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A POSTERIORI ERROR ANALYSIS FOR A DISTRIBUTED OPTIMAL CONTROL PROBLEM GOVERNED BY THE VON KÁRMÁN EQUATIONS

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Abstract. This article discusses the numerical analysis of the distributed optimal control problem governed by the von Kármán equations defined on a polygonal domain in $\mathbb{R}^2$. The state and adjoint variables are discretised using the nonconforming Morley finite element method and the control is discretized using piecewise constant functions. *A priori* and *a posteriori* error estimates are derived for the state, adjoint and control variables. The *a posteriori* error estimates are shown to be efficient. Numerical results that confirm the theoretical estimates are presented.

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1. Introduction

Problem formulation

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and ν denote the outward unit normal vector to the boundary $\partial\Omega$ of Ω. This paper considers the distributed control problem governed by the von Kármán equations stated below:

$$\min_{u\in U_{ad}} \mathcal{J}(\Psi, u) \quad \text{subject to} \tag{1.1a}$$

$$\Delta^2\psi_1 = [\psi_1,\psi_2] + f + \mathcal{C}u,\ \Delta^2\psi_2 = -\frac{1}{2}[\psi_1,\psi_1] \quad \text{in } \Omega, \tag{1.1b}$$

$$\psi_1 = 0,\ \frac{\partial\psi_1}{\partial\nu} = 0 \text{ and } \psi_2 = 0,\ \frac{\partial\psi_2}{\partial\nu} = 0 \quad \text{on } \partial\Omega. \tag{1.1c}$$

Here the cost functional $\mathcal{J}(\Psi,u) := \frac{1}{2}\|\Psi - \Psi_d\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\omega)}$, the state variable $\Psi := (\psi_1,\psi_2)$, where ψ_1 and ψ_2 correspond to the displacement and Airy-stress, $\Psi_d := (\psi_{d,1},\psi_{d,2}) \in \boldsymbol{L}^2(\Omega) := L^2(\Omega)\times L^2(\Omega)$ is the prescribed desired state for Ψ, $\|\Psi-\Psi_d\|^2_{\boldsymbol{L}^2(\Omega)} := \sum_{i=1}^2 \|\psi_i - \psi_{d,i}\|^2_{L^2(\Omega)}$, $\alpha > 0$ is a fixed regularization

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parameter, $U_{ad} \subset L^2(\omega)$, $\omega \subset \Omega$ is a non-empty, closed, convex and bounded set of admissible controls defined by

$$U_{ad} = \big\{u \in L^2(\omega) : u_a \le u(x) \le u_b \ \text{ for almost every } x \text{ in } \omega\big\},$$

$u_a \le u_b \in \mathbb{R}$ are given, Δ^2 denotes the fourth-order biharmonic operator, the von Kármán bracket $[\eta,\chi] := \eta_{xx}\chi_{yy} + \eta_{yy}\chi_{xx} - 2\eta_{xy}\chi_{xy} = \operatorname{cof}\big(D^2\eta\big) : D^2\chi$ with the co-factor matrix $\operatorname{cof}\big(D^2\eta\big)$ of $D^2\eta$, $f \in L^2(\Omega)$, and $\mathcal{C} \in \mathcal{L}(L^2(\omega), L^2(\Omega))$ is the extension operator defined by $\mathcal{C}u(x) = u(x)$ if $x \in \omega$ and $\mathcal{C}u(x) = 0$ if $x \notin \omega$.

Motivation

The von Kármán equations [2–5, 21, 29] that describe the bending of very thin elastic plates offer challenges in its numerical approximation; mainly due to its nonlinearity and higher-order nature. The numerical analysis of von Kármán equations has been studied using conforming finite element methods (FEMs) in [9, 31], nonconforming Morley FEM in [15, 32], mixed FEMs in [17, 34], discontinuous Galerkin methods and C^0 interior penalty methods in [8, 14]. The optimal control problem governed by the von Kármán equations (1.1a)–(1.1c) is analysed in [33] using C^1 conforming finite elements. In [19], the state and adjoint variables are discretised using the Morley FEM and *a priori* error estimates are derived under minimal regularity assumptions on the exact solution. The article [14] discusses reliable and efficient *a posteriori* estimates for the state equations. *To the best of our knowledge, there are no results in literature that discuss a posteriori error analysis for the approximation of regular solutions of optimal control problems governed by von Kármán equations.* Recently, *a posteriori* error analysis for the optimal control problem governed by second-order stationary Navier–Stokes equations is studied in [1] using conforming finite elements *under smallness assumption on the data*. The trilinear form in [1] vanishes whenever the second and third variables are equal, and satisfies the anti-symmetric property with respect to the second and third variables and this aids the *a posteriori* error analysis. *This paper discusses approximation of regular solutions for fourth-order semilinear problems without any smallness assumption on the data. Moreover, the trilinear form for von Kármán equations does not satisfy the properties stated above and hence leads to interesting challenges in the analysis.*

Nonconforming Morley, FEM based on piecewise quadratic polynomials in a triangle is more elegant and simpler for fourth-order problems. However, since the discrete space V_{M} is not a subspace of $H_0^2(\Omega)$, the convergence analysis offers a lot of novelty in the context of control problems governed by semilinear problems with trilinear nonlinearity. The adjoint variable in the control problem satisfies a fourth-order linear problem with lower-order terms and its *a priori* and *a posteriori* analysis with Morley FEM are not available in literature.

Contributions

In continuous formulation (see (2.1)) and the conforming FEM [33], the trilinear form $b(\bullet,\bullet,\bullet)$ is symmetric with respect to all the three variables making the analysis simpler to a certain extent. However, for fourth-order systems, nonconforming Morley FEM is attractive, is *a method of choice* [15], provides optimal order estimates, and these aspects motivated the *a priori* analysis for the optimal control problem considered in [19]. The nonconforming Morley finite elements are based on piecewise quadratic polynomials and are simpler to use. They have lesser number of degrees of freedom in comparison with the conforming Argyris finite elements with 21 degrees of freedom in a triangle or the Bogner–Fox–Schmit finite elements with 16 degrees of freedom in a rectangle. The discrete trilinear form $b_{\mathrm{NC}}(\bullet,\bullet,\bullet) := \frac{1}{2}\sum_{K\in\mathcal{T}}\int_K \operatorname{cof}(D^2\eta_{\mathrm{M}})D\chi_{\mathrm{M}}\cdot D\varphi_{\mathrm{M}}\,\mathrm{d}x$ for all Morley functions $\eta_{\mathrm{M}}, \chi_{\mathrm{M}}$ and φ_{M} utilized in [19]; is obtained after an integration by parts, where $\mathcal{T}$ denotes an admissible triangulation of Ω. This form is symmetric with respect to the second and third variables. Although this choice of trilinear form leads to optimal order error estimates for the optimal control problem (1.1a)–(1.1c), it leads to terms that involve *averages* in the reliability analysis of the state equations (as in the case of Navier–Stokes equation considered in [15]). The *efficiency* estimates are unclear in this context. To overcome this, a more natural trilinear form $b_{\mathrm{NC}}(\eta_{\mathrm{M}},\chi_{\mathrm{M}},\varphi_{\mathrm{M}}) := -\frac{1}{2}\sum_{K\in\mathcal{T}}\int_K[\eta_{\mathrm{M}},\chi_{\mathrm{M}}]\varphi_{\mathrm{M}}\,\mathrm{d}x$ that is symmetric with respect to the first and second variables is chosen in this article. The *a priori* and *a posteriori* analysis for the *state equations* are discussed in [10, 15] for this modified choice of $b_{\mathrm{NC}}(\bullet,\bullet,\bullet)$.

The *a posteriori* analysis for the fully discrete optimal control problem governed by the von Kármán equations addressed in this article is novel. For example, this formulation is different from that in [19], and it is essential to modify the *a priori* error estimates for the discrete optimal problem. The adjoint system of the optimal control problem involves lower-order terms with leading biharmonic operators. *A posteriori* analysis for biharmonic operator with lower-order terms is a problem of independent interest.

Thus the contributions of this article can be summarized as follows.

- Reliable and efficient *a posteriori* error estimates that drive the adaptive refinement for the optimal state and adjoint variables in the energy norm and control variable in the L^2 norm are developed. The approach followed provides a strategy for the nonconforming FEM analysis of distributed optimal control problems governed by higher-order semilinear problems.
- Several auxiliary results that are derived will be of interest in other applications – for example, optimal control problems governed by Navier–Stokes problems in the stream-vorticity formulation.
- The paper illustrates results of computational experiments that validate both theoretical *a priori* and *a posteriori* estimates for the optimal control problem under consideration.
- For a formulation that is different from that in [19], optimal order *a priori* error estimates in energy norm when state and adjoint variables are approximated by Morley FEM and linear order of convergence for control variable in L^2 norm when control is approximated using piecewise constants are outlined.

Organisation

The remaining parts of this paper are organised as follows. Section 2 first presents the weak and Morley finite element formulations for (1.1a)–(1.1c). The main results of this article are also stated. The state and adjoint variables are discretised using Morley finite elements and the control variable is discretised using piecewise constant functions. Section 3 discusses some auxiliary results related to the continuous formulation and Morley FEM. The properties of the interpolation and companion operators that are crucial for the error analysis are discussed in this section. The proofs of the results for the *a posteriori* estimates stated in Section 2 are presented in Sections 4 and 5. Section 4 develops reliable *a posteriori* estimates for the state, adjoint and control variables of the optimal control problem. The efficiency results are discussed in Section 5. Results of numerical experiments that validate theoretical estimates are presented in Section 6. The intermediate results for establishing the *a priori* error estimates for the state, adjoint and control variables under minimal regularity assumption on the exact solution differ from [19] due to a different form of $b_{\mathrm{NC}}(\bullet,\bullet,\bullet)$ and hence are outlined in the appendix.

Notation

Throughout the paper, standard notations on Lebesgue and Sobolev spaces and their norms are employed. The standard seminorm and norm on $H^s(\Omega)$ (resp. $W^{s,p}(\Omega)$) for $s>0$ and $1\le p\le\infty$ are denoted by $|\cdot|_s$ and $\|\cdot\|_s$ (resp. $|\cdot|_{s,p}$ and $\|\cdot\|_{s,p}$) and norm in $L^\infty(\Omega)$ is denoted by $\|\cdot\|_{0,\infty}$. The norm in $H^{-s}(\Omega)$ is denoted by $\|\cdot\|_{-s}$. The standard L^2 inner product and norm are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$. The notation $\|\cdot\|$ is also used to denote the operator norm and should be understood from the context. The notation $\boldsymbol{H}^s(\Omega)$ (resp. $\boldsymbol{L}^p(\Omega)$) is used to denote the product space $H^s(\Omega)\times H^s(\Omega)$ (resp. $L^p(\Omega)\times L^p(\Omega)$). For all $\Phi=(\varphi_1,\varphi_2)\in\boldsymbol{H}^s(\Omega)$ $\big($resp. $\boldsymbol{L}^2(\Omega)\big)$, the product space is equipped with the norm $|\!|\!|\Phi|\!|\!|_s := \left(\|\varphi_1\|_s^2+\|\varphi_2\|_s^2\right)^{1/2}\left(\text{resp. } |\!|\!|\Phi|\!|\!| := \left(\|\varphi_1\|^2+\|\varphi_2\|^2\right)^{1/2}\right)$. The notation $a\lesssim b$ (resp. $a\gtrsim b$) means there exists a generic mesh independent constant C such that $a\le Cb$ (resp. $a\ge Cb$). The positive constants C appearing in the inequalities denote generic constants which do not depend on the mesh size.

Let $\mathcal{T}$ be an admissible and regular triangulation of the domain Ω into simplices in $\mathbb{R}^2$, h_K be the diameter of $K\in\mathcal{T}$ and $h:=\max_{K\in\mathcal{T}} h_K$. Let $\mathbb{T}$ be the set of all admissible triangulations $\mathcal{T}$. Given any $0<\delta<1$, let $\mathbb{T}(\delta)$ be the set of all triangulations $\mathcal{T}$ with mesh size $\le\delta$ for all triangles $K\in\mathcal{T}$ with area $|K|$. Let $\mathcal{E}(\Omega)$ (resp. $\mathcal{E}(\partial\Omega)$) denotes the set of all interior edges (resp. boundary edges) of Ω. The length of any edge E is denoted by h_E. For a nonnegative integer $k\in\mathbb{N}_0$, $\mathcal{P}_k(\mathcal{T})$ denotes the space of piecewise polynomials of degree at most equal to k. Let Π_k denote the L^2 projection onto the space of piecewise polynomials $\mathcal{P}_k(\mathcal{T})$. The mesh size $h_{\mathcal{T}} \in \mathcal{P}_0(\mathcal{T})$ is defined by $h_{\mathcal{T}}|_K := h_K$. The oscillation of f in $\mathcal{T}$ reads $\mathrm{osc}_k(f,\mathcal{T}) = \|h_{\mathcal{T}}^2(f - \Pi_k f)\|$ for $k \in \mathbb{N}_0$. For a nonnegative integer m, and $\Phi = (\varphi_1, \varphi_2) \in W^{m,p}(\mathcal{T})$, where $W^{m,p}(\mathcal{T})$ denotes the broken Sobolev space with respect to $\mathcal{T}$, $\|\Phi\|_{m,p,h}^2 := |\varphi_1|_{m,p,h}^2 + |\varphi_2|_{m,p,h}^2$, and $|\varphi_i|_{m,p,h} = \left(\sum_{K\in\mathcal{T}} |\varphi_i|_{m,p,K}^p\right)^{1/p}$, $i = 1,2$; with $|\cdot|_{m,p,K}$ denoting the usual seminorm in $W^{m,p}(K)$. The notation $\boldsymbol{H}^1(\mathcal{T})$ is used to denote the product space $H^1(\mathcal{T}) \times H^1(\mathcal{T})$.

2. Main results

The weak and Morley FEM formulations corresponding to (1.1) are stated and the main results of this article are presented in this section. The proofs of the reliability and efficiency estimates in Theorem 2.4 are detailed in Sections 4 and 5.

The weak formulation that corresponds to (1.1a)–(1.1c) seeks $(\Psi, u) \in \mathbf{V} \times U_{ad}$ such that

$$\min_{(\Psi,\,u)\in\mathbf{V}\times U_{ad}} \mathcal{J}(\Psi, u) \quad \text{subject to} \tag{2.1a}$$

$$a(\psi_1,\varphi_1) + b(\psi_1,\psi_2,\varphi_1) + b(\psi_2,\psi_1,\varphi_1) = (f + \mathcal{C}u, \varphi_1) \qquad \text{for all } \varphi_1 \in V, \tag{2.1b}$$

$$a(\psi_2,\varphi_2) - b(\psi_1,\psi_1,\varphi_2) = 0 \qquad \text{for all } \varphi_2 \in V, \tag{2.1c}$$

with $V := H_0^2(\Omega)$, $\mathbf{V} = V \times V$, the continuous, V-elliptic bilinear form $a(\bullet,\bullet) : V \times V \to \mathbb{R}$ is defined by $a(\varphi_1,\varphi_2) := \int_\Omega D^2\varphi_1 : D^2\varphi_2 \,\mathrm{d}x$, and the continuous trilinear form $b(\bullet,\bullet,\bullet) : V \times V \times V \to \mathbb{R}$ is defined by $b(\varphi_1,\varphi_2,\varphi_3) := -\frac{1}{2}\int_\Omega [\varphi_1,\varphi_2]\varphi_3 \,\mathrm{d}x$. For a given $u \in L^2(\omega)$, (2.1b) and (2.1c) possesses at least one solution [29].

For all $\boldsymbol{\xi} = (\xi_1,\xi_2)$, $\Phi = (\varphi_1,\varphi_2)$, $\boldsymbol{\eta} = (\eta_1,\eta_2) \in \mathbf{V}$, the operator form for (2.1b) and (2.1c) is

$$\Psi \in \mathbf{V}, \;\; \mathcal{A}\Psi + \mathcal{B}(\Psi) = \mathbf{F} + \mathbf{Cu} \text{ in } \mathbf{V}', \tag{2.2}$$

with $\mathcal{A} \in \mathcal{L}(\mathbf{V},\mathbf{V}')$ defined by $\langle \mathcal{A}\boldsymbol{\xi}, \Phi\rangle$ [1]$= A(\boldsymbol{\xi},\Phi) = a(\xi_1,\varphi_1) + a(\xi_2,\varphi_2)$, $\mathcal{B}$ from $\mathbf{V}$ to $\mathbf{V}'$ defined by $\langle \mathcal{B}(\boldsymbol{\eta}),\Phi\rangle = B(\boldsymbol{\eta},\boldsymbol{\eta},\Phi)$ where $B(\boldsymbol{\eta},\Phi,\boldsymbol{\xi}) = b(\eta_1,\varphi_2,\xi_1) + b(\eta_2,\varphi_1,\xi_1) - b(\eta_1,\varphi_1,\xi_2)$, $\mathbf{F} = \begin{pmatrix} f \\ 0 \end{pmatrix}$, $\mathbf{Cu} = \begin{pmatrix} \mathcal{C}u \\ 0 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} u \\ 0 \end{pmatrix}$, and $(\mathbf{F} + \mathbf{Cu}, \Phi) := (f + \mathcal{C}u, \varphi_1)$.

The linearization of (2.1b) and (2.1c) around Ψ in the direction $\boldsymbol{\xi}$ is given by $\boldsymbol{L\xi} := \mathcal{A}\boldsymbol{\xi} + \mathcal{B}'(\Psi)\boldsymbol{\xi}$, where the operator $\mathcal{B}'(\Psi) \in \mathcal{L}(\mathbf{V},\mathbf{V}')$[2] is defined by $\langle \mathcal{B}'(\Psi)\boldsymbol{\xi},\Phi\rangle := 2B(\Psi,\boldsymbol{\xi},\Phi)$.

Definition 2.1 (Regular solution)**.** For a given $u \in L^2(\omega)$, a solution Ψ of (2.1b) and (2.1c) is said to be regular if the linearized form is well-posed. That is, if $\langle \boldsymbol{L\xi},\Phi\rangle = 0$ for all $\Phi \in \mathbf{V}$, then $\boldsymbol{\xi} = \mathbf{0}$. In this case, the pair (Ψ,u) also is referred to as a regular solution to (1.1b) and (1.1c).

Definition 2.2 (Local solution)**.** [16] The pair $(\bar\Psi,\bar u) \in \mathbf{V} \times U_{ad}$ is a *local solution* to (2.1) if and only if $(\bar\Psi,\bar u)$ satisfies (2.1b) and (2.1c) and there exist neighbourhoods $\mathcal{O}(\bar\Psi)$ of $\bar\Psi$ in $\mathbf{V}$ and $\mathcal{O}(\bar u)$ of $\bar u$ in $L^2(\omega)$ such that $\mathcal{J}(\bar\Psi,\bar u) \le \mathcal{J}(\Psi,u)$ for all pairs $(\Psi,u) \in \mathcal{O}(\bar\Psi) \times (U_{ad} \cap \mathcal{O}(\bar u))$ that satisfy (2.1b) and (2.1c).

Local solutions $(\bar\Psi,\bar u)$ to (2.1) such that the pair is a *regular solution* to (2.2) are approximated in this article. The existence result for (2.1) is stated in Theorem 3.1. The *optimality system* for the optimal control problem (2.1) is

$$A(\bar\Psi,\Phi) + B(\bar\Psi,\bar\Psi,\Phi) = (\mathbf{F} + \mathbf{C\bar u},\Phi) \quad \text{for all } \Phi \in \mathbf{V} \;\; \text{(State equations)} \tag{2.3a}$$

$$A(\Phi,\bar\Theta) + 2B(\bar\Psi,\Phi,\bar\Theta) = (\bar\Psi - \Psi_d,\Phi) \quad \text{for all } \Phi \in \mathbf{V} \;\; \text{(Adjoint equations)} \tag{2.3b}$$

[1] The subscripts in the duality pairings are omitted for notational convenience.

[2] The same notation $'$ is used either to denote the Fréchet derivative of an operator or the dual of a space, but the context helps to clarify its precise meaning.

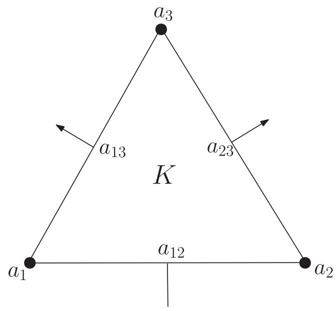


FIGURE 1. Morley triangle.

$$\left(\mathbf{C}^*\bar{\Theta}+\alpha\bar{\mathbf{u}},\mathbf{u}-\bar{\mathbf{u}}\right)_{\boldsymbol{L}^2(\omega)}\geq 0 \qquad \text{for all } \mathbf{u}=(u,0)^T,\ u\in U_{ad}\ \text{(First-order optimality condition)} \tag{2.3c}$$

where $\bar{\Theta}$ is the adjoint state and $\mathbf{C}^*$ denotes the adjoint of $\mathbf{C}$. For almost all $x\in\Omega$, the optimal control $\bar{\mathbf{u}}(x):=(\bar{u}(x),0)$ in (2.3c) satisfies ([35], Thm. 2.28)

$$\bar{\mathbf{u}}(x)=\Pi_{[u_a,u_b]}\left(-\frac{1}{\alpha}\left(\mathbf{C}^*\bar{\Theta}\right)\right), \tag{2.4}$$

where $\bar{\Theta}=\left(\bar{\theta}_1,\bar{\theta}_2\right)$ and the projection operator $\Pi_{[u_a,u_b]}$ is defined by $\Pi_{[u_a,u_b]}(g):=\min\{u_b,\max\{u_a,g\}\}$.

The nonconforming *Morley element space* V_{M} is defined by

$$V_{\mathrm{M}}:=\Big\{v_{\mathrm{M}}\in\mathcal{P}_2(\mathcal{T})|v_{\mathrm{M}}\text{ is continuous at the interior vertices and vanishes at the vertices of }\partial\Omega;$$
$$\text{for all } E\in\mathcal{E}(\Omega),\ \int_E\left[\frac{\partial v_{\mathrm{M}}}{\partial\nu}\right]_E\mathrm{d}s=0;\quad\text{for all } E\in\mathcal{E}(\partial\Omega),\ \int_E\frac{\partial v_{\mathrm{M}}}{\partial\nu}\,\mathrm{d}s=0\Big\},$$

where $[\phi]_E$ denotes the jump of a function ϕ across E and is equipped with the norm $\|\bullet\|_{\mathrm{NC}}$ defined by $\|\varphi\|_{\mathrm{NC}}=\left(\sum_{K\in\mathcal{T}}\|D^2_{\mathrm{NC}}\varphi\|^2_{L^2(K)}\right)^{1/2}$. Throughout the paper, $D_{\mathrm{NC}}\bullet$ and $D^2_{\mathrm{NC}}\bullet$ denote the piecewise gradient and Hessian of the arguments on triangles $K\in\mathcal{T}$. Figure 1 illustrates a Morley triangle $K\in\mathcal{T}$. Let $\mathbf{V}_{\mathrm{M}}:=V_{\mathrm{M}}\times V_{\mathrm{M}}$ and for $\Phi=(\varphi_1,\varphi_2)\in\mathbf{V}_{\mathrm{M}}$, $|\!|\!|\Phi|\!|\!|^2_{\mathrm{NC}}:=\|\varphi_1\|^2_{\mathrm{NC}}+\|\varphi_2\|^2_{\mathrm{NC}}$.

For all $\eta_{\mathrm{M}},\chi_{\mathrm{M}}$ and $\varphi_{\mathrm{M}}\in V_{\mathrm{M}}$, define the discrete bilinear and trilinear forms by

$$a_{\mathrm{NC}}(\eta_{\mathrm{M}},\chi_{\mathrm{M}}):=\sum_{K\in\mathcal{T}}\int_K D^2\eta_{\mathrm{M}}:D^2\chi_{\mathrm{M}}\,\mathrm{d}x \text{ and } b_{\mathrm{NC}}(\eta_{\mathrm{M}},\chi_{\mathrm{M}},\varphi_{\mathrm{M}}):=-\frac{1}{2}\sum_{K\in\mathcal{T}}\int_K[\eta_{\mathrm{M}},\chi_{\mathrm{M}}]\varphi_{\mathrm{M}}\,\mathrm{d}x.$$

Similarly, for $\Xi_{\mathrm{M}}=(\xi_1,\xi_2)$, $\Theta_{\mathrm{M}}=(\theta_1,\theta_2)$, $\Phi_{\mathrm{M}}=(\varphi_1,\varphi_2)\in\mathbf{V}_{\mathrm{M}}$, define

$$A_{\mathrm{NC}}(\Theta_{\mathrm{M}},\Phi_{\mathrm{M}}):=a_{\mathrm{NC}}(\theta_1,\varphi_1)+a_{\mathrm{NC}}(\theta_2,\varphi_2),\ F_{\mathrm{NC}}(\Phi_{\mathrm{M}}):=\sum_{K\in\mathcal{T}}\int_K f\varphi_1\,\mathrm{d}x\ \text{ and}$$
$$B_{\mathrm{NC}}(\Xi_{\mathrm{M}},\Theta_{\mathrm{M}},\Phi_{\mathrm{M}}):=b_{\mathrm{NC}}(\xi_1,\theta_2,\varphi_1)+b_{\mathrm{NC}}(\xi_2,\theta_1,\varphi_1)-b_{\mathrm{NC}}(\xi_1,\theta_1,\varphi_2).$$

The definitions of the discrete bilinear and trilinear forms are meaningful for functions in $V+V_{\mathrm{M}}$ (resp. $\mathbf{V}+\mathbf{V}_{\mathrm{M}}$). Note that for all $\xi,\theta,\varphi\in V$, $a_{\mathrm{NC}}(\xi,\theta)=a(\xi,\theta)$ and $b_{\mathrm{NC}}(\xi,\theta,\varphi)=b(\xi,\theta,\varphi)$.

The *admissible space for discrete controls* is $U_{h,ad}:=\left\{u\in L^2(\omega): u|_K\in\mathcal{P}_0(K),\ u_a\leq u\leq u_b \text{ for all } K\in\mathcal{T}\right\}$. The discrete control problem associated with (2.1) reads

$$\min_{(\Psi_{\mathrm{M}},u_h)\in\mathbf{V}_{\mathrm{M}}\times U_{h,ad}}\mathcal{J}(\Psi_{\mathrm{M}},u_h)\quad\text{subject to} \tag{2.5a}$$

$$A_{\rm NC}(\Psi_{\rm M},\Phi_{\rm M}) + B_{\rm NC}(\Psi_{\rm M},\Psi_{\rm M},\Phi_{\rm M}) = (\mathbf{F}+\mathbf{C}\mathbf{u}_h,\Phi_{\rm M}) \ \text{ for all } \Phi_{\rm M}\in\mathbf{V}_{\rm M}. \tag{2.5b}$$

The existence of a solution to the discrete problem (2.5) follows from Theorem 4.3 of [33]. The *discrete first-order optimality system* that consists of the discrete state and adjoint equations and the first-order optimality condition corresponding to (2.5) is

$$A_{\rm NC}\big(\bar\Psi_{\rm M},\Phi_{\rm M}\big) + B_{\rm NC}\big(\bar\Psi_{\rm M},\bar\Psi_{\rm M},\Phi_{\rm M}\big) = (\mathbf{F}+\mathbf{C}\bar{\mathbf{u}}_h,\Phi_{\rm M}) \qquad \text{for all } \Phi_{\rm M}\in\mathbf{V}_{\rm M} \tag{2.6a}$$

$$A_{\rm NC}\big(\Phi_{\rm M},\bar\Theta_{\rm M}\big) + 2B_{\rm NC}\big(\bar\Psi_{\rm M},\Phi_{\rm M},\bar\Theta_{\rm M}\big) = \big(\bar\Psi_{\rm M}-\Psi_d,\Phi_{\rm M}\big) \qquad \text{for all } \Phi_{\rm M}\in\mathbf{V}_{\rm M} \tag{2.6b}$$

$$\big(\mathbf{C}^*\bar\Theta_{\rm M}+\alpha\bar{\mathbf{u}}_h,\mathbf{u}_h-\bar{\mathbf{u}}_h\big)\ge 0 \qquad \text{for all } \mathbf{u}_h=(u_h,0)^T,\ u_h\in U_{h,ad}, \tag{2.6c}$$

where $\bar\Theta_{\rm M}\in\mathbf{V}_{\rm M}$ (resp. $\bar{\mathbf{u}}_h=(\bar u_h,0)^T,\ \bar u_h\in U_{h,ad}$) denotes the discrete adjoint (resp. control) variable that corresponds to the optimal state variable $\bar\Psi_{\rm M}\in\mathbf{V}_{\rm M}$.

Theorem 2.3 (*A priori* error control)**.** *Given a regular solution $\big(\bar\Psi,\bar u\big)$ to* (2.1)*, there exist $\delta_0,\,\epsilon_0>0$ such that any triangulation $\mathcal{T}\in\mathbb{T}(\delta_0)$ yields a unique discrete solution $\big(\bar\Psi_{\rm M},\bar\Theta_{\rm M},\bar{\mathbf{u}}_h\big)$ to* (2.6) *that satisfies $\big\|\!\big|\bar\Psi-\bar\Psi_{\rm M}\big|\!\big\|_{\rm NC}+\big\|\!\big|\bar\Theta-\bar\Theta_{\rm M}\big|\!\big\|_{\rm NC}+\|\bar u-\bar u_h\|_{L^2(\omega)}\le\varepsilon_0$, where $\bar\Theta$ is the corresponding continuous adjoint variable.*

The proof of the *a priori* estimates differ from that in [19] due to the different expression for the trilinear form. The crucial intermediate steps are outlined in the appendix. The choice of the constants ϵ_0 and δ_0 are discussed in Section 4.

A posteriori error control

Assume that (i) $\omega\subset\Omega$ is a polygonal domain and (ii) $\mathcal{T}$ restricted to ω yields a triangulation for ω. Define the auxiliary variable $\widetilde u_h$ by

$$\widetilde u_h := \Pi_{[u_a,u_b]}\bigg(-\frac{1}{\alpha}\big(\mathcal{C}^*\bar\theta_{\rm M,1}\big)\bigg), \tag{2.7}$$

where $\bar\Theta_{\rm M}=\big(\bar\theta_{\rm M,1},\bar\theta_{\rm M,2}\big)$ is the discrete adjoint variable corresponding to the control $\bar u_h$.

Let $\bar\Psi_{\rm M}=\big(\bar\psi_{\rm M,1},\bar\psi_{\rm M,2}\big)\in\mathbf{V}_{\rm M}$. For $K\in\mathcal{T}$ and $E\in\mathcal{E}(\Omega)$, define the volume estimators as

$$\eta^2_{K,\bar\Psi_{\rm M}} := h_K^4\Big(\big\|f+\mathcal{C}\bar u_h+\big[\bar\psi_{\rm M,1},\bar\psi_{\rm M,2}\big]\big\|^2_{L^2(K)}+\big\|\big[\bar\psi_{\rm M,1},\bar\psi_{\rm M,1}\big]\big\|^2_{L^2(K)}\Big),\quad \eta^2_{K,\bar u_h}:=\|\widetilde u_h-\bar u_h\|^2_{L^2(K)}, \tag{2.8a}$$

$$\eta^2_{K,{\rm res},\bar\Theta_{\rm M}} := h_K^4\Big(\big\|\bar\psi_{\rm M,1}-\psi_{d,1}-\big[\bar\psi_{\rm M,1},\bar\theta_{\rm M,2}\big]+\big[\bar\psi_{\rm M,2},\bar\theta_{\rm M,1}\big]\big\|^2_{L^2(K)}+\big\|\bar\psi_{\rm M,2}-\psi_{d,2}+\big[\bar\psi_{\rm M,1},\bar\theta_{\rm M,1}\big]\big\|^2_{L^2(K)}\Big), \tag{2.8b}$$

$$\eta^2_{K,\mathcal{P}_0,\bar\Theta_{\rm M}} := \big\|D^2\bar\psi_{\rm M,1}(1-\mathcal{P}_0)\bar\theta_{\rm M,2}\big\|^2_{L^2(K)}+\big\|D^2\bar\psi_{\rm M,2}(1-\mathcal{P}_0)\bar\theta_{\rm M,1}\big\|^2_{L^2(K)}+\big\|D^2\bar\psi_{\rm M,1}(1-\mathcal{P}_0)\bar\theta_{\rm M,1}\big\|^2_{L^2(K)}, \tag{2.8c}$$

$$\eta^2_{K,\bar\Theta_{\rm M}} := \eta^2_{K,{\rm res},\bar\Theta_{\rm M}}+\eta^2_{K,\mathcal{P}_0,\bar\Theta_{\rm M}}, \tag{2.8d}$$

and the edge estimators as

$$\eta^2_{E,\bar\Psi_{\rm M}} := h_E\Big(\big\|\big[D^2\bar\psi_{\rm M,1}\tau_E\big]_E\big\|^2_{L^2(E)}+\big\|\big[D^2\bar\psi_{\rm M,2}\tau_E\big]_E\big\|^2_{L^2(E)}\Big), \text{ and} \tag{2.8e}$$

$$\eta^2_{E,\bar\Theta_{\rm M}} := h_E\Big(\big\|\big[D^2\bar\theta_{\rm M,1}\tau_E\big]_E\big\|^2_{L^2(E)}+\big\|\big[D^2\bar\theta_{\rm M,2}\tau_E\big]_E\big\|^2_{L^2(E)}\Big), \tag{2.8f}$$

where τ_E denotes the unit tangential vector to the edge E and $[\phi]_E$ denotes the jump of a function ϕ across E. Further, define the total error estimator η as

$$\eta^2 := \eta^2_{\rm ST}+\eta^2_{\rm AD}+\eta^2_{\rm CON},\ \text{where} \tag{2.9}$$
$$\eta^2_{\rm ST} := \sum_{K\in\mathcal{T}}\eta^2_{K,\bar\Psi_{\rm M}}+\sum_{E\in\mathcal{E}(\Omega)}\eta^2_{E,\bar\Psi_{\rm M}},\ \eta^2_{\rm AD} := \sum_{K\in\mathcal{T}}\eta^2_{K,\bar\Theta_{\rm M}}+\sum_{E\in\mathcal{E}(\Omega)}\eta^2_{E,\bar\Theta_{\rm M}},\ \eta^2_{\rm CON} := \sum_{K\in\mathcal{T}}\eta^2_{K,\bar u_h}.$$

The main result stated next discusses the reliability and efficiency estimates for the control problem.

Theorem 2.4 (*A posteriori* error control)**.** *Given the exact solution* $(\bar{\Psi}, \bar{\Theta}, \bar{\mathbf{u}})$*,* $\varepsilon_0, \delta_0 > 0$ *from a priori error estimate Theorem 2.3, there exist positive constants* C_{rel} *and* C_{eff} *(which depend on* $\mathcal{T}$ *and on the solution* $(\bar{\Psi}, \bar{\Theta}, \bar{\mathbf{u}})$*,* $\varepsilon_0, \delta_0 > 0$*) such that for all* $\mathcal{T} \in \mathbb{T}(\delta_0)$*, the discrete solution* $(\bar{\Psi}_{\text{M}}, \bar{\Theta}_{\text{M}}, \bar{\mathbf{u}}_h)$ *and the error estimator* η *satisfy*

(a) **(Reliability)**

$$\left\|\bar{\Psi} - \bar{\Psi}_{\text{M}}\right\|_{\text{NC}} + \left\|\bar{\Theta} - \bar{\Theta}_{\text{M}}\right\|_{\text{NC}} + \|\bar{u} - \bar{u}_h\|_{L^2(\omega)} \leq C_{\text{rel}}\eta.$$

(b) **(Efficiency)**

$$\eta \leq C_{\text{eff}}\Big(\left\|\bar{\Psi} - \bar{\Psi}_{\text{M}}\right\|_{\text{NC}} + \left\|\bar{\Theta} - \bar{\Theta}_{\text{M}}\right\|_{\text{NC}} + \|\bar{u} - \bar{u}_h\|_{L^2(\omega)} + \text{osc}_0(f, \mathcal{T}) + \text{osc}_0(\Psi_d, \mathcal{T}) + \left\|(1 - I_{\text{M}})\bar{\Theta}\right\|_{\boldsymbol{H}^1(\mathcal{T})}$$
$$+ \left\|(1 - I_{\text{M}})\bar{\Theta}\right\|_{\boldsymbol{L}^\infty(\mathcal{T})} + \left\|(1 - \mathcal{P}_0)\bar{\Theta}\right\|_{\boldsymbol{L}^\infty(\mathcal{T})}\Big).$$

Recall that $\text{osc}_0(f, \mathcal{T}) = \left\|h_{\mathcal{T}}^2(f - \Pi_0 f)\right\|$*; where* Π_0 *denotes the* L^2 *projection onto the space of piecewise constant polynomials and* $I_{\text{M}} : V \to V_{\text{M}}$ *is the Morley interpolation operator defined in Lemma 3.5.*

3. Auxiliary results

This section deals with some auxiliary results in the continuous and discrete frameworks that are useful to establish the error estimates.

The state equations in (2.1b) and (2.1c) can be written as

$$N(\Psi; \Phi) := A(\Psi, \Phi) + B(\Psi, \Psi, \Phi) - (\mathbf{F} + \mathbf{Cu}, \Phi) = 0 \qquad \text{for all } \Phi \in \mathbf{V}.$$

The first and second-order Fréchet derivatives of $N(\Psi)$ at Ψ in the direction $\boldsymbol{\xi}$ are given by $DN(\Psi; \boldsymbol{\xi}, \Phi) := \langle \mathcal{A}\boldsymbol{\xi} + \mathcal{B}'(\Psi)\boldsymbol{\xi}, \Phi\rangle$ and $D^2N(\Psi; \boldsymbol{\xi}, \boldsymbol{\xi}, \Phi) := \langle \mathcal{B}''(\boldsymbol{\xi}, \boldsymbol{\xi}), \Phi\rangle$, where the operator $\mathcal{B}''(\Psi, \boldsymbol{\xi}) \in \mathcal{L}(\mathbf{V} \times \mathbf{V}, \mathbf{V}')$ is defined by $\langle \mathcal{B}''(\Psi, \boldsymbol{\xi}), \Phi\rangle := 2B(\Psi, \boldsymbol{\xi}, \Phi)$.

Define the discrete counterparts $\mathcal{B}_{\text{NC}} : \mathbf{V} + \mathbf{V}_{\text{M}} \to (\mathbf{V} + \mathbf{V}_{\text{M}})'$ as $\langle \mathcal{B}_{\text{NC}}(\Psi), \Phi\rangle = B_{\text{NC}}(\Psi, \Psi, \Phi)$ for all $\Psi, \Phi \in \mathbf{V} + \mathbf{V}_{\text{M}}$. The Fréchet derivative of $\mathcal{B}_{\text{NC}}$ around Ψ at the direction of $\boldsymbol{\xi}$ denoted by $\mathcal{B}'_{\text{NC}}(\Psi)(\boldsymbol{\xi})$ is

$$\langle \mathcal{B}'_{\text{NC}}(\Psi)(\boldsymbol{\xi}), \Phi\rangle = 2B_{\text{NC}}(\Psi, \boldsymbol{\xi}, \Phi) \quad \text{for all } \Psi, \Phi, \boldsymbol{\xi} \in \mathbf{V} + \mathbf{V}_{\text{M}}. \tag{3.1}$$

Theorem 3.1 (Existence result [16])**.** *Let* $(\bar{\Psi}, \bar{u}) \in \mathbf{V} \times L^2(\omega)$ *be a regular solution to* (2.1)*. Then there exist an open ball* $\mathcal{O}(\bar{u})$ *of* $\bar{u}$ *in* $L^2(\omega)$*, an open ball* $\mathcal{O}(\bar{\Psi})$ *of* $\bar{\Psi}$ *in* $\mathbf{V}$*, and a mapping* G *from* $\mathcal{O}(\bar{u})$ *to* $\mathcal{O}(\bar{\Psi})$ *of class* C^∞*, such that, for all* $u \in \mathcal{O}(\bar{u})$*,* $\Psi_u = G(u)$ *is the unique solution in* $\mathcal{O}(\bar{\Psi})$ *to* (2.2)*. Thus,* $G'(u) = (\mathcal{A} + \mathcal{B}'(\Psi_u))^{-1}$ *is uniformly bounded from a smaller ball into a smaller ball* (*these smaller balls are still denoted by* $\mathcal{O}(\bar{u})$ *and* $\mathcal{O}(\bar{\Psi})$ *for notational simplicity*)*. Moreover, if* $G'(u)v =: \mathbf{z}_v \in \mathbf{V}$ *and* $G''(u)v^2 =: \mathbf{w} \in \mathbf{V}$*, then* $\mathbf{z}_v$ *and* $\mathbf{w}$ *satisfy*

$$\mathcal{A}\mathbf{z}_v + \mathcal{B}'(\Psi_u)\mathbf{z}_v = \mathbf{Cv} \quad \text{in } \mathbf{V}', \ \mathcal{A}\mathbf{w} + \mathcal{B}'(\Psi_u)\mathbf{w} + \mathcal{B}''(\mathbf{z}_v, \mathbf{z}_v) = 0 \quad \text{in } \mathbf{V}', \tag{3.2}$$

where $\mathcal{A}+\mathcal{B}'(\Psi_u)$ *is an isomorphism from* $\mathbf{V}$ *into* $\mathbf{V}'$ *for all* $u \in \mathcal{O}(\bar{u})$*. Moreover, there exists a constant* $C_{\text{ub}} > 0$ *such that* $\|(\mathcal{A} + \mathcal{B}'(\Psi_u))^{-1}\|_{\mathcal{L}(\mathbf{V}', \mathbf{V})} \leq C_{\text{ub}}$ *and* $\|\mathbf{z}_v\|_2 \leq \|G'(u)\|_{\mathcal{L}(L^2(\omega), H^2(\Omega))}\|v\|_{L^2(\omega)}$*.*

Remark 3.2. The dependence of Ψ with respect to u is made explicit with the notation Ψ_u only when it is necessary.

Remark 3.3. In this paper, we assume that the exact solution Ψ to the nonlinear problem (2.1) is regular, that is, the linearized form is nonsingular ([28], Def. 2.4, p. 466). Hence the bounded derivative $DN(\Psi)$ of the operator N at the solution Ψ is an isomorphism. That is, the regular solution $\bar{\Psi}$ to (2.1) satisfies the inf-sup condition [22]:

$$0 < \beta := \inf_{\substack{\boldsymbol{\xi} \in \mathbf{V} \\ \|\boldsymbol{\xi}\|_2 = 1}} \sup_{\substack{\Phi \in \mathbf{V} \\ \|\Phi\|_2 = 1}} \langle \mathcal{A}\boldsymbol{\xi} + \mathcal{B}'(\bar{\Psi})\boldsymbol{\xi}, \Phi\rangle, \text{ and this leads to } \left\|\left(\mathcal{A} + \mathcal{B}'(\bar{\Psi})\right)^{-1}\right\|_{\mathcal{L}(\mathbf{V}', \mathbf{V})} = 1/\beta. \tag{3.3}$$

The existence of a solution to (2.1) can be obtained using standard arguments of considering a minimizing sequence, that is bounded in $\mathbf{V} \times L^2(\omega)$, and passing to the limit [26, 30, 35].

Lemma 3.4 (*A priori* bounds, regularity and convergence ([33], Lems. 2.7, 2.9 & 2.10, [10], Thm. 2.1))**.**

(a) *For $f \in H^{-1}(\Omega)$ and $u \in L^2(\omega)$, the solution Ψ of (2.1b) and (2.1c) belongs to $\mathbf{V} \cap \boldsymbol{H}^{2+\gamma}(\Omega)$, $\gamma \in (1/2, 1]$ is the elliptic regularity index, and satisfies the a priori bounds $\|\Psi\|_2 \lesssim (\|f\|_{-1} + \|u\|_{L^2(\omega)})$, $\|\Psi\|_{2+\gamma} \lesssim (\|f\|_{-1}^3 + \|u\|_{L^2(\omega)}^3 + \|f\|_{-1} + \|u\|_{L^2(\omega)})$.*
(b) *The solution $\mathbf{z}_v$ of the linearized problem (3.2) also belongs to $\mathbf{V} \cap \boldsymbol{H}^{2+\gamma}(\Omega)$, and satisfies the a priori bound $\|\mathbf{z}_v\|_{2+\gamma} \lesssim \|v\|_{L^2(\omega)}$.*
(c) *Let $(\bar{\Psi}, \bar{u})$ be a regular solution to (2.1) and $(u_k)_k$ be a sequence in $\mathcal{O}(\bar{u})$ weakly converging to $\bar{u}$ in $L^2(\omega)$. Let Ψ_{u_k} be the solution to (2.2) in $\mathcal{O}(\bar{\Psi})$ that corresponds to u_k. Then $(\Psi_{u_k})_k$ converges to $\bar{\Psi}$ in $\mathbf{V}$.*

When the load function belongs to $H^{-1}(\Omega)$, the solution of the clamped biharmonic plate problem belongs to $H_0^2(\Omega) \cap H^{2+\gamma}(\Omega)$, with $\gamma \in \left(\frac{1}{2}, 1\right]$, when all the interior angles are less than 126.283° ([5], Thm. 2). Note that when Ω is convex, $\gamma = 1$. These regularity results extend to the von Kármán equations ([5], Sect. 6) and to the state and adjoint variables of the control problem [33]. The optimal state and adjoint variables belong to $\mathbf{V} \cap \boldsymbol{H}^{2+\gamma}(\Omega)$, with $\gamma \in \left(\frac{1}{2}, 1\right]$, referred to as the index of elliptic regularity.

The crucial properties of Morley interpolation and companion operators that are useful in the analysis are stated below.

Lemma 3.5 (Morley interpolation operator [12, 13, 24])**.** *For $v \in V$, the Morley interpolation operator $I_{\mathrm{M}} : V \to V_{\mathrm{M}}$ defined by $(I_{\mathrm{M}}v)(z) = v(z)$ for any vertex z of $\mathcal{T}$ and $\int_E \partial I_{\mathrm{M}} v / \partial \nu_E \, \mathrm{d}s = \int_E \partial v / \partial \nu_E \, \mathrm{d}s$ for any edge E of $\mathcal{T}$ satisfies* (a) *the integral mean property $D_{\mathrm{NC}}^2 I_{\mathrm{M}} = \Pi_0 D_{\mathrm{NC}}^2$ of the Hessian,* (b) $\sum_{m=0}^{1} h^{m-2} |(1 - I_{\mathrm{M}})v|_{H^m(K)} \leq C_{\mathrm{I}} |(1 - I_{\mathrm{M}})v|_{H^2(K)} = C_{\mathrm{I}} \left(\|D^2 v\|_{L^2(K)} - \left\|D^2 I_{\mathrm{M}} v\right\|_{L^2(K)} \right)$ *for all $v \in H^2(K)$ and $K \in \mathcal{T}$, and* (c)$\|(1 - I_{\mathrm{M}})v\|_{\mathrm{NC}} \lesssim h^{\gamma} \|v\|_{2+\gamma}$ *for all $v \in V \cap H^{2+\gamma}(\Omega)$.*

Lemma 3.6 (Companion operator [12, 24])**.** *For any $v_{\mathrm{M}} \in V_{\mathrm{M}}$, there exists $J : V_{\mathrm{M}} \to V$ such that*

(a) $I_{\mathrm{M}} J v_{\mathrm{M}} = v_{\mathrm{M}}$ for all $v_{\mathrm{M}} \in V_{\mathrm{M}}$, (b) $\Pi_0((1 - J)v_{\mathrm{M}}) = 0$, (c) $\Pi_0 D_{\mathrm{NC}}^2((1 - J)v_{\mathrm{M}}) = 0$,
(d) $\left\|h_K^{-2}((1 - J)v_{\mathrm{M}})\right\| + \left\|h_K^{-1} D_{\mathrm{NC}}((1 - J)v_{\mathrm{M}})\right\| + \left\|D_{\mathrm{NC}}^2((1 - J)v_{\mathrm{M}})\right\| \leq \Lambda_{\mathrm{J}} \min_{v \in V} \left\|D_{\mathrm{NC}}^2(v_{\mathrm{M}} - v)\right\|$,

$$\text{(e)} \sum_{m=0}^{2} h_K^{2m-4} \|(1 - J)v_{\mathrm{M}}\|_{H^m(K)}^2 \leq C_{\mathrm{J}}^2 \sum_{E \in \mathcal{E}(\Omega(K))} h_E \left\| \left[D_{\mathrm{NC}}^2 v_{\mathrm{M}}\right]_E \tau_E \right\|_{L^2(E)}^2 \lesssim \min_{v \in V} \left\|D_{\mathrm{NC}}^2(v_{\mathrm{M}} - v)\right\|_{L^2(\Omega(K))}^2.$$

Here $\mathcal{N}(K)$ denotes the set of vertices of $K \in \mathcal{T}$ and patch $\Omega(K) := \operatorname{int}\left(\cup_{z \in \mathcal{N}(K)} \cup \mathcal{T}(z)\right)$, $\mathcal{T}(z)$ denotes the triangles that share the vertex z and $\mathcal{E}(\Omega(K))$ denotes the edges in $\Omega(K)$.

For vector-valued functions, the interpolation and companion operators are to be understood componentwise. The bound for discrete trilinear form and lower bounds for discrete norms stated in the next lemma are essential in the analysis.

Lemma 3.7. *For $\boldsymbol{\chi}, \boldsymbol{\lambda}, \Phi \in \mathbf{V} + \mathbf{V}_{\mathrm{M}}$, there exist positive constants C_{dS} and C_{b} such that*

(a) *(Lower bounds for discrete norms)* $\|\Phi\|_{0,\infty} + \|\Phi\|_{1,2,h} \leq C_{\mathrm{dS}} \|\Phi\|_{\mathrm{NC}}$.
(b) *(Bound for $B_{\mathrm{NC}}(\bullet, \bullet, \bullet)$)* $B_{\mathrm{NC}}(\boldsymbol{\chi}, \boldsymbol{\lambda}, \Phi) \leq C_{\mathrm{b}} \|\boldsymbol{\chi}\|_{\mathrm{NC}} \|\boldsymbol{\lambda}\|_{\mathrm{NC}} \|\Phi\|_{\mathrm{NC}}$.

For proofs, see Lemma 4.7 of [15] and Lemma 2.6 of [10].

Optimality conditions

Recall the auxiliary variable $\widetilde{u}_h$ given in (2.7). This computable variable helps to derive the reliability estimate for the control variable.

A key property in favor of $\widetilde{u}_h \in U_{ad}$ is that it satisfies the optimality condition

$$\left(\mathbf{C}^*\bar{\Theta}_{\mathrm{M}} + \alpha\widetilde{\mathbf{u}}_h, \mathbf{u} - \widetilde{\mathbf{u}}_h\right)_{\boldsymbol{L}^2(\omega)} \geq 0 \ \text{ for all } \mathbf{u} = (u,0)^T,\ u \in U_{ad}. \tag{3.4}$$

Define for $u, v \in U_{ad}$, $j'(u)v := (\mathcal{C}^*\theta_{u,1} + \alpha u, v)_{L^2(\omega)}$, where $j : U_{ad} \cap \mathcal{O}(\bar{u}) \to \mathbb{R}$ is the reduced cost functional defined by $j(u) := \mathcal{J}(G(u), u)$ and $G(u) = \Psi_u = (\psi_{u,1}, \psi_{u,2}) \in \mathbf{V}$ is the unique solution to (2.2) corresponding to u. Note that the first-order optimality condition $j'(\bar{u})(u - \bar{u}) \geq 0$ for all $u \in U_{ad}$ translates to (2.3c). The second-order sufficient optimality conditions that ensure the error estimates for this nonlinear control problem are discussed below. For a detailed discussion, we refer to Section 2.3 of [33] and Section 3.2 of [16].

For a *local regular solution* $\left(\bar{\Psi}, \bar{u}\right)$ of (2.1), the reduced control problem seeks a local solution $\bar{u}$ that satisfies $\inf_{u \in U_{ad} \cap \mathcal{O}(\bar{u})} j(u)$, where $j : U_{ad} \cap \mathcal{O}(\bar{u}) \to \mathbb{R}$ is the reduced local cost functional defined by $j(u) := J(G(u), u)$ and $G(u) = \Psi_u = (\psi_{u,1}, \psi_{u,2}) \in \mathbf{V}$ is the unique solution to (2.2) as defined in Theorem 3.1. Since G is of class C^∞ in $\mathcal{O}(\bar{u})$, j is of class C^∞ and for every $u \in \mathcal{O}(\bar{u})$ and $v \in L^2(\Omega)$ ([33], Sect. 2.3),

$$j''(u)v^2 = \int_\Omega \left(|\mathbf{z}_v|^2 + [[\mathbf{z}_v, \mathbf{z}_v]] \cdot \Theta_u\right) \mathrm{d}x + \alpha \int_\Omega |v|^2 \ \mathrm{d}x,$$

where $\mathbf{z}_v = (z_{v,1}, z_{v,2})$ is the solution of (3.2), $[[\mathbf{z}_v, \mathbf{z}_v]] := ([z_{v,1}, z_{v,2}] + [z_{v,2}, z_{v,1}], -[z_{v,1}, z_{v,1}])$, $[\cdot,\cdot]$ being the von Kármán bracket, $\Theta_u = (\theta_{u,1}, \theta_{u,2}) \in \mathbf{V}$ is the solution of the adjoint system and $[[\mathbf{z}_v, \mathbf{z}_v]] \cdot \Theta_u := ([z_{v,1}, z_{v,2}] + [z_{v,2}, z_{v,1}])\theta_{u,1} - [z_{v,1}, z_{v,1}]\theta_{u,2}$. Define the *tangent cone* at $\bar{u}$ to U_{ad} as

$$\mathscr{C}_{U_{ad}}(\bar{u}) := \left\{u \in L^2(\omega) : u(x) \in \mathbb{R} \text{ if } \bar{u}(x) \in (u_a, u_b),\ u(x) \geq 0 \text{ if } \bar{u}(x) = u_a,\ u(x) \leq 0 \text{ if } \bar{u}(x) = u_b\right\}.$$

Introduce the notation $\bar{d}(x) = \mathcal{C}^*\bar{\theta}_1(x) + \alpha\bar{u}(x), x \in \omega$. Associated with $\bar{d}$, we introduce another cone $\mathscr{C}_{\bar{u}} \subset \mathscr{C}_{U_{ad}}(\bar{u})$ defined by

$$\mathscr{C}_{\bar{u}} := \Big\{u \in L^2(\omega) : u(x) = 0 \text{ if } \bar{d}(x) \neq 0,\ u(x) \geq 0 \text{ if } \bar{d}(x) = 0 \text{ and } \bar{u}(x) = u_a,$$
$$u(x) \leq 0 \text{ if } \bar{d}(x) = 0 \text{ and } \bar{u}(x) = u_b\Big\}.$$

Theorem 3.8 (Second-order necessary condition ([33], Thm. 2.14))**.** *Let* $\left(\bar{\Psi}, \bar{u}\right)$ *be a regular local solution of* (2.1)*. Then,*

$$j''(\bar{u})v^2 \geq 0 \qquad \text{for all } v \in \mathscr{C}_{\bar{u}}. \tag{3.5}$$

The optimality condition (3.5) *is equivalent to*

$$\int_\Omega \Big(|\bar{\mathbf{z}}_v|^2 + [[\bar{\mathbf{z}}_v, \bar{\mathbf{z}}_v]]\bar{\Theta}\Big) \,\mathrm{d}x + \alpha \int_\omega |v|^2 \,\mathrm{d}x \geq 0$$

for all $v \in \mathscr{C}_{\bar{u}}$*, where* $\bar{\Theta} = \Theta(\bar{u})$ *is the associated adjoint state and* $\bar{\mathbf{z}}_v = \boldsymbol{z}_v(\bar{u})$ *is the solution to* (3.2) *for* $u = \bar{u}$ *and* $v \in \mathscr{C}_{\bar{u}}$*.*

Theorem 3.9 (Second-order sufficient condition [1,16])**.** *Let* $\left(\bar{\Psi}, \bar{u}\right)$ *be a nonsingular local solution of* (2.1) *and let* $\bar{\Theta} = \Theta(\bar{u})$ *be the associated adjoint state. Let the triplet* $(\bar{\Psi}, \bar{\Theta}, \bar{u}) \in \mathbf{V} \times \mathbf{V} \times L^2(\omega)$ *satisfy the first-order optimality system in* (2.3a)–(2.3c) *and*

$$\int_\Omega \left(|\bar{\mathbf{z}}_v|^2 + [[\bar{\mathbf{z}}_v, \bar{\mathbf{z}}_v]]\bar{\Theta}\right) \mathrm{d}x + \alpha \int_\omega |v|^2 \,\mathrm{d}x > 0 \tag{3.6}$$

for all non zero $v \in \mathscr{C}_{\bar{u}}$. Then, there exist $\epsilon > 0$ and $\mu > 0$ such that, for all $u \in U_{ad}$ satisfying, together with Ψ_u,

$$\|u - \bar{u}\|^2_{L^2(\omega)} + \left\|\!\left|\Psi_u - \bar{\Psi}\right|\!\right\|^2 \leq \epsilon^2,$$

we have

$$\mathcal{J}\big(\bar{\Psi}, \bar{u}\big) + \frac{\mu}{2}\Big(\|u - \bar{u}\|^2_{L^2(\omega)} + \left\|\!\left|\Psi_u - \bar{\Psi}\right|\!\right\|^2\Big) \leq \mathcal{J}(\Psi_u, u).$$

Note that the condition (3.6) is equivalent to the existence of $\widehat{\delta}, \tau > 0$ such that ([16], Cor. 3.11)

$$j''(\bar{u})v^2 > \widehat{\delta}\Big(\|v\|^2_{L^2(\omega)} + \|\bar{\mathbf{z}}_v\|^2\Big), \quad \text{for all } v \in \mathscr{C}^{\tau}_{\bar{u}}, \tag{3.7}$$

where $\bar{\mathbf{z}}_v$ is the solution of (3.2) with $u = \bar{u}$ and

$$\mathscr{C}^{\tau}_{\bar{u}} := \Big\{v \in L^2(\omega) : v(x) = 0 \text{ if } \left|\bar{d}(x)\right| > \tau,\ v(x) \geq 0 \text{ if } \left|\bar{d}(x)\right| \leq \tau \text{ and } \bar{u}(x) = u_a,$$
$$v(x) \leq 0 \text{ if } \left|\bar{d}(x)\right| \leq \tau \text{ and } \bar{u}(x) = u_b\Big\}.$$

The next result follows from Lemma 5 of [1].

Lemma 3.10 (Property of j'')**.** *Let $\mathcal{M} > 0$ be such that $\max\Big\{\|\bar{u} + t(\widetilde{u}_h - \bar{u})\|_{L^\infty(\Omega)}, \|\bar{u} - \widetilde{u}_h\|_{L^\infty(\Omega)}\Big\} \leq \mathcal{M}$ with $t \in (0,1)$. Then, there exists $C_{\mathcal{M}} > 0$ such that $\Big|(j''(\bar{u} + t(\widetilde{u}_h - \bar{u})) - j''(\widetilde{u}_h - \bar{u}))(\widetilde{u}_h - \bar{u})^2\Big| \leq C_{\mathcal{M}}\|\widetilde{u}_h - \bar{u}\|_{L^\infty(\Omega)}\|\widetilde{u}_h - \bar{u}\|^2_{L^2(\Omega)}$.*

4. Reliability analysis

This section deals with the proofs of Theorem 2.4(a) and (b).

The reliability error estimate for the control problem can be expressed as a combination of the reliability results for the state, adjoint and control variables. The individual contributions are presented first. The proof of the main result is presented at the end of this section.

4.1. *A posteriori* error analysis for the state equations

Let $\big(\bar{\Psi}, \bar{u}\big)$ be a regular solution to (2.1) and let $\widehat{\Psi} \in \mathbf{V}$ solve the auxiliary state equation

$$A\Big(\widehat{\Psi}, \Phi\Big) + B\Big(\widehat{\Psi}, \widehat{\Psi}, \Phi\Big) = (\mathbf{F} + \mathbf{C}\bar{\mathbf{u}}_h, \Phi) \text{ for all } \Phi \in \mathbf{V}, \tag{4.1}$$

where $\bar{\mathbf{u}}_h = (\bar{u}_h, 0)^T$ is the discrete control defined in (2.6). Since $\bar{\Psi}$ is a regular solution and $\bar{u}_h$ is sufficiently close to $\bar{u}$ from Theorem 2.3, Theorem 3.1 yields $\widehat{\Psi}$ is regular. That is,

$$0 < \hat{\beta} := \inf_{\substack{\boldsymbol{\xi} \in \mathbf{V} \\ \|\boldsymbol{\xi}\|_2 = 1}} \sup_{\substack{\Phi \in \mathbf{V} \\ \|\Phi\|_2 = 1}} DN\Big(\widehat{\Psi}; \boldsymbol{\xi}, \Phi\Big). \tag{4.2}$$

Note that $\widehat{\Psi}$ solves the von Kármán equations (4.1) and its Morley FE approximation seeks $\bar{\Psi}_{\mathrm{M}}$ as stated in (2.6a). Given the exact solution $\big(\bar{\Psi}, \bar{\Theta}, \bar{\mathbf{u}}\big)$ to (2.3), suppose $\varepsilon_0, \delta_0 > 0$ satisfy Theorem 2.3, and if necessary, are chosen smaller such that, for any $\mathcal{T} \in \mathbb{T}(\delta_0)$, the discrete solution $\big(\bar{\Psi}_{\mathrm{M}}, \bar{\Theta}_{\mathrm{M}}, \bar{\mathbf{u}}_h\big)$ to (2.6) satisfies $\varepsilon_0 \leq \min\Big\{\hat{\beta}/(2C_{\mathrm{b}}(1 + \Lambda_{\mathrm{J}} + C_{\mathrm{ub}})), \beta/(4C_{\mathrm{b}}), \alpha\widehat{\delta}(2C_{\mathcal{M}})^{-1}/C_{\mathrm{dS}}, \tau/(2C_{\mathrm{dS}})\Big\}$ and

$$\left\|\!\left|\bar{\Psi} - \bar{\Psi}_{\mathrm{M}}\right|\!\right\|_{\mathrm{NC}} + \left\|\!\left|\bar{\Theta} - \bar{\Theta}_{\mathrm{M}}\right|\!\right\|_{\mathrm{NC}} + \|\bar{u} - \bar{u}_h\|_{L^2(\omega)} \leq \varepsilon_0, \tag{4.3}$$

where the constants $\widehat{\delta}$, $C_{\mathcal{M}}$ and τ are defined in (3.7) and Lemma 3.10, and β (resp. $\hat{\beta}$) is the inf-sup constant in (3.3) (resp. (4.2)). Note that the constants Λ_{J}, C_{b}, C_{dS} and C_{ub} are from Lemmas 3.6(d), 3.7(a), (b), and Theorem 3.1, respectively.

Theorem 4.1 (Reliability for the state variable)**.** *Let* $\big(\bar{\Psi}, \bar{u}\big) \in \mathbf{V} \times L^2(\omega)$ *be a regular solution to* (2.1) *and* $(\bar{\Psi}_{\mathrm{M}}, \bar{u}_h) \in \mathbf{V}_{\mathrm{M}} \times U_{h,ad}$ *solve* (2.5)*. Then for all* $\mathcal{T} \in \mathbb{T}(\delta_0)$*, there exists an* h*-independent positive constant* $C_{\mathrm{ST,rel}}$ *such that*

$$\left\|\!\left|\bar{\Psi} - \bar{\Psi}_{\mathrm{M}}\right|\!\right\|_{\mathrm{NC}} \le C_{\mathrm{ST,rel}} \left(\sum_{K \in \mathcal{T}} \eta^2_{K,\bar{\Psi}_{\mathrm{M}}} + \sum_{E \in \mathcal{E}(\Omega)} \eta^2_{E,\bar{\Psi}_{\mathrm{M}}} + \|\bar{u} - \bar{u}_h\|^2_{L^2(\omega)} \right)^{1/2}. \tag{4.4}$$

Proof. The terms $\left\|\!\left|\widehat{\Psi} - \bar{\Psi}_{\mathrm{M}}\right|\!\right\|_{\mathrm{NC}}$ and $\left\|\!\left|\bar{\Psi} - \widehat{\Psi}\right|\!\right\|_2$ are estimated and then a triangle inequality completes the proof. Theorem 3.1 for (2.3a) and (4.1) yield $G(\bar{u}) = \bar{\Psi}$, $G(\bar{u}_h) = \widehat{\Psi}$. Also, if $G'(u)v =: \mathbf{z}_v \in \mathbf{V}$, then $\mathbf{z}_v$ satisfies $\mathcal{A}\mathbf{z}_v + \mathcal{B}'(\Psi)\mathbf{z}_v = \mathbf{C}\mathbf{v}$ in $\mathbf{V}'$, where $\Psi = G(u)$ and u, v belong to the interior of $\mathcal{O}(\bar{u})$. Theorem 3.1 proves the uniform boundedness of $\left\|\!\left|(\mathcal{A} + \mathcal{B}'(\Psi_u))^{-1}\right|\!\right\|_{\mathcal{L}(\mathbf{V}',\mathbf{V})}$ whenever $u \in \mathcal{O}(\bar{u})$.

Hence, for $u_t = \bar{u}_h + t(\bar{u} - \bar{u}_h)$ and $\Psi_t = G(u_t)$, mean value theorem, Theorem 3.1 and $\bar{u}_h \in \mathcal{O}(\bar{u})$ show

$$\left\|\!\left|\bar{\Psi} - \widehat{\Psi}\right|\!\right\|_2 = \left\|\!\left|\int_0^1 G'(u_t)(\mathbf{C}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_h))\,\mathrm{d}t\right|\!\right\|_2 = \left\|\!\left|\int_0^1 (\mathcal{A} + \mathcal{B}'(\Psi_t))^{-1}(\mathbf{C}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_h))\,\mathrm{d}t\right|\!\right\|_2 \le C_{\mathrm{ub}}\|\bar{u} - \bar{u}_h\|_{L^2(\omega)}. \tag{4.5}$$

The estimate of $\left\|\!\left|\widehat{\Psi} - \bar{\Psi}_{\mathrm{M}}\right|\!\right\|_{\mathrm{NC}}$ adapts the ideas of [15]. The inf-sup condition (4.2) implies that for any $0 < \epsilon_1 < \hat{\beta}$, there exists some $\Phi \in \mathbf{V}$ with $\|\!|\Phi|\!\|_2 = 1$ and

$$(\hat{\beta} - \epsilon_1)\left\|\!\left|\widehat{\Psi} - J\bar{\Psi}_{\mathrm{M}}\right|\!\right\|_2 \le DN(\widehat{\Psi}; \widehat{\Psi} - J\bar{\Psi}_{\mathrm{M}}, \Phi). \tag{4.6}$$

Since $N(\bullet)$ is quadratic, the finite Taylor series is exact and hence

$$N\big(J\bar{\Psi}_{\mathrm{M}}; \Phi\big) = DN\Big(\widehat{\Psi}; J\bar{\Psi}_{\mathrm{M}} - \widehat{\Psi}, \Phi\Big) + \frac{1}{2}D^2N\Big(\widehat{\Psi}; J\bar{\Psi}_{\mathrm{M}} - \widehat{\Psi}, J\bar{\Psi}_{\mathrm{M}} - \widehat{\Psi}, \Phi\Big).$$

This with $D^2N\Big(\widehat{\Psi}; \widehat{\Psi} - J\bar{\Psi}_{\mathrm{M}}, \widehat{\Psi} - J\bar{\Psi}_{\mathrm{M}}, \Phi\Big) = 2B\Big(\widehat{\Psi} - J\bar{\Psi}_{\mathrm{M}}, \widehat{\Psi} - J\bar{\Psi}_{\mathrm{M}}, \Phi\Big)$, (4.6) and Lemma 3.7(b) show

$$\Big(\hat{\beta} - \epsilon_1\Big)\left\|\!\left|\widehat{\Psi} - J\bar{\Psi}_{\mathrm{M}}\right|\!\right\|_2 \le \big|N\big(J\bar{\Psi}_{\mathrm{M}}; \Phi\big)\big| + C_{\mathrm{b}}\left\|\!\left|\widehat{\Psi} - J\bar{\Psi}_{\mathrm{M}}\right|\!\right\|_2^2. \tag{4.7}$$

A triangle inequality, (4.5), (4.3), Lemma 3.6(d) with $v = \widehat{\Psi}$ and $\varepsilon_0 \le \hat{\beta}/(2C_{\mathrm{b}}(1 + \Lambda_{\mathrm{J}} + C_{\mathrm{ub}}))$ imply

$$\left\|\!\left|\widehat{\Psi} - J\bar{\Psi}_{\mathrm{M}}\right|\!\right\|_{\mathrm{NC}} \le \left\|\!\left|\widehat{\Psi} - \bar{\Psi}\right|\!\right\|_2 + \left\|\!\left|\bar{\Psi} - \bar{\Psi}_{\mathrm{M}}\right|\!\right\|_{\mathrm{NC}} + \left\|\!\left|(1 - J)\bar{\Psi}_{\mathrm{M}}\right|\!\right\|_{\mathrm{NC}} \le (C_{\mathrm{ub}} + 1 + \Lambda_{\mathrm{J}})\varepsilon_0 \le \hat{\beta}/2C_{\mathrm{b}}. \tag{4.8}$$

A substitution of (4.8) in (4.7) leads to

$$\Big(\hat{\beta}/2 - \epsilon_1\Big)\left\|\!\left|\widehat{\Psi} - J\bar{\Psi}_{\mathrm{M}}\right|\!\right\|_2 \le \big|N\big(J\bar{\Psi}_{\mathrm{M}}; \Phi\big)\big|.$$

This eventually shows that

$$\left\|\!\left|\widehat{\Psi} - \bar{\Psi}_{\mathrm{M}}\right|\!\right\|_{\mathrm{NC}} \le 2\hat{\beta}^{-1}\big|N\big(J\bar{\Psi}_{\mathrm{M}}; \Phi\big)\big| + \left\|\!\left|(J - 1)\bar{\Psi}_{\mathrm{M}}\right|\!\right\|_{\mathrm{NC}}. \tag{4.9}$$

The definition of $N(\bullet)$, (2.6a) and rearrangements lead to

$$N\big(J\bar{\Psi}_{\mathrm{M}}; \Phi\big) = A\big(J\bar{\Psi}_{\mathrm{M}}, \Phi\big) + B\big(J\bar{\Psi}_{\mathrm{M}}, J\bar{\Psi}_{\mathrm{M}}, \Phi\big) - (\mathbf{F} + \mathbf{C}\mathbf{u}_h, \Phi)$$

$$= A_{\mathrm{NC}}((J-1)\bar{\Psi}_{\mathrm{M}},\Phi) + A_{\mathrm{NC}}\big(\bar{\Psi}_{\mathrm{M}},(1-I_{\mathrm{M}})\Phi\big) + B_{\mathrm{NC}}\big((J-1)\bar{\Psi}_{\mathrm{M}},J\bar{\Psi}_{\mathrm{M}},\Phi\big)$$
$$+ B_{\mathrm{NC}}\big(\bar{\Psi}_{\mathrm{M}},(J-1)\bar{\Psi}_{\mathrm{M}},\Phi\big) + B_{\mathrm{NC}}\big(\bar{\Psi}_{\mathrm{M}},\bar{\Psi}_{\mathrm{M}},(1-I_{\mathrm{M}})\Phi\big) - (\mathbf{F}+\mathbf{C}\mathbf{u}_h,(1-I_{\mathrm{M}})\Phi) =: \sum_{i=1}^{6} S_i. \tag{4.10}$$

The Cauchy–Schwarz inequality proves $S_1 \le \left\|(J-1)\bar{\Psi}_{\mathrm{M}}\right\|_{\mathrm{NC}}$. Since the piecewise second derivatives of $\bar{\Psi}_{\mathrm{M}}$ are constants, Lemma 3.5(a) implies $S_2 = 0$. The triangle inequalities, Lemma 3.6(d) with $v = \bar{\Psi}$, (4.3) and Lemma 3.4(a) prove

$$\left\|J\bar{\Psi}_{\mathrm{M}}\right\|_2 + \left\|\bar{\Psi}_{\mathrm{M}}\right\|_{\mathrm{NC}} \le \left\|(J-1)\bar{\Psi}_{\mathrm{M}}\right\|_{\mathrm{NC}} + 2\big(\left\|\bar{\Psi}-\bar{\Psi}_{\mathrm{M}}\right\|_{\mathrm{NC}} + \left\|\bar{\Psi}\right\|_2\big) \le (2+\Lambda_{\mathrm{J}})\varepsilon_0 + 2\left\|\bar{\Psi}\right\|_2 := \mathcal{M}_1. \tag{4.11}$$

Lemma 3.7(b) and (4.11) show $S_3 + S_4 \le C_{\mathrm{b}}\mathcal{M}_1\left\|(J-1)\bar{\Psi}_{\mathrm{M}}\right\|_{\mathrm{NC}}$. The definition of $B_{\mathrm{NC}}(\bullet,\bullet,\bullet)$, the Cauchy–Schwarz inequality and Lemma 3.5(b) prove $S_5 + S_6 \le C_{\mathrm{I}}\Big(\sum_{K\in\mathcal{T}} h_K^4\Big(\left\|f + \mathcal{C}u_h + [\bar{\psi}_{\mathrm{M},1},\bar{\psi}_{\mathrm{M},2}]\right\|_{L^2(K)}^2 + \left\|[\bar{\psi}_{\mathrm{M},1},\bar{\psi}_{\mathrm{M},1}]\right\|_{L^2(K)}^2\Big)\Big)^{1/2}$. A substitution of S_1–S_6 in (4.10) and then in (4.9) with Lemma 3.6(e), the definitions (2.8a) and (2.8e) result in

$$\left\|\widehat{\Psi} - \bar{\Psi}_{\mathrm{M}}\right\|_{\mathrm{NC}} \le \tilde{C}_{\mathrm{ST,rel}}\left(\sum_{K\in\mathcal{T}} \eta^2_{K,\bar{\Psi}_{\mathrm{M}}} + \sum_{E\in\mathcal{E}(\Omega)} \eta^2_{E,\bar{\Psi}_{\mathrm{M}}}\right)^{1/2}, \tag{4.12}$$

with $\tilde{C}^2_{\mathrm{ST,rel}} := C_{\mathrm{J}}^2\Big(1 + 2\hat{\beta}^{-1}(1 + C_{\mathrm{b}}\mathcal{M}_1)\Big)^2 + 4\hat{\beta}^{-2}C_{\mathrm{I}}^2$.

A combination of (4.5) and the last displayed result with a triangle inequality concludes the proof. □

4.2. *A posteriori* error analysis for the adjoint equations

The auxiliary problem that corresponds to the adjoint equations seeks $\widehat{\Theta} \in \mathbf{V}$ such that

$$A\Big(\Phi,\widehat{\Theta}\Big) + 2B_{\mathrm{NC}}\Big(\bar{\Psi}_{\mathrm{M}},\Phi,\widehat{\Theta}\Big) = \big(\bar{\Psi}_{\mathrm{M}} - \Psi_d,\Phi\big) \text{ for all } \Phi \in \mathbf{V}, \tag{4.13}$$

where $\bar{\Psi}_{\mathrm{M}} \in \mathbf{V}_{\mathrm{M}}$ is the solution to (2.6a).

Since $\bar{\Psi}$ is a regular solution to (2.1), the adjoint of the operator in (3.3) satisfies the inf-sup condition given by

$$\beta = \inf_{\substack{\boldsymbol{\xi}\in\mathbf{V}\\ \|\boldsymbol{\xi}\|_2=1}} \sup_{\substack{\Phi\in\mathbf{V}\\ \|\Phi\|_2=1}} \big\langle \mathcal{A}\Phi + \mathcal{B}'\big(\bar{\Psi}\big)\Phi,\boldsymbol{\xi}\big\rangle, \quad \left\|\bar{\Theta}\right\|_2 \le \beta^{-1}\left\|\bar{\Psi} - \Psi_d\right\| \tag{4.14}$$

with the last inequality derived from (2.3b).

An introduction of $\bar{\Psi}$ in the second term on the left-hand side of (4.13) yields

$$A\Big(\Phi,\widehat{\Theta}\Big) + 2B_{\mathrm{NC}}\Big(\bar{\Psi}_{\mathrm{M}},\Phi,\widehat{\Theta}\Big) = A\Big(\Phi,\widehat{\Theta}\Big) + 2B\Big(\bar{\Psi},\Phi,\widehat{\Theta}\Big) + 2B_{\mathrm{NC}}\Big(\bar{\Psi}_{\mathrm{M}} - \bar{\Psi},\Phi,\widehat{\Theta}\Big).$$

The first inequality of (4.14), Lemma 3.7(b), (4.3) and $\varepsilon_0 \le \beta/(4C_{\mathrm{b}})$ show that for any $0 < \epsilon_2 < \beta$, there exists some $\Phi \in \mathbf{V}$ with $\|\Phi\|_2 = 1$ such that

$$A\Big(\Phi,\widehat{\Theta}\Big) + 2B_{\mathrm{NC}}\Big(\bar{\Psi}_{\mathrm{M}},\Phi,\widehat{\Theta}\Big) \ge \big(\beta - \epsilon_2 - 2C_{\mathrm{b}}\left\|\bar{\Psi}_{\mathrm{M}} - \bar{\Psi}\right\|_{\mathrm{NC}}\big)\left\|\widehat{\Theta}\right\|_2 \ge (\beta - 2C_{\mathrm{b}}\varepsilon_0)\left\|\widehat{\Theta}\right\|_2 \ge \frac{\beta}{2}\left\|\widehat{\Theta}\right\|_2 \tag{4.15}$$

with $\epsilon_2 \searrow 0$ in the second last step of the inequality above.

This shows the wellposedness of (4.13). A combination of (4.13) and (4.15) leads to a bound for the solution of $\widehat{\Theta}$ of (4.13) as

$$\left\|\widehat{\Theta}\right\|_2 \le 2\beta^{-1}\left\|\bar{\Psi}_{\mathrm{M}} - \Psi_d\right\|. \tag{4.16}$$

For $\Psi, \Phi \in \mathbf{V} + \mathbf{V}_{\mathrm{M}}$, define *linear operators* $\mathcal{F}_{\Psi}$ and $\mathcal{F}_{\Psi,\mathrm{NC}} \in \mathcal{L}(\mathbf{V} + \mathbf{V}_{\mathrm{M}})$ by

$$\mathcal{F}_{\Psi}(\Phi) = \Phi + T[\mathcal{B}'_{\mathrm{NC}}(\Psi)^*(\Phi)] \text{ and } \mathcal{F}_{\Psi,\mathrm{NC}}(\Phi) = \Phi + T_{\mathrm{NC}}[\mathcal{B}'_{\mathrm{NC}}(\Psi)^*(\Phi)], \tag{4.17}$$

where $\mathcal{B}'_{\mathrm{NC}}(\Psi)^*$ is the adjoint operator corresponding to $\mathcal{B}'_{\mathrm{NC}}(\Psi)$. Here the bounded linear operator $T(\bullet)$ (resp. $T_{\mathrm{NC}}(\bullet)$) solves the biharmonic system in the sense that for the load $\boldsymbol{g} \in \mathbf{V}'$ (resp. $\boldsymbol{g} \in \mathbf{V}'_{\mathrm{M}}$), $A(T\boldsymbol{g}, \Phi) = \langle \boldsymbol{g}, \Phi \rangle$ for all $\Phi \in \mathbf{V}$ (resp. $A_{\mathrm{NC}}(T_{\mathrm{NC}}\boldsymbol{g}, \Phi_{\mathrm{M}}) = \langle \boldsymbol{g}, \Phi_{\mathrm{M}} \rangle$ for all $\Phi_{\mathrm{M}} \in \mathbf{V}_{\mathrm{M}}$). A detailed discussion of these operators is provided in the appendix.

The next lemma (proved in the appendix) is utilized in the proof of Theorem 4.3.

Lemma 4.2 (Uniform boundeness of $\mathcal{F}^{-1}_{\Psi_u}$)**.** *If $\bar{\Psi} \in \mathbf{V}$ is a regular solution to (2.1), then $\mathcal{F}_{\Psi_u}$ is an automorphism on $\mathbf{V} + \mathbf{V}_{\mathrm{M}}$, whenever u is sufficiently close to $\bar{u}$. Moreover,* $\|\mathcal{F}^{-1}_{\Psi_u}\|_{\mathcal{L}(\mathbf{V}+\mathbf{V}_{\mathrm{M}})} \leq 1 + 2C_{\mathrm{b}}\left\|\left(\mathcal{A} + \mathcal{B}'(\bar{\Psi})\right)^{-1}\right\|_{\mathcal{L}(\mathbf{V}',\mathbf{V})}\|\!|\bar{\Psi}|\!\|_2$.

Theorem 4.3 (Reliability for the adjoint variable)**.** *Let $\left(\bar{\Psi}, \bar{\Theta}, \bar{\mathbf{u}}\right)$ (resp. $\left(\bar{\Psi}_{\mathrm{M}}, \bar{\Theta}_{\mathrm{M}}, \bar{\mathbf{u}}_h\right)$) solve the optimality system (2.3) (resp. (2.6)). Then for all $\mathcal{T} \in \mathbb{T}(\delta_0)$, there exists an h-independent positive constant $C_{\mathrm{AD,rel}}$ such that*

$$\|\!|\bar{\Theta} - \bar{\Theta}_{\mathrm{M}}|\!\|_{\mathrm{NC}} \leq C_{\mathrm{AD,rel}}\left(\sum_{K\in\mathcal{T}} \eta^2_{K,\bar{\Psi}_{\mathrm{M}}} + \sum_{K\in\mathcal{T}} \eta^2_{K,\bar{\Theta}_{\mathrm{M}}} + \sum_{E\in\mathcal{E}(\Omega)} \eta^2_{E,\bar{\Psi}_{\mathrm{M}}} + \sum_{E\in\mathcal{E}(\Omega)} \eta^2_{E,\bar{\Theta}_{\mathrm{M}}} + \|\bar{u} - \bar{u}_h\|^2_{L^2(\omega)}\right)^{1/2}. \tag{4.18}$$

Proof. The terms $\left\|\!\left|\widehat{\Theta} - \bar{\Theta}_{\mathrm{M}}\right|\!\right\|_{\mathrm{NC}}$ and $\left\|\!\left|\bar{\Theta} - \widehat{\Theta}\right|\!\right\|_2$ are estimated and then a triangle inequality completes the proof. The inf-sup condition (4.14) implies for any $0 < \epsilon_3 < \beta$, there exists some $\Phi \in \mathbf{V}$ with $\|\!|\Phi|\!\|_2 = 1$ and

$$(\beta - \epsilon_3)\left\|\!\left|\widehat{\Theta} - J\bar{\Theta}_{\mathrm{M}}\right|\!\right\|_2 \leq A\left(\widehat{\Theta} - J\bar{\Theta}_{\mathrm{M}}, \Phi\right) + 2B_{\mathrm{NC}}\left(\bar{\Psi}_{\mathrm{M}}, \Phi, \widehat{\Theta} - J\bar{\Theta}_{\mathrm{M}}\right) + 2B_{\mathrm{NC}}\left(\bar{\Psi} - \bar{\Psi}_{\mathrm{M}}, \Phi, \widehat{\Theta} - J\bar{\Theta}_{\mathrm{M}}\right).$$

Since $\|\!|\bar{\Psi} - \bar{\Psi}_{\mathrm{M}}|\!\|_{\mathrm{NC}} \leq \varepsilon_0 \leq \beta/(4C_{\mathrm{b}})$, Lemma 3.7(b) for the last term in the right-hand side of the above inequality shows

$$(\beta/2 - \epsilon_3)\left\|\!\left|\widehat{\Theta} - J\bar{\Theta}_{\mathrm{M}}\right|\!\right\|_2 \leq A\left(\widehat{\Theta} - J\bar{\Theta}_{\mathrm{M}}, \Phi\right) + 2B_{\mathrm{NC}}\left(\bar{\Psi}_{\mathrm{M}}, \Phi, \widehat{\Theta} - J\bar{\Theta}_{\mathrm{M}}\right).$$

This, equations (4.13), (2.6b) and simple manipulation eventually lead to

$$\begin{aligned}(\beta/2 - \epsilon_3)\left\|\!\left|\widehat{\Theta} - J\bar{\Theta}_{\mathrm{M}}\right|\!\right\|_2 &\leq \left(\bar{\Psi}_{\mathrm{M}} - \Psi_d, \Phi\right) - A\left(J\bar{\Theta}_{\mathrm{M}}, \Phi\right) - 2B_{\mathrm{NC}}\left(\bar{\Psi}_{\mathrm{M}}, \Phi, J\bar{\Theta}_{\mathrm{M}}\right) = \left(\bar{\Psi}_{\mathrm{M}} - \Psi_d, (1 - I_{\mathrm{M}})\Phi\right)\\ &\quad - A_{\mathrm{NC}}\left((J-1)\bar{\Theta}_{\mathrm{M}}, \Phi\right) + A_{\mathrm{NC}}\left(\bar{\Theta}_{\mathrm{M}}, (I_{\mathrm{M}} - 1)\Phi\right) - 2B_{\mathrm{NC}}\left(\bar{\Psi}_{\mathrm{M}}, \Phi, J\bar{\Theta}_{\mathrm{M}}\right) + 2B_{\mathrm{NC}}\left(\bar{\Psi}_{\mathrm{M}}, I_{\mathrm{M}}\Phi, \bar{\Theta}_{\mathrm{M}}\right)\\ &= \left(\bar{\Psi}_{\mathrm{M}} - \Psi_d, (1 - I_{\mathrm{M}})\Phi\right) - A_{\mathrm{NC}}\left((J-1)\bar{\Theta}_{\mathrm{M}}, \Phi\right) + A_{\mathrm{NC}}\left(\bar{\Theta}_{\mathrm{M}}, (I_{\mathrm{M}} - 1)\Phi\right) + 2B_{\mathrm{NC}}\left(\bar{\Psi}_{\mathrm{M}}, \Phi, (1-J)\bar{\Theta}_{\mathrm{M}}\right)\\ &\quad + 2B_{\mathrm{NC}}\left(\bar{\Psi}_{\mathrm{M}}, (I_{\mathrm{M}} - 1)\Phi, \bar{\Theta}_{\mathrm{M}}\right) =: \sum_{i=1}^{5} S_i.\end{aligned} \tag{4.19}$$

The Cauchy–Schwarz inequality shows that $S_2 \leq \|\!|(J-1)\bar{\Theta}_{\mathrm{M}}|\!\|_{\mathrm{NC}}$. Since the piecewise second derivatives of $\bar{\Theta}_{\mathrm{M}}$ are constants, Lemma 3.5(a) implies $S_3 = 0$. Lemma 3.7(b) and (4.11) prove $S_4 \leq C_{\mathrm{b}}\mathcal{M}_1\|\!|(J-1)\bar{\Theta}_{\mathrm{M}}|\!\|_{\mathrm{NC}}$. The orthogonality property of J in Lemma 3.6(c) proves $B_{\mathrm{NC}}(\bar{\Psi}_{\mathrm{M}}, (1-J)I_{\mathrm{M}}\Phi, \mathcal{P}_0\bar{\Theta}_{\mathrm{M}}) = 0$. This and elementary algebra lead to

$$S_5/2 = B_{\mathrm{NC}}\left(\bar{\Psi}_{\mathrm{M}}, (1-J)I_{\mathrm{M}}\Phi, (1-\mathcal{P}_0)\bar{\Theta}_{\mathrm{M}}\right) + B_{\mathrm{NC}}\left(\bar{\Psi}_{\mathrm{M}}, (JI_{\mathrm{M}} - 1)\Phi, (1-J)\bar{\Theta}_{\mathrm{M}}\right)$$

$$+ B_{\rm NC}\big((1-J)\bar{\Psi}_{\rm M}, (JI_{\rm M}-1)\Phi, J\bar{\Theta}_{\rm M}\big) + B\big(J\bar{\Psi}_{\rm M}, (JI_{\rm M}-1)\Phi, J\bar{\Theta}_{\rm M}\big). \tag{4.20}$$

Triangle inequalities, Lemma 3.6(d) with $v = \bar{\Theta}$, (4.3), the second inequality of (4.14) and Lemma 3.4(a) show

$$\big\| J\bar{\Theta}_{\rm M}\big\|_2 + \|\bar{\Theta}_{\rm M}\|_{\rm NC} \le \|(J-1)\bar{\Theta}_{\rm M}\|_{\rm NC} + 2\big(\|\bar{\Theta}-\bar{\Theta}_{\rm M}\|_{\rm NC} + \|\bar{\Theta}\|_2\big) \le (2+\Lambda_{\rm J})\varepsilon_0 + 2\|\bar{\Theta}\|_2 := \mathcal{M}_2. \tag{4.21}$$

Lemmas 3.5(b) and 3.6(d) with $v = \Phi$ verify

$$\|(JI_{\rm M}-1)\Phi\|_{\rm NC} \le \|(J-1)I_{\rm M}\Phi\|_{\rm NC} + \|(I_{\rm M}-1)\Phi\|_{\rm NC} \ \le (\Lambda_{\rm J}+1)\|(I_{\rm M}-1)\Phi\|_{\rm NC} \le C_{\rm I}(\Lambda_{\rm J}+1). \tag{4.22}$$

The first three terms in the right-hand side of (4.20) are estimated now. The definition of $B_{\rm NC}(\bullet,\bullet,\bullet)$, the Cauchy–Schwarz inequality, (4.22) and the definitions (2.8c) prove

$$B_{\rm NC}\big(\bar{\Psi}_{\rm M}, (1-J)I_{\rm M}\Phi, (1-\mathcal{P}_0)\bar{\Theta}_{\rm M}\big) \le C_{\rm I}(\Lambda_{\rm J}+1)\Bigg(\sum_{K\in\mathcal{T}} \eta^2_{K,\mathcal{P}_0,\bar{\Theta}_{\rm M}}\Bigg)^{1/2}. \tag{4.23}$$

Lemma 3.7(b), (4.11), (4.21), (4.22), Lemma 3.6(e) and the definitions (2.8e), (2.8f) show

$$B_{\rm NC}\big(\bar{\Psi}_{\rm M}, (JI_{\rm M}-1)\Phi, (1-J)\bar{\Theta}_{\rm M}\big) \le C_{\rm b}C_{\rm I}C_{\rm J}(\Lambda_{\rm J}+1)\mathcal{M}_1\Bigg(\sum_{E\in\mathcal{E}(\Omega)} \eta^2_{E,\bar{\Theta}_{\rm M}}\Bigg)^{1/2}, \tag{4.24}$$

$$B_{\rm NC}\big((1-J)\bar{\Psi}_{\rm M}, (JI_{\rm M}-1)\Phi, J\bar{\Theta}_{\rm M}\big) \le C_{\rm b}C_{\rm I}C_{\rm J}(\Lambda_{\rm J}+1)\mathcal{M}_2\Bigg(\sum_{E\in\mathcal{E}(\Omega)} \eta^2_{E,\bar{\Psi}_{\rm M}}\Bigg)^{1/2}. \tag{4.25}$$

The last term on the right-hand side of (4.20) is estimated in its scalar version and details are provided for clarity. The symmetry of $b(\bullet,\bullet,\bullet)$ with respect to the second and third variables, and an introduction of $\bar{\psi}_{\rm M,1}$ and $\bar{\theta}_{\rm M,1}$ imply that the first term in the expansion can be rewritten as

$$\begin{aligned} b\big(J\bar{\psi}_{\rm M,1}, (JI_{\rm M}-1)\phi_2, J\bar{\theta}_{\rm M,1}\big) = b_{\rm NC}\big((J-1)\bar{\psi}_{\rm M,1}, J\bar{\theta}_{\rm M,1}, (JI_{\rm M}-1)\phi_2\big) + b_{\rm NC}\big(\bar{\psi}_{\rm M,1}, (J-1)\bar{\theta}_{\rm M,1}, (JI_{\rm M}-1)\phi_2\big) \\ + b_{\rm NC}\big(\bar{\psi}_{\rm M,1}, \bar{\theta}_{\rm M,1}, (I_{\rm M}-1)\phi_2\big) \end{aligned} \tag{4.26}$$

with $b_{\rm NC}\big(\bar{\psi}_{\rm M,1}, \bar{\theta}_{\rm M,1}, (J-1)I_{\rm M}\phi_2\big) = 0$ from Lemma 3.6(b) in the last step. Lemma 3.7(b) (in its scalar version), (4.11), (4.21), (4.22), Lemma 3.6(e) and (2.8e), (2.8f) lead to bounds for the first and second terms on the right-hand side of (4.26). The third term in the right-hand side of (4.26) is combined with the scalar form of S_1 as

$$2b_{\rm NC}\big(\bar{\psi}_{\rm M,1}, \bar{\theta}_{\rm M,1}, (I_{\rm M}-1)\phi_2\big) + \big(\bar{\psi}_{\rm M,2}-\psi_{d,2}, (I_{\rm M}-1)\phi_2\big) \le C_{\rm I}h^2\Bigg(\sum_{K\in\mathcal{T}} \big\|\bar{\psi}_{\rm M,2}-\psi_{d,2}+\big[\bar{\psi}_{\rm M,1}, \bar{\theta}_{\rm M,1}\big]\big\|^2_{L^2(K)}\Bigg)^{1/2} \tag{4.27}$$

with the Cauchy–Schwarz inequality and Lemma 3.5(b). The remaining two terms in the expansion of $B_{\rm NC}(\bullet,\bullet,\bullet)$ are dealt with in an analogous way.

The results (4.23)–(4.27) are employed to estimate $S_1 + S_5$ first and then substituted in (4.19) with estimates of S_2 to S_4. This, a triangle inequality with $J\bar{\Theta}_{\rm M}$, Lemma 3.6(e) and (2.8d)–(2.8f) show

$$\big\|\widehat{\Theta} - \bar{\Theta}_{\rm M}\big\|_2 \le \tilde{C}_{\rm AD,rel}\Bigg(\sum_{K\in\mathcal{T}} \eta^2_{K,\bar{\Theta}_{\rm M}} + \sum_{E\in\mathcal{E}(\Omega)} \eta^2_{E,\bar{\Psi}_{\rm M}} + \sum_{E\in\mathcal{E}(\Omega)} \eta^2_{E,\bar{\Theta}_{\rm M}}\Bigg)^{1/2} \tag{4.28}$$

with $\epsilon_3 \searrow 0$, where

$$\tilde{C}^2_{\mathrm{AD,rel}} := 4\beta^{-2}\Big(C_{\mathrm{J}}^2((\beta/2+1+C_{\mathrm{b}}\mathcal{M}_1)+8C_{\mathrm{I}}C_{\mathrm{b}}\mathcal{M}_1(\Lambda_{\mathrm{J}}+1))^2 + C_{\mathrm{J}}^2(8C_{\mathrm{I}}C_{\mathrm{b}}\mathcal{M}_2(\Lambda_{\mathrm{J}}+1))^2 + C_{\mathrm{I}}^2\Big(1+4(\Lambda_{\mathrm{J}}+1)^2\Big)\Big).$$

The uniform boundedness property of $\mathcal{F}_{\bar\Psi}^{-1}$ in Lemma 4.2 implies

$$\left\|\widehat{\Theta}-\bar\Theta\right\|_2 = \left\|\mathcal{F}_{\bar\Psi}^{-1}\mathcal{F}_{\bar\Psi}\left(\widehat{\Theta}-\bar\Theta\right)\right\|_2 \le \|\mathcal{F}_{\bar\Psi}^{-1}\|_{\mathcal{L}(\mathbf{V}+\mathbf{V}_{\mathrm{M}})}\left\|\mathcal{F}_{\bar\Psi}\left(\widehat{\Theta}-\bar\Theta\right)\right\|_{\mathrm{NC}}.$$

The definition of $\mathcal{F}_{\bar\Psi}$ given by (4.17), (2.3b) and (4.13) show

$$\begin{aligned}\mathcal{F}_{\bar\Psi}\left(\bar\Theta-\widehat{\Theta}\right) &= T(\bar\Psi-\Psi_d)-\mathcal{F}_{\bar\Psi}\left(\widehat{\Theta}\right) = T(\bar\Psi-\Psi_d)-\widehat{\Theta}-T\Big[\mathcal{B}'_{\mathrm{NC}}(\bar\Psi)^*\left(\widehat{\Theta}\right)\Big]\\ &= T(\bar\Psi-\bar\Psi_{\mathrm{M}})+T\Big[\mathcal{B}'_{\mathrm{NC}}(\bar\Psi_{\mathrm{M}}-\bar\Psi)^*\left(\widehat{\Theta}\right)\Big].\end{aligned}$$

Hence, Lemma 3.7(a) and (b), (4.16) and Theorem 4.1 prove

$$\left\|\widehat{\Theta}-\bar\Theta\right\|_2 \le C_{\mathrm{ST,rel}}\|\mathcal{F}_{\bar\Psi}^{-1}\|_{\mathcal{L}(\mathbf{V}+\mathbf{V}_{\mathrm{M}})}\|T\|\left(C_{\mathrm{dS}}+2C_{\mathrm{b}}\beta^{-1}\right)\left(\sum_{K\in\mathcal{T}}\eta^2_{K,\bar\Psi_{\mathrm{M}}}+\sum_{E\in\mathcal{E}(\Omega)}\eta^2_{E,\bar\Psi_{\mathrm{M}}}+\|\bar u-\bar u_h\|^2_{L^2(\omega)}\right)^{1/2}. \tag{4.29}$$

The combination of (4.28) and (4.29) concludes the proof. □

Remark 4.4. (a) Note that the terms involving $\mathcal{P}_0$ in the reliability estimate of adjoint equations $\eta^2_{K,\bar\Theta_{\mathrm{M}}}$ of (2.8d) are due to the combined effect of non-conformity of the method plus linear lower-order terms.

(b) It is possible to avoid the terms involving $\mathcal{P}_0$ in the reliability estimator $\eta^2_{K,\bar\Theta_{\mathrm{M}}}$ of (2.8d) which comes from $S_5 = B_{\mathrm{NC}}(\bar\Psi_{\mathrm{M}},(I_{\mathrm{M}}-1)\Phi,\bar\Theta_{\mathrm{M}})$ in (4.19) with piecewise integration by parts. However, this leads to several average terms in the edge estimators that are not residuals (in addition to the volume terms). The efficiency analysis for this is still open. A similar observation for the two-dimensional Navier–Stokes equation in the stream function-vorticity formulation can be found in Remark 4.12 of [15].

4.3. *A posteriori* error analysis for the control variable

Let $\widetilde{\Psi}$ and $\widetilde{\Theta}$ be the auxiliary continuous state and adjoint variables associated with the control $\widetilde{u}_h$. That is, for all $\Phi\in\mathbf{V}$, seek $\left(\widetilde{\Psi},\widetilde{\Theta}\right)\in\mathbf{V}\times\mathbf{V}$ such that

$$A\left(\widetilde{\Psi},\Phi\right)+B\left(\widetilde{\Psi},\widetilde{\Psi},\Phi\right) = (\mathbf{F}+\mathbf{C}\widetilde{\mathbf{u}}_h,\Phi)\ \text{ and }\ A\left(\Phi,\widetilde{\Theta}\right)+2B\left(\widetilde{\Psi},\Phi,\widetilde{\Theta}\right) = \left(\widetilde{\Psi}-\Psi_d,\Phi\right).$$

From Lemmas 3.7(a), 3.10 and (4.3), it follows that

$$\|\bar\Theta-\bar\Theta_{\mathrm{M}}\|_{\boldsymbol{L}^\infty(\Omega)} \le C_{\mathrm{dS}}\varepsilon_0 \le \min\left\{\alpha\widehat{\delta}(2C_{\mathcal{M}})^{-1},\tau/2\right\}.$$

Lemma 4.5 (An auxiliary control estimate ([1], Thm. 8))**.** *Let* $\left(\bar\Psi,\bar u\right)$ *be a regular solution to* (2.1), $(\bar\Psi,\bar\Theta,\bar u)$ *solve* (2.3) *and satisfies the sufficient second-order optimality condition. Recall* $\widetilde{u}_h := \Pi_{[u_a,u_b]}\left(-\frac{1}{\alpha}\left(\mathcal{C}^*\bar\theta_{\mathrm{M},1}\right)\right)$ *from* (2.7) *and the constant* $\widehat{\delta}>0$ *from* (3.7)*. Then for* $\mathcal{T}\in\mathbb{T}(\delta_0)$, $\widehat{\delta}\|\bar u-\widetilde{u}_h\|^2_{L^2(\omega)} \le 2(j'(\widetilde{u}_h)-j'(\bar u))(\widetilde{u}_h-\bar u)$.

Theorem 4.6 (Reliability for the control variable)**.** *Let* $\left(\bar{\Psi}, \bar{\Theta}, \bar{\mathbf{u}}\right)$ *(resp.* $\left(\bar{\Psi}_{\mathrm{M}}, \bar{\Theta}_{\mathrm{M}}, \bar{\mathbf{u}}_h\right)$*) solve the optimality system* (2.3) *(resp.* (2.6)*). Then for all* $\mathcal{T} \in \mathbb{T}(\delta_0)$*, there exists an* h*-independent positive constant* $C_{\mathrm{CON,rel}}$ *such that*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\omega)} \leq C_{\mathrm{CON,rel}} \left(\sum_{K \in \mathcal{T}} \eta^2_{K,\bar{\Psi}_{\mathrm{M}}} + \sum_{K \in \mathcal{T}} \eta^2_{K,\bar{\Theta}_{\mathrm{M}}} + \sum_{K \in \mathcal{T}} \eta^2_{K,\bar{u}_h} + \sum_{E \in \mathcal{E}(\Omega)} \eta^2_{E,\bar{\Psi}_{\mathrm{M}}} + \sum_{E \in \mathcal{E}(\Omega)} \eta^2_{E,\bar{\Theta}_{\mathrm{M}}} \right)^{1/2}. \quad (4.30)$$

Proof. The triangle inequality with $\widetilde{u}_h$ and (2.8a) lead to $\|\bar{u} - \bar{u}_h\|_{L^2(\omega)} \leq \|\bar{u} - \widetilde{u}_h\|_{L^2(\omega)} + \left(\sum_{K \in \mathcal{T}} \eta^2_{K,\bar{u}_h}\right)^{1/2}$. The continuous optimality condition (2.3c) with $u = \widetilde{u}_h$ and (3.4) with $u = \bar{u}$ show

$$j'(\bar{u})(\widetilde{u}_h - \bar{u}) \geq 0, \quad -\left(\mathcal{C}^* \bar{\theta}_{\mathrm{M},1} + \alpha \widetilde{u}_h, \widetilde{u}_h - \bar{u}\right) \geq 0.$$

These bounds, Lemma 4.5 and the definition of $j'(\bullet)$ lead to

$$\begin{aligned}\frac{\widehat{\delta}}{2}\|\bar{u} - \widetilde{u}_h\|^2_{L^2(\omega)} &\leq (j'(\widetilde{u}_h) - j'(\bar{u}))(\widetilde{u}_h - \bar{u}) \leq j'(\widetilde{u}_h)(\widetilde{u}_h - \bar{u}) \\ &\leq j'(\widetilde{u}_h)(\widetilde{u}_h - \bar{u}) - \left(\mathcal{C}^* \bar{\theta}_{\mathrm{M},1} + \alpha \widetilde{u}_h, \widetilde{u}_h - \bar{u}\right) = \left(\mathcal{C}^*\left(\widetilde{\theta} - \bar{\theta}_{\mathrm{M},1}\right), \widetilde{u}_h - \bar{u}\right).\end{aligned}$$

Therefore, the Cauchy–Schwarz inequality results in

$$\widehat{\delta}\|\bar{u} - \widetilde{u}_h\|_{L^2(\omega)} \leq 2\left\|\widetilde{\Theta} - \bar{\Theta}_{\mathrm{M}}\right\|. \quad (4.31)$$

A triangle inequality that introduces $\widehat{\Theta}$, Poincaré inequality with constant C_{p}, Lemma 3.7(a) and (4.28) yield

$$\frac{\widehat{\delta}}{2}\|\bar{u} - \widetilde{u}_h\|_{L^2(\omega)} \leq C_{\mathrm{p}}\left\|\widetilde{\Theta} - \widehat{\Theta}\right\|_{\mathrm{NC}} + C_{\mathrm{dS}}\tilde{C}_{\mathrm{AD,rel}}\left(\sum_{K \in \mathcal{T}} \eta^2_{K,\bar{\Theta}_{\mathrm{M}}} + \sum_{E \in \mathcal{E}(\Omega)} \eta^2_{E,\bar{\Theta}_{\mathrm{M}}} + \sum_{E \in \mathcal{E}(\Omega)} \eta^2_{E,\bar{\Psi}_{\mathrm{M}}} \right)^{1/2}. \quad (4.32)$$

The definitions (2.4), (2.7), the Lipschitz property of operator $\Pi_{[u_a,u_b]}$ and Lemma 3.7(a) show $\|\bar{u} - \widetilde{u}_h\|_{L^2(\omega)} \leq \alpha^{-1}\|\bar{\Theta} - \bar{\Theta}_{\mathrm{M}}\|_{\boldsymbol{L}^2(\omega)} \leq \alpha^{-1}C_{\mathrm{dS}}\|\bar{\Theta} - \bar{\Theta}_{\mathrm{M}}\|_{\mathrm{NC}}$. Hence, (4.3) implies $\|\bar{u} - \widetilde{u}_h\|_{L^2(\omega)} \leq \alpha^{-1}C_{\mathrm{dS}}\varepsilon_0$. This, the estimate in (4.29) with $\left(\bar{\Theta}, \bar{\Psi}, \bar{u}\right)$ replaced by $\left(\widetilde{\Theta}, \bar{\Psi}, \widetilde{u}_h\right)$ and the definition (2.8a) show

$$\left\|\widetilde{\Theta} - \widehat{\Theta}\right\|_{\mathrm{NC}} \leq C_{\mathrm{ST,rel}}\left\|\mathcal{F}^{-1}_{\widetilde{\Psi}}\right\|_{\mathcal{L}(\mathbf{V}+\mathbf{V}_{\mathrm{M}})}\|T\|(C_{\mathrm{dS}} + 2C_{\mathrm{b}}\beta^{-1})\left(\sum_{K \in \mathcal{T}} \eta^2_{K,\bar{\Psi}_{\mathrm{M}}} + \sum_{E \in \mathcal{E}(\Omega)} \eta^2_{E,\bar{\Psi}_{\mathrm{M}}} + \sum_{K \in \mathcal{T}} \eta^2_{K,\bar{u}_h} \right)^{1/2}.$$

A substitution of the last displayed inequality in (4.32) with $C^2_{\mathrm{CON,rel}} := 2 + 8\widehat{\delta}^{-2}\left(C^2_{\mathrm{dS}}\tilde{C}^2_{\mathrm{AD,rel}} + (C_{\mathrm{p}}C_{\mathrm{ST,rel}}\left\|\mathcal{F}^{-1}_{\widetilde{\Psi}}\right\|_{\mathcal{L}(\mathbf{V}+\mathbf{V}_{\mathrm{M}})}\|T\|(C_{\mathrm{dS}} + 2C_{\mathrm{b}}\beta^{-1}))^2\right)$ concludes the proof. □

Proof of Theorem 2.4(a)*.* The proof follows from a combination of Theorems 4.1, 4.3 and 4.6. □

5. Efficiency

Lemma 5.1 (Local efficiency for state estimator)**.** *Let* $\left(\bar{\Psi}, \bar{\Theta}, \bar{\mathbf{u}}\right)$ *(resp.* $\left(\bar{\Psi}_{\mathrm{M}}, \bar{\Theta}_{\mathrm{M}}, \bar{\mathbf{u}}_h\right)$*) solve the optimality system* (2.3) *(resp.* (2.6)*). Then,*

$$\eta_{K,\bar{\Psi}_{\mathrm{M}}} \lesssim \left\|D^2(\bar{\Psi} - \bar{\Psi}_{\mathrm{M}})\right\|_{\boldsymbol{L}^2(K)} + h^2_K\left(\|\bar{u} - \bar{u}_h\|_{L^2(K)} + \|f - f_h\|_{L^2(K)}\right), \quad \eta_{E,\bar{\Psi}_{\mathrm{M}}} \lesssim \left\|D^2_{\mathrm{NC}}(\bar{\Psi} - \bar{\Psi}_{\mathrm{M}})\right\|_{\boldsymbol{L}^2(\Omega(K))},$$

where $K \in \mathcal{T}$, $E \in \mathcal{E}(\Omega(K))$ *and* f_h *denotes the piecewise average of* f*.*

Proof. Recall the volume estimator $\eta^2_{K,\bar{\Psi}_{\rm M}} = h_K^4\Big(\big\|f + \mathcal{C}\bar{u}_h + [\bar{\psi}_{\rm M,1}, \bar{\psi}_{\rm M,2}]\big\|^2_{L^2(K)} + \big\|[\bar{\psi}_{\rm M,1}, \bar{\psi}_{\rm M,1}]\big\|^2_{L^2(K)}\Big)$ from (2.8a). For each element $K \in \mathcal{T}$, it holds that

$$h_K^2\big\|f + \mathcal{C}u_h + [\bar{\psi}_{\rm M,1}, \bar{\psi}_{\rm M,2}]\big\|_{L^2(K)} + h_K^2\big\|[\bar{\psi}_{\rm M,1}, \bar{\psi}_{\rm M,1}]\big\|_{L^2(K)} \lesssim h_K^2\|f - f_h\|_{L^2(K)} + h_K^2\|\bar{u} - \bar{u}_h\|_{L^2(K)} + \big\|D^2(\bar{\Psi} - \bar{\Psi}_{\rm M})\big\|_{\boldsymbol{L}^2(K)} + \big\|D^2\bar{\Psi}\big\|_{\boldsymbol{L}^2(K)}\big\|D^2(\bar{\Psi} - \bar{\Psi}_{\rm M})\big\|_{\boldsymbol{L}^2(K)}. \tag{5.1}$$

The proof of (5.1) follows from the standard bubble functions arguments as in Lemma 5.3 of [14]. In the proof therein for the first term on the left-hand side of (5.1), set $\sigma := \big(f_h + \mathcal{C}u_h + [\bar{\psi}_{\rm M,1}, \bar{\psi}_{\rm M,2}]\big)b_K^2$ in K, and zero in $\Omega \setminus K$, where b_K denotes the standard interior bubble function [36]. Then the state equation (2.3a) with the test function $(\sigma, 0)$, $\Delta^2\bar{\psi}_{\rm M,1} = 0$ and $\sigma \in H^2_0(K)$ prove (5.1). The term $\big\|[\bar{\psi}_{\rm M,1}, \bar{\psi}_{\rm M,1}]\big\|_{L^2(K)}$ can be estimated similar to the above analysis.

For the edge estimator term $\eta^2_{E,\bar{\Psi}_{\rm M}} = h_E\Big(\big\|[D^2\bar{\psi}_{\rm M,1}\tau_E]_E\big\|^2_{L^2(E)} + \big\|[D^2\bar{\psi}_{\rm M,2}\tau_E]_E\big\|^2_{L^2(E)}\Big)$, Lemma 3.6(e) with $v = \bar{\psi}_{\rm M,1}$ implies, for $E \in \mathcal{E}(\Omega(K))$,

$$h_E\big\|[D^2\bar{\psi}_{\rm M,1}\tau_E]_E\big\|^2_{L^2(E)} \lesssim \big\|D^2_{\rm NC}(\bar{\psi}_{\rm M,1} - \bar{\psi}_1)\big\|^2_{L^2(\Omega(K))}. \tag{5.2}$$

Analogous arguments lead to similar result for the edge estimator $\big\|[D^2\bar{\psi}_{\rm M,2}\tau_E]_E\big\|^2_{L^2(E)}$. □

Lemma 5.2 (Local efficiency for adjoint estimator)**.** *Let* $(\bar{\Psi}, \bar{\Theta}, \bar{\mathbf{u}})$ *(resp.* $(\bar{\Psi}_{\rm M}, \bar{\Theta}_{\rm M}, \bar{\mathbf{u}}_h)$*) solve the optimality system* (2.3) *(resp.* (2.6)*). Then,*

$$\eta_{K,\bar{\Theta}_{\rm M}} \lesssim \big\|D^2(\bar{\Psi} - \bar{\Psi}_{\rm M})\big\|_{\boldsymbol{L}^2(K)} + \big\|D^2(\bar{\Theta} - \bar{\Theta}_{\rm M})\big\|_{\boldsymbol{L}^2(K)} + h_K^2\|\Psi_d - \Psi_{d,h}\|_{\boldsymbol{L}^2(K)} + \big\|\bar{\Psi} - \bar{\Psi}_{\rm M}\big\|_{\boldsymbol{L}^2(K)} + \big\|\nabla(\bar{\Theta}_{\rm M} - \bar{\Theta})\big\|_{\boldsymbol{L}^2(K)} + \big\|\nabla(1 - I_{\rm M})\bar{\Theta}\big\|_{\boldsymbol{L}^2(K)} + \big\|(1 - I_{\rm M})\bar{\Theta}\big\|_{\boldsymbol{L}^\infty(K)} + \big\|(1 - \mathcal{P}_0)\bar{\Theta}\big\|_{\boldsymbol{L}^\infty(K)},$$

$$\text{and } \eta_{E,\bar{\Theta}_{\rm M}} \lesssim \big\|D^2_{\rm NC}(\bar{\Theta} - \bar{\Theta}_{\rm M})\big\|_{\boldsymbol{L}^2(\Omega(K))},$$

where $K \in \mathcal{T}$, $E \in \mathcal{E}(\Omega(K))$ *and* $\Psi_{d,h}$ *denotes the piecewise average of* Ψ_d.

Proof. The adjoint volume estimator $\eta_{K,\bar{\Theta}_{\rm M}}$ contains two parts: $\eta_{K,{\rm res},\bar{\Theta}_{\rm M}}$ and $\eta_{K,\mathcal{P}_0,\bar{\Theta}_{\rm M}}$. Recall $\eta^2_{K,{\rm res},\bar{\Theta}_{\rm M}} = h_K^4\Big(\big\|\bar{\psi}_{\rm M,1} - \psi_{d,1} - [\bar{\psi}_{\rm M,1}, \bar{\theta}_{\rm M,2}] + [\bar{\psi}_{\rm M,2}, \bar{\theta}_{\rm M,1}]\big\|^2_{L^2(K)} + \big\|\bar{\psi}_{\rm M,2} - \psi_{d,2} + [\bar{\psi}_{\rm M,1}, \bar{\theta}_{\rm M,1}]\big\|^2_{L^2(K)}\Big)$ from (2.8b). For $\eta_{K,{\rm res},\bar{\Theta}_{\rm M}}$ over each $K \in \mathcal{T}$, the standard bubble function technique shows

$$\begin{aligned} &h_K^2\big\|\bar{\psi}_{\rm M,1} - \psi_{d,1} - [\bar{\psi}_{\rm M,1}, \bar{\theta}_{\rm M,2}] + [\bar{\psi}_{\rm M,2}, \bar{\theta}_{\rm M,1}]\big\|_{L^2(K)} + h_K^2\big\|\bar{\psi}_{\rm M,2} - \psi_{d,2} + [\bar{\psi}_{\rm M,1}, \bar{\theta}_{\rm M,1}]\big\|_{L^2(K)} \\ &\quad\le h_K^2\|\Psi_d - \Psi_{d,h}\|_{\boldsymbol{L}^2(K)} + h_K^2\big\|\bar{\Psi} - \bar{\Psi}_{\rm M}\big\|_{\boldsymbol{L}^2(K)} + \big\|D^2\bar{\Theta}\big\|_{\boldsymbol{L}^2(K)}\big\|D^2(\bar{\Psi} - \bar{\Psi}_{\rm M})\big\|_{\boldsymbol{L}^2(K)} \\ &\qquad + \Big(1 + \big\|D^2\bar{\Psi}\big\|_{\boldsymbol{L}^2(K)}\Big)\big\|D^2(\bar{\Theta} - \bar{\Theta}_{\rm M})\big\|_{\boldsymbol{L}^2(K)}. \end{aligned} \tag{5.3}$$

In the proof therein for the first term on the left-hand side of (5.3), set $\sigma := \big(\bar{\psi}_{\rm M,1} - \psi_{d,h,1} - [\bar{\psi}_{\rm M,1}, \bar{\theta}_{\rm M,2}] + [\bar{\psi}_{\rm M,2}, \bar{\theta}_{\rm M,1}]\big)b_K^2$ in K, and zero in $\Omega \setminus K$. The adjoint system (2.3b) with the test function $(\sigma, 0)$, and the symmetry of $b(\bullet, \bullet, \bullet)$ with respect to the second and third variables show

$$\int_K D^2\bar{\theta}_1 : D^2\sigma \,{\rm d}x - \int_K (\bar{\psi}_1 - \psi_{d,1})\sigma \,{\rm d}x + \int_K \big([\bar{\psi}_1, \bar{\theta}_2] - [\bar{\psi}_2, \bar{\theta}_1]\big)\sigma \,{\rm d}x = 0.$$

The combination of this, $\Delta^2\bar{\theta}_{\rm M,1} = 0$ and the arguments in the proof of Lemma 5.3 from [14] prove (5.3). The estimate for the second term on the left-hand side of (5.3) is analogous to that of the first term.

The second part of the adjoint estimator is $\eta^2_{K,\mathcal{P}_0,\bar{\Theta}_\mathrm{M}} := \|D^2\bar\psi_{\mathrm{M},1}(1-\mathcal{P}_0)\bar\theta_{\mathrm{M},2}\|^2_{L^2(K)} + \|D^2\bar\psi_{\mathrm{M},2}(1-\mathcal{P}_0)\bar\theta_{\mathrm{M},1}\|^2_{L^2(K)} + \|D^2\bar\psi_{\mathrm{M},1}(1-\mathcal{P}_0)\bar\theta_{\mathrm{M},1}\|^2_{L^2(K)}$. Consider $\|D^2\bar\psi_{\mathrm{M},1}(1-\mathcal{P}_0)\bar\theta_{\mathrm{M},1}\|_{L^2(K)}$, $K \in \mathcal{T}$ from (2.8c). The Hölder's inequality shows that

$$\|D^2\bar\psi_{\mathrm{M},1}(1-\mathcal{P}_0)\bar\theta_{\mathrm{M},1}\|_{L^2(K)} \le \|D^2\bar\psi_{\mathrm{M},1}\|_{L^2(K)}\|(1-\mathcal{P}_0)\bar\theta_{\mathrm{M},1}\|_{L^\infty(K)}. \tag{5.4}$$

A triangle inequality with $\mathcal{P}_0 I_\mathrm{M}\bar\theta_1$ leads to

$$\|(1-\mathcal{P}_0)\bar\theta_{\mathrm{M},1}\|_{L^\infty(K)} \le \|(1-\mathcal{P}_0)\big(\bar\theta_{\mathrm{M},1} - I_\mathrm{M}\bar\theta_1\big)\|_{L^\infty(K)} + \|(1-\mathcal{P}_0)I_\mathrm{M}\bar\theta_1\|_{L^\infty(K)}.$$

An inverse inequality ([20], Thm. 3.2.6) for the first term and a triangle inequality with $(1-\mathcal{P}_0)\bar\theta_1$ for the second term lead to

$$\begin{aligned}\|(1-\mathcal{P}_0)\bar\theta_{\mathrm{M},1}\|_{L^\infty(K)} &\lesssim h^{-1}\|(1-\mathcal{P}_0)\big(\bar\theta_{\mathrm{M},1} - I_\mathrm{M}\bar\theta_1\big)\|_{L^2(K)} + \|(I_\mathrm{M}-1)\bar\theta_1\|_{L^\infty(K)} + \|(1-\mathcal{P}_0)\bar\theta_1\|_{L^\infty(K)} \\ &\quad + \|\mathcal{P}_0\big(\bar\theta_1 - I_\mathrm{M}\bar\theta_1\big)\|_{L^\infty(K)} \\ &\lesssim \Big(\|\nabla\big(\bar\theta_{\mathrm{M},1}-\bar\theta_1\big)\|_{L^2(K)} + \|\nabla(1-I_\mathrm{M})\bar\theta_1\|_{L^2(K)}\Big) + \|(1-I_\mathrm{M})\bar\theta_1\|_{L^\infty(K)} \\ &\quad + \|(1-\mathcal{P}_0)\bar\theta_1\|_{L^\infty(K)},\end{aligned}$$

where the last inequality uses the projection estimate for $\mathcal{P}_0$ in $L^2(K)$ ([23], Prop. 1.135) and the boundedness property of $\mathcal{P}_0$. This with (5.4) result in

$$\begin{aligned}\|D^2\bar\psi_{\mathrm{M},1}(1-\mathcal{P}_0)\bar\theta_{\mathrm{M},1}\|_{L^2(K)} &\lesssim \|D^2\bar\psi_{\mathrm{M},1}\|_{L^2(K)}\Big(\|\nabla\big(\bar\theta_{\mathrm{M},1}-\bar\theta_1\big)\|_{L^2(K)} + \|\nabla(1-I_\mathrm{M})\bar\theta_1\|_{L^2(K)} \\ &\quad + \|(1-I_\mathrm{M})\bar\theta_1\|_{L^\infty(K)} + \|(1-\mathcal{P}_0)\bar\theta_1\|_{L^\infty(K)}\Big).\end{aligned} \tag{5.5}$$

From (4.11), $|||D^2\bar\Psi_\mathrm{M}|||_{L^2(K)} \le \mathcal{M}_1$. The estimates for the remaining terms $\|D^2\bar\psi_{\mathrm{M},1}(1-\mathcal{P}_0)\bar\theta_{\mathrm{M},2}\|_{L^2(K)}$, $\|D^2\bar\psi_{\mathrm{M},2}(1-\mathcal{P}_0)\bar\theta_{\mathrm{M},1}\|_{L^2(K)}$ follow from similar arguments and hence the details are omitted for brevity. Lemma 3.6(e) leads to the desired estimate for the edge estimator $\eta_{E,\bar\Theta_\mathrm{M}}$. ☐

Remark 5.3. Analogous terms involving projection operators as the last term on the right-hand side of (5.5) are dealt with in Theorem 4.10 of [18].

Lemma 5.4 (Local efficiency for control estimator)**.** *Let* $\big(\bar\Psi,\bar\Theta,\bar{\mathbf{u}}\big)$ *(resp.* $\big(\bar\Psi_\mathrm{M},\bar\Theta_\mathrm{M},\bar{\mathbf{u}}_h\big)$*) solve the optimality system* (2.3) *(resp.* (2.6)*). Then,* $\eta_{K,\bar u_h} \le \alpha^{-1}|||\bar\Theta-\bar\Theta_\mathrm{M}|||_{\boldsymbol{L}^2(K)} + \|\bar u - \bar u_h\|_{L^2(K)}$.

Proof. The control estimator $\eta_{K,\bar u_h} := \|\widetilde u_h - \bar u_h\|_{L^2(K)}$. The definitions (2.4), (2.7) and the Lipschitz property of operator $\Pi_{[u_a,u_b]}$ show

$$\|\bar u - \widetilde u_h\|_{L^2(K)} \le \|\Pi_{[u_a,u_b]}\big(-\alpha^{-1}\big(\mathcal{C}^*\big(\bar\theta_1 - \bar\theta_{\mathrm{M},1}\big)\big)\big)\|_{L^2(K)} \le \alpha^{-1}|||\bar\Theta-\bar\Theta_\mathrm{M}|||_{\boldsymbol{L}^2(K)}.$$

This and a triangle inequality prove $\|\widetilde u_h - \bar u_h\|_{L^2(K)} \le \alpha^{-1}|||\bar\Theta-\bar\Theta_\mathrm{M}|||_{\boldsymbol{L}^2(K)} + \|\bar u - \bar u_h\|_{L^2(K)}$ and concludes the proof of local efficiency for the control variable. ☐

Proof of Theorem 2.4(b)*.* Recall the definition of the estimator η from (2.9). The summation over all the element and edges of the triangulation $\mathcal{T}$, and the local efficiency results in Lemmas 5.1–5.4 show

$$\begin{aligned}\eta &\lesssim |||\bar\Psi-\bar\Psi_\mathrm{M}|||_\mathrm{NC} + |||\bar\Theta-\bar\Theta_\mathrm{M}|||_\mathrm{NC} + \|\bar u-\bar u_h\|_{L^2(\omega)} + \mathrm{osc}_0(f,\mathcal{T}) + \mathrm{osc}_0(\Psi_d,\mathcal{T}) + |||\bar\Psi-\bar\Psi_\mathrm{M}||| \\ &\quad + |||\bar\Theta-\bar\Theta_\mathrm{M}||| + |||\bar\Theta-\bar\Theta_\mathrm{M}|||_{1,2,h} + |||(1-I_\mathrm{M})\bar\Theta|||_{1,2,h} + |||(1-I_\mathrm{M})\bar\Theta|||_{0,\infty} + |||(1-\mathcal{P}_0)\bar\Theta|||_{0,\infty}.\end{aligned}$$

This and Lemma 3.7(a) result in

$$\eta \lesssim \left\|\bar{\Psi} - \bar{\Psi}_{\mathrm{M}}\right\|_{\mathrm{NC}} + \left\|\bar{\Theta} - \bar{\Theta}_{\mathrm{M}}\right\|_{\mathrm{NC}} + \|\bar{u} - \bar{u}_h\|_{L^2(\omega)} + \mathrm{osc}_0(f, \mathcal{T}) + \mathrm{osc}_0(\Psi_d, \mathcal{T}) + \left\|(1 - I_{\mathrm{M}})\bar{\Theta}\right\|_{1,2,h}$$
$$+ \left\|(1 - I_{\mathrm{M}})\bar{\Theta}\right\|_{0,\infty} + \left\|(1 - \mathcal{P}_0)\bar{\Theta}\right\|_{0,\infty}.$$

Here the constant absorbed in "$\lesssim$" depends on the shape-regularity of $\mathcal{T}$. This concludes the proof. □

6. Numerical results

The results of the numerical experiments that support the *a priori* and *a posteriori* estimates are presented in this section.

6.1. Preliminaries

The state and adjoint variables are discretised using the Morley FE and the control variable is discretised using piecewise constant functions. The discrete solution $(\bar{\Psi}_{\mathrm{M}}, \bar{\Theta}_{\mathrm{M}}, \bar{u}_h)$ is computed using a combination of Newtons' method in an inner loop and primal-dual active set strategy in an outer loop, see Section 6.1 of [19] for the details of the implementation procedure for the *a priori* case and a different choice of the trilinear form. The initial guess for $(\bar{\Psi}_{\mathrm{M}}, \bar{\Theta}_{\mathrm{M}})$ in the Newton's method is chosen as the discrete solution to the biharmonic part of the discrete state and adjoint equations in (2.6a) and (2.6b). At each iteration of primal-dual active set algorithm, the Newtons' method converges in ten iterations when the tolerence level for errors is set as less than 10^{-9}. The primal-dual active set algorithm terminates within four steps.

The numerical experiments are performed over uniform and adaptive refinements. The uniform mesh refinement is done by red-refinement criteria, where each triangle is subdivided into four sub-triangles by connecting the midpoints of the edges. The standard adaptive algorithm Solve-Estimate-Mark-Refine [14, 36] is used for the adaptive refinement, and is described in Section 6.3.

Let $\bar{\Psi}_\ell$ be the discrete solution $\bar{\Psi}_{\mathrm{M}}$ at the ℓth level for $\ell = 1, 2, 3, \ldots$ and define $e_\ell(\bar{\Psi}) := \left\|\bar{\Psi} - \bar{\Psi}_\ell\right\|_{\mathrm{NC}}$. The order of convergence in the energy norm at ℓth level for $\bar{\Psi}$ is computed as $\mathrm{Order}(\ell) := \log\big(e_\ell(\bar{\Psi})/e_{\ell+1}(\bar{\Psi})\big)/\log(h_\ell/h_{\ell+1})$ (resp. $\mathrm{Order}(\ell) := \log\big(e_\ell(\bar{\Psi})/e_{\ell+1}(\bar{\Psi})\big)/\log(\mathrm{NDOF}_\ell/\mathrm{NDOF}_{\ell+1})$) for uniform refinements (resp. adaptive refinements), where h_ℓ and NDOF_ℓ denote the mesh size and number of degrees of freedom at ℓth level triangulation $\mathcal{T}_\ell$. The total number of degrees of freedom is NDOF $:= 2\dim(\mathbf{V}_{\mathrm{M}}) + \dim(U_{h,ad})$. Finally, the total error is a sum of $\left\|\bar{\Psi} - \bar{\Psi}_{\mathrm{M}}\right\|_{\mathrm{NC}}$, $\left\|\bar{\Theta} - \bar{\Theta}_{\mathrm{M}}\right\|_{\mathrm{NC}}$ and $\|\bar{u} - \bar{u}_h\|$.

Two examples are presented to illustrate the *a priori* and *a posteriori* reliability and efficiency estimates with $\omega = \Omega$, that is, $\mathcal{C} = \mathrm{I}$. The first example is considered over unit square domain where the solution of the von Kármán equations is sufficiently smooth and the second example is over an L-shaped domain where the solution of the von Kármán equations belongs to $\mathbf{V} \cap \boldsymbol{H}^{2+\gamma}(\Omega)$ with $\gamma \approx 0.5445$.

6.2. Uniform refinement

Example 6.1 (Convex domain)**.** Let the computational domain be $\Omega = (0,1)^2$. The model problem is constructed in such a way that the exact solution is known. The data in the distributed optimal control problem are chosen as $\bar{\psi}_1 = \bar{\psi}_2 = \sin^2(\pi x)\sin^2(\pi y)$, $\bar{\theta}_1 = \bar{\theta}_2 = x^2y^2(1-x)^2(1-y)^2$, $\bar{u}(x) = \Pi_{[-750,-50]}\big(-1/\alpha\,\bar{\theta}_1(x)\big)$, $\alpha = 10^{-5}$, where $\bar{\Psi} = (\bar{\psi}_1, \bar{\psi}_2)$ and $\bar{\Theta} = (\bar{\theta}_1, \bar{\theta}_2)$ denote the optimal state and adjoint variables. The source terms f, g and observation $\bar{\Psi}_d = (\bar{\psi}_{d,1}, \bar{\psi}_{d,2})$ for $\bar{\Psi}$ are then computed using $f = \Delta^2\bar{\psi}_1 - [\bar{\psi}_1, \bar{\psi}_2] - \bar{u}$, $\quad g = \Delta^2\bar{\psi}_2 + \frac{1}{2}[\bar{\psi}_1, \bar{\psi}_1]$ and $\bar{\psi}_{d,1} = \bar{\psi}_1 - \Delta^2\bar{\theta}_1$, $\quad \bar{\psi}_{d,2} = \bar{\psi}_2 - \Delta^2\bar{\theta}_2 + [\bar{\psi}_1, \bar{\theta}_1]$.

The relative errors and orders of convergence for the state, adjoint and control variables and the combined relative errors and orders of convergence are presented in Table 1. Since Ω is convex, Theorem A.10 predicts linear order of convergence for the state and adjoint variables (resp. control variable) in the energy (resp. L^2)

TABLE 1. Errors and orders of convergence for state, adjoint and control variables in Example 6.1.

h	$\frac{\|\|\|\bar{\Psi}-\bar{\Psi}_{\mathrm{M}}\|\|\|_{\mathrm{NC}}}{\|\|\|\bar{\Psi}\|\|\|_2}$	Order	$\frac{\|\|\|\bar{\Theta}-\bar{\Theta}_{\mathrm{M}}\|\|\|_{\mathrm{NC}}}{\|\|\|\bar{\Theta}\|\|\|_2}$	Order	$\frac{\|\bar{u}-\bar{u}_h\|}{\|\bar{u}\|}$	Order	Total error	Order
0.2500	1.208162	–	1.793806	–	1.638183	–	1.574966	–
0.1250	0.654690	0.88	0.730500	1.30	0.581509	1.49	0.592373	1.41
0.0625	0.357561	0.87	0.377143	0.95	0.175353	1.73	0.202300	1.55
0.0312	0.183915	0.96	0.190428	0.99	0.055375	1.66	0.074381	1.44
0.0156	0.092662	0.99	0.095447	1.00	0.021294	1.38	0.031846	1.22
0.0078	0.046422	1.00	0.047753	1.00	0.009674	1.14	0.015107	1.08

TABLE 2. Errors and orders of convergence for state, adjoint and control variables in Example 6.2.

h	NDOF	$\frac{\|\|\|\bar{\Psi}-\bar{\Psi}_{\mathrm{M}}\|\|\|_{\mathrm{NC}}}{\|\|\|\bar{\Psi}\|\|\|_2}$	Order	$\frac{\|\|\|\bar{\Theta}-\bar{\Theta}_{\mathrm{M}}\|\|\|_{\mathrm{NC}}}{\|\|\|\bar{\Theta}\|\|\|_2}$	Order	$\frac{\|\bar{u}-\bar{u}_h\|}{\|\bar{u}\|}$	Order	Total error	Order
0.3536	156	1.371575	–	1.355646	–	0.760376	–	0.812881	–
0.1768	740	0.875686	0.65	0.906115	0.58	0.498261	0.61	0.532436	0.61
0.0884	3204	0.502780	0.80	0.508682	0.83	0.197497	1.34	0.224325	1.25
0.0442	13316	0.270684	0.89	0.268696	0.92	0.072470	1.45	0.089636	1.32
0.0221	54276	0.143731	0.91	0.141920	0.92	0.029279	1.31	0.039162	1.19

norm. These theoretical rates of convergence are confirmed by the numerical outputs. Thus, the error estimates for $\bar{\Psi}$ and $\bar{\Theta}$ with respect to the energy norm converge at optimal rates. The same applies to the error estimate on $\bar{u}$ with respect to the $L^2(\omega)$.

Example 6.2 (Non-convex domain)**.** Consider the non-convex L-shaped domain $\Omega = (-1,1)^2 \setminus \big([0,1) \times (-1,0]\big)$. The source terms f, g and the observation $\Psi_d = (\psi_{d,1}, \psi_{d,2})$ are chosen such that the model problem has the exact singular solution ([25], Sect. 3.4.1) given by $\bar{\psi}_1 = \bar{\psi}_2 = \bar{\theta}_1 = \bar{\theta}_2 = \big(r^2\cos^2\theta - 1\big)^2\big(r^2\sin^2\theta - 1\big)^2 r^{1+\gamma} g_{\gamma,\omega}(\theta)$ where $\gamma \approx 0.5444837367$ is a non-characteristic root of $\sin^2(\gamma\omega) = \gamma^2\sin^2(\omega)$, $\omega = \frac{3\pi}{2}$, and $g_{\gamma,\omega}(\theta) = \Big(\frac{1}{\gamma-1}\sin((\gamma-1)\omega) - \frac{1}{\gamma+1}\sin((\gamma+1)\omega)\Big)(\cos((\gamma-1)\theta) - \cos((\gamma+1)\theta)) - \Big(\frac{1}{\gamma-1}\sin((\gamma-1)\theta) - \frac{1}{\gamma+1}\sin((\gamma+1)\theta)\Big)(\cos((\gamma-1)\omega) - \cos((\gamma+1)\omega))$. The exact control $\bar{u}$ is chosen as $\bar{u}(x) = \Pi_{[-600,-50]}(-1/\alpha\,\bar{\theta}_1(x))$, where $\alpha = 10^{-3}$.

Table 2 shows error estimates and the convergence rates of the state, adjoint and control variables. Since Ω is non-convex, only suboptimal orders of convergence for the state and adjoint variables in the energy norm are obtained as predicted by Theorem A.10.

6.3. Adaptive mesh refinement

The standard adaptive algorithm: Solve-Estimate-Mark-Refine is used for the adaptive mesh-refinement. The total estimator $\eta^2 := \eta^2_{\mathrm{ST}} + \eta^2_{\mathrm{AD}} + \eta^2_{\mathrm{CON}}$ is considered in the adaptive algorithm. Recall δ_0, ε_0 from Theorem 2.4.

TABLE 3. Estimator and order of convergence for state, adjoint and control variables, plus the data oscillation terms in Example 6.1.

h	η_{ST}	Order	η_{AD}	Order	η_{CON}	Order	η	Order	$\mathrm{osc}_0(f,\mathcal{T})$	$\mathrm{osc}_0(\Psi_d,\mathcal{T})$
0.2500	101.670150	–	1.464635	–	84.529580	–	132.227890	–	5.1290e+01	1.0984e-01
0.1250	46.982720	1.11	0.252129	2.54	31.588806	1.42	56.615300	1.22	5.7172e+00	1.4637e-02
0.0625	25.973543	0.86	0.115093	1.13	13.233906	1.26	29.150891	0.96	7.3786e-01	1.8138e-03
0.0312	13.650921	0.93	0.058224	0.98	6.187437	1.10	14.987842	0.96	9.2981e-02	2.2589e-04
0.0156	6.942815	0.98	0.029375	0.99	3.031687	1.03	7.575927	0.98	1.1643e-02	2.8207e-05
0.0078	3.491406	0.99	0.014750	0.99	1.508851	1.01	3.803520	0.99	1.4561e-03	3.5250e-06

Algorithm 1: Adaptive mesh-refinement algorithm.

Set the initial triangulation $\mathcal{T}_0$ such that $\mathcal{T}_0 \in \mathbb{T}(\delta),\ 0 < \delta \leq \delta_0$ and $0 < \theta \leq 1$;
Set the maximum number of iteration Max_ℓ
while $\ell < \mathrm{Max}_\ell$ **do**
 Solve: Compute the solution $(\bar{\Psi}_{\mathrm{M}}, \bar{\Theta}_{\mathrm{M}}, \bar{u}_h)$ over the triangulation $\mathcal{T}_\ell$ using Newtons' method and primal dual active set strategy
 Estimate: Compute the complete estimator η_ℓ^2 from (2.9)
 Mark: Mark a minimal subset $\mathscr{M}_\ell \subset \mathcal{T}_\ell$ by Dörfler marking criteria

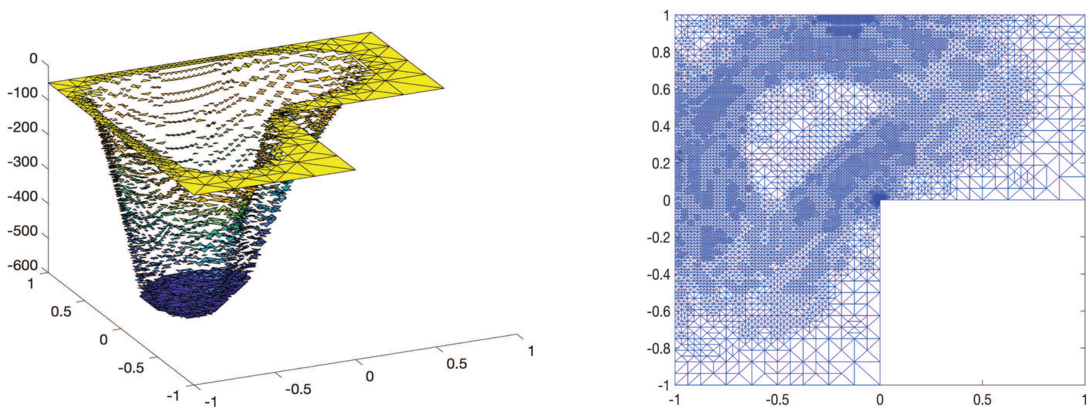
$$\theta \sum_{K\in\mathcal{T}_\ell} \eta_\ell^2(K) \leq \sum_{K\in\mathscr{M}_\ell} \eta_\ell^2(K)$$

 Refine: Compute the closure of $\mathscr{M}_\ell$ and generate new triangulation $\mathcal{T}_{\ell+1}$ using the newest vertex bisection
 Update the triangulation
end

The initial triangulation $\mathcal{T}_0 \in \mathbb{T}(\delta)$ can be obtained by the uniform red-refinement of some admissible triangulation over the domain. In general, it is difficult to quantify the constants $\varepsilon_0,\ \delta_0$ from Theorem 2.4. The initial triangulation used in the following numerical experiments are specified in the respective examples. Further, the marking parameter $\theta = 0.2$ is used.

Convex domain. Consider Example 6.1. This is a test case over the square domain with a smooth exact solution, performed to test the performance of the adaptive estimator for the uniform refinement. The initial triangulation is a criss-cross mesh with one red-refinement, that is, 16 uniform triangles. Table 3 depicts the convergence history of the estimators (defined in (2.8)) for the uniform refinements for the state, adjoint and control estimators. The combined error and estimator convergence rates are also computed. It is observed that the individual errors and estimators as well as the combined error have linear order of convergence. Further, the oscillation terms $\mathrm{osc}_0(f,\mathcal{T})$ and $\mathrm{osc}_0(\Psi_d,\mathcal{T})$ converge with order three (since this is a problem with smooth function over the convex domain). Hence, the theoretical rates of convergence are confirmed by these numerical outputs.

Non-convex domain. This numerical experiment is performed over the non-convex domain (Example 6.2) with the exact solution has a singularity at the origin. The numerical experiment starts on the initial mesh with 24 triangles, and then adaptive refinements are carried out using Algorithm 1. Figure 2 shows that the significant adaptive refinement occurs near the control variable interface and the singularity point of the L-shaped domain. This is expected as the state and adjoint solutions have a singularity at the origin. From Figure 3, it is observed that the control estimator dominates other estimators. This supports the efficiency of the adaptive estimator in the theoretical estimates obtained in the previous section. Figure 3 and Table 4 also indicate that the errors and estimators have optimal convergence in the adaptive refinement. The oscillation terms $\mathrm{osc}_0(f,\mathcal{T})$



(a) (b)

FIGURE 2. Discrete control solution $\bar{u}_h$ (a) and the adaptive mesh-refinement (b) (at level $\ell = 24$) in Example 6.2.

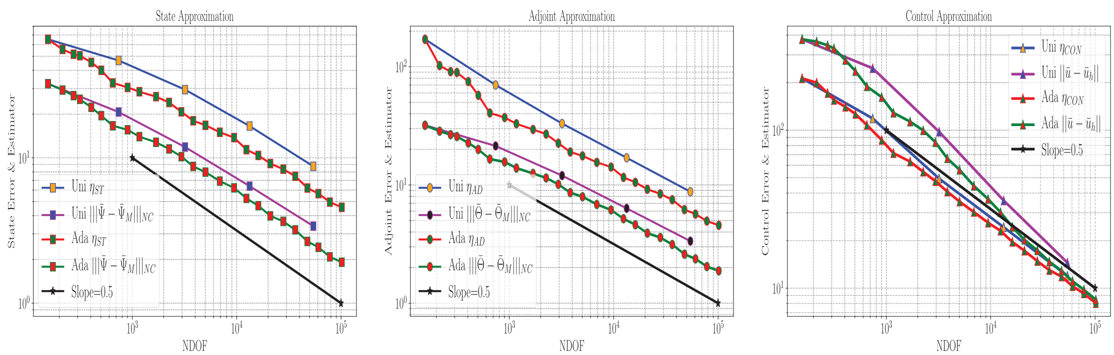


FIGURE 3. Convergence plot of the approximation errors and estimators with adaptive and uniform refinement for state, adjoint and control variables in Example 6.2 (State approximation (*left*), Adjoint approximation (*Middle*), Control approximation (*right*), Uni=Uniform refinement, Ada=Adaptive refinement).

and $\mathrm{osc}_0(\Psi_d, \mathcal{T})$ in the Table 4 have linear order of convergence with respective NDOF, which is higher-order in comparison to the error and estimator.

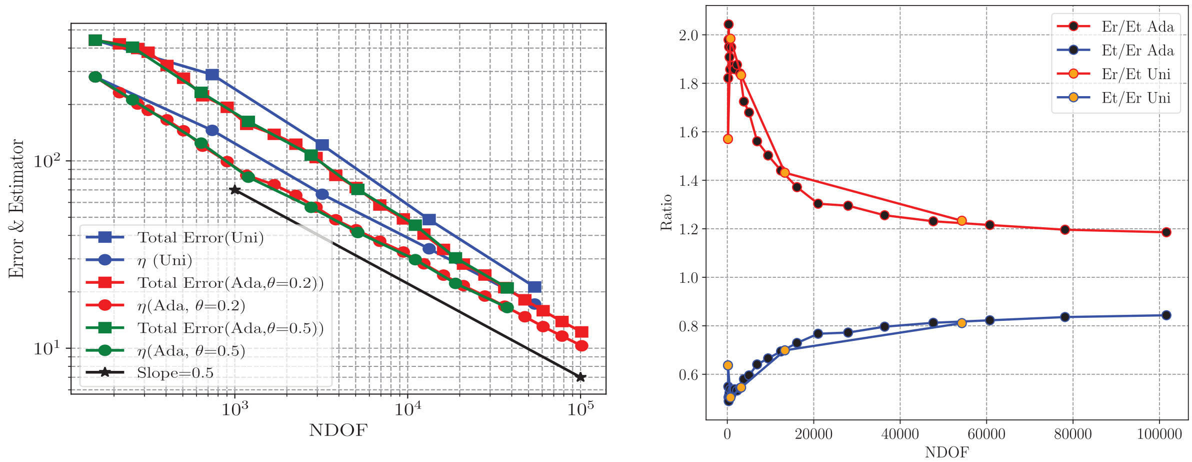
Figure 4a displays the convergence history of the total error and estimator; both achieve optimal convergence in adaptive refinement. Further, it can be observed that the adaptive refinements are performing better in terms of accuracy compared to the uniform refinements. Figure 4b illustrates that reliability and efficiency constants are approaching a constant value with mesh refinement, thus providing a numerical evidence for the efficiency and reliability of *a posteriori* estimator derived in the theory section.

7. CONCLUSIONS

This paper presents reliable and efficient *a posteriori* error estimates for the distributed optimal control problem governed by the von Kármán equations. The *a posteriori* estimator identifies the solution singularity region and the interface of the discrete control, and produces optimal convergence rates. The *a posteriori*

TABLE 4. Errors and orders of convergence for state, adjoint and control variables with adaptive refinement in Example 6.2.

Iter	NDOF	$\frac{\lVert\bar{\Psi}-\bar{\Psi}_{\mathrm{M}}\rVert_{\mathrm{NC}}}{\lVert\bar{\Psi}\rVert_2}$	Order	$\frac{\lVert\bar{\Theta}-\bar{\Theta}_{\mathrm{M}}\rVert_{\mathrm{NC}}}{\lVert\bar{\Theta}\rVert_2}$	Order	$\frac{\lVert\bar{u}-\bar{u}_h\rVert}{\lVert\bar{u}\rVert}$	Order	Error	Order	Er/Et	$\mathrm{osc}_0(f)$	$\mathrm{osc}_0(\Psi_d)$
0	156	1.371575	–	1.355646	–	0.760376	–	0.812881	–	1.47	36.2833	25.8111
4	464	0.842764	0.62	0.866655	0.59	0.488350	0.80	0.520237	0.77	1.93	11.9203	10.4362
8	1643	0.540211	0.32	0.533029	0.32	0.223952	0.47	0.251166	0.44	1.82	3.7135	3.4141
12	5030	0.330837	0.46	0.326537	0.47	0.107915	0.78	0.127130	0.71	1.72	1.2898	1.2332
16	15512	0.195581	0.44	0.191774	0.45	0.050371	0.70	0.062844	0.64	1.41	0.5876	0.5832
20	45948	0.115454	0.56	0.112701	0.58	0.026069	0.54	0.033729	0.55	1.25	0.1031	0.1022
22	73926	0.091240	0.53	0.089210	0.53	0.019682	0.57	0.025822	0.56	1.22	0.0817	0.0815



(a) (b)

FIGURE 4. Convergence plot (*left*), and reliability and efficiency constants (*right*) over uniform and adaptive refinements (*right*) in Example 6.2 (Er=Total Error, Et=Complete Estimator, Uni=Uniform refinement, Ada=Adaptive refinement). (a) Total error and estimator. (b) Efficiency and Reliability.

TABLE 5. Convergence results for post-processed control $\widetilde{u}_h$ for Example 6.1 (Square domain) and Example 6.2 (L-shaped domain).

Square domain			L-shaped domain		
h	$\frac{\lVert\bar{u}-\widetilde{u}_h\rVert}{\lVert\bar{u}\rVert}$	Order	h	$\frac{\lVert\bar{u}-\widetilde{u}_h\rVert}{\lVert\bar{u}\rVert}$	Order
0.250000	0.569226344	–	0.7071068	0.65981424	–
0.125000	0.160066913	1.8303	0.3535534	0.460809013	0.5179
0.062500	0.041173386	1.9589	0.1767767	0.177559399	1.3759
0.031250	0.010363721	1.9902	0.0883883	0.054638969	1.7003
0.015625	0.002595312	1.9975	0.0441941	0.016758562	1.7050

estimator contribution $\eta^2_{K,\mathcal{P}_0,\bar{\Theta}_{\rm M}}$ from the adjoint equations is non-standard due to the combined effect of the chosen trilinear form $b_{\rm NC}(\bullet,\bullet,\bullet)$, non-conformity of the method and linear lower-order terms.

The post-processed control $\widetilde{u}_h$ defined in (2.7) helps to establish *a posteriori* estimates. Table 5 shows *a priori* error estimates and order of convergence for $\widetilde{u}_h$ for Examples 6.1 and 6.2.

Table 5 indicates an improved *a priori* error estimate for $\|\bar{u}-\widetilde{u}_h\|$ in comparison to $\|\bar{u}-\bar{u}_h\|$ in Tables 1 and 2. To justify this theoretically, we could utilize (4.31). However, higher-order convergence rate for $\left|\!\left|\!\left|\widetilde{\Theta}-\bar{\Theta}_{\rm M}\right|\!\right|\!\right|$ needs to be established. A theoretical justification of this superconvergence result is a topic of future research.

Appendix A.

A.1. *A priori* error estimates

This section deals with the *a priori* error estimates for the state, adjoint and control variables under minimal regularity assumptions on the exact solution. The proof of the piecewise H^1 error estimates for the adjoint variable (see Theorem A.8) differs from that of the nonconforming Morley case in [19], since the discrete trilinear form is different, and forms the *main contribution of this subsection.*

Auxiliary results

Some auxiliary results relevant for the *a priori* error estimates are stated and this is followed by the error estimates.

Lemma A.1 (Bounds for $A_{\rm NC}(\bullet,\bullet)$ ([7], Lems. 4.2, 4.3))**.** *If $\boldsymbol{\chi}\in\boldsymbol{H}^{2+\gamma}(\Omega)$, $\Phi\in\mathbf{V}\cap\boldsymbol{H}^{2+\gamma}(\Omega)$ and $\Phi_{\rm M}\in\mathbf{V}_{\rm M}$, then*

(a) $A_{\rm NC}(\boldsymbol{\chi},J\Phi_{\rm M}-\Phi_{\rm M})\lesssim h^\gamma\|\boldsymbol{\chi}\|_{2+\gamma}\|\Phi_{\rm M}\|_{\rm NC}$. (b) $A_{\rm NC}(\boldsymbol{\chi},I_{\rm M}\Phi-\Phi)\lesssim h^{2\gamma}\|\boldsymbol{\chi}\|_{2+\gamma}\|\Phi\|_{2+\gamma}$.
(c) $A_{\rm NC}(\Phi_{\rm M},\Phi_{\rm M})=\|\Phi_{\rm M}\|^2_{\rm NC}$.

Lemma A.2 (Bounds for $B_{\rm NC}(\bullet,\bullet,\bullet)$)**.** *The boundedness properties of $B_{\rm NC}(\bullet,\bullet,\bullet)$ stated below hold.*

(a) $B_{\rm NC}(\boldsymbol{\chi},\boldsymbol{\lambda},\Phi)\lesssim\|\boldsymbol{\chi}\|_{\rm NC}\|\boldsymbol{\lambda}\|_{\rm NC}\|\Phi\|_\infty$ for all $\boldsymbol{\chi},\boldsymbol{\lambda},\Phi\in\mathbf{V}+\mathbf{V}_{\rm M}$.
(b) $B_{\rm NC}(\boldsymbol{\chi},\boldsymbol{\lambda},\Phi)\lesssim\|\boldsymbol{\chi}\|_{2+\gamma}\|\boldsymbol{\lambda}\|_{\rm NC}\|\Phi\|_1$ for all $\boldsymbol{\chi}\in\boldsymbol{H}^{2+\gamma}(\Omega),\boldsymbol{\lambda}\in\mathbf{V}+\mathbf{V}_{\rm M},\Phi\in\boldsymbol{H}^1_0(\Omega)$.
(c) $B_{\rm NC}(\boldsymbol{\chi},\boldsymbol{\lambda},\Phi)\lesssim\|\boldsymbol{\chi}\|_{2+\gamma}\|\boldsymbol{\lambda}\|_{2+\gamma}\|\Phi\|_{0,2,h}$ for all $\boldsymbol{\chi},\boldsymbol{\lambda}\in\boldsymbol{H}^{2+\gamma}(\Omega),\Phi\in\mathbf{V}+\mathbf{V}_{\rm M}$.

Proof. The bound in (a) follows from the definition of $B_{\rm NC}(\bullet,\bullet,\bullet)$ and $b_{\rm NC}(\bullet,\bullet,\bullet)$, and the generalised Hölder's inequality. For $\chi\in V\cap H^{2+\gamma}(\Omega),\lambda$ and $\phi\in V+V_{\rm M}$, (b) follows from the definition of $B_{\rm NC}(\bullet,\bullet,\bullet)$ and $b_{\rm NC}(\bullet,\bullet,\bullet)$, the estimate $\sum_{K\in\mathcal{T}}\int_K[\chi,\lambda]\phi\,{\rm d}x\lesssim\|\chi\|_{2,4}\|\lambda\|_{\rm NC}\|\phi\|_{0,4,h}$, and the Sobolev embeddings $H^{2+\gamma}(\Omega)\hookrightarrow W^{2,4}(\Omega)$ and $H^1(\Omega)\hookrightarrow L^4(\Omega)$. The last inequality follows using the estimate $\sum_{K\in\mathcal{T}}\int_K[\chi,\lambda]\phi\,{\rm d}x\lesssim\|\chi\|_{2,4}\|\lambda\|_{2,4}\|\phi\|_{0,2,h}$, and the continuous embedding $H^{2+\gamma}(\Omega)\hookrightarrow W^{2,4}(\Omega)$ where $\chi,\lambda\in V\cap H^{2+\gamma}(\Omega)$ and $\phi\in V+V_{\rm M}$. □

Lemma A.3. *For $\Psi,\boldsymbol{\chi},\Theta\in\mathbf{V}\cap\boldsymbol{H}^{2+\gamma}(\Omega)$ and $\Psi_{\rm M}\in\mathbf{V}_{\rm M}$,*

$$B_{\rm NC}(\Psi-\Psi_{\rm M},\boldsymbol{\chi},\Theta)\lesssim(h^\gamma\|\Psi-\Psi_{\rm M}\|_{\rm NC}+\|\Psi-\Psi_{\rm M}\|)\|\boldsymbol{\chi}\|_{2+\gamma}\|\Theta\|_{2+\gamma}.$$

Proof. Since the piecewise second derivatives of $I_{\rm M}\boldsymbol{\chi}$ are constants, the definition of $B_{\rm NC}(\bullet,\bullet,\bullet)$ and Lemma 3.6(c) show $B_{\rm NC}((J-1)\Psi_{\rm M},I_{\rm M}\boldsymbol{\chi},\mathcal{P}_0\Theta)=0$. This and elementary algebra lead to

$$\begin{aligned}B_{\rm NC}(\Psi-\Psi_{\rm M},\boldsymbol{\chi},\Theta)&=B(\Psi-J\Psi_{\rm M},\boldsymbol{\chi},\Theta)+B_{\rm NC}((J-1)\Psi_{\rm M},\boldsymbol{\chi},\Theta)\\&=B(\Psi-J\Psi_{\rm M},\boldsymbol{\chi},\Theta)+B_{\rm NC}((J-1)\Psi_{\rm M},(1-I_{\rm M})\boldsymbol{\chi},\Theta)+B_{\rm NC}((J-1)\Psi_{\rm M},I_{\rm M}\boldsymbol{\chi},(1-\mathcal{P}_0)\Theta).\end{aligned}\tag{A.1}$$

The definition of $B(\bullet,\bullet,\bullet)$, the symmetry of $b(\bullet,\bullet,\bullet)$ in the first and third variables, Lemma A.2(c), triangle inequality with $\Psi_{\rm M}$ and Lemma 3.6(d) with $v = \Psi$ lead to

$$B(\Psi - J\Psi_{\rm M}, \boldsymbol{\chi}, \Theta) \lesssim \big(h^2 \|\Psi - \Psi_{\rm M}\|_{\rm NC} + \|\Psi - \Psi_{\rm M}\|\big)\|\boldsymbol{\chi}\|_{2+\gamma}\|\Theta\|_{2+\gamma}.$$

Lemmas 3.7(b), 3.5(c) and 3.6(d) with $v = \Psi$ result in $B_{\rm NC}((J-1)\Psi_{\rm M}, (1-I_{\rm M})\boldsymbol{\chi}, \Theta) \lesssim h^{\gamma}\|\boldsymbol{\chi}\|_{2+\gamma}\|\Theta\|_2 \times \|\Psi - \Psi_{\rm M}\|_{\rm NC}$. Lemmas A.2(a), 3.5(b), 3.6(d) with $v = \Psi$, projection estimate in $L^{\infty}(\mathcal{T})$ ([23], Prop. 1.135) and the global Sobolev embedding $H^{2+\gamma}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ imply $B_{\rm NC}((J-1)\Psi_{\rm M}, I_{\rm M}\boldsymbol{\chi}, (1-\mathcal{P}_0)\Theta) \lesssim h\|\boldsymbol{\chi}\|_2\|\Theta\|_{2+\gamma}\|\Psi - \Psi_{\rm M}\|_{\rm NC}$. A substitution of the last three bounds in (A.1) concludes the proof. ☐

For a given $\mathbf{F}$, fixed control $u \in U_{ad}$ and $\mathbf{u} = (u, 0)$, consider the auxiliary state equation that seeks $\Psi_u \in \mathbf{V}$ such that

$$A(\Psi_u, \Phi) + B(\Psi_u, \Psi_u, \Phi) = (\mathbf{F} + \mathbf{C}\mathbf{u}, \Phi) \text{ for all } \Phi \in \mathbf{V}. \tag{A.2}$$

The nonconforming Morley finite element (FE) approximation to (A.2) seeks $\Psi_{u,\rm M} \in \mathbf{V}_{\rm M}$ such that

$$A_{\rm NC}(\Psi_{u,\rm M}, \Phi_{\rm M}) + B_{\rm NC}(\Psi_{u,\rm M}, \Psi_{u,\rm M}, \Phi_{\rm M}) = (\mathbf{F} + \mathbf{C}\mathbf{u}, \Phi_{\rm M}) \ \text{ for all } \Phi_{\rm M} \in \mathbf{V}_{\rm M}. \tag{A.3}$$

The result on the existence, uniqueness and error estimates of the auxiliary state equation is proved with the help of Lemma A.4. The proofs that are available in [10, 19] are skipped. Note that a modified proof of Lemma A.4 is presented and it utilises the properties of the companion operator to obtain sharper bounds in comparison to Lemma 3.12 of [19].

A linear mapping

For a given $\mathbf{g} = (g_1, g_2) \in \mathbf{V}'$, let the *linear operator* $T \in \mathcal{L}(\mathbf{V}', \mathbf{V})$ defined by $T\mathbf{g} := \boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbf{V}$ solve the biharmonic system $A(\boldsymbol{\xi}, \Phi) = \langle \mathbf{g}, \Phi \rangle$ for all $\Phi \in \mathbf{V}$, that is,

$$\Delta^2 \xi_1 = g_1 \ \text{ in } \Omega,\ \Delta^2 \xi_2 = g_2 \text{ in } \Omega,\ \xi_1 = 0,\ \frac{\partial \xi_1}{\partial \nu} = 0 \text{ and } \xi_2 = 0,\ \frac{\partial \xi_2}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{A.4}$$

Moreover, for $\mathbf{g} \in \boldsymbol{H}^{-1}(\Omega)$, $\boldsymbol{\xi} \in \mathbf{V} \cap \boldsymbol{H}^{2+\gamma}(\Omega)$, $\gamma \in (1/2, 1]$, the elliptic regularity [6] result stated next holds.

$$\|\boldsymbol{\xi}\|_2 \lesssim \|\mathbf{g}\|_{-1}, \quad \|\boldsymbol{\xi}\|_{2+\gamma} \lesssim \|\mathbf{g}\|_{-1}. \tag{A.5}$$

For $\mathbf{g} \in \mathbf{V}_{\rm M}'$, define the bounded discrete operator $T_{\rm NC} : \mathbf{V}_{\rm M}' \to \mathbf{V}_{\rm M}$ by $T_{\rm NC}\mathbf{g} := \boldsymbol{\xi}_{\rm M}$ where $\boldsymbol{\xi}_{\rm M} \in \mathbf{V}_{\rm M}$ solves the discrete problem

$$A_{\rm NC}(\boldsymbol{\xi}_{\rm M}, \Phi_{\rm M}) = \langle \mathbf{g}, \Phi_{\rm M} \rangle \ \text{ for all } \Phi_{\rm M} \in \mathbf{V}_{\rm M}. \tag{A.6}$$

The lemma stated next is utilized to prove the existence and uniqueness of the solution to (A.2).

Lemma A.4 (An intermediate estimate)**.** *Let $\bar{\Psi} \in \mathbf{V} \cap \boldsymbol{H}^{2+\gamma}(\Omega)$ be a regular solution to (2.1). Then $\forall \epsilon > 0$, there exists $\mathcal{T} \in \mathbb{T}(\delta_1)$ with $\delta_1 > 0$ such that* $\big\| T[\mathcal{B}'_{\rm NC}(\bar{\Psi})] - T_{\rm NC}[\mathcal{B}'_{\rm NC}(\Psi)] \big\|_{\mathcal{L}(\mathbf{V}+\mathbf{V}_{\rm M})} < \epsilon$ for all $\Psi \in B_{\rho_\epsilon}(\bar{\Psi})$.

Proof. For $\mathbf{z} \in \mathbf{V} + \mathbf{V}_{\rm M}$, (3.1) and Lemma 3.7(b) show $\mathcal{B}'_{\rm NC}(\bar{\Psi})(\mathbf{z}) \in \mathbf{V}'$ and $\mathcal{B}'_{\rm NC}(\bar{\Psi})(\mathbf{z}) \in \mathbf{V}'_{\rm M}$. For $\Psi \in \mathbf{V} + \mathbf{V}_{\rm M}$, the definitions of $T(\bullet)$ and $T_{\rm NC}(\bullet)$, and (A.5) imply that $\boldsymbol{\theta}(\bar{\Psi}) =: T[\mathcal{B}'_{\rm NC}(\bar{\Psi})(\mathbf{z})] \in \mathbf{V} \cap \boldsymbol{H}^{2+\gamma}(\Omega)$ and $\boldsymbol{\theta}_{\rm M}(\Psi) =: T_{\rm NC}[\mathcal{B}'_{\rm NC}(\Psi)(\mathbf{z})] \in \mathbf{V}_{\rm M}$ solve

$$A(\boldsymbol{\theta}(\bar{\Psi}), \Phi) = \langle \mathcal{B}'_{\rm NC}(\bar{\Psi})(\mathbf{z}), \Phi \rangle \qquad \text{for all } \Phi \in \mathbf{V}, \tag{A.7}$$

$$A_{\rm NC}(\boldsymbol{\theta}_{\rm M}(\Psi), \Phi_{\rm M}) = \langle \mathcal{B}'_{\rm NC}(\Psi)(\mathbf{z}), \Phi_{\rm M} \rangle \qquad \text{for all } \Phi_{\rm M} \in \mathbf{V}_{\rm M}. \tag{A.8}$$

Let $\boldsymbol{\theta}_{\rm M}(\bar{\Psi})$ and $\boldsymbol{\theta}^J_{\rm M}(\bar{\Psi}) \in \mathbf{V}_{\rm M}$ solve the discrete problems

$$A_{\rm NC}(\boldsymbol{\theta}_{\rm M}(\bar{\Psi}), \Phi_{\rm M}) = \langle \mathcal{B}'_{\rm NC}(\bar{\Psi})(\mathbf{z}), \Phi_{\rm M} \rangle \qquad \text{for all } \Phi_{\rm M} \in \mathbf{V}_{\rm M}, \tag{A.9}$$

$$A_{\mathrm{NC}}\Big(\boldsymbol{\theta}_{\mathrm{M}}^{J}(\bar{\Psi}),\Phi_{\mathrm{M}}\Big) = \big\langle \mathcal{B}'_{\mathrm{NC}}(\bar{\Psi})(\mathbf{z}), J\Phi_{\mathrm{M}}\big\rangle \qquad \text{for all } \Phi_{\mathrm{M}}\in \mathbf{V}_{\mathrm{M}}. \tag{A.10}$$

A triangle inequality yields

$$\|\boldsymbol{\theta}(\bar{\Psi}) - \boldsymbol{\theta}_{\mathrm{M}}(\Psi)\|_{\mathrm{NC}} \le \Big\|\boldsymbol{\theta}(\bar{\Psi}) - \boldsymbol{\theta}_{\mathrm{M}}^{J}(\bar{\Psi})\Big\|_{\mathrm{NC}} + \Big\|\boldsymbol{\theta}_{\mathrm{M}}^{J}(\bar{\Psi}) - \boldsymbol{\theta}_{\mathrm{M}}(\bar{\Psi})\Big\|_{\mathrm{NC}} + \|\boldsymbol{\theta}_{\mathrm{M}}(\bar{\Psi}) - \boldsymbol{\theta}_{\mathrm{M}}(\Psi)\|_{\mathrm{NC}}. \tag{A.11}$$

Notice that $\boldsymbol{\theta}_{\mathrm{M}}^{J}(\bar{\Psi})$ is the Morley nonconforming solution to (A.7) for a modified right-hand side $\mathcal{B}'_{\mathrm{NC}}(\bar{\Psi})(\mathbf{z})\circ J \in \mathbf{V}'_{\mathrm{M}}$.

The best approximation result from Theorem 3.2 of [11] shows

$$\Big\|\boldsymbol{\theta}(\bar{\Psi}) - \boldsymbol{\theta}_{\mathrm{M}}^{J}(\bar{\Psi})\Big\|_{\mathrm{NC}} \le \sqrt{1+\Lambda_{\mathrm{J}}^2}\,\|(1-I_{\mathrm{M}})\boldsymbol{\theta}(\bar{\Psi})\|_{\mathrm{NC}}. \tag{A.12}$$

This together with the interpolation estimate from Lemma 3.5(c), (A.5), (A.7) and Lemma A.2(b) imply

$$\Big\|\boldsymbol{\theta}(\bar{\Psi}) - \boldsymbol{\theta}_{\mathrm{M}}^{J}(\bar{\Psi})\Big\|_{\mathrm{NC}} \lesssim h^{\gamma}\sqrt{1+\Lambda_{\mathrm{J}}^2}\,\|\boldsymbol{\theta}(\bar{\Psi})\|_{2+\gamma} \lesssim h^{\gamma}\|\mathcal{B}'_{\mathrm{NC}}(\bar{\Psi})(\mathbf{z})\|_{-1} \lesssim h^{\gamma}\|\bar{\Psi}\|_{2+\gamma}\|\mathbf{z}\|_{\mathrm{NC}}. \tag{A.13}$$

The combination of (A.9) and (A.10), and (3.1) show

$$A_{\mathrm{NC}}\Big(\boldsymbol{\theta}_{\mathrm{M}}(\bar{\Psi}) - \boldsymbol{\theta}_{\mathrm{M}}^{J}(\bar{\Psi}),\Phi_{\mathrm{M}}\Big) = 2B_{\mathrm{NC}}(\bar{\Psi},\mathbf{z},(1-J)\Phi_{\mathrm{M}}).$$

The inverse inequality and Lemma 3.6(d) prove $\|(1-J)\Phi_{\mathrm{M}}\|_{0,\infty} \lesssim h^{-1}\|(1-J)\Phi_{\mathrm{M}}\| \lesssim h\|\Phi_{\mathrm{M}}\|_{\mathrm{NC}}$. This, Lemma A.1(c) with test function $\Phi_{\mathrm{M}} := \boldsymbol{\theta}_{\mathrm{M}}(\bar{\Psi}) - \boldsymbol{\theta}_{\mathrm{M}}^{J}(\bar{\Psi})$ and Lemma A.2(a) imply

$$\Big\|\boldsymbol{\theta}_{\mathrm{M}}^{J}(\bar{\Psi}) - \boldsymbol{\theta}_{\mathrm{M}}(\bar{\Psi})\Big\|_{\mathrm{NC}} \lesssim h\|\bar{\Psi}\|_{2}\|\mathbf{z}\|_{\mathrm{NC}}. \tag{A.14}$$

The combination of (A.8) and (A.9), and (3.1), Lemma A.1(c) with test function $\boldsymbol{\theta}_{\mathrm{M}}(\Psi) - \boldsymbol{\theta}_{\mathrm{M}}(\bar{\Psi})$, and Lemma 3.7(b) prove $\|\boldsymbol{\theta}_{\mathrm{M}}(\Psi) - \boldsymbol{\theta}_{\mathrm{M}}(\bar{\Psi})\|_{\mathrm{NC}} \lesssim \|\mathbf{z}\|_{\mathrm{NC}}\|\Psi-\bar{\Psi}\|_{\mathrm{NC}}$. A substitution of this and (A.13), (A.14) in (A.11) lead to the result that for any preassigned $\epsilon > 0$, h_1 and the radius $\rho_\epsilon > 0$ can be chosen small such that for all $\Psi \in B_{\rho_\epsilon}(\bar{\Psi})$, $\|T[\mathcal{B}'_{\mathrm{NC}}(\bar{\Psi})(\mathbf{z})] - T_{\mathrm{NC}}[\mathcal{B}'_{\mathrm{NC}}(\Psi)(\mathbf{z})]\|_{\mathrm{NC}} < \epsilon\|\mathbf{z}\|_{\mathrm{NC}}$, leading to the desired estimate. □

Recall from the notation that $\mathrm{osc}_k(f,\mathcal{T}) = \|h_{\mathcal{T}}^2(f-\Pi_k f)\|$ for $k\in\mathbb{N}_0$.

Theorem A.5 (Existence, uniqueness and error estimates)**.**

(i) *Let* $(\bar{\Psi},\bar{u}) \in \mathbf{V}\times L^2(\omega)$ *be a regular solution to* (2.1)*. Then, there exist* $\delta_1,\rho_1,\rho_2 > 0$ *with* $\delta_2 \le \delta_1$ *such that, for* $\mathcal{T}\in\mathbb{T}(\delta_2)$ *and* $u\in B_{\rho_2}(\bar{u})$, (A.3) *admits a unique solution in* $B_{\rho_1}(\bar{\Psi})$*, where* $u \in B_{\rho_2}(\bar{u})$ *(resp.* $\Psi\in B_{\rho_1}(\bar{\Psi})$*) implies* $\|u-\bar{u}\|_{L^2(\omega)} \le \rho_2$ *(resp.* $\|\Psi-\bar{\Psi}\|_{\mathrm{NC}}\le\rho_1$*).*

(ii) *Let* $(\bar{\Psi},\bar{u}) \in \mathbf{V}\times L^2(\omega)$ *be a regular solution to* (2.1)*. Then, for* $u\in B_{\rho_2}(\bar{u})$ *and* $\mathcal{T}\in\mathbb{T}(\delta_2)$*, the solutions* Ψ_u *and* $\Psi_{u,\mathrm{M}}$ *to* (A.2) *and* (A.3) *satisfy the energy and broken* H^1 *norm estimates* (a) $\|\Psi_u-\Psi_{u,\mathrm{M}}\|_{\mathrm{NC}} \lesssim \|(1-I_{\mathrm{M}})\Psi_u\|_{\mathrm{NC}} + \mathrm{osc}_1(f+\mathcal{C}u+[\psi_{u,1},\psi_{u,2}],\mathcal{T}) + \mathrm{osc}_1([\psi_{u,1},\psi_{u,1}],\mathcal{T})$, (b) $\|\Psi_u-\Psi_{u,\mathrm{M}}\|_{1,2,h} \lesssim h^{\gamma}(\|\Psi_u-\Psi_{u,\mathrm{M}}\|_{\mathrm{NC}} + \mathrm{osc}_m(f+\mathcal{C}u,\mathcal{T}))$ *for each* $m\in\mathbb{N}_0$*.*

(iii) *For* u, $\hat{u}\in B_{\rho_2}(\bar{u})$*, and* $\mathcal{T}\in\mathbb{T}(\delta_2)$*, the solutions* Ψ_u *and* $\Psi_{\hat{u},\mathrm{M}}$ *to* (A.2) *and* (A.3)*, with controls chosen as* u *and* $\hat{u}$ *respectively, satisfy* $\|\Psi_u-\Psi_{\hat{u},\mathrm{M}}\|_{\mathrm{NC}} \lesssim h^{\gamma} + \|u-\hat{u}\|_{L^2(\omega)}$*.*

Here $\gamma\in(1/2,1]$ *is the elliptic regularity index.*

The proof of (i) and (iii) can be found in Theorem 3.8(i) and Lemma 3.9 of [19]. The error estimate in energy and piecewise H^1 norms given by (ii)(a) and (b) are established in Theorem 3.1 of [10].

Remark A.6. The well-known result for the biharmonic problem for the approximation using Morley nonconforming FEM which states that the L^2 error estimate cannot be further improved than that of H^1 error estimate [27] extends to von Kármán equations and thus Theorem A.5(ii), and Lemma 3.5(c) show
(a) $\|\Psi_u - \Psi_{u,\mathrm{M}}\|_{\mathrm{NC}} \lesssim h^\gamma$,
(b) $\|\Psi_u - \Psi_{u,\mathrm{M}}\|_{1,2,h} \lesssim h^{2\gamma}$, and (c) $\|\Psi_u - \Psi_{u,\mathrm{M}}\| \lesssim h^{2\gamma}$.

The auxiliary problem corresponding to the adjoint equations seeks $\Theta_u \in \mathbf{V}$ such that

$$A(\Phi,\Theta_u) + 2B(\Psi_u,\Phi,\Theta_u) = (\Psi_u - \Psi_d, \Phi) \text{ for all } \Phi \in \mathbf{V}, \tag{A.15}$$

where $\Psi_u \in \mathbf{V}$ is the solution to (A.2). A Morley FE discretization corresponding to (A.15) seeks $\Theta_{u,\mathrm{M}} \in \mathbf{V}_\mathrm{M}$ such that, for all $\Phi_\mathrm{M} \in \mathbf{V}_\mathrm{M}$,

$$A_{\mathrm{NC}}(\Phi_\mathrm{M},\Theta_{u,\mathrm{M}}) + 2B_{\mathrm{NC}}(\Psi_{u,\mathrm{M}},\Phi_\mathrm{M},\Theta_{u,\mathrm{M}}) = (\Psi_{u,\mathrm{M}} - \Psi_d, \Phi_\mathrm{M}). \tag{A.16}$$

The existence, uniqueness and convergence results stated in the next theorem follow analogous to that of Theorems 4.1 and 4.2(a) from [19] and are skipped for brevity.

Theorem A.7 (Existence, uniqueness and energy error estimate)**.** *Let* $(\bar\Psi,\bar u) \in \mathbf{V}\times L^2(\omega)$ *be a regular solution to* (2.1)*. Then,* (i) *there exist* $0<\rho_3\le\rho_2$ *and* $\delta_3\le\delta_2$ *such that, for all* $\mathcal{T}\in\mathbb{T}(\delta_3)$ *and* $u\in B_{\rho_3}(\bar u)$*,* (A.16) *admits a unique solution,* (ii) *for* $u\in B_{\rho_3}(\bar u)$ *and* $\mathcal{T}\in\mathbb{T}(\delta_3)$*, the solutions* Θ_u *and* $\Theta_{u,\mathrm{M}}$ *of* (A.15) *and* (A.16) *satisfy the energy norm error estimate:* $\|\Theta_u-\Theta_{u,\mathrm{M}}\|_{\mathrm{NC}} \lesssim \|\Psi_u-\Psi_{u,\mathrm{M}}\|_{\mathrm{NC}} + h^\gamma(\|\psi_{u,1}-\psi_{d,1}-[\psi_{u,1},\theta_{u,2}]+[\psi_{u,2},\theta_{u,1}]\| + \|\psi_{u,2}-\psi_{d,2}+[\psi_{u,1},\theta_{u,1}]\|)$*, where* Ψ_u *(resp.* $\Psi_{u,\mathrm{M}}$*) solves* (A.2) *(resp.* (A.3)*) and* $\gamma\in\left(\frac12,1\right]$.

The proof of *a priori* H^1 error estimate stated in the next theorem for adjoint variables is a non-trivial modification of the corresponding result in [19]. The form of the error estimate will be useful in the adaptive convergence study that is planned for future.

Theorem A.8 (Piecewise H^1 error estimate)**.** *Let* $(\bar\Psi,\bar u)\in\mathbf{V}\times L^2(\omega)$ *be a regular solution to* (2.1)*. Then for* $\mathcal{T}\in\mathbb{T}(\delta_3)$*, the solutions* Θ_u *and* $\Theta_{u,\mathrm{M}}$ *of* (A.15) *and* (A.16) *satisfy*

$$\begin{aligned}\|\Theta_u-\Theta_{u,\mathrm{M}}\|_{1,2,h} \lesssim h^\gamma\Big(&\|\Psi_u-\Psi_{u,\mathrm{M}}\|_{\mathrm{NC}} + \|\Theta_u-\Theta_{u,\mathrm{M}}\|_{\mathrm{NC}} + \|(1-\mathcal{P}_0)\Theta_u\|_{0,\infty}\Big)\\ &+ \|\Psi_u-\Psi_{u,\mathrm{M}}\|_{\mathrm{NC}}\|\Theta_u-\Theta_{u,\mathrm{M}}\|_{\mathrm{NC}} + \|\Psi_u-\Psi_{u,\mathrm{M}}\| + h^{2+\gamma}\mathrm{osc}_0(\Psi_u-\Psi_d,\mathcal{T}),\end{aligned}$$

where $\mathrm{osc}_0(f,\mathcal{T}) = \|h_{\mathcal{T}}^2(f-\Pi_0 f)\|$*,* Ψ_u *(resp.* $\Psi_{u,\mathrm{M}}$*) solves* (A.2) *(resp.* (A.3)*) and* $\gamma\in\left(\frac12,1\right]$.

Proof. **Step 1: Isolates a crucial term.** Let $\boldsymbol{\rho}_\mathrm{M} := I_\mathrm{M}\Theta_u - \Theta_{u,\mathrm{M}} \in \mathbf{V}_\mathrm{M}$. The triangle inequality leads to

$$\|\Theta_u-\Theta_{u,\mathrm{M}}\|_{1,2,h} \le \|(1-I_\mathrm{M})\Theta_u\|_{1,2,h} + \|(1-J)\boldsymbol{\rho}_\mathrm{M}\|_{1,2,h} + \|J\boldsymbol{\rho}_\mathrm{M}\|_1. \tag{A.17}$$

Lemma 3.5(a) shows that $\Theta_u - I_\mathrm{M}\Theta_u$ is orthogonal to Φ_M for all $\Phi_\mathrm{M}\in V_\mathrm{M}$ and so Lemma 3.5(b) and the Pythagoras theorem prove

$$h^{-2}\|\Theta_u - I_\mathrm{M}\Theta_u\|_{1,2,h}^2 \le C_\mathrm{I}^2\|\Theta_u - I_\mathrm{M}\Theta_u\|_{\mathrm{NC}}^2 = C_\mathrm{I}^2\Big(\|\Theta_u-\Theta_{u,\mathrm{M}}\|_{\mathrm{NC}}^2 - \|\boldsymbol{\rho}_\mathrm{M}\|_{\mathrm{NC}}^2\Big) \le C_\mathrm{I}^2\|\Theta_u-\Theta_{u,\mathrm{M}}\|_{\mathrm{NC}}^2. \tag{A.18}$$

Lemma 3.6(d) with $v=0$ and the Pythagoras theorem in the above displayed inequality show

$$h^{-1}\|\boldsymbol{\rho}_\mathrm{M} - J\boldsymbol{\rho}_\mathrm{M}\|_{1,2,h} \le \Lambda_\mathrm{J}\|\boldsymbol{\rho}_\mathrm{M}\|_{\mathrm{NC}} \lesssim \|\Theta_u-\Theta_{u,\mathrm{M}}\|_{\mathrm{NC}}, \tag{A.19}$$

where "$\lesssim$" absorbs Λ_J and C_I. (A.17)–(A.19) conclude the first step and show

$$\|\Theta_u-\Theta_{u,\mathrm{M}}\|_{1,2,h} \lesssim h\|\Theta_u-\Theta_{u,\mathrm{M}}\|_{\mathrm{NC}} + \|J\boldsymbol{\rho}_\mathrm{M}\|_1. \tag{A.20}$$

Step 2: Estimates $\|J\boldsymbol{\rho}_{\mathrm{M}}\|_{1}$ in (A.20). For a given $\mathbf{g}\in\boldsymbol{H}^{-1}(\Omega)$, consider the dual problem that seeks $\boldsymbol{\chi_g}\in\mathbf{V}$ such that

$$A(\boldsymbol{\chi_g},\Phi)+2B(\Psi_u,\boldsymbol{\chi_g},\Phi)=\langle\mathbf{g},\Phi\rangle\qquad\text{for all }\Phi\in\mathbf{V}.\tag{A.21}$$

The existence of solution to (A.21) and the regularity results stated below follow from Theorem 3.1. Note that $\boldsymbol{\chi_g}\in\mathbf{V}\cap\mathbf{H}^{2+\gamma}(\Omega)$ and

$$\|\boldsymbol{\chi_g}\|_{2}\lesssim\|\mathbf{g}\|_{-1}\text{ and }\|\boldsymbol{\chi_g}\|_{2+\gamma}\lesssim\|\mathbf{g}\|_{-1}.\tag{A.22}$$

Choose $\mathbf{g}=-\Delta J\boldsymbol{\rho}_{\mathrm{M}}\in\boldsymbol{L}^2(\Omega)$ and $\Phi=J\boldsymbol{\rho}_{\mathrm{M}}\in\mathbf{V}$. This and elementary algebra eventually lead to

$$\begin{aligned}\|\nabla J\boldsymbol{\rho}_{\mathrm{M}}\|^2&=A_{\mathrm{NC}}(\boldsymbol{\chi_g},(J-1)\boldsymbol{\rho}_{\mathrm{M}})+2\big(B_{\mathrm{NC}}(\Psi_u,\boldsymbol{\chi_g},(J-1)\boldsymbol{\rho}_{\mathrm{M}})+B_{\mathrm{NC}}(\Psi_u,\boldsymbol{\chi_g},(I_{\mathrm{M}}-1)\Theta_u)\big)\\&\quad+\big(A_{\mathrm{NC}}(\boldsymbol{\chi_g},(I_{\mathrm{M}}-1)\Theta_u)+A_{\mathrm{NC}}(\boldsymbol{\chi_g},\Theta_u-\Theta_{u,\mathrm{M}})\big)+2B_{\mathrm{NC}}(\Psi_u,\boldsymbol{\chi_g},\Theta_u-\Theta_{u,\mathrm{M}})=\sum_{i=1}^{4}T_i.\end{aligned}\tag{A.23}$$

Step 3: Estimates the terms $T_1,\cdots,T_4$. Lemma A.1(a) shows that

$$T_1:=A_{\mathrm{NC}}(\boldsymbol{\chi_g},(J-1)\boldsymbol{\rho}_{\mathrm{M}})\lesssim h^{\gamma}\|\boldsymbol{\rho}_{\mathrm{M}}\|_{\mathrm{NC}}\|\boldsymbol{\chi_g}\|_{2+\gamma}\lesssim h^{\gamma}\|\Theta_u-\Theta_{u,\mathrm{M}}\|_{\mathrm{NC}}\|\boldsymbol{\chi_g}\|_{2+\gamma}$$

with (A.19) in the end. Lemmas A.2(c), 3.6(d) with $v=\boldsymbol{\rho}_{\mathrm{M}}$, 3.5(b), (A.18) and (A.19) imply

$$\frac{1}{2}T_2:=B_{\mathrm{NC}}(\Psi_u,\boldsymbol{\chi_g},(J-1)\boldsymbol{\rho}_{\mathrm{M}})+B_{\mathrm{NC}}(\Psi_u,\boldsymbol{\chi_g},(I_{\mathrm{M}}-1)\Theta_u)\lesssim h^2\|\Psi_u\|_{2+\gamma}\|\boldsymbol{\chi_g}\|_{2+\gamma}\|\Theta_u-\Theta_{u,\mathrm{M}}\|_{\mathrm{NC}}.$$

Simple manipulations lead to

$$\begin{aligned}T_3:=&A_{\mathrm{NC}}(\boldsymbol{\chi_g},(I_{\mathrm{M}}-1)\Theta_u)+A_{\mathrm{NC}}((1-I_{\mathrm{M}})\boldsymbol{\chi_g},\Theta_u-\Theta_{u,\mathrm{M}})+A_{\mathrm{NC}}((1-J)I_{\mathrm{M}}\boldsymbol{\chi_g},\Theta_u-\Theta_{u,\mathrm{M}})\\&+A_{\mathrm{NC}}(JI_{\mathrm{M}}\boldsymbol{\chi_g},\Theta_u-\Theta_{u,\mathrm{M}}).\end{aligned}\tag{A.24}$$

Lemma 3.5(a) shows $A_{\mathrm{NC}}(\Phi_{\mathrm{M}},I_{\mathrm{M}}\Theta_u-\Theta_u)=0=A_{\mathrm{NC}}(\boldsymbol{\chi_g}-I_{\mathrm{M}}\boldsymbol{\chi_g},\Phi_{\mathrm{M}})$ for all $\Phi_{\mathrm{M}}\in V_{\mathrm{M}}$. This shows that for the first two terms in (A.24) it holds

$$A_{\mathrm{NC}}(\boldsymbol{\chi_g}-I_{\mathrm{M}}\boldsymbol{\chi_g},I_{\mathrm{M}}\Theta_u-\Theta_u)+A_{\mathrm{NC}}(\boldsymbol{\chi_g}-I_{\mathrm{M}}\boldsymbol{\chi_g},\Theta_u-\Theta_{u,\mathrm{M}})=0.$$

The boundedness of $A_{\mathrm{NC}}(\bullet,\bullet)$, Lemma 3.6(d) with $v=\boldsymbol{\chi_g}$ and Lemma 3.5(c) result in an estimate for the third term in (A.24) as

$$A_{\mathrm{NC}}((1-J)I_{\mathrm{M}}\boldsymbol{\chi_g},\Theta_u-\Theta_{u,\mathrm{M}})\le h^{\gamma}\|\boldsymbol{\chi_g}\|_{2+\gamma}\|\Theta_u-\Theta_{u,\mathrm{M}}\|_{\mathrm{NC}}.\tag{A.25}$$

Lemma 3.6(c) shows $A_{\mathrm{NC}}(JI_{\mathrm{M}}\boldsymbol{\chi_g}-I_{\mathrm{M}}\boldsymbol{\chi_g},\Theta_{u,\mathrm{M}})=0$. This, (A.15) and (A.16) lead to an expression for the last term in (A.24) as

$$\begin{aligned}A_{\mathrm{NC}}(JI_{\mathrm{M}}\boldsymbol{\chi_g},\Theta_u-\Theta_{u,\mathrm{M}})&=A_{\mathrm{NC}}(JI_{\mathrm{M}}\boldsymbol{\chi_g},\Theta_u)-A_{\mathrm{NC}}(I_{\mathrm{M}}\boldsymbol{\chi_g},\Theta_{u,\mathrm{M}})\\&=(\Psi_u-\Psi_d,JI_{\mathrm{M}}\boldsymbol{\chi_g})-2B(\Psi_u,JI_{\mathrm{M}}\boldsymbol{\chi_g},\Theta_u)-(\Psi_{u,\mathrm{M}}-\Psi_d,I_{\mathrm{M}}\boldsymbol{\chi_g})+2B_{\mathrm{NC}}(\Psi_{u,\mathrm{M}},I_{\mathrm{M}}\boldsymbol{\chi_g},\Theta_{u,\mathrm{M}})\\&=(\Psi_u-\Psi_d,(J-1)I_{\mathrm{M}}\boldsymbol{\chi_g})-(\Psi_{u,\mathrm{M}}-\Psi_u,I_{\mathrm{M}}\boldsymbol{\chi_g})-2B(\Psi_u,JI_{\mathrm{M}}\boldsymbol{\chi_g},\Theta_u)+2B_{\mathrm{NC}}(\Psi_{u,\mathrm{M}},I_{\mathrm{M}}\boldsymbol{\chi_g},\Theta_{u,\mathrm{M}}).\end{aligned}\tag{A.26}$$

Lemma 3.6(b) shows $\Pi_0\boldsymbol{z}=0$ for $z=(J-1)I_{\mathrm{M}}\boldsymbol{\chi_g}$. This, the Cauchy–Schwarz inequality, Lemmas 3.6(d) with $v=\boldsymbol{\chi_g}$, 3.7(a) and 3.5(b), (c) lead to the estimate for the first two terms of (A.26) as

$$\begin{aligned}(\Psi_u-\Psi_d-\Pi_0(\Psi_u-\Psi_d),(J-1)I_{\mathrm{M}}\boldsymbol{\chi_g})&\lesssim h^{2+\gamma}\mathrm{osc}_0(\Psi_u-\Psi_d)\|\boldsymbol{\chi_g}\|_{2+\gamma},\\(\Psi_{u,\mathrm{M}}-\Psi_u,I_{\mathrm{M}}\boldsymbol{\chi_g})&\lesssim\|\Psi_{u,\mathrm{M}}-\Psi_u\|\|\boldsymbol{\chi_g}\|_{2}.\end{aligned}\tag{A.27}$$

The terms involving trilinear forms in (A.26) are estimated now. The orthogonality property of J in Lemma 3.6(c) shows that $B_{\mathrm{NC}}(\Psi_{u,\mathrm{M}}, \boldsymbol{\chi_g} - JI_{\mathrm{M}}\boldsymbol{\chi_g}, \mathcal{P}_0\Theta_u) = 0$. This and a simple manipulation (omitting a factor 2) lead to an expression for the last two terms of (A.26) combined with T_4 as

$$\begin{aligned} &B_{\mathrm{NC}}\big(\Psi_u - \Psi_{u,\mathrm{M}}, (1 - JI_{\mathrm{M}})\boldsymbol{\chi_g}, \Theta_u\big) + B_{\mathrm{NC}}\big(\Psi_{u,\mathrm{M}}, (1 - JI_{\mathrm{M}})\boldsymbol{\chi_g}, (1 - \mathcal{P}_0)\Theta_u\big) \\ &\quad + B_{\mathrm{NC}}\big(\Psi_{u,\mathrm{M}}, I_{\mathrm{M}}\boldsymbol{\chi_g}, \Theta_{u,\mathrm{M}}\big) - B_{\mathrm{NC}}\big(\Psi_u, \boldsymbol{\chi_g}, \Theta_{u,\mathrm{M}}\big). \end{aligned} \tag{A.28}$$

The triangle inequality with $I_{\mathrm{M}}\boldsymbol{\chi}_g$, Lemmas 3.7(b), 3.5(c) and 3.6(d) with $v = \boldsymbol{\chi}_g$ result in

$$B_{\mathrm{NC}}\big(\Psi_u - \Psi_{u,\mathrm{M}}, (1 - JI_{\mathrm{M}})\boldsymbol{\chi_g}, \Theta_u\big) \lesssim h^\gamma \big\|\boldsymbol{\chi_g}\big\|_{2+\gamma} \|\Theta_u\|_{2+\gamma} \|\Psi_u - \Psi_{u,\mathrm{M}}\|_{\mathrm{NC}}. \tag{A.29}$$

Lemmas A.2(a), 3.5(c) and 3.6(d) with $v = \boldsymbol{\chi}_g$ show

$$B_{\mathrm{NC}}\big(\Psi_{u,\mathrm{M}}, (1 - JI_{\mathrm{M}})\boldsymbol{\chi_g}, (1 - \mathcal{P}_0)\Theta_u\big) \lesssim h^\gamma \big\|\boldsymbol{\chi_g}\big\|_{2+\gamma} \|\Psi_u\|_{2+\gamma} \|\Theta_u - \mathcal{P}_0\Theta_u\|_{0,\infty}. \tag{A.30}$$

The integral mean property of I_{M} in Lemma 3.5(a) shows $B_{\mathrm{NC}}(\Psi_{u,\mathrm{M}}, I_{\mathrm{M}}\boldsymbol{\chi_g} - \boldsymbol{\chi_g}, \mathcal{P}_0\Theta_u) = 0$. This and a simple manipulation show that the last two terms in (A.28) can be rewritten as

$$\begin{aligned} &B_{\mathrm{NC}}\big(\Psi_u - \Psi_{u,\mathrm{M}}, (1 - I_{\mathrm{M}})\boldsymbol{\chi_g}, \Theta_{u,\mathrm{M}}\big) + B_{\mathrm{NC}}\big(\Psi_u, (1 - I_{\mathrm{M}})\boldsymbol{\chi_g}, \Theta_u - \Theta_{u,\mathrm{M}}\big) + B_{\mathrm{NC}}\big(\Psi_u - \Psi_{u,\mathrm{M}}, (I_{\mathrm{M}} - 1)\boldsymbol{\chi_g}, \Theta_u\big) \\ &\quad + B_{\mathrm{NC}}\big(\Psi_{u,\mathrm{M}}, (I_{\mathrm{M}} - 1)\boldsymbol{\chi_g}, (1 - \mathcal{P}_0)\Theta_u\big) + B_{\mathrm{NC}}\big(\Psi_u - \Psi_{u,\mathrm{M}}, \boldsymbol{\chi_g}, \Theta_u - \Theta_{u,\mathrm{M}}\big) + B_{\mathrm{NC}}\big(\Psi_{u,\mathrm{M}} - \Psi_u, \boldsymbol{\chi_g}, \Theta_u\big) \\ &\quad = \sum_{i=1}^{6} \mathbb{T}_i. \end{aligned} \tag{A.31}$$

The terms $\mathbb{T}_1, \cdots, \mathbb{T}_6$ are estimated next. The boundedness and interpolation estimates in Lemmas A.2(a), 3.7(b) and 3.5(c) prove

$$\begin{aligned} &\mathbb{T}_1 \lesssim h^\gamma \big\|\boldsymbol{\chi_g}\big\|_{2+\gamma} \|\Theta_u\|_{2+\gamma} \|\Psi_u - \Psi_{u,\mathrm{M}}\|_{\mathrm{NC}},\ \mathbb{T}_2 \lesssim h^\gamma \big\|\boldsymbol{\chi_g}\big\|_{2+\gamma} \|\Psi_u\|_{2+\gamma} \|\Theta_u - \Theta_{u,\mathrm{M}}\|_{\mathrm{NC}} \\ &\mathbb{T}_3 \lesssim h^\gamma \big\|\boldsymbol{\chi_g}\big\|_{2+\gamma} \|\Theta_u\|_{2+\gamma} \|\Psi_u - \Psi_{u,\mathrm{M}}\|_{\mathrm{NC}},\ \mathbb{T}_4 \lesssim h^\gamma \big\|\boldsymbol{\chi_g}\big\|_{2+\gamma} \|\Psi_u\|_{2+\gamma} \|\Theta_u - \mathcal{P}_0\Theta_u\|_{0,\infty} \\ &\mathbb{T}_5 \lesssim \big\|\boldsymbol{\chi_g}\big\|_{2+\gamma} \|\Psi_u - \Psi_{u,\mathrm{M}}\|_{\mathrm{NC}} \|\Theta_u - \Theta_{u,\mathrm{M}}\|_{\mathrm{NC}}. \end{aligned}$$

Lemma A.3 shows $\mathbb{T}_6 \lesssim \big(h^\gamma \|\Psi_u - \Psi_{u,\mathrm{M}}\|_{\mathrm{NC}} + \|\Psi_u - \Psi_{u,\mathrm{M}}\|\big) \big\|\boldsymbol{\chi_g}\big\|_{2+\gamma} \|\Theta_u\|_{2+\gamma}$. A substitution of $\mathbb{T}_1$–$\mathbb{T}_6$ in (A.31) and the resulting estimate with (A.29) and (A.30) in (A.28) lead to a bound for the terms involving the trilinear form $B_{\mathrm{NC}}(\bullet, \bullet, \bullet)$ as

$$\begin{aligned} &\big\|\boldsymbol{\chi_g}\big\|_{2+\gamma} \Big(\|\Theta_u\|_{2+\gamma} \big(h^\gamma \|\Psi_u - \Psi_{u,\mathrm{M}}\|_{\mathrm{NC}} + \|\Psi_u - \Psi_{u,\mathrm{M}}\|\big) + \|\Psi_u - \Psi_{u,\mathrm{M}}\|_{\mathrm{NC}} \|\Theta_u - \Theta_{u,\mathrm{M}}\|_{\mathrm{NC}} \\ &\quad + h^\gamma \|\Psi_u\|_{2+\gamma} \Big(\|\Theta_u - \Theta_{u,\mathrm{M}}\|_{\mathrm{NC}} + \|\Theta_u - \mathcal{P}_0\Theta_u\|_{0,\infty} \Big) \Big). \end{aligned}$$

This expression and (A.27) are first substituted in (A.26), the resulting expression and (A.25) are substituted in (A.24) and utilized in (A.23) with bounds for T_1 and T_2. In combination with $\big\|\boldsymbol{\chi_g}\big\|_{2+\gamma} \lesssim \|\mathbf{g}\|_{-1} \lesssim \|\nabla J\boldsymbol{\rho}_{\mathrm{M}}\|$ from (A.22), this yields

$$\begin{aligned} \|\nabla J\boldsymbol{\rho}_{\mathrm{M}}\| &\lesssim h^\gamma \|\Theta_u\|_{2+\gamma} \big\|\Psi_u - \Psi_{u,\mathrm{M}}\big\|_{\mathrm{NC}} + \big\|\Psi_u - \Psi_{u,\mathrm{M}}\big\|_{\mathrm{NC}} \big\|\Theta_u - \Theta_{u,\mathrm{M}}\big\|_{\mathrm{NC}} + \Big(1 + \|\Theta_u\|_{2+\gamma}\Big) \big\|\Psi_u - \Psi_{u,\mathrm{M}}\big\| \\ &\quad + h^\gamma \Big(1 + \|\Psi_u\|_{2+\gamma}\Big) \big\|\Theta_u - \Theta_{u,\mathrm{M}}\big\|_{\mathrm{NC}} + h^\gamma \|\Psi_u\|_{2+\gamma} \|\Theta_u - \mathcal{P}_0\Theta_u\|_{0,\infty} + h^{2+\gamma} \mathrm{osc}_0(\Psi_u - \Psi_d). \end{aligned}$$

The last displayed inequality, $\|\Theta_u\|_{2+\gamma}$, $\|\Psi_u\|_{2+\gamma} \lesssim 1$ and (A.20) lead to the desired estimate. ∎

Remark A.9. The projection estimate in $L^\infty(\mathcal{T})$ ([23], Prop. 1.135) and global Sobolev embedding $H^{2+\gamma}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ imply $\|(1-\mathcal{P}_0)\Theta_u\|_{0,\infty} \lesssim h\|\Theta_u\|_{2+\gamma}$. This, Theorems A.7 and A.8, and Remark A.6 show (a) $\|\Theta_u - \Theta_{u,\mathrm{M}}\|_{\mathrm{NC}} \lesssim h^\gamma$, (b) $\|\Theta_u - \Theta_{u,\mathrm{M}}\|_{1,2,h} \lesssim h^{2\gamma}$, and (c) $\|\Theta_u - \Theta_{u,\mathrm{M}}\| \lesssim h^{2\gamma}$.

The proof of the error estimates of the nonlinear control problem follows from the second-order sufficient optimality conditions in Theorem 3.9.

Theorem A.10 (*A priori* error estimates ([19], Thm. 5.1))**.** *Let* $(\bar\Psi, \bar u)$ *be a regular solution to* (2.1) *and* $\{(\bar\Psi_{\mathrm{M}}, \bar u_h)\}$ *be a solution to* (2.5) *converging to* $(\bar\Psi, \bar u)$ *in* $\mathbf{V} \times L^2(\omega)$*, for* $\mathcal{T} \in \mathbb{T}(\delta_3)$ *with* $\bar u_h \in B_{\rho_3}(\bar u)$ *as in Theorem A.7. Let* $\bar\Theta$ *and* $\bar\Theta_{\mathrm{M}}$ *be the corresponding continuous and discrete adjoint variables, respectively. Then, there exist* $0 < \delta_0 \leq \delta_3$, ϵ_0 *such that for all* $\mathcal{T} \in \mathbb{T}(\delta_0)$*, it holds* (a) $\|\bar\Psi - \bar\Psi_{\mathrm{M}}\|_{\mathrm{NC}} + \|\bar\Theta - \bar\Theta_{\mathrm{M}}\|_{\mathrm{NC}} + \|\bar u - \bar u_h\|_{L^2(\omega)} \leq \varepsilon_0$*; the solutions* $(\bar\Psi, \bar\Theta, \bar{\mathbf{u}})$ *and* $(\bar\Psi_{\mathrm{M}}, \bar\Theta_{\mathrm{M}}, \bar{\mathbf{u}}_h)$ *satisfy,* (b) $\|\bar\Psi - \bar\Psi_{\mathrm{M}}\|_{\mathrm{NC}} \lesssim h^\gamma$, $\|\bar\Theta - \bar\Theta_{\mathrm{M}}\|_{\mathrm{NC}} \lesssim h^\gamma$*, and* $\|\bar u - \bar u_h\|_{L^2(\omega)} \lesssim h$, $\gamma \in (1/2, 1]$.

The next lemma is a standard result in Banach spaces that helps to prove Lemma 4.2.

Lemma A.11. *Let* X *be a Banach space,* $A \in \mathcal{L}(X)$ *be invertible and* $B \in \mathcal{L}(X)$*. If* $\|A - B\|_{\mathcal{L}(X)} < 1/\|A^{-1}\|_{\mathcal{L}(X)}$*, then* B *is invertible. If* $\|A - B\|_{\mathcal{L}(X)} < 1/\left(2\|A^{-1}\|_{\mathcal{L}(X)}\right)$*, then* $\|B^{-1}\|_{\mathcal{L}(X)} \leq 2\|A^{-1}\|_{\mathcal{L}(X)}$.

The uniform boundedness result for the inverse of the linear mapping $\mathcal{F}_{\Psi_u}$ with a bound independent of the discretization parameter h *without assuming the extra regularity of* $\bar\Psi$ is proved next. This result was used to derive the *a posteriori* error estimates for the adjoint variable.

Proof of Lemma 4.2. Lemma 4.3 of [19] shows that $\mathcal{F}_{\bar\Psi}$ is an automorphism on $\mathbf{V} + \mathbf{V}_{\mathrm{M}}$ if $\bar\Psi \in \mathbf{V}$ is a regular solution to (2.1). Also, for $\boldsymbol{\xi} + \boldsymbol{\xi}_{\mathrm{M}} \in \mathbf{V} + \mathbf{V}_{\mathrm{M}}$, the invertibility of $\mathcal{F}_{\bar\Psi}$ leads to $\mathcal{F}_{\bar\Psi}^{-1}(\boldsymbol{\xi} + \boldsymbol{\xi}_{\mathrm{M}}) = \boldsymbol{\xi} + \boldsymbol{\xi}_{\mathrm{M}} - \left(\mathcal{A} + \mathcal{B}'(\bar\Psi)^*\right)^{-1} \mathcal{B}'_{\mathrm{NC}}(\bar\Psi)^*(\boldsymbol{\xi} + \boldsymbol{\xi}_{\mathrm{M}})$. This, and Lemma 3.7(b) imply $\|\mathcal{F}_{\bar\Psi}^{-1}\|_{\mathcal{L}(\mathbf{V}+\mathbf{V}_{\mathrm{M}})} \leq 1 + 2C_{\mathrm{b}}\left\|\left(\mathcal{A} + \mathcal{B}'(\bar\Psi)^*\right)^{-1}\right\|_{\mathcal{L}(\mathbf{V}', \mathbf{V})} \times \|\bar\Psi\|_2$. Since $\mathcal{A}^* = \mathcal{A}$ and the operator norm of an operator and its adjoint are equal, $\left\|\left(\mathcal{A} + \mathcal{B}'(\bar\Psi)^*\right)^{-1}\right\| = \left\|\left(\mathcal{A} + \mathcal{B}'(\bar\Psi)\right)^{-1}\right\|$ and hence, there exists a constant C independent of h such that $\frac{1}{\|\mathcal{F}_{\bar\Psi}^{-1}\|_{\mathcal{L}(\mathbf{V}+\mathbf{V}_{\mathrm{M}})}} \geq C$.

For $\Phi \in \mathbf{V} + \mathbf{V}_{\mathrm{M}}$, the definition of $\mathcal{F}_\Psi$ in (4.17), the boundedness property of $T(\bullet)$ and $B_{\mathrm{NC}}(\bullet, \bullet, \bullet)$ in Lemma 3.7(b) imply

$$\|\mathcal{F}_{\bar\Psi}(\Phi) - \mathcal{F}_{\Psi_u}(\Phi)\|_{\mathcal{L}(\mathbf{V}+\mathbf{V}_{\mathrm{M}})} \leq \left\|T\left[\mathcal{B}'_{\mathrm{NC}}(\bar\Psi - \Psi_u)^*(\Phi)\right]\right\| \lesssim \|\bar\Psi - \Psi_u\|_2. \tag{A.32}$$

Theorem 3.1 shows $G(\bar u) = \bar\Psi$, $G(u) = \Psi_u$ and the uniform boundedness of $\left\|(\mathcal{A} + \mathcal{B}'(\Psi_u))^{-1}\right\|_{\mathcal{L}(\mathbf{V}', \mathbf{V})}$ whenever $u \in \mathcal{O}(\bar u)$. Hence, for $u_t = u + t(\bar u - u)$ and $\Psi_t = G(u_t)$, mean value theorem, Theorem 3.1 and $u \in \mathcal{O}(\bar u)$ prove

$$\|\bar\Psi - \Psi_u\|_2 = \left\|\int_0^1 G'(u_t)(\mathbf{C}(\bar{\mathbf{u}} - \mathbf{u}))\,\mathrm{d}t\right\|_2 = \left\|\int_0^1 (\mathcal{A} + \mathcal{B}'(\Psi_t))^{-1}(\mathbf{C}(\bar{\mathbf{u}} - \mathbf{u}))\,\mathrm{d}t\right\|_2 \lesssim \|\bar u - u\|_{L^2(\omega)}.$$

Since u is sufficently close to $\bar u$, (A.32) leads to

$$\|\mathcal{F}_{\bar\Psi} - \mathcal{F}_{\Psi_u}\|_{\mathcal{L}(\mathbf{V}+\mathbf{V}_{\mathrm{M}})} \leq C \leq \frac{1}{\|\mathcal{F}_{\bar\Psi}^{-1}\|_{\mathcal{L}(\mathbf{V}+\mathbf{V}_{\mathrm{M}})}}.$$

An application of Lemma A.11 concludes the proof. □

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