

A MULTI-DIMENSIONAL CENTRAL LIMIT BOUND AND ITS APPLICATION TO THE EULER APPROXIMATION FOR LÉVY-SDES

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Abstract. In the one-dimensional case Rio (*Ann. Inst. Henri Poincaré Probab. Stat.* **45** (2009) 802–817) gave a concise bound for the central limit theorem in the Vaserstein distances, which is a ratio between some higher moments and some powers of the variance. As a corollary, it gives an estimate for the normal approximation of the small jumps of Lévy processes, and Fournier (*ESAIM: PS* **15** (2011) 233–248) applied that to the Euler approximation of stochastic differential equations driven by the Lévy noise. It will be shown in this article that following Davie’s idea in (Polynomial Perturbations of Normal Distributions. Available at: www.maths.ed.ac.uk/~sandy/polg.pdf (2016)), one can generalise Rio’s result to the multidimensional case, and have higher-order approximation *via* the perturbed normal distributions, if Cramér’s condition and a slightly stronger moment condition are assumed. Fournier’s result can then be partially recovered.

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1. INTRODUCTION

Given¹ $d, q, q_1 \in \mathbb{Z}^+$, let $a \in \mathbb{R}^q$, $B \in \mathbb{R}^{q \times q_1}$ and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ generated by a q_1 -dimensional Wiener process $\{W_t\}$ and an independent Poisson random measure $N(dz, ds)$ on $\mathbb{R}^q \setminus \{0\} \times [0, \infty)$ with intensity $\nu(dz)ds$. Consider the q -dimensional Lévy process on a bounded interval $[0, T]$:

$$Z_t = at + BW_t + \int_0^t \int_{\mathbb{R}^q \setminus \{0\}} z \tilde{N}(dz, ds), \quad (1.1)$$

where $\tilde{N}(dz, ds)$ is the compensated Poisson measure. Assume the second moment of the Lévy measure $\int_{\mathbb{R}^q \setminus \{0\}} |z|^2 \nu(dz) < \infty$, where $|\cdot|$ denotes the modulus. For $x_0 \in \mathbb{R}^q$ and a bounded Lipschitz function

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¹Throughout this article \mathbb{Z}^+ and \mathbb{N} denote the sets of positive and non-negative integers, respectively.

$\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times q}$, consider the d -dimensional SDE driven by the Lévy process above:

$$x_t = x_0 + \int_0^t \sigma(x_{s-}) dZ_s. \quad (1.2)$$

For $h \in (0, 1)$ and $t_k = kh$, $k = 1, \dots, [T/h]$, where $[\cdot]$ denotes the integer part, it is known that the standard Euler's approximation,

$$X_{k+1} := X_k + \sigma(X_k) (Z_{t_{k+1}} - Z_{t_k}), X_0 = x_0,$$

converges as $h \rightarrow 0$, with rate $1/2$, to the solution of (1.2) in mean-square: there exists a constant C s.t. $\forall h \in (0, 1)$,

$$\left(\mathbb{E} \max_k |x_{t_k} - X_k|^2 \right)^{1/2} \leq Ch^{1/2}.$$

See *e.g.* [12–14]. Although the increments $Z_{t_k} - Z_{t_{k-1}}$ are hard to generate, one may simply ignore the small jumps

$$Z_t^\epsilon := \int_0^t \int_{0 < |z| \leq \epsilon} z \tilde{N}(dz, ds), \quad (1.3)$$

for some $\epsilon \in (0, 1)$, and show that the mean-square convergence rate is preserved. However, that is not a very economical way of simulation, as pointed out by Fournier [9]. Indeed, when the small jumps are completely ignored, the expected computational cost

$$E_\nu(h) = O(h^{-1} + \nu(\{|z| > \epsilon\})),$$

can be considerably large. This happens, *e.g.*, when the Lévy measure ν behaves like α -stable near 0, *i.e.* there exist $\tau > 0$ and $\alpha \in (0, 2)$ s.t. the following condition holds:

Assumption $\mathbf{H}(\tau, \alpha)$. $\nu(dz) \simeq |z|^{-q-\alpha} dz$, $\forall 0 < |z| \leq \tau$.

The symbol \simeq is used where both sides are bounded by each other up to a constant factor depending only on q . Given condition $\mathbf{H}(\tau, \alpha)$, the set of big jumps has measure $\nu(\{|z| > \epsilon\}) \simeq \epsilon^{-\alpha}$, and one has to choose $\epsilon = h^{1/(2-\alpha)}$ to ensure the order $1/2$ of mean-square convergence. As a result $E_\nu(h) = O(h^{-1} + h^{\alpha/(\alpha-2)})$ explodes when α is close to 2.

As a remedy, one may consider approximating the small jumps (1.3) with a normal random variable using the central limit theorem, on which some classical theorems can be found in several books such as [2, 16]. Asmussen and Rosiński [1] adopted this idea and derived some Berry-Esseen bounds for the normal approximation of the small jumps Z_1^ϵ ; they also gave conditions for the weak convergence in the Skorohod space. But their method only works for $q = 1$, and the Berry-Esseen-type bounds are not very useful for the strong L^p -approximation of Lévy-SDEs as they only concern the uniform distance between the c.d.f.'s. Aiming at the Euler approximation of (1.2), Fournier [9] proved that by adding this normal random variable to the Euler scheme the expected computational cost can be controlled (no explosion of $E_\alpha(h)$ near $\alpha = 2$), while the $1/2$ convergence rate is still preserved. However, as pointed out himself, the method is also restricted to the case $q = 1$.

Such a restriction of dimension only emerged at a key step in [9] (Cor. 4.2), borrowed from a result by Rio [17] (Cor. 4.2) on the central limit theorem. The latter ensures that, for a sequence of i.i.d., mean-0 random

variables² $X_j \in \mathbb{R}$ and $Y_m := m^{-1/2} \sum_{j=1}^m X_j$ for any $m \in \mathbb{Z}^+$, there is an absolute constant C s.t.

$$\mathbb{W}_2(\mathbb{P}_m, \mathcal{N}(0, \text{var}X)) \leq C \left(\frac{\mathbb{E}|X|^4}{\text{var}X} \right)^{\frac{1}{2}} m^{-\frac{1}{2}}, \quad (1.4)$$

where \mathbb{P}_m denotes the distribution of Y_m and $\mathbb{W}_p(\cdot, \cdot)$ is the p -Vaserstein (or ‘‘Wasserstein’’) distance. For probability measures \mathbb{P}, \mathbb{Q} on \mathbb{R}^q , such a distance is defined by

$$\mathbb{W}_p(\mathbb{P}, \mathbb{Q}) := \inf_{\pi \in \Pi(\mathbb{P}, \mathbb{Q})} \left(\int_{\mathbb{R}^q \times \mathbb{R}^q} |x - y|^p \pi(\mathrm{d}x, \mathrm{d}y) \right)^{\frac{1}{p}},$$

where $\Pi(\mathbb{P}, \mathbb{Q})$ is the set of all joint probability measures on $\mathbb{R}^q \times \mathbb{R}^q$ with marginal laws \mathbb{P} and \mathbb{Q} . Rio [17] (Thm. 4.1) in fact only assumed the independence of $\{X_j\}$, but regarding central limit approximations and the simulation of Lévy processes one only considers the i.i.d. case. The constant C in (1.4) would vary in p for a bound in \mathbb{W}_p and is later optimised in [18]. Apart from the restriction $q = 1$, Rio’s effective bounds only hold for $p \leq 4$. But this has been improved by Bobkov [3] (Thm. 1.1), allowing the \mathbb{W}_p -convergence of order $O(m^{-1/2})$ for any $p \geq 1$, assuming $\mathbb{E}|X|^{p+2} < \infty$.

The dimensional restriction in Rio and Bobkov’s results comes from the fact that when $q = 1$, for $p \geq 1$ the \mathbb{W}_p distance between two probability measures \mathbb{P}, \mathbb{Q} on \mathbb{R} is explicitly given (see Thm. 2.18 and Rem. 2.19 in [20]):

$$\mathbb{W}_p(\mathbb{P}, \mathbb{Q}) = \left(\int_0^1 |F^{-1}(t) - G^{-1}(t)|^p \mathrm{d}t \right)^{\frac{1}{p}}, \quad (1.5)$$

where $p \geq 1$, F, G are the c.d.f.’s of \mathbb{P}, \mathbb{Q} , and F^{-1}, G^{-1} are their generalised inverses, respectively. For $p = 1$ there is a further equality $\mathbb{W}_1(\mathbb{P}, \mathbb{Q}) = \int_{\mathbb{R}} |F(x) - G(x)| \mathrm{d}x$; in general there is no explicit formula for $q \geq 2$. However, if two probability distributions \mathbb{P} and \mathbb{Q} on \mathbb{R}^q have densities f and g , respectively, instead of the precise formula (1.5) there is the inequality

$$\mathbb{W}_p(\mathbb{P}, \mathbb{Q}) \leq C_p \left(\int_{\mathbb{R}^q} |x|^p |f(x) - g(x)| \mathrm{d}x \right)^{\frac{1}{p}}, \quad (1.6)$$

for all $p \geq 1$, as a variant of Proposition 7.10 in [20].

This article presents an attempt to handle the normal approximation for the small jumps (1.3) for $q \geq 2$ using the bound (1.6), and give a positive answer to Fournier’s question.

Davie [7] sketched an asymptotic approach via Edgeworth expansion of the density of Y_m , and proved (as a corollary to Prop. 2 therein) the rate $O(m^{-1/2})$ under the assumption that all moments of X are bounded. Moreover, he in fact showed a coupling between Y_m and the normal distribution perturbed by polynomials. Section 1 of this article basically follows Davie’s approach, but expounds detailed calculations and specify the range of p and precisely how many moments of X are needed – see Theorem 2.7 below.

The rate of convergence for the multi-dimensional central limit theorem has been studied using different methods. A strong result by Zaitsev (summarised as Thm. 2 in [22] and proved as Thm. 1.3 in [21]) gives a sharp Chernoff-type bound, and by Chebyshev’s inequality the central limit theorem follows in a stronger sense: for independent $\{X_j\}$ each having identity covariance and independent standard Gaussian $\{\xi_j\}$ with partial sums $\mathcal{Y}_m := m^{-1/2} \sum_{j=1}^m \xi_j$, if the law of each X_j satisfies certain analyticity conditions (see the definition of the class $\mathcal{A}_q(\tau)$ in [22]), then the distance $\max_{k \leq m} |Y_k - \mathcal{Y}_k|$ is of order $O(m^{-1/2} \log m)$ in probability. The

²When only the distribution of the X_j ’s is considered, the subscript j is omitted for simplicity.

logarithmic factor emerges because the method is based on the dyadic approximation by Komlós, Major and Tusnády (KMT) [15]. The KMT method is much stronger than the usual central limit theorem since it considers the simultaneous approximation between Y_1, Y_2, \dots, Y_m and $\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_m$. Einmahl [8] generalised the original KMT method to the multi-dimensional case, and Zaitsev's theorem [21] is an improved version of that, albeit it requires the local existence of the moment generating function.

Since the central limit theorem only concerns the coupling between Y_m and \mathcal{Y}_m , one should expect the $\log m$ factor to be removed as in the one-dimensional result of Rio. This has indeed been achieved by Bobkov [4] (Thm. 6.1) under the assumption that $\mathbb{E}|X|^5 < \infty$; given only $\mathbb{E}|X|^4 < \infty$, his result is weakened to $O(m^{-1/2}(\log m)^{q/4-1})$. It is worth mentioning here that, shortly after this article had been submitted, using Stein's method Bonis [5] (Thm. 8) managed to achieve the optimal rate $O(m^{-1/2})$ given only $\mathbb{E}|X|^4 < \infty$, which is a significant improvement. However, both approaches only work for $p = 2$ since their arguments rely on some entropic transport inequalities for the W_2 distance. In this special case (normal approximation for Y_m in W_2) the result derived in this article is not optimal, as it requires $\mathbb{E}|X|^{4+\tau} < \infty$ for some $\tau \in (0, 1)$ and Cramér's condition $\overline{\lim}_{|s| \rightarrow \infty} |\mathbb{E} \exp(isX)| < 1$.

Nevertheless, given that $\mathbb{E}|X|^{6+\tau} < \infty$ and Cramér's condition, the result here would give a coupling for Y_m of order $O(m^{-1})$ in W_p for a positive even integer p . The proof is based on the Edgeworth expansion, which Bobkov used in [3] (Cor. 9.2) to find a closer coupling Y_m in the one-dimensional case: he achieved W_p bounds of order $O(m^{-\eta})$ with $\eta > 1/2$, but in return Cramér's condition and some higher moments (depending on p, η) are needed. Theorem 2.7 here can be regarded as a generalisation of that.

In Section 2, the central limit bound in W_p is applied to the normal approximation for the small jumps (1.3). This is done by viewing Z_t^ϵ as a compound Poisson process, assuming Cramér's condition and that the Lévy measure ν is sufficiently singular at 0 (Thm. 3.1). A desired coupling $W_p(Z_t^\epsilon, \sqrt{t}\mathcal{N}(0, \Sigma_\epsilon)) = O(\epsilon)$ is then achieved for $t = \epsilon$ and $\Sigma_\epsilon = \int_{0 < |z| \leq \epsilon} zz^\top \nu(dz)$, which covers the case $\mathbf{H}(\tau, \alpha)$. However, those assumptions can all be removed if one compromises for a suboptimal rate, as is proved in the appendices of Godinho's paper [10] (Prop. A.2), where only bounded jumps are considered. Again, there is a logarithmic factor because the proof directly uses the aforementioned result of Zaitsev.

Finally, the significance of using the Vaserstein distances instead of other ones is that, when generating numerical approximations for an SDE, the convergence in W_p is equivalent to the usual strong L^p -convergence. The reader is referred to the last section of [6] for a discussion on the contexts where such a substitution holds or fails. Unlike some of the results therein, the method to be introduced here is applicable to the simulation of stochastic flows defined by a Lévy SDE, since it only aims at a coupling for the increments $Z_{t_k} - Z_{t_{k-1}}$.

Throughout this article the generic positive constants C and c may change their values, with subscripts indicating their dependence of parameters. The notations \lesssim and \gtrsim indicate inequalities that hold with a factor C_q . The notation ξ_Σ always stands for an \mathbb{R}^q -random variable following $\mathcal{N}(0, \Sigma)$. The symbol $|\cdot|$, depending on the object it acts on, stands for the modulus of vectors on \mathbb{R}^q , the absolute value for scalars, and the 1-norm of multi-indices on \mathbb{N}^q . For any $\rho \in \mathbb{N}^q$, it would be convenient to introduce the notation $|\rho|_* := \rho_1 + 2\rho_2 + \dots + q\rho_q$. In the context of matrices, I stands for the identity matrix and $\|\cdot\|$ denotes the 2-norm, which is equivalent to any matrix norm.

2. A COUPLING FOR THE CENTRAL LIMIT THEOREM

This section follows Davie's asymptotic approach via Edgeworth expansion briefly sketched in [7], and elaborates each step rigorously. The goal is to achieve a good W_p bound for the central limit theorem using inequality (1.6), and for that one may first approximate the Fourier transform of the central limit sum.

2.1. Asymptotic estimates of the characteristic function

Let $q \in \mathbb{Z}^+$ and $\{X_j\}_{j \geq 1}$ be a sequence of i.i.d. \mathbb{R}^q -random variables with mean 0, covariance Σ and characteristic function χ , and define the weighted sum $Y_m = m^{-1/2} \sum_{i=1}^m X_j$, $m \in \mathbb{Z}^+$. Denote by ψ_m and \mathbb{P}_m the characteristic function and distribution of Y_m , respectively. Then one has asymptotic expansion (using the

multi-index $\alpha \in \mathbb{N}^q$, $s^\alpha = s_1^{\alpha_1} \cdots s_q^{\alpha_q}$):

$$\log \chi(s) \sim -\frac{1}{2}s \cdot \Sigma s + \sum_{|\alpha| \geq 3} \frac{i^{|\alpha|}}{\alpha!} \mu_\alpha s^\alpha,$$

where $\mu_\alpha = \mu_\alpha(X) = i^{-|\alpha|} \partial^\alpha \log \chi(0)$ is the α -th cumulant of X . This gives a formal expansion for $\log \psi_m(z) = m \log \chi(m^{-1/2}z) \sim -\frac{1}{2}z \cdot \Sigma z + \sum_{|\alpha| \geq 3} \frac{i^{|\alpha|}}{\alpha!} m^{1-|\alpha|/2} \mu_\alpha z^\alpha$, implying that

$$\psi_m(z) \sim e^{-\frac{1}{2}z \cdot \Sigma z} \left(1 + \sum_{k=1}^{\infty} m^{-\frac{k}{2}} P_k(z) \right), \quad (2.1)$$

where $P_k(z)$ is a polynomial whose monomials have highest degree $3k$ and lowest degree $k+2$, with coefficients bounded by $C_k(\mathbb{E}|X|^{k+2})^k$ – see Lemma 7.1 in [2]. The inverse Fourier transform of (2.1) gives the Edgeworth expansion for the density f_m of Y_m , if it exists. Detailed derivation for $q=1$ can also be found in [16] (Chap. VI).

In this section, the shorthand notations $\varepsilon := m^{-1/2}$, $\mathcal{P}_{\varepsilon,r} := 1 + \sum_{k=1}^r \varepsilon^k P_k$, $\forall r \in \mathbb{Z}^+$, and $\mathcal{P}_\varepsilon := \mathcal{P}_{\varepsilon,\infty}$ are used, and ε and m may be frequently interchanged. Denote by $\lambda_1 \leq \cdots \leq \lambda_q$ the eigenvalues of Σ , and assume $\lambda_1 \leq 1 \leq \lambda_q$ without loss of generality. Furthermore, $\forall M > 0$ denote $\kappa_M := 1 \vee \mathbb{E}|X|^M$, then $\kappa_M^{1/M}$ increases in M by Hölder's inequality, and so does κ_M . By Lemma 6.3 in [2], $|\mu_\alpha| \leq C_\alpha \kappa_{|\alpha|}$, $\forall \alpha \in \mathbb{N}^q$.

Lemma 2.1. *Suppose that Σ is non-singular, $\mathbb{E}|X|^{n+\tau} < \infty$ for a fixed integer $n \geq 3$ and $\tau \in (0,1)$. Let $\beta \in (0,1/3)$ and $\delta := \min\{\lambda_1/\kappa_3, \kappa_n^{-1/n}/2\}$. Then for $m \in \mathbb{Z}^+$ and $|z| \leq m^{1/2}\delta$, $|\psi_m(z)| \leq \exp(-\frac{1}{4}z \cdot \Sigma z)$; for $m > \max\{(\kappa_3/\lambda_1)^3, \kappa_{n+\tau}^{\max\{4,6/(n(1-3\beta))\}}\}$ and $|z| \leq m^{\beta/2}$,*

$$\left| \psi_m(z) - e^{-\frac{1}{2}z \cdot \Sigma z} \mathcal{P}_{\varepsilon,n-2}(z) \right| \leq C_{n,\tau} \kappa_{n+\tau}^{n-2} \left(|z|^{n+\tau} + |z|^{3(n-1)} \right) e^{-\frac{1}{4}z \cdot \Sigma z} \varepsilon^{n-2+\tau}. \quad (2.2)$$

Proof. First of all Taylor's theorem gives the identity

$$\chi(s) = 1 - \frac{1}{2}s \cdot \Sigma s + \mathbb{E} \int_0^1 \frac{1}{2} e^{i\theta(s \cdot X)} (1-\theta)^2 (is \cdot X)^3 d\theta. \quad (2.3)$$

Then for $|s| \leq \delta_1 := \lambda_1/\kappa_3 \leq \sqrt{2/\lambda_q}$, the inequality $\log u \leq u-1$, $\forall u > 0$, implies that

$$\begin{aligned} \log |\chi(s)| &\leq \log \left(1 - \frac{1}{2}s \cdot \Sigma s + \frac{1}{6} \mathbb{E}|X|^3 |s|^3 \right) \leq -\frac{1}{2}s \cdot \Sigma s + \frac{1}{6} \delta_1 \mathbb{E}|X|^3 |s|^2 \\ &\leq -\frac{1}{2}s \cdot \Sigma s + \frac{1}{4} \lambda_1 |s|^2 \leq -\frac{1}{4}s \cdot \Sigma s, \end{aligned}$$

and the first claim $|\psi_m(z)| \leq \exp(-\frac{1}{4}z \cdot \Sigma z)$ holds for $|z| \leq m^{1/2}\delta_1$.

On the other hand, for $|s| \leq \kappa_3^{-1/3}/2 \leq \lambda_q^{-1/2}/2$, from (2.3) one sees that

$$\operatorname{Re} \chi(s) \geq 1 - \frac{1}{2} \lambda_q |s|^2 - \frac{1}{6} \mathbb{E}|X|^3 |s|^3 > \frac{1}{2},$$

and hence the principle branch of $\log \chi(s)$ is well-defined, and $|\chi(s)| > 1/2$. For fixed $n \geq 3$, define, $\forall s \in \mathbb{R}^q$,

$$S_n(s) := \sum_{|\alpha|=2}^n \frac{i^{|\alpha|}}{\alpha!} \mu_\alpha s^\alpha, \quad T_n(s) := \sum_{|\alpha|=2}^n \frac{i^{|\alpha|}}{\alpha!} s^\alpha \mathbb{E} X^\alpha = \sum_{j=2}^n \frac{1}{j!} \mathbb{E}(i s \cdot X)^j.$$

Then $\forall u \in \mathbb{R}$, the bound $|e^{iu} - 1| \leq 2 \wedge |u| \leq 2^{1-\tau} |u|^\tau$, $\forall \tau \in (0, 1)$, and the identity

$$e^{iu} = \sum_{k=0}^n \frac{(iu)^k}{k!} + \frac{i^n}{(n-1)!} \int_0^1 (1-\theta)^{n-1} u^n (e^{i\theta u} - 1) d\theta,$$

lead to, by the substitution $u = s \cdot X$ and taking expectation,

$$|\chi(s) - 1 - T_n(s)| \leq C_{n,\tau} \kappa_{n+\tau} |s|^{n+\tau}. \quad (2.4)$$

Meanwhile one can write the following expansion (with Taylor remainder $R_n(s)$):

$$\log(1 + T_n(s)) = \sum_{l=1}^n \frac{(-1)^{l+1}}{l} T_n^l(s) + R_n(s) = S_n(s) + \tilde{S}_n(s) + R_n(s),$$

where $\tilde{S}_n(s)$ is a polynomial of which each monomial has degree at least $n+1$. The fact that the first few terms agree with $S_n(s)$ is due to the relation between the cumulants μ_α and the moments $\mathbb{E} X^\alpha$ (see [2], Sect. 6, p. 46). By the multinomial theorem, for $l = 1, \dots, n$ each monomial $\sigma_{\rho,l}(s)$ in $T_n^l(s)$ takes the form

$$\sigma_{\rho,l}(s) = C_{n,l,\rho} \prod_{j=1}^{n-1} (\mathbb{E}(s \cdot X)^{j+1})^{\rho_j},$$

for some $\rho \in \mathbb{N}^{n-1}$, $|\rho| = l$. Then the monomials $\tilde{\sigma}_{\rho,l}(s)$ of \tilde{S}_n correspond to those with ρ satisfying $\sum_{j=1}^{n-1} (j+1)\rho_j = |\rho|_* + l \geq n+1$. If one further chooses $\delta_2 := \kappa_n^{-1/n}/2 < 1$, then for $|s| \leq \delta_2$,

$$\begin{aligned} |\tilde{\sigma}_{\rho,l}(s)| &\leq C_{n,l} |s|^{|\rho|_*+l} \prod_{j=1}^{n-1} \kappa_{j+1}^{\rho_j} \leq C_{n,l} |s|^{n+1} \kappa_n^{-(|\rho|_*+l-(n+1))/n} \prod_{j=1}^{n-1} \kappa_{j+1}^{(|\rho|_*+l)/(j+1)} \\ &= C_{n,l} |s|^{n+1} \kappa_n^{(n+1)/n} \prod_{j=1}^{n-1} \left(\kappa_n^{-1/n} \kappa_{j+1}^{1/(j+1)} \right)^{|\rho|_*+l} \leq C_{n,l} \kappa_n^{1+1/n} |s|^{n+1}, \end{aligned}$$

where Hölder's inequality is used in the last step. Therefore $|\tilde{S}_n(s)| \leq C_n \kappa_n^{1+1/n} |s|^{n+1}$. Also notice that, for $|s| \leq \delta_2$ and $j = 2, \dots, n$, one has $|s|^{j-1} \kappa_j \leq \kappa_n^{1/n} (\kappa_n^{-1/n} \kappa_j^{1/j})^j \leq \kappa_n^{1/n}$. This implies that $|T_n(s)| \leq \sum_{j=2}^n |s|^{j-1} \kappa_j |s|/j! \leq (e-2) \kappa_n^{1/n} |s|/2 < 1/2$, and that $|1 + \theta T_n(s)| \geq 1/2$ for any $\theta \in [0, 1]$. Therefore

$$|R_n(s)| \leq \frac{1}{n!} \int_0^1 (1-\theta)^n \left| \frac{T_n(s)}{1 + \theta T_n(s)} \right|^{n+1} d\theta \leq C_n |T_n(s)|^{n+1} \leq C_n \kappa_n^{1+1/n} |s|^{n+1}, \quad (2.5)$$

which leads to the bound $|\log(1 + T_n(s)) - S_n(s)| \leq C_n \kappa_n^{1+1/n} |s|^{n+1}$.

Notice that for $|s| < \delta_2 < 1$ both $\operatorname{Re}\chi(s)$ and $\operatorname{Re}(1 + T_n(s))$ are no less than $1/2$, implying that $|\theta\chi(s) + (1 - \theta)(1 + T_n(s))| \geq 1/2$ for any $\theta \in [0, 1]$. Thus, by the triangle inequality and (2.4), one deduces

$$\begin{aligned} |\log \chi(s) - S_n(s)| &\leq |\log \chi(s) - \log(1 + T_n(s))| + |\log(1 + T_n(s)) - S_n(s)| \\ &\leq 2|\chi(s) - 1 - T_n(s)| + |\log(1 + T_n(s)) - S_n(s)| \\ &\leq C_{n,\tau} \kappa_{n+\tau}^{1+1/n} |s|^{n+\tau}. \end{aligned}$$

Returning to ψ_m , as $\log \psi_m(z) = \varepsilon^{-2} \log \chi(\varepsilon z)$, from the estimate above one has

$$|\log \psi_m(z) - \varepsilon^{-2} S_n(\varepsilon z)| \leq C_{n,\tau} \kappa_{n+\tau}^{1+1/n} \varepsilon^{n-2+\tau} |z|^{n+\tau}. \quad (2.6)$$

Moreover, writing $U_n(z) := \frac{1}{2}z \cdot \Sigma z + \varepsilon^{-2} S_n(\varepsilon z)$, one can apply Taylor's theorem again to the exponential $\exp(U_n(z))$ (recall the notation $\mathcal{P}_{\varepsilon,\cdot}$):

$$\begin{aligned} \exp\left(\sum_{|\alpha|=3}^n \frac{i^{|\alpha|}}{\alpha!} \varepsilon^{|\alpha|-2} \mu_\alpha z^\alpha\right) &= 1 + U_n(z) + \frac{1}{2!} U_n^2(z) + \cdots + \frac{1}{(n-2)!} U_n^{n-2}(z) + V(z) \\ &= 1 + \mathcal{P}_{\varepsilon,n-2}(z) + \tilde{P}(z) + V(z), \end{aligned}$$

where $\tilde{P}(z) = 0$ for $n = 3$ (i.e. $P_1(z) = U_3(z)$ contains all the cubic terms) and otherwise a polynomial of degree $n(n-2)$ with complex coefficients that contain products of the cumulants μ_α up to $|\alpha| = n$ and powers of ε at least $n-1$; the Taylor remainder $V(z)$ is given by

$$V(z) = \frac{1}{(n-2)!} \int_0^1 (1-\theta)^{n-2} U_n^{n-1}(z) e^{\theta U_n(z)} d\theta.$$

For $|z| \leq m^{1/6} = \varepsilon^{-1/3}$, one claims the following bound:

$$|\tilde{P}(z)| \leq C_n \kappa_n^{n-2} \varepsilon^{n-1} (|z|^{n+3} + |z|^{3(n-1)}).$$

This can be seen by checking the powers of ε and z in each $U_n^l(z)$, $l = 1, \dots, n-2$. For each l , the multinomial theorem gives (with multi-indices $\rho \in \mathbb{N}^{n-2}$, $\alpha \in \mathbb{N}^q$)

$$U_n^l(z) = (-1)^l \sum_{|\rho|=l} \binom{l}{\rho} (i\varepsilon)^{|\rho|_*} \prod_{j=1}^{n-2} \left(\sum_{|\alpha|=j+2} \frac{1}{\alpha!} \mu_\alpha z^\alpha \right)^{\rho_j}.$$

Then each monomial of $U_n^l(z)$ is bounded by $C_{n,l} \kappa_n^l \varepsilon^{|\rho|_*} |z|^{|\rho|_*+2l}$, and the monomials $\tilde{p}_{\rho,l}(z)$ of $\tilde{P}(z)$ correspond to those with $|\rho|_* \geq n-1$ and $l \geq 2$. When $|\rho|_* + 2l \leq 3(n-1)$ the claim follows immediately from interpolating the powers of $|z|$; when $|\rho|_* + 2l > 3(n-1)$, note that $|\rho|_* > |\rho| = l$, and so for $|z| \leq \varepsilon^{-1/3}$,

$$|\tilde{p}_{\rho,l}(z)| \leq C_{n,l} \kappa_n^l \varepsilon^{\frac{2}{3}(|\rho|_*-l)+n-1} |z|^{3(n-1)} \leq C_n \kappa_n^{n-2} \varepsilon^{n-1} |z|^{3(n-1)}.$$

Regarding the Taylor remainder $V(z)$, notice that for $|z| \leq \varepsilon^{-\beta}$, $\forall \beta \in (0, 1/3)$, and $\varepsilon < \kappa_n^{-1}$,

$$\begin{aligned} |U_n(z)| &\leq \sum_{j=1}^{n-2} \varepsilon^j |z|^{j+2} \kappa_{j+2} \leq \sum_{j=1}^{n-2} \varepsilon^{j-\beta(j-1)} |z|^3 \kappa_{j+2} \leq \sum_{j=1}^{n-2} \varepsilon^{\frac{2}{3}(j-1)} \kappa_n^{(j+2)/n} \varepsilon |z|^3 \\ &\leq \sum_{j=1}^{n-2} \kappa_n^{\frac{3}{n} + (\frac{1}{n} - \frac{2}{3})(j-1)} \varepsilon |z|^3 \leq (n-2) \kappa_n^{3/n} \varepsilon |z|^3, \end{aligned}$$

and furthermore $|U_n(z)| \leq (n-2)\kappa_n^{3/n}\varepsilon^{1-3\beta}$. Thus one arrives at

$$|V(z)| \leq C_n \kappa_n^3 \exp\left((n-2)\varepsilon^{1-3\beta}\kappa_n^{3/n}\right) \varepsilon^{n-1}|z|^{3(n-1)}.$$

Now recall that $\varepsilon^{-2}S_n(\varepsilon z) = -\frac{1}{2}z \cdot \Sigma z + U_n(z)$, and by the triangle inequality,

$$\begin{aligned} \left| \psi_m(z) - e^{-\frac{1}{2}z \cdot \Sigma z} \mathcal{P}_{\varepsilon, n-2}(z) \right| &\leq \left| e^{\log \psi_m(z)} - e^{-\frac{1}{2}z \cdot \Sigma z + U_n(z)} \right| + \left| e^{-\frac{1}{2}z \cdot \Sigma z + U_n(z)} - e^{-\frac{1}{2}z \cdot \Sigma z} \mathcal{P}_{\varepsilon, n-2}(z) \right| \\ &\leq |\psi_m(z)| \left| 1 - \exp(-\log \psi_m(z) + \varepsilon^{-2}S_n(\varepsilon z)) \right| + e^{-\frac{1}{2}z \cdot \Sigma z} \left(\left\| \tilde{P}(z) \right\| + |V(z)| \right). \end{aligned}$$

Using the estimate (2.6), the inequality $|1 - e^u| \leq e^{|u|}|u|$, $\forall u \in \mathbb{C}$, and the bounds for $\tilde{P}(z)$ and $V(z)$ derived above, one deduces that for $|z| \leq \varepsilon^{-\beta} < \varepsilon^{-1/3}$,

$$\begin{aligned} \left| \psi_m(z) - e^{-\frac{1}{2}z \cdot \Sigma z} \mathcal{P}_{\varepsilon, n-2}(z) \right| &\leq C_{n, \tau} |\psi_m(z)| \exp\left(\varepsilon^{2(n-2)/3} \kappa_{n+\tau}^{1+1/n}\right) \varepsilon^{n-2+\tau} |z|^{n+\tau} \kappa_{n+\tau}^{1+1/n} \\ &\quad + C_n \kappa_n^{n-2} \exp\left((n-2)\varepsilon^{1-3\beta}\kappa_n^{3/n}\right) \varepsilon^{n-1} (|z|^{n+3} + |z|^{3(n-1)}) e^{-\frac{1}{2}z \cdot \Sigma z}. \end{aligned}$$

Now with $\delta := \delta_1 \wedge \delta_2$ fixed, for m large one has $m^{\beta/2} < m^{1/2}\delta$. Also, for fixed $\beta \in (0, 1/3)$ and $\tau \in (0, 1)$, one may further choose $m > \kappa_{n+\tau}^{3(1+1/n)/(n-2)} \vee \kappa_n^{6/(n(1-3\beta))}$ s.t. the exponents in the coefficients above are bounded by 1. This is satisfied when $m > \kappa_{n+\tau}^{\max\{4, 6/(n(1-3\beta))\}}$. For $m > \delta^{-3} > \delta^{2/(\beta-1)}$ the first claim still holds, and so the second claim follows. \square

In order to bound the integral of the left-hand side term in (2.2) over all of \mathbb{R}^q , one may assume **Cramér's condition**:

$$\overline{\lim}_{|s| \rightarrow \infty} |\chi(s)| < 1,$$

or equivalently, there exist $\rho > 0$ and $\gamma \in (0, 1)$ s.t. the following condition holds:

Assumption CC(ρ, γ). $|\chi(s)| \leq \gamma$, $\forall |s| \geq \rho$.

As explained in [2] (p. 207), if χ satisfies Cramér's condition, then $|\chi(s)| < 1$, $\forall s \neq 0$; it is satisfied when X has a density by the Riemann-Lebesgue theorem. Discrete distributions are excluded, but some singular and yet non-lattice distributions are also allowed, such as the distribution on the Cantor middle-third set that gives mass 2^{-j} to each interval on the j -th level.

Given the X_j 's satisfying Cramér's condition, the following lemma shows that it is also satisfied for the weighted sum Y_m .

Lemma 2.2. *Let χ satisfy **CC**(ρ, γ) with ρ, γ explicitly known and $\delta \in (0, \rho \wedge 1)$. Then $\exists \bar{\gamma} = \bar{\gamma}(\rho, \gamma, \delta) \in (0, 1)$ s.t. $|\psi_m(z)| < \bar{\gamma}^m$ for $|z| > m^{1/2}\delta$.*

Proof. Let $N \in \mathbb{Z}^+$ and write $\chi(Ns) = |\chi(Ns)|e^{i\theta_1}$, $\chi(s) = |\chi(s)|e^{i\theta_0}$, where θ_1, θ_0 depend on s . Then, with F being the distribution of X , one gets $\int_{\mathbb{R}^q} \sin(s \cdot x - \theta_0) F(dx) = 0$ and

$$\begin{aligned} 1 - |\chi(s)| &= \int_{\mathbb{R}^q} (1 - \cos(s \cdot x - \theta_0)) F(dx) = \int_{\mathbb{R}^q} 2 \sin^2 \frac{1}{2}(s \cdot x - \theta_0) F(dx) \\ &\geq \frac{1}{N^2} \int_{\mathbb{R}^q} 2 \sin^2 \frac{N}{2}(s \cdot x - \theta_0) F(dx) = \frac{1}{N^2} \int_{\mathbb{R}^q} (1 - \cos(Ns \cdot x - N\theta_0)) F(dx), \end{aligned}$$

where the inequality $|\sin(N\phi)| \leq N|\sin\phi|$, $\forall N \in \mathbb{N}$, $\phi \in \mathbb{R}$, is used. Meanwhile,

$$|\chi(Ns)| = e^{-i\theta_1} \int_{\mathbb{R}^q} e^{iNs \cdot x} F(dx) = e^{i(N\theta_0 - \theta_1)} \int_{\mathbb{R}^q} e^{i(Ns \cdot x - N\theta_0)} F(dx),$$

which implies

$$\begin{aligned} 1 - |\chi(s)| &\geq \frac{1}{N^2} - \frac{1}{N^2} \operatorname{Re} \int_{\mathbb{R}^q} e^{i(Ns \cdot x - N\theta_0)} F(dx) \\ &\geq \frac{1}{N^2} - \frac{1}{N^2} \left| \int_{\mathbb{R}^q} e^{i(Ns \cdot x - N\theta_0)} F(dx) \right| = \frac{1}{N^2} - \frac{1}{N^2} |\chi(Ns)|. \end{aligned}$$

Choose $N = \lceil (\rho + 1)/\delta \rceil > \rho/\delta$, then $|\chi(s)| \leq 1 - (1 - \gamma)\delta^2/(\rho + 1)^2 =: \bar{\gamma}$ for $\delta < |s| < \rho$. Clearly $\bar{\gamma} \geq \gamma$, and $|\psi_m(z)| = |\chi(m^{-1/2}z)|^m < \bar{\gamma}^m < 1$ for $|z| > m^{1/2}\delta$. \square

From now on the following bounds will be frequently used: $\forall M, c > 0$,

$$\int_{\mathbb{R}^q} |x|^M e^{-cx \cdot \Sigma x} dx = \int_{\mathbb{R}^q} \left| \Sigma^{-\frac{1}{2}} y \right|^M e^{-c|y|^2} \det(\Sigma^{-\frac{1}{2}}) dy \leq C_{q,c,M} (\det \Sigma)^{-\frac{1}{2}} \lambda_1^{-\frac{M}{2}}, \quad (2.7)$$

and

$$\int_{\mathbb{R}^q} |x|^M \phi_\Sigma(x) dx = C_q \int_{\mathbb{R}^q} \left| \Sigma^{\frac{1}{2}} y \right|^M e^{-\frac{1}{2}|y|^2} dy \leq C_{q,M} \lambda_q^{\frac{M}{2}}, \quad (2.8)$$

where the inverse and the square root of Σ are well-defined since it is positive definite.

Although Cramér's condition gives some restriction on the law of X , it does not require the smoothness or the existence of the density f_m of Y_m . In order to see how close the law of Y_m is to the perturbed normal distributions from polynomial expansions, one may use a smoothing argument. Let \tilde{f}_m and $\tilde{\psi}_m$ be the density and characteristic function of the mollified measure $\mathbb{P}_m * \theta_m$, where θ_m is a measure with smooth density, still denoted by θ_m or θ_ε :

$$\theta_\varepsilon(x) = \varepsilon^{-q(n+1)} h(\varepsilon^{-n-1}x), \quad (2.9)$$

for some function $0 \leq h \in C_0^\infty(\mathbb{R}^q)$ supported on the open unit ball and $\int_{\mathbb{R}^q} h(x) dx = 1$. Thus θ_ε is a probability density supported on $\{|x| < \varepsilon^{n+1}\}$. Write \hat{h} and $\hat{\theta}_\varepsilon$ as their respective Fourier transforms.

Proposition 2.3. *Under the assumptions in Lemmas 2.1 and 2.2, for any integer $n \geq 3$, $\tau \in (0, 1)$, $\beta \in (0, 1/3)$ and m sufficiently large, it holds true that*

$$\int_{\mathbb{R}^q} \left| \tilde{\psi}_m(z) - e^{-\frac{1}{2}z \cdot \Sigma z} \mathcal{P}_{\varepsilon, n-2}(z) \right| dz \leq C_{q,n,\tau} (\det \Sigma)^{-\frac{1}{2}} \lambda_1^{-\frac{n-1}{2\beta}} \kappa_{n+\tau}^{n-2} \varepsilon^{n-2+\tau}.$$

Proof. Note that $\tilde{\psi}_m = \psi_m \hat{\theta}_\varepsilon$, and by the first estimate in Lemma 2.1, for $|z| \leq m^{1/2}\delta$,

$$\begin{aligned} \left| \tilde{\psi}_m(z) - \psi_m(z) \right| &= |\psi_m(z)| \left| \hat{\theta}_\varepsilon(z) - 1 \right| \leq |\psi_m(z)| \int_{|x| < \varepsilon^{n+1}} |e^{iz \cdot x} - 1| \theta_\varepsilon(x) dx \\ &\leq |\psi_m(z)| |z| \varepsilon^{n+1} \leq \varepsilon^{n+1} |z| e^{-\frac{1}{4}z \cdot \Sigma z}. \end{aligned}$$

Hence by the second estimate in Lemma 2.1 and the triangle inequality,

$$\left| \tilde{\psi}_m(z) - e^{-\frac{1}{2}z \cdot \Sigma z} \mathcal{P}_{\varepsilon, n-2}(z) \right| \leq C_{n, \tau} \kappa_{n+\tau}^{n-2} \varepsilon^{n-2+\tau} \left(|z|^{n+\tau} + |z|^{3(n-1)} \right) e^{-\frac{1}{4}z \cdot \Sigma z},$$

for $|z| \leq m^{\beta/2} \leq m^{1/2} \delta$ for m large. Also for all $z \in \mathbb{R}^q$,

$$\begin{aligned} \left| \hat{\theta}_\varepsilon(z) \right| &= \left| \int_{|x| < \varepsilon^{n+1}} e^{iz \cdot x} \theta_\varepsilon(x) dx \right| = \left| \int_{|x| < \varepsilon^{n+1}} e^{iz \cdot x} \varepsilon^{-q(n+1)} h(\varepsilon^{-n-1} x) dx \right| \\ &= \left| \int_{|y| < 1} e^{i\varepsilon^{n+1} z \cdot y} h(y) dy \right| = \left| \hat{h}(\varepsilon^{n+1} z) \right| \leq C_q \varepsilon^{-K(n+1)} |z|^{-K}, \end{aligned} \quad (2.10)$$

for any $K > 0$, since $h \in C_0^\infty(\mathbb{R}^q)$ with all the derivatives in $L^1(\mathbb{R}^q)$. Then for $|z| > m^{1/2} \delta$, as $|\psi_m(z)| < \bar{\gamma}^m$ by Lemma 2.2, one may choose $K = q + 1$ for convenience and $|\tilde{\psi}_m(z)| \leq \bar{\gamma}^m \min\{1, C_q \varepsilon^{-(q+1)(n+1)} |z|^{-q-1}\}$. For $|z| \leq m^{1/2} \delta$ one still has $|\tilde{\psi}_m(z)| \leq \exp(-\frac{1}{4}z \cdot \Sigma z)$.

Given all the estimates for $\tilde{\psi}_m(z)$ on different domains, one can split the integral in question into three parts:

$$\begin{aligned} \tilde{I} &:= \int_{\mathbb{R}^q} \left| \tilde{\psi}_m(z) - e^{-\frac{1}{2}z \cdot \Sigma z} \mathcal{P}_{\varepsilon, n-2}(z) \right| dz \\ &= \left(\int_{|z| \leq m^{\beta/2}} + \int_{m^{\beta/2} < |z| \leq m^{1/2} \delta} + \int_{|z| > m^{1/2} \delta} \right) \left| \tilde{\psi}_m(z) - e^{-\frac{1}{2}z \cdot \Sigma z} \mathcal{P}_{\varepsilon, n-2}(z) \right| dz. \end{aligned}$$

Then by virtue of Lemma 2.1 and Lemma 2.2,

$$\begin{aligned} \tilde{I} &\leq C_{n, \tau} \kappa_{n+\tau}^{n-2} \varepsilon^{n-2+\tau} \int_{|z| \leq m^{\beta/2}} \left(|z|^{n+\tau} + |z|^{3(n-1)} \right) e^{-\frac{1}{4}z \cdot \Sigma z} dz + \int_{m^{\beta/2} < |z| \leq m^{\frac{1}{2}} \delta} e^{-\frac{1}{4}z \cdot \Sigma z} dz \\ &\quad + \int_{|z| > m^{1/2} \delta} \bar{\gamma}^m \left(1 \wedge C_q \varepsilon^{-(q+1)(n+1)} |z|^{-q-1} \right) dz + \int_{|z| > m^{\beta/2}} e^{-\frac{1}{2}z \cdot \Sigma z} |\mathcal{P}_{\varepsilon, n-2}(z)| dz. \end{aligned}$$

Use (2.7) for the first integral, combine the second and the fourth, and split the third into the set where $|z|$ is large and its complement to get (Λ denotes the Lebesgue measure on \mathbb{R}^q)

$$\begin{aligned} \tilde{I} &\leq C_{q, n, \tau} (\det \Sigma)^{-\frac{1}{2}} \lambda_1^{-\frac{3}{2}(n-1)} \kappa_{n+\tau}^{n-2} \varepsilon^{n-2+\tau} + \bar{\gamma}^m \Lambda(\{|z| \leq C_q \varepsilon^{-n-1}\}) \\ &\quad + C_q \bar{\gamma}^m \varepsilon^{-(q+1)(n+1)} \int_{|z| > C_q \varepsilon^{-n-1}} |z|^{-q-1} dz + 2 \int_{|z| > m^{\beta/2}} e^{-\frac{1}{4}z \cdot \Sigma z} \left(1 + \sum_{k=1}^{n-2} \varepsilon^k |P_k(z)| \right) dz \\ &\leq C_{q, n, \tau} (\det \Sigma)^{-\frac{1}{2}} \lambda_1^{-\frac{3}{2}(n-1)} \kappa_{n+\tau}^{n-2} \varepsilon^{n-2+\tau} + C_q \bar{\gamma}^m \varepsilon^{-(q+1)(n+1)} \\ &\quad + C_{q, n} \int_{|z| > m^{\beta/2}} e^{-\frac{1}{4}z \cdot \Sigma z} \kappa_n^{n-2} \left(1 + \sum_{k=1}^{n-2} \varepsilon^k |z|^{3k} \right) dz. \end{aligned}$$

The second term can be absorbed by the first term if m is sufficiently large s.t. it satisfies the criterion for m in Lemma 2.1 and that

$$\bar{\gamma}^m m^{\frac{1}{2}(q(n+1)+1-\tau)} \leq (\det \Sigma)^{-\frac{1}{2}} \lambda_1^{-\frac{3}{2}(n-1)} \kappa_{n+\tau}^{n-2}. \quad (2.11)$$

For the third term, notice that $|z| > 1$ and that $1 < \varepsilon|z|^{1/\beta}$, $\forall \beta \in (0, 1/3)$. Thus

$$\tilde{I} \leq C_{q,n,\tau} (\det \Sigma)^{-\frac{1}{2}} \lambda_1^{-\frac{3}{2}(n-1)} \kappa_{n+\tau}^{n-2} \varepsilon^{n-2+\tau} + C_{q,n} \kappa_n^{n-2} \int_{\mathbb{R}^q} e^{-\frac{1}{4}z \cdot \Sigma z} (\varepsilon|z|^{1/\beta})^{n-1} dz,$$

and the result follows from (2.7) again. \square

2.2. Perturbed normal distributions

Now given Proposition 2.3, one can approximate the density \tilde{f}_m by the inverse Fourier transform \mathcal{F}^{-1} of $\exp(-\frac{1}{2}z \cdot \Sigma z) \mathcal{P}_\varepsilon(z)$. Define, $\forall x \in \mathbb{R}^q$, the Edgeworth polynomials $\{Q_k\}$ by

$$\phi_\Sigma(x) Q_k(x) := \mathcal{F}^{-1} \left\{ e^{-\frac{1}{2}z \cdot \Sigma z} P_k(z) \right\} (x), \quad \forall k \in \mathbb{Z}^+, \quad (2.12)$$

and accordingly $\mathcal{Q}_{\varepsilon,r} := 1 + \sum_{k=1}^r \varepsilon^k Q_k$, $\forall r \in \mathbb{Z}^+$. Then each monomial of Q_k has the same degree as that of P_k . In fact, if $\Sigma = \text{diag}(\lambda_1, \dots, \lambda_q)$, one can explicitly show that

$$Q_k(x) = \sum_{|\alpha|=k+2}^{3k} (-1)^{|\alpha|} b_\alpha \prod_{j=1}^q \lambda_j^{-\alpha_j/2} H_{\alpha_j} \left(\lambda_j^{-1/2} x_j \right), \quad (2.13)$$

where $b_\alpha = b_\alpha(\mu_\beta : |\beta| \leq k+2)$ is the real coefficient of $(iz)^\alpha$ in $P_k(z)$ satisfying $|b_\alpha| \leq \kappa_{k+2}^k$, and H_j is the Hermite polynomial of degree j . See [16] (Chap. VI, Sect. 1) for the precise values.

Remark 2.4. Since $\exp(-\frac{1}{2}z \cdot \Sigma z) \mathcal{P}_{\varepsilon,n-2}(z)$ and $\psi_m(z)$ have the same derivatives at 0 up to order n , the Edgeworth sum $\phi_\Sigma \mathcal{Q}_{\varepsilon,n-2}$ and Y_m have the same moments up to order n .

For a positive-definite $q \times q$ matrix Σ , let \mathcal{P}_Σ be the set of polynomials $S : \mathbb{R}^q \rightarrow \mathbb{R}$ s.t. $\int_{\mathbb{R}^q} S_j(x) \phi_\Sigma(x) dx = 0$ and \mathcal{P}_G be the set of polynomials $U : \mathbb{R}^q \rightarrow \mathbb{R}^q$ s.t. $U = \nabla u$ for some polynomial $u : \mathbb{R}^q \rightarrow \mathbb{R}$. Furthermore let $\mathcal{P}_\Sigma^\infty$ be the set of sequences (S_1, S_2, \dots) , $S_j \in \mathcal{P}_\Sigma$, and \mathcal{P}_G^∞ be the set of sequences (U_1, U_2, \dots) , $U_j \in \mathcal{P}_G$.

Given polynomials $U_j : \mathbb{R}^q \rightarrow \mathbb{R}^q$, $j = 1, \dots, k$, define $\mathbf{U}_{\varepsilon,k}(x) := x + \sum_{j=1}^k \varepsilon^j U_j(x)$, $\forall \varepsilon > 0$. Then for a ξ_Σ following $\mathcal{N}(0, \Sigma)$, the random variable $\mathbf{U}_{\varepsilon,k}(\xi_\Sigma)$ is said to have a perturbed normal distribution, whose density can be formally expressed as

$$\zeta_{\varepsilon,k}(y) = \det \left(D\mathbf{U}_{\varepsilon,k}^{-1}(y) \right) \phi_\Sigma \left(\mathbf{U}_{\varepsilon,k}^{-1}(y) \right).$$

Davie [7] (Sect. 2) showed, using a recursive construction, that one can approximate $\zeta_{\varepsilon,k}(y)$ by $\phi_\Sigma(y) \mathcal{S}_{\varepsilon,l}(y) := \phi_\Sigma(y) \left(1 + \sum_{j=1}^l \varepsilon^j S_j(y) \right)$ up to order $O(\varepsilon^{l+1})$, where for each $j \leq l$, $S_j : \mathbb{R}^q \rightarrow \mathbb{R}$ is a polynomial uniquely determined by U_1, \dots, U_j only. Since l is arbitrary, for each k the polynomials U_1, \dots, U_k uniquely determine a sequence (S_1, S_2, \dots) , and hence the map $\mathfrak{S}_\Sigma : (U_1, U_2, \dots) \mapsto (S_1, S_2, \dots)$ is well-defined. Moreover, each $S_j \in \mathcal{P}_\Sigma$ by Lemma 1 in [7].

A given sequence $(S_1, S_2, \dots) \in \mathcal{P}_\Sigma^\infty$ can have several preimages under \mathfrak{S}_Σ . But according to Lemma 2 in [7], if one restricts \mathfrak{S}_Σ on \mathcal{P}_G^∞ then it is a bijection.³ As is shown in the preceding paragraphs therein, this follows from the bijectivity of the linear map

$$\mathcal{L}_\Sigma : \mathcal{P}_G \rightarrow \mathcal{P}_\Sigma, \quad U(x) \mapsto \nabla \cdot U(x) - x \cdot \Sigma^{-1} U(x).$$

³This is motivated by Brenier's transport theorem for the quadratic cost – see Theorem 2.12 in [20] for the general statement and Lemma 5 in [7] for a special case.

The preimages of the bijection \mathfrak{S}_Σ are defined inductively in the following way: given a sequence $(S_1, S_2, \dots) \in \mathcal{P}_\Sigma^\infty$, suppose $U_1, \dots, U_k \in \mathcal{P}_G$ are found with $\mathfrak{S}_\Sigma(U_1, \dots, U_k) = (S_1, \dots, S_k, \tilde{S}_{k+1}, \dots)$, then adding an additional U_{k+1} gives a different sequence

$$\mathfrak{S}_\Sigma(U_1, \dots, U_k, U_{k+1}) = (S_1, \dots, S_k, \tilde{S}_{k+1} - \mathcal{L}_\Sigma U_{k+1}, \dots).$$

This means that $U_{k+1} \in \mathcal{P}_G$ is determined by the equation $\tilde{S}_{k+1} - \mathcal{L}_\Sigma U_{k+1} = S_{k+1}$. Writing $U_{k+1} = \nabla u_{k+1}$, one looks for a polynomial u_{k+1} that solves the Hermite-type equation

$$-\Delta u_{k+1}(x) + x \cdot \Sigma^{-1} \nabla u_{k+1}(x) = S_{k+1}(x) - \tilde{S}_{k+1}(x), \quad x \in \mathbb{R}^q. \quad (2.14)$$

For the initial step set $\tilde{S}_1 \equiv 0$ and solve the PDE by induction on the degree of u_1 ; at each step, first compute \tilde{S}_{k+1} from u_1, \dots, u_k and then solve the PDE again by induction on the degree of u_{k+1} – see similar arguments presented in the proof of Lemma 1 in [6].

The computation of $\tilde{S}_{k+1}(x)$ can be done in the following formal way. First write

$$\phi_\Sigma(x) = \zeta_{\varepsilon, k}(\mathbf{U}_{\varepsilon, k}(x)) \det(\mathbf{D}\mathbf{U}_{\varepsilon, k}(x)), \quad (2.15)$$

by a change of variables. With $U_j = \nabla u_j$, $\mathbf{D}\mathbf{U}_{\varepsilon, k}(x) = I + \sum_{j=1}^k \varepsilon^j \mathbf{D}^2 u_j(x)$, and so the determinant above can be expressed as $1 + \varepsilon v_1(x) + \dots + \varepsilon^{qk} v_{qk}(x)$, where for each $l \leq qk$, v_l is the sum of $(\partial_{i_1 j_1}^2 u_1)^{\rho_1} \dots (\partial_{i_k j_k}^2 u_k)^{\rho_k}$ over all the second derivatives and all multi-indices $\rho \in \mathbb{N}^k$ s.t. $|\rho|_* = l$. Then by formally writing $\zeta_{\varepsilon, k}(y) = \phi_\Sigma(y) \tilde{S}_\varepsilon(y)$ with $y = \mathbf{U}_{\varepsilon, k}(x)$ and $\tilde{S}_\varepsilon(y) = 1 + \sum_{j=1}^k \varepsilon^j S_j(y) + \sum_{j=k+1}^\infty \varepsilon^j \tilde{S}_j(y)$, one can rearrange (2.15) to get

$$\begin{aligned} & 1 + \varepsilon S_1(y) + \dots + \varepsilon^k S_k(y) + \varepsilon^{k+1} \tilde{S}_{k+1}(y) + \dots \\ &= \frac{\exp \left\{ \sum_{j=1}^k \varepsilon^j x \cdot \Sigma^{-1} \nabla u_j(x) + \frac{1}{2} \sum_{j_1, j_2=1}^k \varepsilon^{j_1+j_2} \nabla u_{j_1}(x) \cdot \Sigma^{-1} \nabla u_{j_2}(x) \right\}}{1 + \varepsilon v_1(x) + \dots + \varepsilon^{qk} v_{qk}(x)} \\ &= 1 + \varepsilon T_1(x) + \varepsilon^2 T_2(x) + \dots, \end{aligned}$$

where the series on the right-hand side is obtained by multiplying out the Maclaurin series for e^z and $1/(1+z)$. Since differentiating a polynomial only changes its coefficients by a constant and reduces its degree, $|T_{k+1}(x)| \leq C_{q, k} \|\Sigma^{-1}\|^{k+1} \sum_{|\rho|_* = k+1} (1 + |u_1(x)|)^{\rho_1} \dots (1 + |u_k(x)|)^{\rho_k}$.

On the left-hand side, each polynomial $S_j(y)$ with degree $d_j \geq 1$ can be expressed as $S_j(x) + \varepsilon w_{j,1}(x) + \dots + \varepsilon^{d_j k} w_{j, d_j k}(x)$ by its Taylor expansion about x . Since all the derivatives of $S_j(x)$ have their norms bounded by $C_{q, j} (1 + |S_j(x)|)$, one has, for each $j \leq k$ and $l \leq d_j k$, that $|w_{j, l}(x)| \leq C_{q, j, l} \sum_{s=1}^l \sum_{|\rho|_* = s} (1 + |S_j(x)|) |U_1(x)|^{\rho_1} \dots |U_s(x)|^{\rho_s}$. Thus, by expanding out $\tilde{S}_{k+1}(y)$ in terms of x and matching the coefficients of ε^{k+1} on both sides, one gets

$$\tilde{S}_{k+1}(x) = T_{k+1}(x) - w_{k,1}(x) - w_{k-1,2}(x) - \dots - w_{2, k-1}(x) - w_{1, k}(x). \quad (2.16)$$

Although the calculation for \tilde{S}_{k+1} above is completely formal, it is equivalent to Davie's construction in [7] due to the uniqueness of the power series expansion. For a rigorous proof of such an approximation of $\zeta_{\varepsilon, k}$, the reader is referred to Proposition 1 in [7].

Remark 2.5. The set \mathcal{P}_G is invariant under orthogonal transformation: given $U(x) \in \mathcal{P}_G$ and an orthogonal matrix A , the polynomial $G(x) = A^{-1}U(Ax)$ also lies in \mathcal{P}_G .

To see this, notice that if $U(x) = \nabla u(x)$ and A is a $q \times q$ matrix, then $g(x) := u(Ax)$ has gradient $A^\top U(Ax)$ and so $G(x) = \nabla u(Ax)$ if A is orthogonal.

The following lemma is a quantitative application of Proposition 1 in [7].

Lemma 2.6. *The real polynomials $\{Q_k\}_{k=1}^\infty$ uniquely determine a sequence of polynomials $\{p_k\}_{k=1}^\infty \in \mathcal{P}_G^\infty$ s.t. $\forall r \in \mathbb{Z}^+$ and ε sufficiently small,*

- (i) $|p_k(x)| \leq C_{q,k} \lambda_1^{-5k(k+2)} \lambda_q^{1+\frac{5}{2}k(k+2)} \kappa_{r+2}^{k^2} (1 + |x|^{3k})$ for all $k = 1, \dots, r$ and $x \in \mathbb{R}^q$;
- (ii) The random variable $\mathbf{p}_{\varepsilon,r}(\xi_\Sigma) := \xi_\Sigma + \sum_{k=1}^r \varepsilon^k p_k(\xi_\Sigma)$ has density $\zeta_{\varepsilon,r}(x) = \phi_\Sigma(x) \mathcal{Q}_{\varepsilon,r}(x) + R_{\varepsilon,r}(x)$, where for any $M \geq 1$,

$$\int_{\mathbb{R}^q} |x|^M |R_{\varepsilon,r}(x)| dx \leq C_{q,r,M} \lambda_1^{-5(r+1)(r+2)} \lambda_q^{\frac{5}{2}(r+1)(r+3) + \frac{M}{2}} \kappa_{r+2}^{(r+1)^2} \varepsilon^{r+1}.$$

Proof. First of all, the Edgeworth polynomials $\{Q_k\}$ defined by (2.12) are orthogonal to ϕ_Σ :

$$\int_{\mathbb{R}^q} \phi_\Sigma(x) Q_k(x) dx = \widehat{\phi_\Sigma Q_k}(0) = 1 \cdot P_k(0) = 0.$$

Thus $\{Q_k\} \in \mathcal{P}_\Sigma^\infty$, and hence $\{p_k\} := \mathfrak{S}_\Sigma^{-1}(\{Q_k\})$ gives the sequence sought after; for a fixed r , p_1, \dots, p_r are determined by Q_1, \dots, Q_r only. Moreover, if $\mathfrak{S}_\Sigma(p_1, \dots, p_r) = (Q_1, \dots, Q_r, \tilde{Q}_{r+1}, \dots)$, then the density $\zeta_{\varepsilon,r}$ of $\mathbf{p}_{\varepsilon,r}(\xi_\Sigma)$ can be approximated by the expansion $\phi_\Sigma(\mathcal{Q}_{\varepsilon,r} + \varepsilon^{r+1} \tilde{Q}_{r+1})$ according to Proposition 1 in [7]. More precisely, $\forall M \geq 1$,

$$\int_{\mathbb{R}^q} |x|^M |\zeta_{\varepsilon,r}(x) - \phi_\Sigma(x)(\mathcal{Q}_{\varepsilon,r}(x) + \varepsilon^{r+1} \tilde{Q}_{r+1}(x))| dx \leq C_{q,r,M} K_r^{N_r} \varepsilon^{r+2}, \quad (2.17)$$

where K_r is an upper bound for $\|\Sigma\|, \|\Sigma^{-1}\|$ and the absolute value of the coefficients of p_1, \dots, p_r , and $N_r = N_r(q, M) > 0$ is a constant depending on the maximum degree of p_1, \dots, p_r . Then for $\varepsilon \leq K_r^{-N_r}$ this bound can be brought down to $C_{q,r,M} \varepsilon^{r+1}$, and it remains to find an upper bound for \tilde{Q}_{r+1} to derive the estimates in question.

For all $k \leq r$, write $p_k = \nabla u_k$ where u_k satisfies (2.14) with $S_k \equiv Q_k$ and $\tilde{S}_k \equiv \tilde{Q}_k$. Assume that Σ is diagonal. Then by (2.13), $\forall k, x$ one has $|Q_k(x)| \leq C_{q,k} \lambda_1^{-3k} \kappa_{k+2}^k (1 + |x|^{3k})$. Now one can bound the polynomials $\{\tilde{Q}_k\}$ and $\{u_k\}$ inductively. For each $k \leq r-1$ suppose that (i) holds true for all $j \leq k$:

$$|u_j(x)| \leq C_{q,j} \lambda_1^{-5j(j+2)} \lambda_q^{1+\frac{5}{2}j(j+2)} \kappa_{r+2}^{j^2} (1 + |x|^{3j}).$$

From the construction of T_{k+1} and $\{w_{j,l}\}$ one sees that,

$$\begin{aligned} |T_{k+1}(x)| &\leq C_{q,k} \|\Sigma^{-1}\|^{k+1} \lambda_1^{-5 \sum^* j(j+2) \rho_j} \lambda_q^{k+1+\frac{5}{2} \sum^* j(j+2) \rho_j} \kappa_{r+2}^{\sum^* j^2 \rho_j} \left(1 + |x|^{\sum^* 3j \rho_j}\right) \\ &\leq C_{q,k} \lambda_1^{-(k+1)(5k+11)} \lambda_q^{(k+1)(\frac{5}{2}k+6)} \kappa_{r+2}^{(k+1)^2} \left(1 + |x|^{3(k+1)}\right), \\ |w_{j,l}(x)| &\leq C_{q,j,l} \lambda_1^{-3j-5 \sum^\dagger s(s+2) \rho_s} \lambda_q^{l+\frac{5}{2} \sum^\dagger s(s+2) \rho_s} \kappa_{r+2}^{j+\sum^\dagger s^2 \rho_s} \left(1 + |x|^{3j+\sum^\dagger 3s \rho_s}\right) \\ &\leq C_{q,j,l} \lambda_1^{-3j-5l(l+2)} \lambda_q^{l+\frac{5}{2}l(l+2)} \kappa_{r+2}^{j+l(l+1)} \left(1 + |x|^{3(j+l)}\right), \end{aligned} \quad (2.18)$$

where \sum^* denotes the summation over $j = 1 \dots, k$ and all multi-indices $\rho \in \mathbb{N}^k$ s.t. $|\rho|_* = k+1$, and \sum^\dagger denotes the summation over $s = 1, \dots, l$ and all $\rho \in \mathbb{N}^l$ s.t. $|\rho|_* = l$. Then $\left| \sum_{j+l=k+1} w_{j,l}(x) \right| \leq$

$C_{q,k} \lambda_1^{-5(k+1)(k+2)} \lambda_q^{k(\frac{5}{2}k+6)} \kappa_{r+2}^{(k+1)^2} (1 + |x|^{3(k+1)})$, which is no more than (2.18), and hence by (2.16) \tilde{Q}_{k+1} has the same bound as (2.18).

For each $\alpha \in \mathbb{N}^q$, it is known that the Hermite-type polynomial

$$H_{\alpha,\Sigma}(x) = \frac{1}{\sqrt{\alpha!}} \prod_{j=1}^q H_{\alpha_j}(\lambda_j^{-1/2} x_j)$$

is the eigenfunction of the differential operator of the equation (2.14) corresponding to the eigenvalue $\nu_\alpha := \sum_{j=1}^q \alpha_j \lambda_j^{-1} \leq |\alpha|/\lambda_q$. Since $\{H_{\alpha,\Sigma}\}$ form an orthonormal basis for the Hilbert space $L^2(\mathbb{R}^q, \phi_\Sigma d\Lambda)$, the polynomial u_{k+1} can be expressed as

$$u_{k+1}(x) = \sum_{|\alpha| \leq 3(k+1)} c_\alpha \nu_\alpha^{-1} H_{\alpha,\Sigma}(x),$$

where $c_\alpha = \int_{\mathbb{R}^q} (Q_{k+1}(z) - \tilde{Q}_{k+1}(z)) H_{\alpha,\Sigma}(z) \phi_\Sigma(z) dz$. Then by the Cauchy-Schwartz inequality and (2.8), the above estimate for \tilde{Q}_{k+1} implies that

$$\begin{aligned} |u_{k+1}(x)| &\leq C_{q,k} \sum_{|\alpha| \leq 3(k+1)} C_\alpha \left(\int_{\mathbb{R}^q} |Q_{k+1}(z) - \tilde{Q}_{k+1}(z)|^2 \phi_\Sigma(z) dz \right)^{\frac{1}{2}} \lambda_q \lambda_1^{-\frac{|\alpha|}{2}} (1 + |x|^{|\alpha|}) \\ &\leq C_{q,k} \lambda_1^{-(k+1)(5k+11) - \frac{3}{2}(k+1)} \lambda_q^{(k+1)(\frac{5}{2}k+6) + \frac{3}{2}(k+1)+1} \kappa_{r+2}^{(k+1)^2} (1 + |x|^{3(k+1)}) \\ &\leq C_{q,k} \lambda_1^{-5(k+1)(k+3)} \lambda_q^{1 + \frac{5}{2}(k+1)(k+3)} \kappa_{r+2}^{(k+1)^2} (1 + |x|^{3(k+1)}), \end{aligned}$$

which agrees with the induction hypothesis; the initial step for u_1 also holds true as $\tilde{Q}_1 \equiv 0$. Therefore the bound in (i) holds true for each u_k , and it holds true for its gradient p_k , too. The induction step also gives the bound (2.18) for \tilde{Q}_{r+1} , and hence (ii) follows from the triangle inequality and (2.8) again.

For a general positive-definite Σ , one diagonalises it with an orthogonal matrix A and applies the same arguments above to the scaled polynomials $p_k^*(x) := A^\top p_k(Ax)$. By Remark 2.5 the p_k^* 's still lie in \mathcal{P}_G , and the results still hold. \square

The proof above takes a compromise approach by introducing \tilde{Q}_{r+1} in (2.17): the condition “ ε sufficiently small” is not needed for Lemma 2.6, as Proposition 1 in [7] allows an $O(\varepsilon^{r+1})$ estimate for $\int_{\mathbb{R}^q} |x|^M |\zeta_{\varepsilon,r}(x) - \phi_\Sigma(x) \mathcal{Q}_{\varepsilon,r}(x)| dx$ for all $\varepsilon > 0$. However, whilst practically $K_r = \lambda_1^{-5r(r+2)} \lambda_q^{1+5r(r+2)/2} \kappa_{r+2}^{r^2}$ by (i), it is rather complicated to compute N_r explicitly.

Before proceeding to the main result, given fixed parameters $\beta \in (0, 1/3)$ and $\bar{\gamma}, \tau \in (0, 1)$, it would be convenient to combine all the criteria for ε together: for any integer $r \geq 3$ the statement “ m **sufficiently large w.r.t. r** ” or “ ε **sufficiently small w.r.t. r** ” refers to that $m > \kappa_{r+\tau}^{\max\{4, 6/(r(1-3\beta))\}} \vee K_{r-3}^{2N_{r-3}}$ with $K_0, N_0 := 1$ and that (2.11) holds for $n = r$.

2.3. Main result and some special cases

Given Lemma 2.6, it will be shown in the following theorem that the normal distribution $\mathcal{N}(0, \Sigma)$ perturbed by the polynomials $\{p_k\}$ is close to the law \mathbb{P}_m in the Vaserstein distances. The proof is a more detailed and quantitative version of what is exhibited in Section 4 in [7], and specifies the dependence on Σ and certain moments of X .

Theorem 2.7. *Suppose Σ is non-singular and χ satisfies $\mathbf{CC}(\rho, \gamma)$ for some $\rho > 0, \gamma \in (0, 1)$. Fix an integer $n \geq 3$, an even integer $p \in 2\mathbb{Z}^+$ and $\beta \in (0, 1/3)$. If $\mathbb{E}|X|^{p(n-2)+2+\tau} < \infty$ for some $\tau \in (0, 1)$, then for m sufficiently large w.r.t. $p(n-2) + 2$,*

$$\mathbb{W}_p(Y_m, \mathbf{p}_{m,n-3}(\xi_\Sigma)) \leq C_{p,q,n,\tau} \Xi_X m^{-(n-2)/2},$$

where $\mathbf{p}_{m,n-3}$ is the polynomial defined by Lemma 2.6, and Ξ_X is a constant depending on p, q, n, β, Σ and moments of X up to order $p(n-2) + 2 + \tau$.

Proof. Denote $r = p(n-2) + 2$. Taking the inverse Fourier transform, Proposition 2.3 implies that for all $x \in \mathbb{R}^q$ and for m sufficiently large w.r.t. r ,

$$\begin{aligned} |F_{r-2}(\varepsilon, x)| &:= \left| \tilde{f}_m(x) - \phi_\Sigma(x) \mathcal{Q}_{\varepsilon, r-2}(x) \right| \leq C_q \int_{\mathbb{R}^q} \left| \tilde{\psi}_m(z) - e^{-\frac{1}{2}z \cdot \Sigma z} \mathcal{P}_{\varepsilon, r-2}(z) \right| dz \\ &\leq C_{q,r,\tau} (\det \Sigma)^{-\frac{1}{2}} \lambda_1^{-\frac{r-1}{2\beta}} \kappa_{r+\tau}^{r-2} \varepsilon^{r-2+\tau}. \end{aligned} \quad (2.19)$$

The goal is to use the inequality (1.6) to bound the \mathbb{W}_p distance, for which one first writes

$$\int_{\mathbb{R}^q} |x|^p |F_{r-2}(\varepsilon, x)| dx \leq \int_{\mathbb{R}^q} |x|^p \left(\tilde{f}_m(x) + \phi_\Sigma(x) |\mathcal{Q}_{\varepsilon, r-2}(x)| \right) dx \leq I_1 + 2I_2 + I_3,$$

where, for any $\eta \in (0, \tau)$,

$$\begin{aligned} I_1 &:= \int_{|x| \leq \varepsilon^{-\eta/(p+q)}} |x|^p |F_{r-2}(\varepsilon, x)| dx, & I_2 &:= \int_{|x| > \varepsilon^{-\eta/(p+q)}} |x|^p \phi_\Sigma(x) |\mathcal{Q}_{\varepsilon, r-2}(x)| dx, \\ I_3 &:= \int_{|x| > \varepsilon^{-\eta/(p+q)}} |x|^p F_{r-2}(\varepsilon, x) dx. \end{aligned}$$

For any fixed $p \geq 2$ and $\eta \in (0, \tau)$, one finds, by virtue of (2.19),

$$I_1 \leq C_{q,r,\tau} (\det \Sigma)^{-\frac{1}{2}} \lambda_1^{-\frac{r-1}{2\beta}} \kappa_{r+\tau}^{r-2} \varepsilon^{r-2+\tau-\eta}.$$

For the integral on the complement $\{x : |x| > \varepsilon^{-\eta/(p+q)}\} = \{1 < \varepsilon |x|^{(p+q)/\eta}\}$, one gets

$$\begin{aligned} I_2 &\leq \int_{|x| > \varepsilon^{-\eta/(p+q)}} |x|^p \phi_\Sigma(x) \kappa_r^{r-2} \left(1 + \sum_{k=1}^{r-2} \varepsilon^k |x|^{3k} \right) dx \\ &\leq C_r \int_{|x| > \varepsilon^{-\eta/(p+q)}} |x|^p \phi_\Sigma(x) \kappa_r^{r-2} \varepsilon^{r-1} |x|^{\frac{p+q}{\eta}(r-1)} dx \leq C_r \lambda_q^{\frac{p}{2} + \frac{p+q}{2\eta}(r-1)} \kappa_r^{r-2} \varepsilon^{r-1}, \end{aligned} \quad (2.20)$$

due to the fact that $(p+q)/\eta > 3$ and (2.8). Also observe that

$$I_3 \leq \int_{\mathbb{R}^q} |x|^p \left(\tilde{f}_m(x) - \phi_\Sigma(x) \mathcal{Q}_{\varepsilon, r-2}(x) \right) dx + I_1 =: I_4 + I_1,$$

by the triangle inequality. In order to get a good estimate for I_4 , first observe that $\forall p \geq 2$ by Rosenthal's inequality – see *e.g.* Lemma 1 in [11],

$$\int_{\mathbb{R}^q} |x|^p \mathbb{P}_m(dx) = \mathbb{E}|Y_m|^p = m^{-\frac{p}{2}} \mathbb{E} \left| \sum_{j=1}^m X_j \right|^p \leq C_p \left(m^{1-\frac{p}{2}} \mathbb{E}|X|^p + (\mathbb{E}|X|^2)^{\frac{p}{2}} \right). \quad (2.21)$$

Also, from the construction of θ_ε (2.9) (now supported on $\{|x| < \varepsilon^{r+1}\}$),

$$\int_{\mathbb{R}^q} |x|^p \theta_\varepsilon(dx) = \int_{|y| < 1} |y|^p \varepsilon^{p(r+1)} h(y) dy < \varepsilon^{p(r+1)}, \quad (2.22)$$

by a change of variables. For an even $p \geq 4$, as $p < r$ observe that all the moments up to p of the expansion $\phi_\Sigma \mathcal{Q}_{\varepsilon, r-2}$ match those of Y_m by Remark 2.4. Hence by (2.21) and (2.22),

$$\begin{aligned} I_4 &\leq \int_{\mathbb{R}^q} \int_{\mathbb{R}^q} (|x+y|^p - |x|^p) \mathbb{P}_m(dx) \theta_\varepsilon(dy) \leq C_{p,q} \sum_{k=1}^p \int_{\mathbb{R}^q} \int_{\mathbb{R}^q} |x|^{p-k} |y|^k \mathbb{P}_m(dx) \theta_\varepsilon(dy) \\ &\leq C_{p,q} \varepsilon^{r+1} \int_{\mathbb{R}^q} |x|^{p-1} \mathbb{P}_m(dx) \leq C_{p,q} \varepsilon^{r+1} \left(\varepsilon^{p-3} \mathbb{E}|X|^{p-1} + \lambda_q^{\frac{p-1}{2}} \right); \end{aligned}$$

for $p = 2$ the bound is reduced to $C_{p,q} \varepsilon^{r+1} (\mathbb{E}|X|^2)^{1/2}$ by (2.21) and Hölder's inequality. Therefore, altogether one arrives at, for $p \leq r$,

$$\int_{\mathbb{R}^q} |x|^p |F_{r-2}(\varepsilon, x)| dx \leq C_{p,q,r,\tau} \left((\det \Sigma)^{-\frac{1}{2}} \lambda_1^{-\frac{r-1}{2\beta}} + \lambda_q^{\frac{p}{2} + \frac{p+q}{2\eta}(r-1)} \right) \kappa_{r+\tau}^{r-2} \varepsilon^{r-2+\tau-\eta}.$$

Finally by the triangle inequality one removes the $(r-2)$ -th term in $\mathcal{Q}_{\varepsilon, r-2}$:

$$\int_{\mathbb{R}^q} |x|^p |F_{r-2}(\varepsilon, x)| dx \geq \int_{\mathbb{R}^q} |x|^p |F_{r-3}(\varepsilon, x)| dx - C_{q,r} \varepsilon^{r-2} \kappa_r^{r-2} \int_{\mathbb{R}^q} |x|^p (|x|^r + |x|^{3(r-2)}) \phi_\Sigma dx,$$

and uses (2.8) again to deduce the following estimate for the Edgeworth approximation:

$$\int_{\mathbb{R}^q} |x|^p |F_{r-3}(\varepsilon, x)| dx \leq C_{p,q,r} \left((\det \Sigma)^{-\frac{1}{2}} \lambda_1^{-\frac{r-1}{2\beta}} + \lambda_q^{\frac{p}{2} + \frac{p+q}{2\eta}(r-1)} \right) \kappa_{r+\tau}^{r-2} \varepsilon^{r-2}.$$

Since the smooth measure θ_ε is also supported on $\{x : |x| < \varepsilon^{r-2}\}$, the estimate above implies that the Edgeworth polynomials $\{Q_k\} \in \mathcal{P}_\Sigma$ form an \mathcal{A}_Σ -sequence for the family of probability measures $\{\mathbb{P}_m\}$ – see Definition 1 in [7]. Then one can extend the expansion $\mathcal{Q}_{\varepsilon, r-3}$ to a larger value of r and take the p th root to get a \mathbb{W}_p estimate, as in done in the proof of Theorem 4 in [7].

With the polynomials $\{p_k\} = \mathfrak{G}_\Sigma^{-1}(\{Q_k\})$ and $\mathbf{p}_{\varepsilon, r-3}$, $R_{\varepsilon, r-3}$ defined as in Lemma 2.6, using the triangle inequality and the inequality (1.6), one can deduce the following estimate for an integer $n \geq 3$ by replacing $r = p(n-2) + 2$ in the estimate:

$$\begin{aligned} \mathbb{W}_p(Y_m, \mathbf{p}_{\varepsilon, p(n-2)-1}(\xi_\Sigma)) &\leq C_p \left(\int_{\mathbb{R}^q} |x|^p |F_{p(n-2)-1}| dx + \int_{\mathbb{R}^q} |x|^p |R_{\varepsilon, p(n-2)-1}(x)| dx \right)^{\frac{1}{p}} \\ &\leq C_{p,q,n,\tau} \left((\det \Sigma)^{-\frac{1}{2p}} \lambda_1^{-\frac{1}{2\beta}(n-2+\frac{1}{p})} + \lambda_q^{\frac{1}{2} + \frac{p+q}{2\eta}(n-2+\frac{1}{p})} \right) \kappa_{p(n-2)+2+\tau}^{n-2} \\ &\quad + C_{p,q,n} \lambda_1^{-5(n-2)(p(n-2)+1)} \lambda_q^{\frac{1}{2} + \frac{5}{2}(n-2)(p(n-2)+1)} \kappa_{p(n-2)+1}^{p(n-2)^2} \varepsilon^{n-2}, \end{aligned}$$

whilst the excess terms from $n - 2$ to $p(n - 2) - 1$ can be handled by (i) of Lemma 2.6 and (2.8) again:

$$\begin{aligned} \mathbb{W}_p(\mathbf{p}_{\varepsilon, p(n-2)-1}(\xi_\Sigma), \mathbf{p}_{\varepsilon, n-3}(\xi_\Sigma)) &\leq C_p \left(\mathbb{E} \left| \sum_{k=n-2}^{p(n-2)-1} \varepsilon^k p_k(\xi_\Sigma) \right|^p \right)^{\frac{1}{p}} \\ &\leq C_{p,q,n} \lambda_1^{-5p^2(n-2)^2+5} \lambda_q^{\frac{5}{2}p^2(n-2)^2+\frac{3}{2}p(n-2)-3} \kappa_{p(n-2)+1}^{(p(n-2)-1)^2} \varepsilon^{n-2}. \end{aligned}$$

Thus the claimed result follows from the triangle inequality. \square

Remark 2.8. The highest moment of X needed for Theorem 2.7 is independent of the dimension q .

The optimal result for the central limit theorem for $q = 1$ is already given in [3], which is not fully recovered by Theorem 2.7 as the inequality (1.6) is rather crude compared to (1.5). For $q \geq 2$, Theorem 2.7 immediately implies the following (by choosing $n = 3$):

Corollary 2.9. *Suppose the i.i.d. random variables $\{X_j\}$ have non-singular covariance Σ and satisfy $\mathbf{CC}(\rho, \gamma)$ for some $\rho > 0, \gamma \in (0, 1)$, and let $p \in 2\mathbb{Z}^+$. If $\mathbb{E}|X|^{p+2+\tau} < \infty$ for some $\tau \in (0, 1)$, then by taking e.g. $\beta = 1/6, \eta = \tau/2$, the following holds for m sufficiently large w.r.t. $p + 2$:*

$$\begin{aligned} \mathbb{W}_p(Y_m, \xi_\Sigma) &\leq C_{p,q,\tau} \left((\det \Sigma)^{-1/(2p)} \lambda_1^{-3(1+1/p)} + \lambda_q^{(p+q)(1+1/p)/\tau+1/2} \right) \kappa_{p+2+\tau} m^{-1/2} \\ &\quad + C_{p,q} \lambda_1^{-5p^2+5} \lambda_q^{(5p^2+3p)/2-3} \kappa_{p+1}^{2V(p-1)^2} m^{-1/2}. \end{aligned}$$

As mentioned in the introduction, for the special case $p = 2$ this corollary is weaker than the results of Bobkov [4] and Bonis [5]. Although the condition $\mathbb{E}|X|^{4+\tau} < \infty$ is slightly better than that of Bobkov, he does not require Cramér's condition as he aimed at estimating the relative entropy $\mathbb{H}(\mathbb{P}_m \| \mathcal{N}(0, \Sigma))$, which in turn gives a bound for the \mathbb{W}_2 distance according to Talagrand's transport inequality [19]. On the other hand, Bonis' optimal result relies on a differential estimate for the \mathbb{W}_2 distance in terms of the Fisher information. As the \mathbb{W}_2 distance enjoys those special properties, the inequality (1.6) does not provide a sharp bound.

However, Theorem 2.7 can potentially give higher-order convergence if one considers a non-trivial expansion ($n > 3$). For example, when choosing $n = 4$, one gets a rate $O(m^{-1})$ under Cramér's condition and that $\mathbb{E}|X|^{6+\tau} < \infty$. The task is to find the polynomial p_1 using the method described in the discussion before Lemma 2.6: given $Q_1(x)$ defined in (2.13), one looks for the unique (up to an additive constant) polynomial solution $u_1 : \mathbb{R}^q \rightarrow \mathbb{R}$ satisfying (2.14) for the initial step:

$$-\Delta u_1(x) + x \cdot \Sigma^{-1} \nabla u_1(x) = Q_1(x).$$

To illustrate that consider the simplest case where $q = 2$ and $\Sigma = I$. The cubic polynomial $6iP_1(z) = \mu_{(3,0)} z_1^3 + 3\mu_{(2,1)} z_1^2 z_2 + 3\mu_{(1,2)} z_1 z_2^2 + \mu_{(0,3)} z_2^3$ gives

$$6Q_1(x) = \mu_{(3,0)} H_3(x_1) + 3\mu_{(2,1)} H_2(x_1) H_1(x_2) + 3\mu_{(1,2)} H_1(x_1) H_2(x_2) + \mu_{(0,3)} H_3(x_2).$$

Notice that $x \cdot \nabla u(x) = ku(x)$ for any monomial u of degree k , and so the polynomial solution to the PDE above is cubic with no quadratic terms. Then using the property $H'_j = jH_{j-1}$ and matching the coefficients on both sides of the equation, one gets

$$\begin{aligned} u_1(x) &= \frac{1}{18} \mu_{(3,0)} H_3(x_1) + \frac{1}{6} \mu_{(2,1)} H_2(x_1) H_1(x_2) + \frac{1}{6} \mu_{(1,2)} H_1(x_1) H_2(x_2) + \frac{1}{18} \mu_{(0,3)} H_3(x_2) \\ &\quad + \frac{1}{3} (\mu_{(3,0)} + \mu_{(1,2)}) H_1(x_1) + \frac{1}{3} (\mu_{(0,3)} + \mu_{(2,1)}) H_1(x_2) + C, \end{aligned}$$

and hence the perturbing polynomial $p_1 = \nabla u_1$ is found.

Under certain stronger conditions, one can also obtain higher-order convergence without specifying the perturbing polynomials p_k . For $q = 1$, Bobkov [3] (Thm. 1.3) proved that if $\mathbb{E}X^k = \mathbb{E}\xi_\Sigma^k$ for $k = 1, \dots, n-1$, $3 \leq n \in \mathbb{Z}^+$, and $\mathbb{E}|X|^{p(n-2)+2} < \infty$, then under Cramér's condition one has $\mathbb{W}_p(Y_m, \xi_\Sigma) = O(m^{-(n-2)/2})$. This can be readily generalised to $q \geq 2$ by Theorem 2.7: if the first $n-1$ moments match those of $\mathcal{N}(0, \Sigma)$, the cumulants $\mu_\alpha(X) = \mu_\alpha(\xi_\Sigma) = 0$ for all $|\alpha| = 3, \dots, n-1$, implying that $P_k = Q_k \equiv 0$. This immediately implies that $\mathcal{L}_\Sigma p_k \equiv 0$ in (2.14) for all $k = 1, \dots, n-3$, forcing $p_k \equiv 0$. Therefore one asserts the following:

Corollary 2.10. *Suppose the i.i.d. random variables $\{X_j\}$ with non-singular covariance Σ satisfy Cramér's condition and let $p \in 2\mathbb{Z}^+$. If $\exists 3 \leq n \in \mathbb{Z}^+$ s.t. $\mathbb{E}X^\alpha = \mathbb{E}\xi_\Sigma^\alpha$ for all $|\alpha| = 1, \dots, n-1$, and $\mathbb{E}|X|^{p(n-2)+2+\tau} < \infty$ for some $\tau \in (0, 1)$, then there exists a constant C depending on p, q, τ, Σ and the moments of X up to order $p(n-2) + 2 + \tau$ s.t.*

$$\mathbb{W}_p(Y_m, \xi_\Sigma) \leq Cm^{-(n-2)/2},$$

for m sufficiently large w.r.t. $p(n-2) + 2$.

3. APPLICATION TO THE EULER APPROXIMATION FOR LÉVY SDES

Consider the d -dimensional SDE (1.2) driven by a q -dimensional Lévy process (1.1). Assume that the Lévy measure ν has finite second moment, and the function $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times q}$ is bounded and Lipschitz. It will be shown in this section that the q -dimensional small jumps (1.3) can also be approximated by a normal random variable with rate 1 while the computational cost $E_\nu(h)$ remains controlled for ν satisfying certain stable-like conditions, in particular $\mathbf{H}(\tau, \alpha)$.

3.1. Normal approximation of the small jumps

The way both Fournier [9] and Godinho [10] applied the central limit theorem for the small jumps Z_t^ϵ is to split the time interval into m subdivisions and view Z_t^ϵ as the sum of the i.i.d. random variables $\int_{(j-1)t/m}^{jt/m} \int_{0 < |z| \leq \epsilon} z \tilde{N}(dz, ds)$, $j = 1, \dots, m$. Here an alternative approach is considered: one may decompose the range of the jumps $\{0 < |z| \leq \epsilon\}$ into countably many annuli and represent the small jumps as a sum:

$$Z_t^\epsilon = \sum_{r=r_0}^{\infty} \int_0^t \int_{\Omega_r} z \tilde{N}(dz, ds) =: \sum_{r=r_0}^{\infty} V_t^r, \quad (3.1)$$

where $\Omega_r = \{2^{-r-1} < |z| \leq 2^{-r}\}$ and $r_0 = -\log_2 \epsilon > 0$. Assume ν to be σ -finite and denote $\nu_r := \nu(\Omega_r)$. By the Lévy-Itô decomposition one knows that each V_t^r is a compensated compound Poisson process with parameter $t\nu_r$:

$$V_t^r = \sum_{j=1}^{N_t^r} X_j^r - t\nu_r \mathbb{E}X^r, \quad (3.2)$$

where $\{X_j^r\}$ are i.i.d. random variables bounded within Ω_r and N_t^r is an independent $\text{Poi}(t\nu_r)$ -random variable.

Instead of directly working with V_t^r , one may first consider a general compensated compound Poisson process V_t of the form (3.2) with parameter μ , $N_t \sim \text{Poi}(t\mu)$ and the jumps X_j on the interval $[0, 1]$. Expecting μ to be large, one can write

$$Y = \mu^{-\frac{1}{2}} V_1 = \mu^{-\frac{1}{2}} \sum_{j=1}^{N_1} X_j - \mu^{\frac{1}{2}} \mathbb{E}X,$$

and approximate it by Edgeworth-type polynomials using the same recipe just as before.

Let ψ and χ be the characteristic functions of Y and the X_j 's, respectively, and Σ_X be the covariance matrix of X , with eigenvalues $\lambda_{1,X} \leq \dots \leq \lambda_{q,X}$, and similarly let $\kappa_{M,X} := 1 \vee \mathbb{E}|X|^M$, $\forall M > 0$. Then, by conditioning on the value of N_1 and taking expectation, one can write down the following simple relation between the distributions of X and Y :

$$\psi(z) = \exp \left\{ \mu \left(\chi \left(\mu^{-\frac{1}{2}} z \right) - 1 \right) - i \mu^{\frac{1}{2}} z \cdot \mathbb{E}X \right\}.$$

Given this convenient expression, instead of taking the logarithm one may directly apply Taylor expansion to χ and find a formal expansion identical to (2.1) (with μ in place of m):

$$\psi(z) \sim e^{-\frac{1}{2}z \cdot \Sigma_X z} \left(1 + \sum_{k=1}^{\infty} \mu^{-\frac{k}{2}} P_k(z) \right).$$

Then the $(n-2)$ -th truncation of the series above satisfies the second bound in Lemma 2.1 for $|z| \leq \mu^{\beta/2}$, $\beta \in (0, 1/3)$. Note that the P_k here are slightly different (in fact simpler): since no logarithm is taken, the cumulants μ_α are replaced with just $\mathbb{E}X^\alpha$. For $|z| \leq \mu^{1/2}\delta = \mu^{1/2} \min\{\lambda_{1,X}/\kappa_{3,X}, \kappa_{n,X}^{-1/n}/2\}$ the first bound in Lemma 2.1 still holds since

$$\begin{aligned} |\psi(z)| &= \left| \exp \left\{ \mu \left(\chi \left(\mu^{-\frac{1}{2}} z \right) - 1 \right) - i \mu^{\frac{1}{2}} z \cdot \mathbb{E}X \right\} \right| \\ &\leq e^{-\frac{1}{2}z \cdot \Sigma_X z + \frac{1}{6}\mu^{-\frac{1}{2}}\mathbb{E}|X|^3|z|^3} \leq e^{-\frac{1}{4}z \cdot \Sigma_X z}. \end{aligned}$$

Moreover, by imposing Cramér's condition $\mathbf{CC}(\rho, \gamma)$ on the distribution of X , one can still achieve a similar bound for $|\psi|$:

$$\begin{aligned} |\psi(z)| &= \left| \exp \left\{ \chi \left(\mu^{-\frac{1}{2}} z \right) - 1 - i \mu^{-\frac{1}{2}} z \cdot \mathbb{E}X \right\} \right|^\mu \\ &= \left(\exp \left\{ \operatorname{Re} \chi \left(\mu^{-\frac{1}{2}} z \right) - 1 \right\} \right)^\mu \leq (e^{\bar{\gamma}-1})^\mu \in (0, 1), \end{aligned} \quad (3.3)$$

for $|z| > \mu^{1/2}\delta$ and some $\bar{\gamma} \in (0, 1)$ according to Lemma 2.2. Thus one arrives at virtually the same estimate as in Proposition 2.3, and therefore Theorem 2.7 still holds true for $\varepsilon = \mu^{-1/2}$ sufficiently small w.r.t. $p+2$, and Corollary 2.9 applies with μ in place of m and $\exp(\bar{\gamma}-1)$ in place of $\bar{\gamma}$.

For the normal approximation ($n=3$), since no perturbing polynomials p_k are concerned, one can scale the jumps $\widehat{X} := \Sigma_X^{-1/2}X$ and deduce, $\forall \tau \in (0, 1)$,

$$\mathbb{W}_p \left(V_1, \mu^{\frac{1}{2}} \xi_{\Sigma_X} \right) \leq \|\Sigma_X^{\frac{1}{2}}\| \mathbb{W}_p \left(\Sigma_X^{-\frac{1}{2}} V_1, \mu^{\frac{1}{2}} \xi_I \right) \leq C_{p,q,\tau} \kappa_{p+2+\tau, \widehat{X}}^{2\vee(p-1)^2} \lambda_{q, \widehat{X}}^{1/2}. \quad (3.4)$$

One can apply the above arguments to the jump process (3.2) by scaling the jump sizes. For the jumps X_j^r on each annulus Ω_r , define $X_j := 2^r X_j^r$ and $\widehat{X}_j := \Sigma_X^{-1/2} X_j$ accordingly. For each fixed r , the X_j^r 's are i.i.d. with characteristic function

$$\chi^r(s) = \nu_r^{-1} \int_{\Omega_r} e^{is \cdot x} \nu(dx).$$

This implies that X has scaled covariance $\Sigma_X = \nu_r^{-1} 2^{2r} \int_{\Omega_r} x x^\top \nu(dx)$ with eigenvalues $\lambda_{j,X} = \nu_r^{-1} 2^{2r} \lambda_{j,r}$, where $\lambda_{1,r} \leq \dots \leq \lambda_{q,r}$ are the eigenvalues of $\Sigma_r := \int_{\Omega_r} x x^\top \nu(dx)$. Also notice that $\mathbb{E}|X|^M = \nu_r^{-1} 2^{rM} \int_{\Omega_r} |x|^M \nu(dx) \leq 1$, $\forall M > 0$, implying that $\mathbb{E}|\widehat{X}|^M \leq \lambda_{1,X}^{-M/2}$.

Thus, if Σ_r is non-singular for each r , then (assuming $\lambda_{1,X} \leq 1$ w.l.o.g.) one can apply (3.4) with parameter $\mu = t\nu_r$:

$$\begin{aligned} \mathbb{W}_p \left(V_t^r, \sqrt{t} \xi_{\Sigma_r} \right) &= 2^{-r} \mathbb{W}_p \left(2^r V_t^r, \sqrt{t\nu_r} \xi_{\Sigma_X} \right) \\ &\leq C_{p,q,\tau} \lambda_{1,X}^{-\left(1 + \frac{(p-1)^2}{2}\right)(p+2+\tau)} \lambda_{q,r}^{1/2} \nu_r^{-1/2}. \end{aligned} \quad (3.5)$$

Denote further $\Sigma_\epsilon := \int_{0 < |x| \leq \epsilon} x x^\top \nu(dx)$, then from this bound one can find a coupling between Z_t^ϵ and $\mathcal{N}(0, t\Sigma_\epsilon)$ under suitable conditions.

Theorem 3.1. *Suppose $\xi_r(s) := \chi^r(2^r s)$ satisfies **CC**(ρ, γ) for some $\rho > 0, \gamma \in (0, 1)$ uniformly for all $r \geq r_0$. If $\nu_r^{-1} = o(2^{-r})$ as $r \rightarrow \infty$, then $\forall p \in 2\mathbb{Z}^+, t \geq \epsilon$ and ϵ sufficiently small,*

$$\mathbb{W}_p \left(Z_t^\epsilon, \sqrt{t} \xi_{\Sigma_\epsilon} \right) \leq C_{p,q} \epsilon.$$

Proof. Note that on each Ω_r it is always true that $\lambda_{q,r} \leq \text{tr} \Sigma_r \leq 2^{-2r} \nu_r$ and $\lambda_{q,r} \geq q^{-1} \text{tr} \Sigma_r \geq q^{-1} 2^{-2(r+1)} \nu_r$. Write $\xi_r(s) = |\xi_r(s)| e^{i\theta}$, where $\theta = \theta(r, s)$ satisfies $\int_{\Omega_r} \sin(2^r s \cdot x - \theta) \nu(dx) = 0$. Then $\int_{\Omega_r} \sin(2^r s \cdot x) \nu(dx) = \tan \theta \int_{\Omega_r} \cos(2^r s \cdot x) \nu(dx)$ if $\theta \not\equiv \pm\pi/2 \pmod{\pi}$, and otherwise $\int_{\Omega_r} \cos(2^r s \cdot x) \nu(dx) = 0$. By the uniform Cramér's condition for $\xi_r(s)$, there exist $\rho > 0, \gamma \in (0, 1)$ s.t. $\forall r \geq r_0$ and $|s| \geq \rho$,

$$|\xi_r(s)| = \nu_r^{-1} \int_{\Omega_r} \cos(2^r s \cdot x - \theta) \nu(dx) \in [0, \gamma].$$

If $\theta \not\equiv \pm\pi/2 \pmod{\pi}$, expand out the integrand using the identity $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$, $\forall \alpha, \beta \in \mathbb{R}$, replace the term $\int_{\Omega_r} \sin(2^r s \cdot x) \nu(dx)$ and rearrange to get

$$(\nu_r \cos \theta)^{-1} \int_{\Omega_r} \cos(2^r s \cdot x) \nu(dx) \in [0, \gamma].$$

Therefore, regardless of the values of θ , one always has $\left| \int_{\Omega_r} \cos(2^r s \cdot x) \nu(dx) \right| \leq \gamma \nu_r$ for $|s| \geq \rho$. Write $s = |s|v$ where $v \in \mathbb{S}^{q-1}$ is a unit vector. Then for $|s| \geq \rho$,

$$\begin{aligned} v \cdot \Sigma_r v &= \int_{\Omega_r} |v \cdot x|^2 \nu(dx) \geq 2^{-2r+2} |s|^{-2} \int_{\Omega_r} \sin^2(2^{r-1} s \cdot x) \nu(dx) \\ &= 2^{-2r+1} |s|^{-2} \int_{\Omega_r} (1 - \cos(2^r s \cdot x)) \nu(dx) \geq 2^{-2r+1} \rho^{-2} (1 - \gamma) \nu_r. \end{aligned}$$

This means $\lambda_{1,r} \gtrsim 2^{-2r} \nu_r$ by choosing v to be the eigenvector of $\lambda_{1,r}$. Hence $\lambda_{1,r} \simeq \lambda_{q,r} \simeq 2^{-2r} \nu_r$ and $\lambda_{1,X} = \nu_r^{-1} 2^{2r} \lambda_{1,r} \simeq 1$, $\forall r \geq r_0$.

Since $\xi_r(s)$ is the characteristic function of $X = 2^r X^r$, the uniform Cramér's condition validates the bound (3.3) with a uniform $\bar{\gamma} = \bar{\gamma}(\rho, \gamma)$ and (3.4) holds with $\mu = t\nu_r \geq \epsilon \nu_r \geq 2^{-r} \nu_r$ sufficiently large w.r.t. $p+2$. More precisely, one can choose ϵ sufficiently small s.t. for all $r \geq r_0$, similar to (2.11),

$$(e^{\bar{\gamma}-1})^{2^{-r} \nu_r} (2^{-r} \nu_r)^{(q+1)(p+3)/2} \lesssim 1.$$

Thus, all the arguments leading towards (3.5) are justified, where the bound is immediately reduced to $C_{p,q}2^{-r}$, and therefore

$$\mathbb{W}_p\left(Z_t^\epsilon, \sqrt{t}\xi_{\Sigma_t}\right) = \mathbb{W}_p\left(\sum_{r=r_0}^{\infty} V_t^r, \sum_{r=r_0}^{\infty} \sqrt{t}\xi_{\Sigma_r}\right) \leq C_{p,q} \sum_{r=r_0}^{\infty} 2^{-r} = C_{p,q}\epsilon.$$

□

Together with the finite second moment of ν , the theorem above requires the order of ν_r is between $O(2^{r+})$ and $O(2^{2r})$ as $r \rightarrow \infty$, *i.e.* the Lévy measure needs to be sufficiently singular near 0. The uniform Cramér condition requires certain comparability between ν and the Lebesgue measure Λ , conditional on Ω_r . The following lemma gives a sufficient condition.

Lemma 3.2. *If there exist $a, b \in (0, 1)$ s.t. for each $r \geq r_0$, any measurable subset Γ_r of Ω_r satisfying $\Lambda(\Gamma_r)/\Lambda(\Omega_r) \geq a$ must have that $\nu(\Gamma_r)/\nu(\Omega_r) \geq b$, then $\xi_r(s) = \chi^r(2^r s)$ satisfies $\mathbf{CC}(\rho, \gamma)$ for some $\rho > 0, \gamma \in (0, 1)$ uniformly for all $r \geq r_0$.*

Proof. For any $a' \in (a, 1)$ denote $b' = \sin^2 \frac{\pi}{2}(1 - a') \in (0, 1)$. For any $\theta \in \mathbb{R}$, $v \in \mathbb{R}^q$, consider

$$D_k = D_k(v, \theta) := \{x \in \Omega_r : 2k\pi + (1 - a')\pi \leq v \cdot x - \theta \leq 2(k + 1)\pi - (1 - a')\pi\}, \quad k \in \mathbb{Z},$$

on each of which $\sin^2 \frac{1}{2}(v \cdot x - \theta) \geq b'$. They are parallel “stripes” across the annulus Ω_r with width $2a'\pi/|v|$ equidistantly away from each other by $2(1 - a')\pi/|v|$. This can be seen by rotating so that v lies on one axis. Thus for $|v| > \pi 2^{r+1}$ there is at least one non-empty D_k . Denote $\Gamma_r = \bigcup_{k \in \mathbb{Z}} D_k$, then the ratio $\Lambda(\Gamma_r)/\Lambda(\Omega_r)$ approaches a' as $|v| \rightarrow \infty$, regardless of the translation θ . Therefore one can find some constants $\rho > 0$ and $\gamma' = \gamma'(\rho, q) \in (0, a')$ s.t. for all $|v| \geq 2^r \rho$, $\Lambda(\Gamma_r)/\Lambda(\Omega_r) \geq \gamma'$. Choose $\gamma' = a$ as given in the assumption, then $\nu(\Gamma_r)/\nu(\Omega_r) \geq b$ for all $|v| \geq 2^r \rho$. Write $\xi_r(s) = |\xi_r(s)|e^{i\theta}$ for some $\theta = \theta(r, s)$, then

$$1 - |\xi_r(s)| \geq 2\nu_r^{-1} \int_{\Gamma_r} \sin^2 \frac{1}{2}(2^r s \cdot x - \theta) \nu(dx) \geq 2b' \nu(\Gamma_r)/\nu(\Omega_r) \geq 2b'b,$$

for all $r \geq r_0$, and the result follows by setting $v = 2^r s$ and $\gamma = 1 - 2b'b \in (0, 1)$. □

Corollary 3.3. *If ν satisfies $\mathbf{H}(\tau, \alpha)$ for $\alpha \in (1, 2)$, then Theorem 3.1 holds for $\epsilon \in (0, \tau)$ sufficiently small.*

Proof. One just needs to check that the assumptions in Theorem 3.1 are satisfied. First of all, for $\alpha \in (1, 2)$ and $\forall r \geq r_0$,

$$\nu_r \simeq \int_{\Omega_r} |x|^{-q-\alpha} dx = C_q \frac{2^\alpha - 1}{\alpha} 2^{\alpha r}.$$

Then $2^r \nu_r^{-1} = o(1)$ for $\alpha \in (1, 2)$. For any measurable subset Γ_r of Ω_r ,

$$\frac{\nu(\Gamma_r)}{\nu(\Omega_r)} \geq \frac{\int_{\Gamma_r} |x|^{-q-\alpha} dx}{\int_{\Omega_r} |x|^{-q-\alpha} dx} \geq 2^{-q-\alpha} \frac{\Lambda(\Gamma_r)}{\Lambda(\Omega_r)},$$

which validates Lemma 3.2. □

It is worth mentioning that if the condition $\mathbf{H}(\tau, \alpha)$ is assumed a priori, then one could directly use the Lévy-Khintchine formula to study the global behaviour of the characteristic function of Z_t^ϵ , which would greatly simplify the analysis of Section 1, but the same arguments used in the proof of Theorem 2.7 would still be needed for the coupling result.

3.2. A coupling for Euler's approximation

Given the coupling result above, one finally arrives at the stage of recovering Fournier's results [9] on the Euler approximation of the SDE (1.2) driven by the Lévy process (1.1):

$$x_t = x_0 + \int_0^t \sigma(x_{s-}) dZ_s, \quad t \in [0, T], \quad (3.6)$$

$$Z_t = at + BW_t + \int_0^t \int_{\mathbb{R}^q \setminus \{0\}} z \tilde{N}(dz, ds). \quad (3.7)$$

For fixed $h, \epsilon \in (0, 1)$ introduce the following random variable

$$\bar{\Delta}_1 := \bar{a}h + \bar{B}\sqrt{h}\xi_I + \sum_{j=1}^{N_h^\epsilon} Y_j^\epsilon, \quad (3.8)$$

and take independent copies $\bar{\Delta}_2, \dots, \bar{\Delta}_{[T/h]}$, where $\{Y_j^\epsilon\}$ are i.i.d. random variables having distribution $\mathbb{1}_{|z|>\epsilon}\nu(dz)/\nu(|z|>\epsilon)$, N_h^ϵ is an independent Poisson random variable with parameter $h\nu(\{|z|>\epsilon\})$, and the coefficients $\bar{a} = a - \int_{|z|>\epsilon} z\nu(dz)$, $\bar{B} := (BB^\top + \Sigma_\epsilon)^{1/2}$, $\Sigma_\epsilon = \int_{0<|z|\leq\epsilon} zz^\top \nu(dz)$. For $t_k = kh$, $k = 1, \dots, [T/h]$, write the increments $\Delta_k := Z_{t_k} - Z_{t_{k-1}}$. Then one may attempt to find a coupling between the standard Euler's approximation

$$X_{k+1} := X_k + \sigma(X_k)\Delta_{k+1}, \quad X_0 = x_0,$$

and the numerical scheme

$$\bar{X}_{k+1} := \bar{X}_k + \sigma(\bar{X}_k)\bar{\Delta}_{k+1}, \quad \bar{X}_0 = x_0. \quad (3.9)$$

For that one claims the following statement as an analogue to Lemma 5.2 in [9]:

Proposition 3.4. *If ν satisfies the conditions of Theorem 3.1, then for ϵ sufficiently small there exist on the same probability space two sequences of i.i.d. random variables $\{\Delta'_k\}$, $\{\bar{\Delta}'_k\}$, with the same distributions as $\{\Delta_k\}$ and $\{\bar{\Delta}_k\}$ respectively, s.t.*

$$\left(\mathbb{E}|\Delta'_k - \bar{\Delta}'_k|^p\right)^{\frac{1}{p}} \leq C_q \epsilon,$$

for all $k \in \mathbb{Z}^+$ and $p \in 2\mathbb{Z}^+$, and $\mathbb{E}\Delta'_k = \mathbb{E}\bar{\Delta}'_k = ah$, $\text{var}\Delta'_k = \text{var}\bar{\Delta}'_k = (BB^\top + \Sigma_\epsilon)h$.

Proof. By Theorem 3.1, for ϵ sufficiently small there is a standard normal random variable ξ'_I on the same probability space s.t.

$$\left(\mathbb{E}\left|\int_0^h \int_{0<|z|\leq\epsilon} z \tilde{N}(dz, ds) - h\Sigma_\epsilon^{\frac{1}{2}}\xi'_I\right|^p\right)^{\frac{1}{p}} \leq C_q \epsilon,$$

according to the definition of the W_p distance. If one sets

$$\begin{aligned} \Delta'_1 &:= ah + BW_h + \int_0^h \int_{0<|z|\leq\epsilon} z \tilde{N}(dz, ds) + \int_0^h \int_{|z|>\epsilon} z \tilde{N}(dz, ds), \\ \bar{\Delta}'_1 &:= ah + BW_h + h\Sigma_\epsilon^{\frac{1}{2}}\xi'_I + \int_0^h \int_{|z|>\epsilon} z \tilde{N}(dz, ds), \end{aligned}$$

then Δ'_1 has the same law as Δ_1 , and $\bar{\Delta}'_1$ has the same law as $\bar{\Delta}_1$. Thus the result follows by taking independent copies. \square

Proposition 3.4 can be immediately used to partially recover the main results in [9] (Thm. 2.2): the proof is independent of the key coupling result (Lem. 5.2), so one can replace the latter with the proposition above. Hence one can restate those results as follows:

Theorem 3.5. *Suppose $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times q}$ is bounded and Lipschitz, and the Lévy measure ν for the Lévy process (3.7) satisfies conditions of Theorem 3.1. Let $\epsilon, h \in (0, 1)$, and x be the unique solution to the SDE (3.6) for $t \in [0, T]$. Then for ϵ sufficiently small, there exists a coupling between $\{x_{t_k}\}_k$ and $\{\bar{X}_k\}_k$ defined by (3.9) and (3.8) s.t.*

$$\mathbb{E} \max_k |x_{t_k} - \bar{X}_k|^2 \leq C_1(h + \epsilon).$$

Moreover, if $\nu(\{|z| > \epsilon\}) = 0$, i.e. $Z_t = Z_t^\epsilon$ as in (3.7), and $\{\tilde{x}_t^\epsilon\}$ is the unique solution to the continuous SDE $\tilde{x}_t^\epsilon = x_0 + \int_0^t \sigma(\tilde{x}_t^\epsilon) d\tilde{Z}_t^\epsilon$ where $\tilde{Z}_t^\epsilon = at + (BB^\top + \Sigma_\epsilon)^{1/2}W_t$, then there exists a coupling between x and \tilde{x}^ϵ s.t.

$$\mathbb{E} \sup_{t \in [0, T]} |x_t - \tilde{x}_t^\epsilon|^2 \leq C_2\epsilon.$$

The constants C_1, C_2 depend on $d, q, T, |a|, \|B\|, \|\sigma\|_\infty, \Sigma_\epsilon$.

Instead of repeating the same arguments of Fournier [9], the reader is referred to the proof of Theorem 2.2 therein. Note that Proposition 3.4 above allows one to replace the $\beta_\epsilon(\nu)$ in Lemma 5.2 with ϵ^2 , and the rest of the calculations can be readily generalised to the multi-dimensional case. In particular, under the assumption $\mathbf{H}(\tau, \alpha)$ for some $\alpha \in (1, 2)$, by choosing $\epsilon = h$ one recovers the mean-square convergence rate $O(h)$ and the computational cost $E_\nu(h) = O(h^{-1} + h^{-\alpha})$ is controlled. The second statement corresponds to Corollary 3.2 in [9]. For that, one simply takes $\bar{\Delta}_1 = \bar{a}h + \bar{B}\sqrt{h}\xi_I$ instead of (3.8) and $h = \epsilon$, and runs the same argument as in Proposition 3.4, omitting the big-jump part.

The general case where σ is locally Lipschitz with linear growth and only $\int_{\mathbb{R}^q \setminus \{0\}} 1 \wedge |z|^2 \nu(dz) < \infty$ is assumed can be treated by the same localisation argument as in Theorem 7.1 in [9], and the mean-square convergence could be generalised to the strong L^p -convergence for $p \in 2\mathbb{Z}^+$ without much trouble. Nevertheless, it needs to be pointed out that the rate of convergence here is optimal for coupling the small jumps only - it might not be so if one can couple the entire Lévy increment. For the same reason the results achieved in this article cannot be applied to recover Theorem 3.1 in [9]. Finally, I believe the conditions of Theorem 3.1 can be relaxed to some extent. *e.g.*, one may take a hint from Proposition A.2 in [10] that it possibly suffices for ν to give a suitable portion of mass to the biggest annulus Ω_{r_0} .

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