

ON THE RATE OF CONVERGENCE IN THE CENTRAL LIMIT THEOREM FOR HIERARCHICAL LAPLACIANS[☆]

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Abstract. Let (X, d) be a proper ultrametric space. Given a measure m on X and a function $B \mapsto C(B)$ defined on the set of all non-singleton balls B we consider the hierarchical Laplacian $L = L_C$. Choosing a sequence $\{\varepsilon(B)\}$ of i.i.d. random variables we define the perturbed function $C(B, \omega)$ and the perturbed hierarchical Laplacian $L^\omega = L_{C(\omega)}$. We study the arithmetic means $\bar{\lambda}(\omega)$ of the L^ω -eigenvalues. Under certain assumptions the normalized arithmetic means $(\bar{\lambda} - \mathbb{E}\bar{\lambda}) / \sigma(\bar{\lambda})$ converge in law to the standard normal distribution. In this note we study convergence in the total variation distance and estimate the rate of convergence.

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1. INTRODUCTION

The concept of hierarchical lattices and hierarchical distances was proposed by F.J. Dyson in his famous paper on the phase transition for **1D** ferromagnetic model with long range interaction [11]. The notion of the hierarchical Laplacian L , which is closely related to the Dyson's model was studied in several mathematical papers [2, 3, 6, 7, 14, 15, 16, 17]. These papers contain some basic information about L (the spectrum, the Markov semigroup, resolvent etc). When the space and the hierarchical lattice satisfy some symmetry conditions (homogeneity, self-similarity etc.) the space can be identified with a totally disconnected, locally compact Abelian group G equipped with a translation invariant ultrametric d and with a Haar measure m . The Markov semigroup $P^t = \exp(-tL)$ acting on $L^2(G, m)$ becomes then symmetric, translation invariant and isotropic. In particular, $\text{Spec}(L)$ is pure point and all eigenvalues have infinite multiplicity.

The paper [4] studies a class of random perturbations of the hierarchical Laplacian L . Each outcome $L^\omega, \omega \in \Omega$, of the perturbed hierarchical Laplacian is by itself a hierarchical Laplacian whence its spectrum $\text{Spec}(L^\omega)$ is still pure point (with compactly supported eigenfunctions). Using the classical average procedure one defines the integrated density of states. Contrary to the deterministic case it may admit a continuous density w.r.t. m , the density of states. The density of states detects the spectral bifurcation from the pure point spectrum to

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the continuous one. The eigenvalues form locally a Poisson point process with intensity given by the density of states. The normalized sequence of arithmetic means of L^ω -eigenvalues converges in law to the standard normal distribution. In this note we study the convergence in relative entropy, in particular in the total variation distance. Under certain mild assumptions we establish the rate of convergence.

2. PRELIMINARIES

2.1. Hierarchical lattice.

Let (X, d) be a proper non-compact ultrametric space. Recall that it means that all closed balls are compact, and that d is an *ultrametric*, that is

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

A basic consequence is that any two balls are either disjoint or one is contained in the other. The collection of all balls with a fixed positive radius forms a countable partition of X , and decreasing the radius leads to a refined partition. This is consistent with the structure of the “hierarchical lattice” as in the old papers, going back to [11].

Let m be a Radon measure on X such that $m(X) = \infty$ and $m(B) > 0$ for each closed ball which is not a singleton, and $m(\{a\}) > 0$ if and only if a is an isolated point of X . Let \mathcal{B} be the collection of all balls with $m(B) > 0$. Each $B \in \mathcal{B}$ has a unique *predecessor* or *parent* $B' \in \mathcal{B} \setminus \{B\}$ which contains B and is such that $B \subseteq D \subseteq B'$ for $D \in \mathcal{B}$ implies $D \in \{B, B'\}$. In this case, B is called a *successor* of B' . Since X is proper, each non-singleton ball has only finitely many (and at least 2) successors. Their number is the *degree* of the ball.

2.2. Hierarchical Laplacian.

We consider a function $C : \mathcal{B} \rightarrow (0, \infty)$ which satisfies, for all $B \in \mathcal{B}$ and all non-isolated $a \in X$,

$$\lambda(B) := \sum_{D \in \mathcal{B} : D \supseteq B} C(D) < \infty, \quad \text{and} \quad \sum_{B \in \mathcal{B} : B \ni a} C(B) = \infty. \quad (2.1)$$

Let \mathcal{F} be the set of all locally constant functions having compact support. It is known that \mathcal{F} consists of continuous functions and is dense in all $L^p(X, m)$. Given the space X , the measure m and the function $C : \mathcal{B} \rightarrow (0, \infty)$, we define (pointwise) the hierarchical Laplacian L_C : for each f in \mathcal{F} and $x \in X$ we set

$$L_C f(x) := \sum_{B \in \mathcal{B} : B \ni x} C(B) \left(f(x) - \frac{1}{m(B)} \int_B f \, dm \right).$$

The operator (L_C, \mathcal{F}) acts in $L^2(X, m)$, is symmetric and admits a complete system of eigenfunctions $\{f_B : B \in \mathcal{B}\}$ given by

$$f_B = \frac{\mathbf{1}_B}{m(B)} - \frac{\mathbf{1}_{B'}}{m(B')}.$$

The eigenvalue corresponding to f_B depends only on B' and is $\lambda(B')$, as given in (2.1). Since all f_B belong to \mathcal{F} and the system $\{f_B : B \in \mathcal{B}\}$ is complete we conclude that (L_C, \mathcal{F}) is an essentially self-adjoint operator. By a slight abuse of notation, we shall write (L_C, Dom_{L_C}) for its unique self-adjoint extension. For all of this we refer to [5, 6, 7].

2.3. Homogeneous hierarchical Laplacian.

For the analysis undertaken in this paper, we require that the ultrametric measure space (X, d, m) and the hierarchical Laplacian L_C are *homogeneous*, that is there exists a group of isometries of (X, d) which

- acts transitively on X , and
- leaves both the reference measure m and the function $C(B)$ invariant.

The first assumption implies that (X, d) is either discrete or perfect. Basic examples which we have in mind are

1. $X = \mathbb{Q}_p$ – the ring of p -adic numbers, where $p \geq 2$ (integer).
2. $X = \bigoplus_{j=1}^{\infty} \mathbb{Z}/p_j \mathbb{Z}$ – the direct sum of countably many cyclic groups.
3. $X = S_{\infty}$ – the infinite symmetric group, that is, the group of all permutations of the positive integers that fix all but finitely many elements.

The homogeneity assumptions and the fact that X is non-compact imply that we have the following two cases.

Case 1. X is perfect, and $\{d(x, y) : y \in X\} = \{0\} \cup \{r_k : k \in \mathbb{Z}\}$, where $r_k < r_{k+1}$ with $\lim_{k \rightarrow \infty} r_k = \infty$ and $\lim_{k \rightarrow -\infty} r_k = 0$;

Case 2. X is countable, and $\{d(x, y) : y \in X\} = \{r_k : k \in \mathbb{N}_0\}$, where $r_0 = 0$, $r_k < r_{k+1}$ with $\lim_{k \rightarrow \infty} r_k = \infty$.

In both cases, we let \mathcal{B}_k be the collection of all closed balls of diameter r_k . This is a partition of X , and it is finer than \mathcal{B}_{k+1} . By homogeneity, all balls in \mathcal{B}_k are isometric. In particular, the number n_k of successor balls is the same for each ball in \mathcal{B}_k , where $k \in \mathbb{Z}$ in Case 1, and $k \in \mathbb{N}_0$ in Case 2. We notice that the degree sequence (n_k) satisfies $2 \leq n_k < \infty$.

It is useful to associate an infinite *tree* with X , see Figure 1. Its vertex set is \mathcal{B} , and there is an edge between any $B \in \mathcal{B}$ and its predecessor B' . In this situation, \mathcal{B}_k is the *horocycle* H_k of the tree with index k , and X is the (lower) *boundary* of that tree. For more details see [6, 9, 10].

For having homogeneity, the reference measure m is also uniquely defined up to a constant factor. If we set $m(B) = 1$, for each $B \in \mathcal{B}_0$, then, for any $k \in \mathbb{Z}$ (Case 1), resp. $k \in \mathbb{N}_0$ (Case 2), and for $B \in \mathcal{B}_k$,

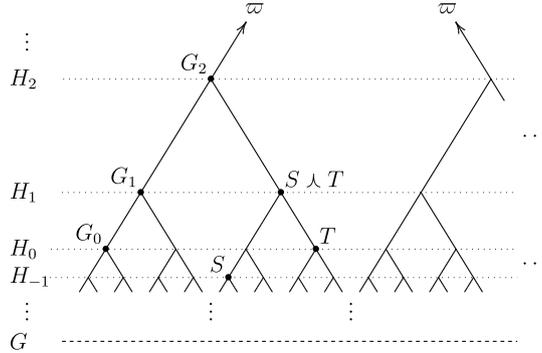
$$m(B) = \nu_k, \quad \text{where} \quad \nu_k = \begin{cases} n_1 n_2 \dots n_k & \text{for } k > 0, \text{ and} \\ 1/(n_{k+1} n_{k+2} \dots n_0) & \text{for } k < 0 \text{ in Case 1.} \end{cases}$$

This determines m uniquely as a measure on the Borel σ -algebra of X . Note that $\nu_k \geq 2^k$ for $k > 0$. Regarding the hierarchical Laplacian, homogeneity means that $C(B) = C_k$ is the same for each $B \in \mathcal{B}_k$. Along with the function $C(B)$, also the eigenvalues of (2.1) depend only on k :

$$\lambda(B) = \lambda_k \text{ for all } B \in \mathcal{B}_k, \quad \text{where} \quad \lambda_k = \sum_{\ell \geq k} C_{\ell}.$$

As noticed in [9, 10], the homogeneous ultrametric measure space (X, m) can then be identified with a locally compact totally disconnected group G equipped with its Haar measure. In fact, we may even identify it with an abelian group. If (r_k) is the sequence of distances defined above then $G_k = B(e, r_k)$ is a compact-open subgroup of $G \equiv X$,

$$G = \bigcup_k G_k, \quad \text{and} \quad n_k = [G_k : G_{k-1}]$$

FIGURE 1. Tree of balls $\mathcal{T}(X)$ with forward degree $n_l = 2$.

gives the degree sequence. The collection \mathcal{B}_k of balls with diameter r_k consists of all cosets of G_k in G . We usually normalize the Haar measure m such that $m(G_0) = 1$.

2.4. Random perturbations.

Let L_C be the homogeneous hierarchical Laplacian. Let $\{\varepsilon(B)\}_{B \in \mathcal{B}}$ be a sequence of symmetric i.i.d. random variables defined on the probability space (Ω, \mathbb{P}) and taking values in some small interval $[-\epsilon, \epsilon] \subset (-1, 1)$. We define the perturbed function $C(B, \omega)$ and the perturbed hierarchical Laplacian as follows:

$$C(B, \omega) = C(B)(1 + \varepsilon(B, \omega))$$

and

$$L^\omega f(x) = L_{C(\omega)} f(x) = \sum_{B \in \mathcal{B}: x \in B} C(B, \omega) \left(f(x) - \frac{1}{m(B)} \int_B f \, dm \right).$$

Evidently L^ω may well be non-homogeneous for some $\omega \in \Omega$. Still it has a pure point spectrum for all ω but the structure of the closed set $\text{Spec}(L^\omega)$ can be quite complicated, see [7] for various examples.

Let us fix a horocycle H and compute the eigenvalue $\lambda(B, \omega)$ for B in H . Without loss of generality we may assume that $H = H_0$. Let ϖ be the Alexandrov point and let $\{B_k\}_{k \geq 0}$ be the unique infinite geodesic path in $\mathcal{T}(X)$ from B to ϖ . We have

$$\begin{aligned} \lambda(B, \omega) &= \sum_{k \geq 0} C(B_k, \omega) = \sum_{k \geq 0} C_k (1 + \varepsilon(B_k, \omega)) \\ &= \lambda_0 \left(1 + \sum_{k \geq 0} a_k \varepsilon(B_k, \omega) \right) = \lambda_0 (1 + U(B, \omega)), \end{aligned}$$

where $a_k = C_k / \lambda_0$, and

$$U(B, \omega) = \sum_{k \geq 0} a_k \varepsilon(B_k, \omega). \quad (2.2)$$

Notice that $\sum_{k \geq 0} a_k = 1$ and that $\{U(B)\}_{B \in H}$ are (dependent) identically distributed symmetric random variables taking values in $[-\epsilon, \epsilon] \subset (-1, 1)$. In particular, $\mathbb{E}\lambda(B, \cdot) = \lambda_0$.

2.5. Normal approximation.

Let us choose a reference point $o \in X$, say the neutral element in our group-identification, and let \mathcal{O} denote the family of all balls O centred at o . Let $\mathcal{B}_0(O)$, $O \in \mathcal{O}$, be the set of all balls $B \subset O$ each of which belongs to the horocycle H_0 . We set

$$\bar{\lambda}_O(\omega) = \frac{1}{|\mathcal{B}_0(O)|} \sum_{B \in \mathcal{B}_0(O)} \lambda(B, \omega)$$

and¹

$$\Lambda_O(\omega) = \frac{\bar{\lambda}_O(\omega) - \lambda_0}{\sigma(\bar{\lambda}_O)},$$

where $|\mathcal{B}_0(O)|$ stands for cardinality of the finite set $\mathcal{B}_0(O)$.

It was recently proved that Λ_O converges in law to the standard normal random variable Z , as $O \rightarrow \varpi$, whenever the following condition holds

$$1/\kappa \leq C(B) (\text{diam}(B))^{\delta/2} \leq \kappa, \quad (2.3)$$

or equivalently,

$$1/2\kappa \leq \lambda(B) (\text{diam}(B))^{\delta/2} \leq 2\kappa, \quad (2.4)$$

for some $\kappa > 0$ and $\delta \geq 1$, see [4], Theorem 3.2.

The main aim of this paper is to strengthen this result, namely we want to prove that under certain conditions the convergence $\Lambda_O \rightarrow Z$ holds in the total variation distance, see Theorem 3.1.

2.6. Metric matters.

Given two probability distributions P and Q on the real line the total variation distance from P to Q is defined as

$$\|P - Q\|_{TV} = 2 \sup_{A \in \mathcal{B}(\mathbb{R})} |P(A) - Q(A)|.$$

Notice that if P and Q are absolutely continuous with respect to some measure ν and f and g are their densities then

$$\|P - Q\|_{TV} = 2\|f - g\|_{L^1(\nu)}.$$

The relative entropy (also called the Kullback-Leibler distance) of P with respect to Q is defined as

$$D(P \| Q) = \int_{\mathbb{R}} \log_2 \left(\frac{dP}{dQ} \right) dP,$$

when P is absolutely continuous with respect to Q and $D(P \| Q) = +\infty$ otherwise.

¹We use both $\text{Var}(X)$ and $\sigma(X)^2$ to denote the variance of the random variable X .

Let X be a random variable with the density p with respect to the Lebesgue measure. Its differential entropy is defined as

$$h(X) = - \int_{\mathbb{R}} p(x) \log_2 p(x) dx.$$

We also consider the quantity

$$D(X) = h(W) - h(X) = \int_{\mathbb{R}} p(x) \log_2 \frac{p(x)}{q(x)} dx,$$

where W is a normal random variable, $q(x)$ its density, $\mathbb{E}(W) = \mathbb{E}(X)$ and $\text{Var}(W) = \text{Var}(X)$. Observe that $D(X) = D(P_X \| P_W)$, where P_X denotes the distribution of the random variable X . We have $D(a + bX) = D(X)$. Moreover, if $P_X = N(0, \sigma^2)$ then

$$h(X) = \frac{\log_2(2\pi e\sigma^2)}{2} \quad \text{and} \quad D(X) = 0. \quad (2.5)$$

We recall that by the Pinsker inequality [18] we have

$$\|P_X - P_W\|_{TV}^2 \leq 2D(X).$$

We shall need the following bound on the relative entropy expressed in terms of densities ([8], Lem. 2.2). Let X be a random variable whose distribution function $F_X(x)$ is absolutely continuous with $F'_X(x) = p(x)$, assume further that the first absolute moment of X is finite. Let Z be the standard normal random variable and $F'_Z(x) = \phi(x)$ be its density. For any $T \geq 0$ we have

$$D(P_X \| P_Z) \leq e^{-T^2/2} + \sqrt{2\pi} \int_{-T}^T (p(x) - \phi(x))^2 e^{x^2/2} dx + \frac{1}{2} \int_{|x| \geq T} x^2 p(x) dx + \int_{|x| \geq T} p(x) \log_2 p(x) dx. \quad (2.6)$$

2.7. Berry-Essen bounds.

Let X_1, X_2, \dots, X_n be independent centred random variables. Assuming that for some $s > 2$ and all $1 \leq k \leq n$, $\mathbb{E}|X_k|^s < \infty$. Let us denote

$$\sigma_k^2 = \text{Var}(X_k), \quad B_n = \sum_{k=1}^n \sigma_k^2, \quad S_n = \frac{X_1 + \dots + X_n}{B_n^{1/2}},$$

and

$$\mathbb{L}_s = \frac{1}{B_n^{s/2}} \sum_{k=1}^n \mathbb{E}|X_k|^s.$$

Let P_n be the distribution of the random variable S_n and $N(0, 1)$ be the standard normal distribution. One of the main ingredients in our analysis is the following theorem, see [8], Theorems 1.1 and 1.2.

Theorem 2.1. *Assume that $D(X_k) \leq A$ for some positive constant A , and all $1 \leq k \leq n$. The following statements hold true*

1. $\mathbb{L}_3 < +\infty$ implies $\|P_n - N(0, 1)\|_{TV} \leq C\mathbb{L}_3$,

2. $\mathbb{L}_4 < +\infty$ implies $D(S_n) \leq C'\mathbb{L}_4$,

where the constants C and C' depend only on A .

We shall need the following technical result which we prove here for completeness.

Lemma 2.2. *Assume that X, Y are independent and $\text{Var}(X) + \text{Var}(Y) = 1$, then*

$$D(X + Y) \leq \text{Var}(X)D(X) + \text{Var}(Y)D(Y).$$

In particular, for any collection of independent random variables $\{X_k\}$,

$$D\left(\sum_{k=1}^n X_k\right) \leq \max_{1 \leq k \leq n} D(X_k).$$

Proof. We apply the following inequality, see [12], Theorem D.1,

$$2^{2h(X)} + 2^{2h(Y)} \leq 2^{2h(X+Y)}. \quad (2.7)$$

According to (2.5), we have

$$D(X + Y) = h(Z) - h(X + Y) \quad \text{and} \quad h(Z) = \log_2 \sqrt{2\pi e}.$$

Similarly, we write

$$D(X) = h(Z_1) + h(X) \quad \text{and} \quad D(Y) = h(Z_2) + h(Y),$$

where Z_1 and Z_2 are normal random variables such that $\text{Var}(Z_1) = \text{Var}(X)$ and $\text{Var}(Z_2) = \text{Var}(Y)$. Using this in (2.7) we get

$$2^{h(Z_1)} \cdot 2^{-2D(X)} + 2^{h(Z_2)} \cdot 2^{-2D(Y)} \leq 2^{2h(Z_1+Z_2)} \cdot 2^{-2D(X+Y)},$$

whence

$$\text{Var}(X) \cdot 2^{-2D(X)} + \text{Var}(Y) \cdot 2^{-2D(Y)} \leq 2^{-2D(X+Y)}.$$

Since $x \mapsto 2^{-2x}$ is convex, we get

$$2^{-2[\text{Var}(X)D(X)+\text{Var}(Y)D(Y)]} \leq \text{Var}(X) \cdot 2^{-2D(X)} + \text{Var}(Y) \cdot 2^{-2D(Y)}.$$

The result follows. □

3. CENTRAL LIMIT THEOREM

The main result of the article is the following theorem.

Theorem 3.1. *In notation of Section 2, suppose that condition (2.3) holds with $\delta \geq 1$. Assume that the i.i.d. random variable $\epsilon(B)$ has a bounded density with respect to the Lebesgue measure. For $O \in H_N$ the following inequality holds*

$$\|P_{\Lambda_O} - N(0, 1)\|_{TV} \leq \begin{cases} C N^{-\frac{1}{2}} & \text{for } \delta = 1, \\ C v_N^{-\frac{3}{2} \min\{\delta-1, \frac{1}{3}\}} & \text{for } \delta > 1, \end{cases}$$

for some $C > 0$ and all $N \geq 1$.

Proof. For $O \in H_N$ we write $\bar{\lambda}_N(\omega)$ for $\bar{\lambda}_O(\omega)$ and $\Lambda_N(\omega)$ for $\Lambda_O(\omega)$, then

$$\begin{aligned}\bar{\lambda}_N(\omega) &= \lambda_0 (1 + \bar{U}_N(\omega)), \\ \bar{U}_N(\omega) &= \frac{1}{v_N} \sum_{B \in \mathcal{B}_0(O)} U(B, \omega), \\ \Lambda_N(\omega) &= \frac{\bar{U}_N(\omega)}{\sigma(\bar{U}_N)}.\end{aligned}\tag{3.1}$$

To estimate the quantity $\|P_{\Lambda_O} - \mathcal{N}\|_{TV}$ we apply Theorem 2.1. We distinguish two cases ($\delta > 1$) and ($\delta = 1$).

The case ($\delta > 1$). Let $\{O_k\}_{k \geq N}$ be the infinite geodesic path from O to ϖ . Similarly, for each $B \in H_0$ we pick the infinite geodesic path $\{B_k\}_{k \geq 0}$ from B to ϖ . Applying (2.2) to equation (3.1) we obtain

$$\begin{aligned}\bar{U}_N &= \frac{1}{n_1 \dots n_N} \sum_{B \in H_0: B \subset O} \sum_{k \geq 0} a_k \varepsilon(B_k) = \frac{1}{n_1 \dots n_N} \sum_{k \geq 0} a_k \sum_{B \in H_0: B \subset O} \varepsilon(B_k) \\ &= \frac{1}{n_1 \dots n_N} \left(a_0 \sum_{B_0 \in H_0: B_0 \subseteq O} \varepsilon(B_0) + a_1 n_1 \sum_{B_1 \in H_1: B_1 \subseteq O} \varepsilon(B_1) \right. \\ &\quad \left. + a_2 n_1 n_2 \sum_{B_2 \in H_2: B_2 \subseteq O} \varepsilon(B_2) + \dots + a_N n_1 n_2 \dots n_N \varepsilon(O_N) \right) \\ &\quad + a_{N+1} \varepsilon(O_{N+1}) + a_{N+2} \varepsilon(O_{N+2}) + \dots .\end{aligned}$$

Let us introduce two random variables

$$\mathcal{U}_N = \frac{a_0}{n_1 \dots n_N} \sum_{B_0 \in H_0: B_0 \subseteq O} \varepsilon(B_0) + \frac{a_1}{n_2 \dots n_N} \sum_{B_1 \in H_1: B_1 \subseteq O} \varepsilon(B_1) + \dots + a_N \varepsilon(O_N)\tag{3.2}$$

and

$$X_N = a_{N+1} \varepsilon(O_{N+1}) + a_{N+2} \varepsilon(O_{N+2}) + \dots .\tag{3.3}$$

The random variables \mathcal{U}_N and X_N are independent, have zero mean and

$$\bar{U}_N = \mathcal{U}_N + X_N.$$

Let us denote by $\mathcal{B}_k(O)$ the set of all balls $B \subset O$ which belong to the horocycle H_k , and let $\mathcal{B}^*(O) = \bigcup_{k=0}^N \mathcal{B}_k(O)$. Then

$$\Lambda_N = \frac{\sum_{B \in \mathcal{B}^*(O)} X_B + X_N}{B_N^{1/2}},$$

where

$$X_B = \frac{a_k}{n_{k+1} \dots n_N} \varepsilon(B), \text{ for } B \in \mathcal{B}_k(O),$$

and

$$B_N = \sum_{B \in \mathcal{B}^*(O)} \sigma^2(X_B) + \sigma^2(X_N).$$

As $\epsilon(B)$ are bounded i.i.d. having bounded density, $D(X_B) = D(\epsilon(B)) < \infty$ for all balls $B \in \mathcal{B}^*(O)$. We claim that, for some $A > 0$,

$$D(X_N) \leq A, \quad \text{for all } N.$$

Indeed, let $\widetilde{X}_N = X_N / \sqrt{\sigma(X_N)}$. Then $D(X_N) = D(\widetilde{X}_N)$. Random variable \widetilde{X}_N is bounded because, by (2.3),

$$\left| \widetilde{X}_N \right| \leq \frac{\sum_{k>N} a_k}{\sqrt{\sum_{k>N} a_k^2}} \leq C \frac{a_{N+1}}{\sqrt{a_{N+1}^2}} = C < \infty,$$

for some $C > 0$. It follows that the density \widetilde{p}_N of \widetilde{X}_N is continuous and compactly supported function. Applying (2.6) we obtain²

$$D(\widetilde{X}_N) = D(\widetilde{X}_N \| Z) \leq e^{-\frac{c^2}{2}} + \sqrt{2\pi} e^{\frac{c^2}{2}} \|\widetilde{p}_N - \phi\|_2^2,$$

where ϕ is the standard normal density. Let $\Phi_{\widetilde{X}_N}$ and Φ be the characteristic functions of random variables \widetilde{X}_N and Z respectively. The Plancherel formula yields

$$\|\widetilde{p}_N - \phi\|_2^2 = \frac{1}{2\pi} \|\Phi_{\widetilde{X}_N} - \Phi\|_2^2 \leq \frac{1}{\pi} \left(\|\Phi_{\widetilde{X}_N}\|_2^2 + \|\Phi\|_2^2 \right).$$

Let Φ_ε be the characteristic function of $\varepsilon(B)$. We have

$$|\Phi_{\widetilde{X}_N}(\xi)| = \prod_{k=N+1}^{\infty} \left| \Phi_\varepsilon \left(\frac{a_k}{\sqrt{\sigma(X_N)}} \xi \right) \right| \leq \left| \Phi_\varepsilon \left(\frac{a_{N+1}}{\sqrt{\sigma(X_N)}} \xi \right) \right|,$$

whence

$$\begin{aligned} \|\Phi_{\widetilde{X}_N}\|_2^2 &\leq \int \left| \Phi_\varepsilon \left(\frac{a_{N+1}}{\sqrt{\sigma(X_N)}} \xi \right) \right|^2 d\xi = \frac{\sqrt{\sigma(X_N)}}{a_{N+1}} \|\Phi_\varepsilon\|_2^2 = \frac{\sqrt{\sum_{k>N} a_k^2}}{a_{N+1}} \|\Phi_\varepsilon\|_2^2 \\ &\leq C' \|\Phi_\varepsilon\|_2^2, \end{aligned}$$

where C' does not depend on N .

Next we want to estimate the Lyapunov ratio

$$\mathbb{L}_s = \frac{\sum_{B \in \mathcal{B}(O)} \mathbb{E}|X_B|^s + \mathbb{E}|X_N|^s}{B_N^{s/2}}.$$

Since \mathcal{U}_N and X_N are independent,

$$B_N = \sigma(\overline{U}_N)^2 = \sigma(\mathcal{U}_N)^2 + \sigma(X_N)^2.$$

² $\|f\|_2$ is the norm in the space of square integrable functions w.r.t. the Lebesgue measure.

By condition (2.3),³

$$\sigma(X_N)^2 \asymp (n_0 \dots n_N n_{N+1})^{-\delta}$$

and

$$\sigma(\mathcal{U}_N)^2 \asymp \begin{cases} (n_0 \dots n_N)^{-1} & \text{if } \delta > 1, \\ N \cdot (n_0 \dots n_N)^{-1} & \text{if } \delta = 1, \\ (n_0 \dots n_N)^{-\delta} & \text{if } \delta < 1, \end{cases} \quad (3.4)$$

for N large enough, see [4], Claim 1. Hence, for $\delta \geq 1$ and N big enough,

$$B_N \asymp \sigma(\mathcal{U}_N)^2. \quad (3.5)$$

Next, direct computation shows that

$$\begin{aligned} \sum_{B \in \mathcal{B}_0(O)} \mathbb{E}|X_B|^s + \mathbb{E}|X_N|^s &= \left(\frac{a_0}{n_1 \dots n_N} \right)^s \sum_{B_0 \in H_0: B_0 \subset O} \mathbb{E}|\varepsilon(B_0)|^s + \left(\frac{a_1}{n_2 \dots n_N} \right)^s \sum_{B_1 \in H_1: B_1 \subset O} \mathbb{E}|\varepsilon(B_1)|^s \\ &\quad + \dots + a_N^s \mathbb{E}|\varepsilon(O_N)|^s + \mathbb{E} \left| \sum_{k=N+1}^{\infty} a_k \varepsilon(O_k) \right|^s \\ &\asymp \frac{1}{v_N^{s-1}} \left(1 + v_1^{s-1-s\delta/2} + \dots + v_N^{s-1-s\delta/2} \right) \end{aligned}$$

whence

$$\sum_{B \in \mathcal{B}_0(O)} \mathbb{E}|X_B|^s + \mathbb{E}|X_N|^s \asymp \begin{cases} \frac{1}{v_N^{s-1}} & \text{if } s-1-s\delta/2 < 0, \\ \frac{1}{v_N^{s/2-1}} & \text{if } s-1-s\delta/2 \geq 0. \end{cases}$$

Combining this with (3.5) we obtain

$$\mathbb{L}_s \asymp \begin{cases} \frac{1}{v_N^{s/2-1}} & \text{if } s-1-s\delta/2 < 0, \\ \frac{1}{v_N^{s/2(\delta-1)}} & \text{if } s-1-s\delta/2 \geq 0. \end{cases}$$

Thus, for $s = 3$ and $\delta > 1$, we get the desired result.

The case ($\delta = 1$). Let us introduce auxiliary notation

$$\begin{aligned} Y_0 &= \frac{a_0}{n_1 \dots n_N} \sum_{B \in H_0: B \subset O} \varepsilon(B), \\ Y_1 &= \frac{a_1}{n_2 \dots n_N} \sum_{B \in H_1: B \subset O} \varepsilon(B), \\ &\dots \\ Y_N &= a_N \varepsilon(O_N) \quad \text{and} \quad Y_{N+1} = X_N. \end{aligned}$$

³ $f(x) \asymp g(x)$ means that there are some constants $c, C > 0$ such that $f(x) \leq cg(x)$ and $g(x) \leq Cf(x)$.

Clearly, each of Y_k has mean zero and

$$\Lambda_N = \frac{Y_0 + Y_1 + \dots + Y_{N+1}}{B_{N+1}^{1/2}}, \quad B_{N+1} = \sum_{k=0}^{N+1} \sigma(Y_k)^2.$$

We again apply Theorem 2.1 because, by Lemma 2.2,

$$D(Y_k) = D\left(\sum_{B \in H_k: B \subseteq O} \varepsilon(B)\right) \leq D(\varepsilon(B)) < \infty, \quad 0 \leq k \leq N.$$

To estimate the Lyapunov ratios we use the Marcinkiewicz-Zygmund inequality [19], Chapter VII, Section 3, for independent random variables $\xi_1, \xi_2, \dots, \xi_n$ with mean zero and for all $p > 1$ there is a constant $C(p) > 0$ such that

$$\mathbb{E}\left|\sum_{k=1}^n \xi_k\right|^p \leq C(p) \mathbb{E}\left(\sum_{k=1}^n \xi_k^2\right)^{p/2}.$$

Setting

$$\xi_k = \sum_{B \in H_k: B \subseteq O} \varepsilon(B),$$

the Marcinkiewicz-Zygmund inequality yields

$$\mathbb{E}|Y_k|^s = \left(\frac{a_k}{n_{k+1} \dots n_N}\right)^s \mathbb{E}|\xi_k|^s \leq C(s) \text{Var}(\varepsilon(B)) \frac{a_k^s}{(n_{k+1} \dots n_N)^{s/2}}.$$

It follows, see (2.3), that

$$\begin{aligned} \sum_{k=0}^N \mathbb{E}|Y_k|^s &\leq C_1(s) \left(\frac{a_0^s}{(n_1 \dots n_N)^{s/2}} + \frac{a_1^s}{(n_2 \dots n_N)^{s/2}} + \dots + a_N^s\right) \\ &= \frac{C_1(s)}{v_N^{s/2}} \left(a_0^s + a_1^s n_1^{s/2} + \dots + a_N^s v_N^{s/2}\right) \leq C_2(s) \frac{N}{v_N^{s/2}}. \end{aligned}$$

Hence, using (3.5) and (3.4) with $\delta = 1$, we finally get

$$\mathbb{L}_s \leq \frac{C}{N^{s/2-1}}.$$

By Theorem 2.1, the result follows. \square

Remark 3.2. Applying similar reasoning we can also estimate the relative entropy distance

$$D(A_O \| \mathcal{N}) \leq \begin{cases} C N^{-1} & \text{for } \delta = 1, \\ C v_N^{-2 \min\{\delta-1, \frac{1}{2}\}} & \text{for } \delta > 1, \end{cases}$$

for some $C > 0$ and all $N > 1$.

4. AN EXAMPLE

As an example we consider the space $X = \mathbb{Q}_p$ equipped with its standard ultrametric $d(x, y) = |x - y|_p$ and its normalized Haar measure. Let \mathcal{D}^α , $\alpha > 0$, be the homogeneous hierarchical Laplacian uniquely defined by its eigenvalues

$$\lambda(B) = \left(\frac{p}{\text{diam}(B)} \right)^\alpha, \quad B \in \mathcal{B}.$$

Let $\mathcal{D}^\alpha(\omega)$ be its random perturbation by i.i.d. $\{\varepsilon(B)\}_{B \in \mathcal{B}}$ as defined in Section 2. As in the previous section we assume that $\varepsilon(B)$ admits a bounded density. We notice that \mathcal{D}^α satisfies condition (2.4) with $\delta = 2\alpha$. In particular, for any $\alpha \geq 1/2$ the normalized arithmetic means $\Lambda_O(\omega)$ converge as $O \rightarrow \varpi$ to the standard normal random variable Z in the sense of the total variation distance. In this section we study convergence assuming that $0 < \alpha < 1/2$.

Theorem 4.1. *For any $0 < \alpha < 1/2$ there exists a non-gaussian random variable Λ such that the normalized arithmetic means Λ_O converge to Λ in the sense of the total variation distance.*

Proof. Following line-by-line the proof of the Theorem 3.1 we write

$$\sigma(\bar{U}_N)^2 = \sigma(\mathcal{U}_N)^2 + \sigma(X_N)^2.$$

As $\lambda_k = p^{-\alpha(k-1)}$ and $c_k = (p^\alpha - 1)p^{-\alpha k}$, we get

$$a_k = c_k/\lambda_0 = (p^\alpha - 1)p^{-\alpha(k+1)}, \quad k \geq 0.$$

Using the above data and setting $\sigma^2 = \text{Var}(\varepsilon)$ we estimate $\sigma(\mathcal{U}_N)$ and $\sigma(X_N)$ at ∞ as follows

$$\begin{aligned} \sigma(\mathcal{U}_N)^2 &= \sigma^2 \left(\frac{a_0^2}{p^N} + \frac{a_1^2}{p^{N-1}} + \dots + a_N^2 \right) = \sigma^2 (p^\alpha - 1)^2 \sum_{0 \leq l \leq N} p^{-N+l-2\alpha(l+1)} \\ &\sim \frac{\sigma^2 (p^\alpha - 1)^2}{1 - p^{2\alpha-1}} p^{-2\alpha(N-1)} = \frac{\sigma^2}{1 - p^{2\alpha-1}} a_N^2 \end{aligned} \quad (4.1)$$

and

$$\begin{aligned} \sigma(X_N)^2 &= \sigma^2 (a_{N+1}^2 + a_{N+2}^2 + \dots) = \sigma^2 (p^\alpha - 1)^2 \sum_{l \geq N+1} p^{-2\alpha(l+1)} \\ &\sim \frac{\sigma^2 (p^\alpha - 1)^2}{1 - p^{-2\alpha}} p^{-2\alpha(N+2)} = \frac{\sigma^2}{1 - p^{-2\alpha}} a_{N+1}^2. \end{aligned} \quad (4.2)$$

Let $\{\varepsilon_i\}_{i \geq 0}$ be i.i.d. random variables independent of $\{\varepsilon(B)\}_{B \in \mathcal{B}}$ and having the same common distribution as $\{\varepsilon(B)\}_{B \in \mathcal{B}}$. By (3.3) and (4.2) the random variable $X_N/\sigma(X_N)$ converges in law to the random variable

$$X = \sqrt{1 - p^{-2\alpha}} \left(\frac{\varepsilon_0}{\sigma(\varepsilon_0)} + p^{-\alpha} \frac{\varepsilon_1}{\sigma(\varepsilon_2)} + \dots + p^{-k\alpha} \frac{\varepsilon_k}{\sigma(\varepsilon_k)} + \dots \right).$$

Let $\{\varepsilon_{ij}\}_{i,j \geq 0}$ be i.i.d. random variables independent of both $\{\varepsilon_i\}_{i \geq 0}$ and $\{\varepsilon(B)\}_{B \in \mathcal{B}}$ and having the same common distribution as $\{\varepsilon(B)\}_{B \in \mathcal{B}}$. Define the random variables

$$S_k = \sum_{0 \leq j \leq p^k} \varepsilon_{kj}, \quad k = 0, 1, 2, \dots$$

By (3.2) and (4.1) the random variable $\mathcal{U}_N/\sigma(\mathcal{U}_N)$ converges in law to the random variable

$$U = \sqrt{1 - p^{2\alpha-1}} \sum_{k \geq 0} p^{(2\alpha-1)k/2} \frac{S_k}{\sigma(S_k)}.$$

Finally, the random variable

$$A_N = \frac{\bar{U}_N}{\sigma(\bar{U}_N)} = \frac{\sigma(\mathcal{U}_N)}{\sigma(\bar{U}_N)} \frac{\mathcal{U}_N}{\sigma(\mathcal{U}_N)} + \frac{\sigma(X_N)}{\sigma(\bar{U}_N)} \frac{X_N}{\sigma(X_N)}$$

converges in law to the random variable

$$A = \sqrt{\frac{1 - p^{-2\alpha}}{1 - p^{-1}}} U + \sqrt{\frac{p^{-2\alpha} - p^{-1}}{1 - p^{-1}}} V.$$

By Cramér's theorem U and V , and therefore A are not Gaussian.

We claim that A_N converges to A in the total variation distance. To prove the claim we work with characteristic functions. As before for a random variable X we denote by $\Phi_X(\xi) = \mathbb{E}(\exp(i\xi X))$ its characteristic function. The following inequality holds

$$\left| \Phi_{A_N} \left(\frac{\sigma(\bar{U}_N)}{a_{N+1}} \xi \right) \right| \leq |\Phi_\varepsilon(\xi)| |\Phi_\varepsilon(p^{-\alpha}\xi)|. \quad (4.3)$$

Equations (4.1) and (4.2) yield

$$A_N = \frac{\sigma(\bar{U}_N)}{a_{N+1}} \rightarrow \left(\frac{\sigma^2(1 - p^{-1})}{(p^{-2\alpha} - p^{-1})(1 - p^{-2\alpha})} \right)^{1/2} := A.$$

Moreover $\Phi_{A_N}(\xi) \rightarrow \Phi_A(\xi)$ pointwise. As ε has a bounded density, $\Phi_\varepsilon \in L^2(\mathbb{R})$. Therefore the function in the right-hand side of (4.3) is in $L^1(\mathbb{R})$. This in turn implies that $\Phi_{A_N}(\xi) \rightarrow \Phi_A(\xi)$ in $L^1(\mathbb{R})$. It follows that the density p_N of A_N converges pointwise to the density p_A of A . Finally Scheffé's lemma [21], section 2.9, yields that $A_N \rightarrow A$ in the total variation distance, as desired. \square

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