 2.9. *For $1 \leq p \leq 2$, $x \in (0, \frac{1}{2}]$*

$$\frac{\log 2}{2}\frac{2-p}{p} \leq 1 - x^{\frac{2-p}{p}} \leq \frac{2-p}{p}\log\frac{1}{x}.$$

Proof. Let $\beta := \frac{2-p}{p} \in [0,1]$. First, we have

$$1 - x^\beta \geq 1 - \left(\frac{1}{2}\right)^\beta \geq \frac{\log 2}{2}\beta.$$

Second, for any $0 < x < 1$, $1 - x^\beta - \beta\log\frac{1}{x}$ is decreasing in $\beta > 0$.

□

Lemma 2.10. *For $\sqrt{3}/2 < a < 1, 1 \leq p \leq 2$, let $f(t) := \dfrac{t}{(\alpha+t)^2}\left(1 - (1+t^{-\gamma})^{-\beta}\right)$,*
$t \in (0,1)$, $\alpha = \frac{1-a^2}{a^2} \in (0,1/3)$, $\beta = \frac{(2-p)}{p} \in [0,1]$, $\gamma = \frac{n-1}{2} \geq 1$, we have

$$\sup_{t \in (0,1)} f(t) = \sup_{t \in (0,\alpha)} f(t) \leq \frac{(1 - (1+\alpha^{-\gamma})^{-\beta})}{\alpha}.$$

Proof. Indeed the derivative of f is

$$f'(t) = \frac{(\alpha - t)\left(1 - (1 + t^{-\gamma})^{-\beta}\right) - (\alpha + t)\gamma\beta t^{-\gamma}\left(1 - (1 + t^{-\gamma})^{-\beta - 1}\right)}{(\alpha + t)^3} =: \frac{g_1(t)}{(\alpha + t)^3}.$$

Step 1: Clearly, if $t \geq \alpha$, $g_1(t) < 0$, then $f' < 0$, that is, $f(t)$ is decreasing in $t \in (\alpha, 1)$. Thus we have

$$\sup_{t \in (0,1)} f(t) = \sup_{t \in (0,\alpha)} f(t) \leq \frac{1}{\alpha^2} \sup_{t \in (0,\alpha)} t\left(1 - \left(1 + t^{-\gamma}\right)^{-\beta}\right).$$

Step 2: Let $f_1(t) := t\left(1 - (1 + t^{-\gamma})^{-\beta}\right)$, $t \in (0, \alpha) \subset (0, 1/3)$. Next we will finish the proof by verifying

$$\sup_{t \in (0,\alpha)} f_1(t) = f_1(\alpha).$$

So we just need to show f_1 is increasing in $(0, \alpha)$ for any $\beta \in [0, 1]$ and $\gamma \geq 1$.

$$f_1'(t) = \left(1 + t^{-\gamma}\right)^{-\beta - 1}\left\{\left(1 + t^{-\gamma}\right)^{\beta + 1} - 1 - (1 + \gamma\beta)t^{-\gamma}\right\}.$$

Next we will verify $g_2(u) := (1 + u)^{\beta + 1} - 1 - (1 + \gamma\beta)u \geq 0$ for $u := t^{-\gamma} \in (\alpha^{-\gamma}, +\infty) \subset (3^\gamma, +\infty)$, which can be simply obtained by the following

$$\begin{aligned}
g_2'(u) &= (\beta + 1)(1 + u)^\beta - (1 + \gamma\beta) \\
&\geq (\beta + 1)(1 + 3^\gamma)^\beta - (1 + \gamma\beta) \\
&\geq e^{\beta \log(1 + 3^\gamma)} - (1 + \gamma\beta) \\
&\geq e^{\gamma\beta} - (1 + \gamma\beta) \geq 0,
\end{aligned}$$

and

$$\begin{aligned}
g_2(3^\gamma) &= (1 + 3^\gamma)^{\beta + 1} - 1 - (1 + \gamma\beta)3^\gamma \\
&= (1 + 3^\gamma)e^{\beta \log(1 + 3^\gamma)} - 1 - (1 + \gamma\beta)3^\gamma \\
&\geq (1 + 3^\gamma)e^{\gamma\beta} - 1 - (1 + \gamma\beta)3^\gamma \geq 0.
\end{aligned}$$

$\square$

3. Main results and proof

The first result is on an image measure of μ_x^n, stated as follows:

Theorem 3.1. *Let μ_x^n be the Moebius measure given by* (1.1) *on the unit sphere S^{n-1} with $n \geq 3$, and $\nu_{|x|,n}$ be the image measure of μ_x^n under the map $x \mapsto d(x, e_1)$. For any $x \in \mathbb{R}^n$ with $0 \leq |x| < 1$, the optimal Beckner constant $C_p(\nu_{|x|,n})$ satisfies:*
(1) For $0 \leq |x| \leq \sqrt{3}/2$,

$$\frac{\log 2}{32(n - 1)} \leq C_p(\nu_{|x|,n}) \leq \frac{20(2 + \sqrt{3})^2}{n - 1};$$

(2) For $\sqrt{3}/2 < |x| < 1$, there exist constants $C_1, C_2, C_3, C_4 > 0$ such that,

$$\max\left\{\frac{C_1(1-|x|)}{n-1}, \frac{C_2}{n^2}\left(1-\left(1+\frac{1}{(1-|x|^2)^{\frac{n-1}{2}}}\right)^{\frac{p-2}{p}}\right)\frac{p}{2-p}\right\}$$

$$\leq C_p(\nu_{|x|,n}) \leq$$

$$\max\left\{\frac{C_3(1-|x|)}{n-1}, \frac{C_4}{n^2}\left(1-\left(1+\frac{1}{(1-|x|^2)^{\frac{n-1}{2}}}\right)^{\frac{p-2}{p}}\right)\frac{p}{2-p}\right\}.$$

Remark 3.2. Notice that for the case $p = 1$, as x is close to the sphere S^{n-1}, an interesting phenomenon appears: the order of the optimal Poincaré constant $C_P(\nu_{|x|,n})$ in n is transferred from $1/n$ to $1/n^2$. By the way, the barrier $\sqrt{3}/2$ is just a number to show the difference in the neighbourhood of 1. It can be replaced by any fixed number $\varepsilon \in (0,1)$ at the price of the modification of some constants C_1, C_2 and so forth.

By Lemma 2.1, one can obtain the following conclusion on μ_x^n immediately:

Theorem 3.3. *Let μ_x^n be the Moebius measure given by* (1.1) *on the unit sphere S^{n-1} with $n \geq 3$. For any $x \in \mathbb{R}^n$ with $0 \leq |x| < 1$, the optimal Beckner constant $C_p(\mu_x^n)$ satisfies:*
(1) For $0 \leq |x| \leq \sqrt{3}/2$,

$$\frac{\log 2}{32(n-1)} \leq C_p(\mu_x^n) \leq \frac{20(2+\sqrt{3})^2}{n-1};$$

(2) For $\sqrt{3}/2 < |x| < 1$, there exist constants $C_1, C_2, C_3, C_4 > 0$ such that,

$$\max\left\{\frac{C_1(1-|x|)}{n-1}, \frac{C_2}{n^2}\left(1-\left(1+\frac{1}{(1-|x|^2)^{\frac{n-1}{2}}}\right)^{\frac{p-2}{p}}\right)\frac{p}{2-p}\right\}$$

$$\leq C_p(\mu_x^n) \leq$$

$$\max\left\{\frac{C_3(1-|x|)}{n-1}, \frac{C_4}{n^2}\left(1-\left(1+\frac{1}{(1-|x|^2)^{\frac{n-1}{2}}}\right)^{\frac{p-2}{p}}\right)\frac{p}{2-p}, \frac{1}{n-2}\right\}.$$

Remark 3.4. For the case $p = 1$, $C_P(\mu_x^n)$ is bounded from above by K_1/n for some constant $K_1 > 0$ independent of x, and if x is in the neighbourhood of 0, $C_P(\mu_x^n)$ also can be bounded from below by K_2/n for another constant $K_2 > 0$ independent of x. By [13], one can get immediately that μ_x^n also satisfies Talagrand's transportation inequality uniformly in n and x.

Remark 3.5. For the case $p \to 2$, by the following simple calculation,

$$\lim_{p\uparrow 2} \frac{1-\left(1+\frac{1}{(1-|x|^2)^{\frac{n-1}{2}}}\right)^{\frac{p-2}{p}}}{\frac{2-p}{p}} = \log\left(1+\frac{1}{(1-|x|^2)^{\frac{n-1}{2}}}\right),$$

we can see that $C_{LS}(\mu_x^n)$ has the order $\frac{1}{n}\log(1+\frac{1}{1-|x|})$ in n and x, which implies that $C_{LS}(\mu_x^n)$ explodes with speed $\log(1+\frac{1}{1-|x|})$ as $|x| \to 1$ for n fixed, and then logarithmic Soboelv inequality is strictly stronger than Talagrand's transportation inequality. In fact, we also can prove that for any $p \in [1,2)$ and $|x| < 1$ from

Lemma 2.9,

$$1 - \left(1 + \frac{1}{(1 - |x|^2)^{\frac{n-1}{2}}}\right)^{\frac{p-2}{p}} \leq \frac{2-p}{p} \log\left(1 + \frac{1}{(1 - |x|^2)^{\frac{n-1}{2}}}\right).$$

But we fail to reach a lower bound with the same order up to a constant, and it is the failure that shows the gap between Poincaré inequality and logarithmic Sobolev inequality.

Remark 3.6. From the two remarks above, one can see that our estimates recover the results in [5]. Besides, for the case $n = 2$, it is simpler and the similar result is obtained in [9] by a similar lemma with Lemma 2.1 (see [7]).

Proof of Theorem 3.1. By rotational invariance of the sphere, without loss of generality, we take $x = ae_1$ with $0 < a < 1$, recall $\nu_{a,n}$ the image of $\mu_{ae_1}^n$ by the map $y \to d(y, e_1)$, which is a probability measure on $[0, \pi]$ with density

$$h_{a,n} := \frac{d\nu_{a,n}}{d\theta} = \frac{s_{n-2}}{s_{n-1}} \frac{(1 - a^2)^{\frac{n-1}{2}}}{(1 - a\cos\theta)^{n-1}} \sin^{n-2}\theta.$$

We choose $\theta_a = \arccos(a)$, which is the median of $\nu_{a,n}$ (see (3.1)), and define the following parameters,

$$b_-(a, n, p) = \sup_{\alpha \in (0, \theta_a)} \int_0^\alpha h_{a,n} \left(1 - \left(1 + \frac{1}{2\int_0^\alpha h_{a,n}}\right)^{\frac{p-2}{p}}\right) \int_\alpha^{\theta_a} \frac{1}{h_{a,n}},$$

$$b_+(a, n, p) = \sup_{\alpha \in (\theta_a, \pi)} \int_\alpha^\pi h_{a,n} \left(1 - \left(1 + \frac{1}{2\int_\alpha^\pi h_{a,n}}\right)^{\frac{p-2}{p}}\right) \int_{\theta_a}^\alpha \frac{1}{h_{a,n}}$$

Notice that the case $a = 0$ is corresponding to the uniform probability measures on S^{n-1}. By symmetry $b_-(0, n, p) = b_+(0, n, p)$. Moreover, $b_\pm(0, n, p)$ is just $b(n, p)$ in Lemma 2.8 with the order $\frac{1}{n-1}\frac{2-p}{p}$.

Obviously, the parameters above involve two important items $\int_\alpha^\beta h_{a,n}$ and $\int_\alpha^\beta \frac{1}{h_{a,n}}$ for any α, β in $[0, \pi]$.

Make a change of variables $\phi = \Phi(\theta)$ such that $\Phi([0, \theta_a]) = [0, \pi/2]$, $\Phi([\theta_a, \pi]) = [\pi/2, \pi]$, which is the angular version of the one introduced by Schechtman and Schmuckenschläger in [11]. The change is as follows:

$$\cos\phi = \frac{\cos\theta - a}{1 - a\cos\theta} = \frac{\cos\theta - a}{1 - a^2 - a(\cos\theta - a)},$$

$$\sin\phi = \frac{\sqrt{1 - a^2}\sin\theta}{1 - a\cos\theta}.$$

So $\Phi(\theta) = \arccos(F(\cos\theta - a))$, where

$$F(t) = \frac{t}{1 - a^2 - at}, \quad t \in [-1 - a, 1 - a]$$

is an increasing function, and its inverse function is

$$F^{-1}(c) = \frac{(1 - a^2)c}{1 + ac}, \quad c \in [-1, 1],$$

whose derivative is

$$(F^{-1})'(c) = \frac{1-a^2}{(1+ac)^2}.$$

Note that $\cos\phi = F(\cos\theta - a)$, so we have $\sin\theta \mathrm{d}\theta = \sin\phi(F^{-1})'(\cos\phi)\mathrm{d}\phi$. Then we have for any $0 < \theta_1 \leq \theta_2 < \pi$,

$$
\begin{aligned}
\int_{\theta_1}^{\theta_2} h_{a,n} &= \int_{\theta_1}^{\theta_2} \frac{(1-a^2)^{\frac{n-1}{2}}}{(1-a\cos\theta)^{n-1}} \frac{s_{n-2}}{s_{n-1}}(\sin\theta)^{n-2}\mathrm{d}\theta \\
&= \int_{\theta_1}^{\theta_2} \frac{1-a^2}{(1-a\cos\theta)^2} \frac{s_{n-2}}{s_{n-1}} \left(\frac{\sqrt{1-a^2}\sin\theta}{1-a\cos\theta}\right)^{n-3} \sin\theta \mathrm{d}\theta \\
&= \int_{\Phi(\theta_1)}^{\Phi(\theta_2)} (1-a^2) \left(\frac{F^{-1}(\cos\phi)}{\cos\phi}\right)^{-2} \frac{s_{n-2}}{s_{n-1}}(\sin\phi)^{n-3}\sin\phi(F^{-1})'(\cos\phi)\mathrm{d}\phi \\
&= \int_{\Phi(\theta_1)}^{\Phi(\theta_2)} h_{0,n}(\phi)\mathrm{d}\phi,
\end{aligned}
\tag{3.1}
$$

and

$$
\begin{aligned}
\int_{\theta_1}^{\theta_2} \frac{1}{h_{a,n}} &= \int_{\theta_1}^{\theta_2} \frac{(1-a\cos\theta)^{n-1}}{(1-a^2)^{\frac{n-1}{2}}} \frac{s_{n-1}}{s_{n-2}}(\sin\theta)^{2-n}\mathrm{d}\theta \\
&= \int_{\theta_1}^{\theta_2} \frac{s_{n-1}}{s_{n-2}} \left(\frac{1-a\cos\theta}{\sqrt{1-a^2}\sin\theta}\right)^{n-1} \sin\theta \mathrm{d}\theta \\
&= \int_{\Phi(\theta_1)}^{\Phi(\theta_2)} \frac{s_{n-1}}{s_{n-2}}(\sin\phi)^{1-n}\sin\phi(F^{-1})'(\cos\phi)\mathrm{d}\phi \\
&= \int_{\Phi(\theta_1)}^{\Phi(\theta_2)} \frac{1-a^2}{(1+a\cos\phi)^2} \frac{1}{h_{0,n}(\phi)}\mathrm{d}\phi.
\end{aligned}
\tag{3.2}
$$

By (3.1), (3.2), $b_-(a,n,p)$ can be rewritten immediately as

$$
\begin{aligned}
b_-(a,n,p) = \sup_{\psi\in(0,\pi/2)} &\left(\int_0^\psi (\sin\phi)^{n-2}\mathrm{d}\phi\right)\left(1 - \left(1 + \frac{s_{n-1}}{2s_{n-2}\int_0^\psi (\sin\phi)^{n-2}\mathrm{d}\phi}\right)^{\frac{p-2}{p}}\right) \\
&\times \int_\psi^{\pi/2} \frac{1-a^2}{(1+a\cos\phi)^2}\frac{1}{(\sin\phi)^{n-2}}\mathrm{d}\phi
\end{aligned}
$$

while $b_+(a,n,p)$ turns out to be

$$
\begin{aligned}
b_+(a,n,p) = \sup_{\varphi\in(\pi/2,\pi)} &\left(\int_\varphi^\pi (\sin\phi)^{n-2}\mathrm{d}\phi\right)\left(1 - \left(1 + \frac{s_{n-1}}{2s_{n-2}\int_\varphi^\pi (\sin\phi)^{n-2}\mathrm{d}\phi}\right)^{\frac{p-2}{p}}\right) \\
&\times \left(\int_{\pi/2}^\varphi \frac{1-a^2}{(1+a\cos\phi)^2}\frac{1}{(\sin\phi)^{n-2}}\mathrm{d}\phi\right).
\end{aligned}
$$

Clearly, for $b_-(a, n, p)$ we have,

$$\frac{1-a}{2} b_-(0, n, p) \le b_-(a, n, p) \le (1 - a^2) b_-(0, n, p). \tag{3.3}$$

Next we plan to give the estimate of $b_+(a, n, p)$:

Case I: If $0 \le a \le \sqrt{3}/2$, then $\frac{1}{4} b_+(0, n, p) \le b_+(a, n, p) \le (2 + \sqrt{3})^2 b_+(0, n, p)$.

Case II: If $\sqrt{3}/2 < a < 1$, then for $n = 3$,

$$\frac{1}{2e^2} \left(1 - \left(1 + \frac{1}{1-a} \right)^{\frac{p-2}{p}} \right) \le b_+(a, 3, p) \le 6 \left(1 - \left(1 + \frac{1}{1-a} \right)^{\frac{p-2}{p}} \right);$$

and for $n > 3$,

$$\frac{15}{16(n-1)(n+1)} \left(1 - \left(1 + \frac{1}{(1-a^2)^{\frac{n-1}{2}}} \right)^{\frac{p-2}{p}} \right)$$

$$\le b_+(a, n, p) \le$$

$$\max \left\{ \frac{256\sqrt{3}}{9(n-1)(n-3)} \left(1 - \left(1 + \frac{1}{(1-a^2)^{\frac{n-1}{2}}} \right)^{\frac{p-2}{p}} \right), 4(2 + \sqrt{3})^2 (1 - a^2) b_+(0, n, p) \right\}.$$

First, we consider the case $n = 3$: by simple calculus, we have

$$\int_\varphi^\pi \sin \phi \, \mathrm{d}\phi = 1 + \cos \varphi,$$

and

$$\frac{1}{2} \int_0^{-\cos\varphi} \frac{\mathrm{d}t}{(1-at)^2(1-t)} \le \int_{\pi/2}^\varphi \frac{1}{(1 + a \cos \phi)^2} \frac{1}{\sin \phi} \mathrm{d}\phi \le \int_0^{-\cos\varphi} \frac{\mathrm{d}t}{(1-at)^2(1-t)}.$$

Furthermore,

$$\int_0^{-\cos\varphi} \frac{\mathrm{d}t}{(1-at)^2(1-t)} = \int_{1+\cos\varphi}^1 \frac{\mathrm{d}u}{(1-a+au)^2 u}$$

$$= \frac{1}{(1-a)^2} \left\{ \log \frac{1-a+au}{u} + \frac{au}{1-a+au} - a \right\} |_{u=1+\cos\varphi}.$$

Therefore, the estimate on $b_+(a, 3, p)$ is reduced to that on the following up to $1/2$ and 1,

$$\frac{1+a}{1-a} \sup_{u \in (0,1)} u \left(1 - \left(1 + \frac{1}{u} \right)^{\frac{p-2}{p}} \right) \left\{ \log \frac{1-a+au}{u} + \frac{au}{1-a+au} - a \right\}.$$

On one hand,

$$b_+(a,3,p) \geq \frac{1+a}{2(1-a)} \cdot \frac{1-a}{e-a} \left(1 - \left(1 + \frac{e-a}{1-a}\right)^{\frac{p-2}{p}}\right) \times \left\{\log \frac{1-a+a\cdot\frac{1-a}{e-a}}{\frac{1-a}{e-a}} + \frac{a\cdot\frac{1-a}{e-a}}{1-a+a\cdot\frac{1-a}{e-a}} - a\right\}$$

$$= \frac{1+a}{2(e-a)} \left(1 - \left(1 + \frac{e-a}{1-a}\right)^{\frac{p-2}{p}}\right) \left(1 + \frac{a}{e} - a\right)$$

$$\geq \frac{1}{2e^2} \left(1 - \left(1 + \frac{1}{1-a}\right)^{\frac{p-2}{p}}\right).$$

On the other hand,

$$b_+(a,3,p) \leq \frac{1+a}{1-a} \sup_{u\in(0,1)} u \left(1 - \left(1 + \frac{1}{u}\right)^{\frac{p-2}{p}}\right) \log \frac{1-a+au}{u}$$

$$= (1+a) \sup_{t\in(0,1/(1-a))} t \left(1 - \left(1 + \frac{1}{t} \cdot \frac{1}{1-a}\right)^{\frac{p-2}{p}}\right) \log \frac{1+at}{t}$$

$$\leq 2 \sup_{t>0} t \left(1 - \left(1 + \frac{1}{t}\right)^{\frac{p-2}{p}}\right) \log \left(1 + \frac{1}{t}\right) + 2 \sup_{t>0} t \log \left(1 + \frac{1}{t}\right) \left(1 - \left(1 + \frac{1}{1-a}\right)^{\frac{p-2}{p}}\right)$$

$$\leq \frac{2(2-p)}{p} \sup_{t>0} t \log^2 \left(1 + \frac{1}{t}\right) + 2\left(1 - \left(1 + \frac{1}{1-a}\right)^{\frac{p-2}{p}}\right)$$

$$= \frac{2-p}{p} \frac{2}{e-1} + 2\left(1 - \left(1 + \frac{1}{1-a}\right)^{\frac{p-2}{p}}\right)$$

$$\leq 6 \left(1 - \left(1 + \frac{1}{1-a}\right)^{\frac{p-2}{p}}\right)$$

The second inequality above is derived from Lemma 2.6, and the later two inequalities are from Lemma 2.9.

Second, for the case $n > 3$:

Part 1: On the lower bound of $b_+(a,n,p)$ for $\sqrt{3}/2 < a < 1$ and $n \geq 3$. Taking $\varphi = \arccos(-a)$, we get

$$b_+(a,n,p) \geq \left(\int_{\arccos(-a)}^{\pi} (\sin\phi)^{n-2} d\phi\right) \left(1 - \left(1 + \frac{s_{n-1}}{2s_{n-2}\int_{\arccos(-a)}^{\pi}(\sin\phi)^{n-2}d\phi}\right)^{\frac{p-2}{p}}\right)$$

$$\times \left(\int_{\pi/2}^{\arccos(-a)} \frac{1-a^2}{(1+a\cos\phi)^2} \frac{1}{(\sin\phi)^{n-2}} d\phi\right).$$

With simple calculus, the two items above have bounds from below as

$$\int_{\arccos(-a)}^{\pi} (\sin\phi)^{n-2} d\phi \geq \int_{\arccos(-a)}^{\pi} (\sin\phi)^{n-2}(-\cos\phi) d\phi = \frac{(1-a^2)^{\frac{n-1}{2}}}{n-1}, \tag{3.4}$$

and

$$\int_{\pi/2}^{\arccos(-a)} \frac{1}{(1+a\cos\phi)^2} \frac{1}{(\sin\phi)^{n-2}} \mathrm{d}\phi \geq \int_{\pi/2}^{\arccos(-a)} \frac{1}{(1-\cos^2\phi)^2} \frac{1}{(\sin\phi)^{n-2}} \mathrm{d}\phi$$

$$\geq \int_{\pi/2}^{\arccos(-a)} \frac{-\cos\phi}{(\sin\phi)^{n+2}} \mathrm{d}\phi \tag{3.5}$$

$$= \frac{(1-a^2)^{-\frac{n+1}{2}} - 1}{n+1}.$$

By the monotonicity of the function $f(r) := r(1 - (1 + \frac{C_1}{r})^{-C_2})$ in $r > 0$ for any C_1, $C_2 > 0$ fixed from Lemma 2.7, we have that for any $a \in (\sqrt{3}/2, 1)$,

$$b_+(a,n,p) \geq \frac{1 - (1-a^2)^{\frac{n+1}{2}}}{(n-1)(n+1)} \left(1 - \left(1 + \frac{(n-1)s_{n-1}}{2s_{n-2}(1-a^2)^{\frac{n-1}{2}}} \right)^{\frac{p-2}{p}} \right)$$

$$\geq \frac{15}{16(n-1)(n+1)} \left(1 - \left(1 + \frac{1}{(1-a^2)^{\frac{n-1}{2}}} \right)^{\frac{p-2}{p}} \right)$$

where the second inequality is true since

$$\frac{(n-1)s_{n-1}}{s_{n-2}} = (n-1) \int_0^\pi (\sin\theta)^{n-2} \mathrm{d}\theta \geq 2(n-1) \int_0^{\frac{\pi}{2}} (\sin\theta)^{n-2} \cos\theta \mathrm{d}\theta = 2.$$

Part 2: On the upper bound of $b_+(a,n,p)$ for $\sqrt{3}/2 < a < 1$ and $n > 3$.

$$b_+(a,n,p) = \max \left\{ \sup_{\varphi \in (\pi/2, 5\pi/6)} I(\varphi,a,p), \sup_{\varphi \in (5\pi/6, \pi)} I(\varphi,a,p) \right\},$$

where

$$I(\varphi,a,p) = \left(\int_\varphi^\pi (\sin\phi)^{n-2} \mathrm{d}\phi \right) \left(1 - \left(1 + \frac{s_{n-1}}{2s_{n-2} \int_\varphi^\pi (\sin\phi)^{n-2} \mathrm{d}\phi} \right)^{\frac{p-2}{p}} \right)$$

$$\times \left(\int_{\pi/2}^\varphi \frac{1-a^2}{(1+a\cos\phi)^2} \frac{1}{(\sin\phi)^{n-2}} \mathrm{d}\phi \right).$$

Step 1:

$$\sup_{\varphi \in (\frac{\pi}{2}, \frac{5\pi}{6})} I(\varphi,a,p) \leq 4(2+\sqrt{3})^2(1-a^2) \sup_{\varphi \in (\frac{\pi}{2}, \frac{5\pi}{6})} I(\varphi,0,p)$$

$$\leq 4(2+\sqrt{3})^2(1-a^2)b_+(0,n,p). \tag{3.6}$$

Step 2: For $\varphi \in (5\pi/6, \pi)$, we have

$$\int_\varphi^\pi (\sin\phi)^{n-2}\mathrm{d}\phi \leq \frac{2}{\sqrt{3}} \int_\varphi^\pi (\sin\phi)^{n-2}(-\cos\phi)\mathrm{d}\phi \leq \frac{2(\sin\varphi)^{n-1}}{\sqrt{3}(n-1)}$$

and

$$\int_{\pi/2}^\varphi \frac{1-a^2}{(1+a\cos\phi)^2} \frac{1}{(\sin\phi)^{n-2}}\mathrm{d}\phi \leq \frac{1-a^2}{(1+a\cos\varphi)^2} \int_{\pi/2}^\varphi \frac{1}{(\sin\phi)^{n-2}}\mathrm{d}\phi,$$

with

$$\int_{\pi/2}^\varphi \frac{1}{(\sin\phi)^{n-2}}\mathrm{d}\phi \leq 2 \int_{2\pi/3}^\varphi \frac{1}{(\sin\phi)^{n-2}}\mathrm{d}\phi \leq 4 \int_{\pi/2}^\varphi \frac{-\cos\phi}{(\sin\phi)^{n-2}}\mathrm{d}\phi \leq \frac{4(\sin\varphi)^{3-n}}{n-3}.$$

By Lemma 2.7, we have

$$\sup_{\varphi\in(\frac{5\pi}{6},\pi)} I(\varphi, a, p)$$

$$\leq \frac{8(1-a^2)}{\sqrt{3}(n-1)(n-3)} \sup_{\varphi\in(\frac{5\pi}{6},\pi)} \frac{(\sin\varphi)^2}{(1+a\cos\varphi)^2} \left(1 - \left(1 + \frac{\sqrt{3}(n-1)s_{n-1}}{4s_{n-2}(\sin\varphi)^{n-1}}\right)^{\frac{p-2}{p}}\right)$$

$$\leq \frac{8(1-a^2)}{\sqrt{3}(n-1)(n-3)} \sup_{\varphi\in(\frac{5\pi}{6},\pi)} \frac{(\sin\varphi)^2}{(1+a\cos\varphi)^2} \left(1 - \left(1 + \left(\frac{1}{\sin^{(n-1)}\varphi}\right)^2\right)^{\frac{p-2}{p}}\right)$$

$$\leq \frac{16(1-a^2)}{\sqrt{3}(n-1)(n-3)} \sup_{\varphi\in(\frac{5\pi}{6},\pi)} \frac{(\sin\varphi)^2}{(1+a\cos\varphi)^2} \left(1 - \left(1 + \frac{1}{(\sin\varphi)^{(n-1)}}\right)^{\frac{p-2}{p}}\right)$$

$$\leq \frac{64(1-a^2)}{\sqrt{3}(n-1)(n-3)a^4} \sup_{t\in(0,1)} \frac{t}{(\frac{1-a^2}{a^2}+t)^2} \left(1 - \left(1 + \frac{1}{t^{\frac{n-1}{2}}}\right)^{\frac{p-2}{p}}\right)$$

$$\leq \frac{64}{\sqrt{3}(n-1)(n-3)a^2} \left(1 - \left(1 + \left(\frac{a^2}{1-a^2}\right)^{\frac{n-1}{2}}\right)^{\frac{p-2}{p}}\right)$$

$$\leq \frac{256\sqrt{3}}{9(n-1)(n-3)} \left(1 - \left(1 + \frac{1}{(1-a^2)^{\frac{n-1}{2}}}\right)^{\frac{p-2}{p}}\right)$$

The second inequality above is obtained by $\frac{\sqrt{3}(n-1)s_{n-1}}{4s_{n-2}} \leq \frac{1}{(\sin\varphi)^{n-1}}$ if $\varphi \in \left(\frac{5\pi}{6}, \pi\right)$ from Lemma 2.5. The third and fifth ones come respectively from Lemma 2.6 and Lemma 2.10. □

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