ESAIM: M2AN 56 (2022) 969–1005
https://doi.org/10.1051/m2an/2022028

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

CONVERGENCE ANALYSIS OF A FULLY DISCRETE FINITE ELEMENT METHOD FOR THERMALLY COUPLED INCOMPRESSIBLE MHD PROBLEMS WITH TEMPERATURE-DEPENDENT COEFFICIENTS

Qianqian Ding[1], Xiaonian Long[2] and Shipeng Mao[3,*]

Abstract. In this paper, we study a fully discrete finite element scheme of thermally coupled incompressible magnetohydrodynamic with temperature-dependent coefficients in Lipschitz domain. The variable coefficients in the MHD system and possible nonconvex domain may cause nonsmooth solutions. We propose a fully discrete Euler semi-implicit scheme with the magnetic equation approximated by Nédélec edge elements to capture the physical solutions. The fully discrete scheme only needs to solve one linear system at each time step and is unconditionally stable. Utilizing the stability of the numerical scheme and the compactness method, the existence of weak solution to the thermally coupled MHD model in three dimensions is established. Furthermore, the uniqueness of weak solution and the convergence of the proposed numerical method are also rigorously derived. Under the hypothesis of a low regularity for the exact solution, we rigorously establish the error estimates for the velocity, temperature and magnetic induction unconditionally in the sense that the time step is independent of the spacial mesh size.

Mathematics Subject Classification. 65M60, 65M15, 76W05.

Received August 4, 2021. Accepted March 12, 2022.

1. Introduction

Magnetohydrodynamic (MHD) is the theory of macroscopic interaction of conductive fluid and electromagnetic induction. It consists of a viscous, incompressible fluid which has the property of electric current conduction and interacting with electromagnetic inductions. There are lots of applications in astronomy and geophysics as well as engineering problems, such as metallurgical engineering, electromagnetic pumping, stirring of liquid metals, liquid metal cooling of nuclear reactors, refer to [16, 21, 36, 42], and the references therein. However, in many cases, the effect of temperature can not be ignored. Especially, the change of temperature will cause the change of fluid coefficients [52, 54] as well as the magnetic field coefficients [10, 34].

Keywords and phrases. Magnetohydrodynamics, temperature-dependent coefficients, finite element method, well-posedness, convergence, error estimates.

[1] School of Mathematics, Shandong University, Jinan 250100, P.R. China.

[2] College of Mathematics and Information Science, Henan University of Economics and Law, Zhengzhou 450047, P.R. China.

[3] NCMIS, LSEC, Institute of Computational Mathematics and Scientific/Enginnering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, School of Mathematical Science, University of Chinese Academy of Sciences, Beijing 100190, P.R. China.

*Corresponding author: maosp@lsec.cc.ac.cn

In this work, we consider the following transient incompressible Navier–Stokes equations and Maxwell's equations coupled to the heat equation with temperature-dependent coefficients in $\mathbb{R}^3$ [3,10,16,42] as follows,

$$\begin{cases} \boldsymbol{u}_t - \operatorname{div}[\nu(\theta)\nabla \boldsymbol{u}] + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p + \mu \boldsymbol{B}\times \mathbf{curl}\,\boldsymbol{B} - \boldsymbol{\beta}(\theta)\theta = \boldsymbol{f} & \text{in } Q_T, \quad (1.1)\\ \boldsymbol{B}_t + \mathbf{curl}\,[\sigma(\theta)\mathbf{curl}\,\boldsymbol{B}] - \mathbf{curl}\,(\boldsymbol{u}\times\boldsymbol{B}) = \boldsymbol{g} & \text{in } Q_T, \quad (1.2)\\ \theta_t - \operatorname{div}[\kappa(\theta)\nabla\theta] + \boldsymbol{u}\cdot\nabla\theta = \psi & \text{in } Q_T, \quad (1.3)\\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } Q_T, \quad (1.4)\\ \operatorname{div}\boldsymbol{B} = 0 & \text{in } Q_T, \quad (1.5)\end{cases}$$

where $Q_T = \Omega\times(0,T)$, $T>0$ is a given finite final time, Ω is a bounded, simply-connected and Lipschitz polyhedral domain. $\boldsymbol{u}$ denotes the velocity field, p the pressure, $\boldsymbol{B}$ the magnetic induction, θ the temperature, ν the kinematic viscosity, σ the electric conductivity, μ the magnetic permeability, κ the thermal conductivity, ψ a given heat source, $\boldsymbol{\beta}$ the thermal expansion coefficient, $\boldsymbol{f}$ a forcing term for the magnetic induction, $\boldsymbol{g}$ the known applied current with $\operatorname{div}\boldsymbol{g} = 0$. The system is considered in conjunction with the following initial values and boundary conditions,

$$\boldsymbol{u}(x,0) = \boldsymbol{u}^0, \quad \boldsymbol{B}(x,0) = \boldsymbol{B}^0, \quad \theta(x,0) = \theta^0 \qquad \forall x\in\Omega, \tag{1.6}$$

$$\boldsymbol{u} = \boldsymbol{u}_D, \quad \theta = \theta_D, \quad \boldsymbol{B}\times\boldsymbol{n} = 0 \qquad \text{on } S_T, \tag{1.7}$$

where $S_T = \partial\Omega\times(0,T)$, $\boldsymbol{n}$ is the outer unit normal of $\partial\Omega$ and the initial magnetic induction $\boldsymbol{B}^0$ satisfies $\operatorname{div}\boldsymbol{B}^0 = 0$. Our results in this paper are also valid for another frequently used set of boundary conditions of the magnetic induction $\boldsymbol{B}$ for (1.1)–(1.5) given by

$$\boldsymbol{B}\cdot\boldsymbol{n} = 0, \quad \boldsymbol{n}\times\mathbf{curl}\,\boldsymbol{B} = 0 \qquad \text{on } S_T.$$

The velocity $\boldsymbol{u}$, pressure p, temperature θ and magnetic induction $\boldsymbol{B}$

$$(\boldsymbol{u}, p, \theta, \boldsymbol{B}) : Q_T \to \mathbb{R}^3\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^3$$

are unknown functions. The following functions

$$(\boldsymbol{f}, \psi, \boldsymbol{g}) : Q_T \to \mathbb{R}^3\times\mathbb{R}\times\mathbb{R}^3$$

are given functions. Those functions ν, $\boldsymbol{\beta}$, σ and κ

$$(\nu, \boldsymbol{\beta}, \sigma, \kappa) : Q_T\times\mathbb{R} \to \mathbb{R}^+\times\mathbb{R}^3\times\mathbb{R}^+\times\mathbb{R}^+$$

are continuously differentiable functions in (x,t,θ).

In the last several decades, various finite element methods for MHD problems regardless of heat effects have been extensively developed in the literature. Let us review the references and try to summarize them, which is unlikely to be complete and accurate, of course. We can mainly classify them into two formulations based on the chosen discrete finite element spaces. The first formulation was proposed by Gunzburger *et al.* [26] and they made use of standard Lagrange $\boldsymbol{H}^1$ finite element spaces to approximate both the hydrodynamic unknowns and the magnetic induction. It is therefore easy to implement and has been extensively employed in computational MHD, see *e.g.* [20,25,35,55] for stationary models and [4,17,27,58] for time-dependent models. However, it is known that the nodal finite element method discretizations of the magnetic operator cannot be correctly approximated when the magnetic induction components may have regularity below $\boldsymbol{H}^1(\Omega)$, *cf.* [14,30], which may be frequently encountered in non-convex polyhedral or with a non $C^{1,1}$ boundary. A possible way to overcome these difficulties was proposed in [48] by virtue of Nédélec finite elements for the magnetic induction $\boldsymbol{B}$, which leads to another natural formulation and is valid for non-smooth magnetic solution. This variational formulation seems to be attractive and has been employed in [19,24,45,50] and the references therein. We also mention the recent papers [31, 32], where different formulations were proposed to maintain the divergence free magnetic solutions in the numerical schemes.

When buoyancy effects cannot be neglected in the momentum equation due to temperature differences in the conductive flow, the incompressible MHD is usually coupled to the heat equation by the well-known Boussinesq approximation. For instance, as a pioneer work, Lagrange continuous finite element methods for the stationary heat coupled MHD equations with constant coefficients had been studied in [39, 40]. On the other hand, in many practical applications of MHD problems, the change of temperature will affect the coefficients in the fluid field as well as the electromagnetic filed. It is of great significance to study reliable finite element methods for the coupled MHD system with the coefficients depend on the temperature. The coupled fluid systems or electromagnetic models with temperature-dependent coefficients are faced with nonlinear PDEs with great mathematical challenges, which has attracted attention by both physicists and mathematicians, see *e.g.* [10, 33, 43, 54] and the references therein. As far as we know, the first work to study error estimates of finite element methods for the thermally coupled MHD equations with temperature-dependent coefficients is given in [47], where a fully discrete Crank–Nicolson scheme is proposed and investigated. More recently, the fully discrete Euler semi-implicit scheme has been studied in [46]. The proposed schemes in the above two papers are based on the magnetic induction approximated by Lagrange $\boldsymbol{H}^1$ finite element method and all the error estimates are conducted under sufficiently smooth assumption on the exact solutions. However, in view of the highly nonlinearity brought by the temperature-dependent coefficients and the Lorentz terms in the magnetic equation, as well as a possible non-convex domain or a non $C^{1,1}$ boundary, we can not expect to a smooth solution for the magnetic induction belong to $\boldsymbol{H}^1(\Omega)$ in such situations (see [12,13]). Thus $\boldsymbol{H}(\mathbf{curl})$-conforming Nédélec edge element is a natural choice to approximate the magnetic equation in order to capture the physical solutions. Furthermore, Nédélec finite elements seems to be a better choice since it can treat the boundary conditions of magnetic induction easier than Lagrange finite element discretization.

In this work, we will give a rigorous convergence analysis and error estimates of a fully discrete finite element method for the MHD system described by (1.1)–(1.7) based on the magnetic induction approximated by $\boldsymbol{H}(\mathbf{curl})$-conforming Nédélec edge element. The time discretization is based on a backward Euler semi-implicit scheme and the stable Taylor-Hood type finite elements are used to approximate the fluid field and Lagrange finite element to approximate the heat equation. In the first half of the paper, we show that the numerical solution converges to a weak solution of the continuous system without any further regularity assumption as both meshwidth and timestep tend to zero. In fact, the first convergence result of finite element discretization for MHD is attributed to [45], where some weak and strong convergence of the subsequence of discrete velocity and magnetic field are proved by the standard compact argument. In this paper, we will extend the results to MHD models with temperature-dependent coefficients. Strong convergence of all the discrete fields (velocity, magnetic induction and heat) is proved rigorously. So the result of this paper is also an improvement of [31], which has not proved any strong convergence of the discrete fields. Furthermore, we show the uniqueness of weak solution for incompressible MHD models with temperature-dependent coefficients provided it satisfies a smoother condition, which seems to be new in the literature. Then we can prove that the whole sequence of the discrete solution converges (strongly) to the unique weak solution. In the second half of the paper, under a weak regularity hypothesis on the exact solution, a rigorous error estimate of the velocity, temperature and magnetic induction are established unconditionally in the sense that the timestep is independent on the spacial meshwidth. In this paper, we will confront with two major difficulties in the analysis. The first difficulty arises from the nonlinearity in the model caused by variable temperature-dependent coefficients, and the other is the very low regularity of the exact solution caused by the nonlinearity structure or non-convex domain. In order to deal with these difficult problems, some technical tools need to be developed in this paper.

A brief overview of this paper is provided as follows. In the next section, we describe the notations and some preliminary knowledge to be used throughout the paper. In Section 3, we propose a fully finite element discrete method with a backward Euler semi-implicit scheme for the system (1.1)–(1.7), and some basic lemmas and theorems are recalled. In Section 4, the well-posedness and stability of the fully discrete scheme are presented. We show that the fully discrete solution converges to a weak solution of the continuous problem as Δt and h

tend to zero. The uniqueness of the continuous problem is also established under a slight smoother assumption on the weak solution. In Section 5, we prove error estimates for all variables under a weak regularity assumption. In Section 6, we present two numerical examples to illustrate our theoretical results. Finally, we close the paper with some concluding remarks in Section 7.

2. Functional setting for the magneto-heat coupling model

For mathematical setting of problem (1.1)–(1.5) with the initial values and boundary conditions (1.6)–(1.7), we first introduce some notations that will be used throughout the paper. For all $m \in \mathbb{N}^+$, $1 \le p \le \infty$, let $W^{m,p}(\Omega)$ denote the standard Sobolev space and it is written as $H^m(\Omega)$ when $p = 2$. The norm in $W^{m,p}(\Omega)$ is defined by $\|\cdot\|_{m,p}$ such that

$$\begin{aligned} \|v\|_{m,p} &= \left(\sum_{|\alpha|\le m} \|D^\alpha v\|_{0,p}^p\right)^{1/p} && \text{with } \|v\|_{0,p} = \left(\int_\Omega |v|^p \,\mathrm{d}x\right)^{1/p} \quad 1 \le p < \infty, \\ \|v\|_{m,\infty} &= \max_{|\alpha|\le m} \|D^\alpha v\|_{0,\infty} && \text{with } \|v\|_{0,\infty} = \operatorname*{ess\,sup}_{x\in\Omega} |v(x)|, \end{aligned}$$

where

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}},$$

for the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, with $\alpha_1, \alpha_2, \alpha_3 \ge 0$. For the function spaces $L^p(0, T; X)$, $1 \le p \le \infty$, the norms are denoted as

$$\begin{aligned} \|v\|_{L^p(0,T;X)} &= \left(\int_0^T \|v(t)\|_X^p \,\mathrm{d}t\right)^{1/p} \qquad \text{for } 1 \le p < +\infty, \\ \|v\|_{L^\infty(0,T;X)} &= \operatorname*{ess\,sup}_{0\le t\le T} \|v(t)\|_X, \end{aligned}$$

where X is a real Banach space with the norm $\|\cdot\|_X$. The inner product will be denoted by $(\cdot,\cdot)$, that is $(\phi, \psi) = \int_\Omega \phi\psi \,\mathrm{d}x$, the norm in $L^2(\Omega)$ defined by $\|\cdot\|_0$. $\langle\cdot,\cdot\rangle$ is for the dual product between a Banach space and the dual space. Vector-valued quantities will be denoted in boldface notations, such as $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{L}^2(\Omega) = \left(L^2(\Omega)\right)^3$. We use C and c, with or without subscripts, bars, tildes or hats, to denote generic positive constants independent of the discretization parameters, which may take different values at different places.

We introduce the following classical Sobolev spaces:

$$\begin{aligned} &\boldsymbol{X} = \boldsymbol{H}^1(\Omega), \quad \boldsymbol{X}_0 = \left\{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \operatorname{div} \boldsymbol{v} = 0\right\}, \quad Y = H^1(\Omega), \quad Y_0 = H_0^1(\Omega), \\ &M = L^2(\Omega), \quad Q = \left\{q \in M, \int_\Omega q(x)\,\mathrm{d}x = 0\right\}, \\ &\boldsymbol{W} = \left\{\boldsymbol{C} \in \boldsymbol{L}^2(\Omega), \mathbf{curl}\,\boldsymbol{C} \in \boldsymbol{L}^2(\Omega)\right\}, \quad \boldsymbol{W}_0 = \{\boldsymbol{C} \in \boldsymbol{W},\ \boldsymbol{C} \times \boldsymbol{n}|_{\partial\Omega} = 0\}. \end{aligned}$$

Here and what in follows, we define the following norm

$$\|\boldsymbol{C}\|_{\boldsymbol{W}} = \|\boldsymbol{C}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} = \left(\|\boldsymbol{C}\|_0^2 + \|\mathbf{curl}\,\boldsymbol{C}\|_0^2\right)^{1/2} \qquad \forall \boldsymbol{C} \in \boldsymbol{W}.$$

We also need to define the following Sobolev spaces

$$\boldsymbol{H}(\mathrm{div};\Omega) = \left\{\boldsymbol{b} \in \boldsymbol{L}^2(\Omega), \operatorname{div} \boldsymbol{b} \in L^2(\Omega)\right\}, \quad \boldsymbol{H}\left(\mathrm{div}^0;\Omega\right) = \{\boldsymbol{b} \in \boldsymbol{H}(\mathrm{div};\Omega), \operatorname{div} \boldsymbol{b} = 0\}$$

and $\boldsymbol{\mathcal{H}}(\Omega) = \boldsymbol{W}_0 \cap \boldsymbol{H}(\mathrm{div};\Omega)$, which is equipped with the following norm

$$\|\boldsymbol{v}\|_{\boldsymbol{\mathcal{H}}(\Omega)} = \Big(\|\mathbf{curl}\,\boldsymbol{v}\|_0^2 + \|\mathrm{div}\,\boldsymbol{v}\|_0^2\Big)^{1/2} \qquad \forall\,\boldsymbol{v} \in \boldsymbol{\mathcal{H}}(\Omega).$$

We recall the following embedding result (see *e.g.*, Prop. 3.7 of [2] or [23]) which is valid for a Lipschitz polyhedron.

Lemma 2.1. *There exists a parameter $\delta_1 = \delta_1(\Omega) > 0$ such that the embedding $\boldsymbol{\mathcal{H}}(\Omega) \hookrightarrow \boldsymbol{L}^{3+\delta_1}(\Omega)$ is compact.*

It is well known that the following Poincaré type and embedding inequalities are valid in bounded polyhedral domains (see Chapter 3 of [41] for more details),

$$\|\boldsymbol{v}\|_{0,m} \le c_1\|\nabla \boldsymbol{v}\|_0, \qquad m \in [1,6] \qquad \forall\,\boldsymbol{v} \in \boldsymbol{X}_0, \tag{2.1}$$

$$\|\boldsymbol{v}\|_0 \le c_2\|\mathbf{curl}\,\boldsymbol{v}\|_0 \qquad \forall\,\boldsymbol{v} \in \boldsymbol{W}_0 \cap \boldsymbol{H}\big(\mathrm{div}^0;\Omega\big), \tag{2.2}$$

$$\|\boldsymbol{v}\|_{0,\infty} \le c_3\|\boldsymbol{v}\|_{1+l,2} \qquad \forall\, l > 1/2, \tag{2.3}$$

$$\|\boldsymbol{v}\|_{1,3} \le c_4\|\boldsymbol{v}\|_{1+l,2}, \quad \|\mathbf{curl}\,\boldsymbol{v}\|_{0,3} \le c_5\|\mathbf{curl}\,\boldsymbol{v}\|_{l,2} \qquad \forall\, l > 1/2. \tag{2.4}$$

For every $\omega \in H^{\frac{1}{2}}(\partial\Omega)$, let $Y_0(\omega)$ be an affine space of Y defined by

$$Y_0(\omega) \equiv \{\xi \in Y; \xi - \theta_\omega \in Y_0\},$$

where $\theta_\omega \in Y$ is an extension of ω. Similarly, let $\boldsymbol{V} = \boldsymbol{X}_0$, for every $\boldsymbol{w} \in \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$, let $\boldsymbol{V}(\boldsymbol{w})$ be an affine space of $\boldsymbol{X}$ defined by

$$\boldsymbol{V}(\boldsymbol{w}) \equiv \{\boldsymbol{v} \in \boldsymbol{X}; \boldsymbol{v} - \boldsymbol{u}_{\boldsymbol{w}} \in \boldsymbol{V}\},$$

where $\boldsymbol{u}_{\boldsymbol{w}} \in \boldsymbol{X}$ is an extension of $\boldsymbol{w}$.

We denote by $\mathscr{C}^m(\Omega)$ the space of functions m times continuously differentiable in Ω. The space $\mathscr{C}^m\big(\overline{\Omega}\big)$ consists of functions in $\mathscr{C}^m(\Omega)$ bounded, and uniformly continuous in Ω with derivatives up to the m th order. The space $\mathscr{C}^{m,1}\big(\overline{\Omega}\big)$ consists of functions in $\mathscr{C}^m\big(\overline{\Omega}\big)$ that are Lipschitz continuous in $\overline{\Omega}$ with derivatives up to the mth order, refer to [23].

In order to describe our scheme concisely, for all $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \in (\boldsymbol{X}\times\boldsymbol{X}\times\boldsymbol{X})$, $(\theta,\varphi) \in (Y \times Y)$, we denote the trilinear terms as

$$\mathscr{O}_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \frac{1}{2}\bigg\{\int_\Omega [(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}]\,\boldsymbol{w}\,\mathrm{d}x - \int_\Omega [(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}]\,\boldsymbol{v}\,\mathrm{d}x\bigg\},$$

$$\mathscr{O}_2(\boldsymbol{u},\theta,\varphi) = \frac{1}{2}\bigg\{\int_\Omega (\boldsymbol{u}\cdot\nabla\theta)\varphi\,\mathrm{d}x - \int_\Omega (\boldsymbol{u}\cdot\nabla\varphi)\theta\,\mathrm{d}x\bigg\}.$$

Now, we introduce the definition of weak solution to the heat coupled MHD system (1.1)–(1.7).

Definition 2.2. Suppose that

$$\boldsymbol{f} \in L^2\big(0,T;\boldsymbol{L}^2(\Omega)\big), \quad \psi \in L^2\big(0,T;L^2(\Omega)\big), \quad \boldsymbol{g} \in L^2\big(0,T;\boldsymbol{L}^2(\Omega)\big), \quad \boldsymbol{B}^0 \in \boldsymbol{W},$$
$$\boldsymbol{u}_D \in H^1\Big(0,T;\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)\Big), \quad \boldsymbol{u}^0 \in \boldsymbol{V}(\boldsymbol{u}_D(\cdot,0)), \quad \theta_D \in H^1\Big(0,T;H^{\frac{1}{2}}(\partial\Omega)\Big), \quad \theta^0 \in Y_0(\theta_D(\cdot,0)).$$

We say that $(\boldsymbol{u},p,\theta,\boldsymbol{B})$ is the weak solution of (1.1)–(1.7), if there holds (i)

$$\boldsymbol{u} \in L^\infty\big(0,T;\boldsymbol{L}^2(\Omega)\big) \cap L^2(0,T;\boldsymbol{V}(\boldsymbol{u}_D)), \quad \theta \in L^\infty\big(0,T;L^2(\Omega)\big) \cap L^2(0,T;Y_0(\theta_D)),$$
$$\boldsymbol{B} \in L^\infty\big(0,T;\boldsymbol{L}^2(\Omega)\big) \cap L^2(0,T;\boldsymbol{W}_0), \quad p \in L^\infty(0,T;Q).$$

(ii) For any $(\boldsymbol{v}, q, \varphi, \boldsymbol{C}) \in (\boldsymbol{X}_0 \times Q \times Y_0 \times \boldsymbol{W}_0)$, the weak formulation holds

$$\begin{cases} \langle \boldsymbol{u}_t, \boldsymbol{v}\rangle + \mathcal{A}_1(\nu(\theta), \boldsymbol{u}, \boldsymbol{v}) + \mathcal{O}_1(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + \mu(\boldsymbol{B} \times \mathbf{curl}\,\boldsymbol{B}, \boldsymbol{v}) + b(\boldsymbol{v}, p) - b(\boldsymbol{u}, q) - (\boldsymbol{\beta}(\theta)\theta, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), & (2.5)\\ \langle \boldsymbol{B}_t, \boldsymbol{C}\rangle + (\sigma(\theta)\mathbf{curl}\,\boldsymbol{B}, \mathbf{curl}\,\boldsymbol{C}) - (\boldsymbol{u} \times \boldsymbol{B}, \mathbf{curl}\,\boldsymbol{C}) = (\boldsymbol{g}, \boldsymbol{C}), & (2.6)\\ \langle \theta_t, \varphi\rangle + \mathcal{A}_2(\kappa(\theta), \theta, \varphi) + \mathcal{O}_2(\boldsymbol{u}, \theta, \varphi) = (\psi, \varphi), & (2.7)\end{cases}$$

where

$$\begin{aligned} \mathcal{A}_1(\nu(\theta), \boldsymbol{u}, \boldsymbol{v}) &= \int_\Omega \nu(\theta)\nabla \boldsymbol{u} : \nabla \boldsymbol{v}\,\mathrm{d}x, \qquad b(\boldsymbol{v}, q) = -\int_\Omega q\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}x,\\ \mathcal{A}_2(\kappa(\theta), \theta, \varphi) &= \int_\Omega \kappa(\theta)\nabla\theta \cdot \nabla\varphi\,\mathrm{d}x. \end{aligned}$$

(iii) For all $t \in [0, T]$, there holds

$$\begin{aligned} &\frac{1}{2}\left[\|\boldsymbol{u}(t,\cdot)\|_0^2 + \mu\|\boldsymbol{B}(t,\cdot)\|_0^2 + 0\|\theta(t,\cdot)\|_0^2\right] + \int_0^T \left[\left\|\sqrt{\nu(\theta)}\nabla\boldsymbol{u}\right\|_0^2 + \mu\left\|\sqrt{\sigma(\theta)}\mathbf{curl}\,\boldsymbol{B}\right\|_0^2 + \left\|\sqrt{\kappa(\theta)}\nabla\theta\right\|_0^2\right]\mathrm{d}t\\ &= \frac{1}{2}\left[\|\boldsymbol{u}(0)\|_0^2 + \mu\|\boldsymbol{B}(0)\|_0^2 + \|\theta(0)\|_0^2\right] + \int_0^T [(\boldsymbol{f}, \boldsymbol{u}) + \mu(\boldsymbol{g}, \boldsymbol{B}) + (\psi, \theta) + (\boldsymbol{\beta}(\theta)\theta, \boldsymbol{u})]\mathrm{d}t. \end{aligned}$$

Remark 2.3. Since $\nabla\phi \in \boldsymbol{W}_0$ for all $\phi \in H_0^1(\Omega)$, by choosing $\boldsymbol{C} = \nabla\phi$ in (2.6), we can deduce $(\boldsymbol{B}_t, \nabla\phi) = 0$. Due to the orthogonal decomposition $\boldsymbol{L}^2(\Omega) = \boldsymbol{H}(\mathrm{div}^0; \Omega) \oplus \nabla H_0^1(\Omega)$, see [18, 41], it can be checked that $\mathrm{div}\,\boldsymbol{B}_t = 0$, together with $\mathrm{div}\,\boldsymbol{B}^0 = 0$, then we are able to obtain that $\mathrm{div}\,\boldsymbol{B}(t) = 0$, for all $t \in (0, T]$.

Remark 2.4. Under the external forces $(\boldsymbol{f}, \boldsymbol{u})$, $(\boldsymbol{g}, \boldsymbol{B})$ and (ψ, θ), the total energy includes the fluid kinetic energy $\frac{1}{2}\|\boldsymbol{u}(t)\|_0^2$, the magnetic energy $\frac{1}{2}\mu\|\boldsymbol{B}(t)\|_0^2$ and thermal energy $\frac{1}{2}\|\theta(t)\|_0^2$, while the dissipation of energy contains the friction losses $\left\|\sqrt{\nu(\theta)}\nabla\boldsymbol{u}\right\|_0^2$, the Ohmic losses $\mu\left\|\sqrt{\sigma(\theta)}\mathbf{curl}\,\boldsymbol{B}\right\|_0^2$ and heat losses $\left\|\sqrt{\kappa(\theta)}\nabla\theta\right\|_0^2$.

3. A fully discrete finite element method based on Euler scheme

In this section, we introduce a mixed finite element method which describes a spatial discretization of the problem (2.5)–(2.7) based on the backward Euler scheme.

Let Ω be a polyhedral domain, and the domain is partitioned into a mesh $\mathcal{T}_h$, where h is the diameter of the element. Each tetrahedron K is supposed to be the image of a reference tetrahedron $\hat{K}$ under an affine map F_K. The family of meshes $\{\mathcal{T}_h\}_{h>0}$ is assumed to be regular and quasi-uniform. Let $P_k(K)$ be the space of polynomials of total degree at most $k \geq 0$ on K and $\widetilde{P}_k(K)$ the space of homogeneous polynomials k on K. We first introduce the generalized Taylor-Hood element $\left(\boldsymbol{X}_h^k, Q_h^{k-1}\right)$ with $k \geq 2$, where $\boldsymbol{X}_h^k$ is the k order vectorial Lagrange finite element subspace of $\boldsymbol{X}$, Q_h^{k-1} is the $k-1$ order scalar Lagrange finite element subspace of Q. And Y_h^k is the k order scalar Lagrange finite element subspace of Y, refer to [6, 23]. For the case $k = 1$, we use the well-known stable mini-elements to approximate velocity and pressure, *cf.* [6, 8, 23].

The space $\mathcal{D}_k(K)$ denotes the polynomials $\boldsymbol{q}$ in $\widetilde{\boldsymbol{P}}_k(K)$ that satisfy $\boldsymbol{q}(x) \cdot \boldsymbol{x} = 0$ on K. For $1 \leq k$, we define the space

$$\mathcal{N}_k(K) = \boldsymbol{P}_{k-1}(K) \oplus \mathcal{D}_k(K).$$

To approximate the magnetic induction, we use Nédélec $\boldsymbol{H}(\mathbf{curl})$-conforming finite element space (see [41, 44]), which is defined by

$$\boldsymbol{W}_h^k = \{\boldsymbol{C} \in \boldsymbol{W}_0,\, \boldsymbol{C}|_K \in \mathcal{N}_k(K) \quad \forall\, K \in \mathcal{T}_h\}.$$

Note that the above definition is the first family $\boldsymbol{H}(\mathbf{curl})$-conforming discrete space. We can also employ the second family finite element spaces with $\mathcal{N}_k(K)$ is chosen by $\boldsymbol{P}_k(K)$.

Setting $S_h = \{C \in H^1(\Omega) \cap L_0^2(\Omega),\, C \in P_k(K),\, \forall\, K \in \mathcal{T}_h\}$, we introduce the discretely solenoidal function space

$$\boldsymbol{W}_{0h}^k = \{\boldsymbol{C} \in \boldsymbol{W}_h^k, (\boldsymbol{C}, \nabla S) = 0 \quad \forall\, S \in S_h\}.$$

Furthermore, the space $\boldsymbol{W}_{0h}^k$ is known to satisfy the following discrete Poincaré-Friedrichs inequality (see [48] or Thm. 4.7 of [30]),

$$\|\boldsymbol{c}_h\|_0 \le C_* \|\mathbf{curl}\,\boldsymbol{c}_h\|_0 \qquad \forall\, \boldsymbol{c}_h \in \boldsymbol{W}_{0h}^k, \tag{3.1}$$

with a constant $C_* > 0$ independent of the mesh-size h.

The link between the spaces $\boldsymbol{W}_{0h}^k$ and $\boldsymbol{\mathcal{W}}(\Omega)$ is accomplished by the Hodge mapping $Z\colon \boldsymbol{H}(\mathbf{curl};\Omega) \to \boldsymbol{\mathcal{W}}(\Omega)$, refer to [30], where $\boldsymbol{\mathcal{W}}(\Omega) = \{\boldsymbol{C} \in \boldsymbol{\mathcal{H}}(\Omega),\ \operatorname{div} \boldsymbol{C} = 0 \ \text{ in } \Omega\}$ such that

$$\mathbf{curl}\, Z(\boldsymbol{C}) = \mathbf{curl}\,\boldsymbol{C} \qquad \forall\, \boldsymbol{C} \in \boldsymbol{W}.$$

Furthermore, the Hodge mapping satisfies the following approximation property, there exists $l = l(\Omega) > 0$,

$$\|\boldsymbol{C}_h - Z(\boldsymbol{C}_h)\|_0 \le c h^{\frac{1}{2}+l} \|\mathbf{curl}\,\boldsymbol{C}_h\|_0 \quad \forall\, \boldsymbol{C}_h \in \boldsymbol{W}_{0h}^k. \tag{3.2}$$

We use Q_h^{k-1} to approximate the pressure p, and use a finite element affine space

$$Y_{0h}^k(\omega) \equiv \{\xi_h \in Y_h^k;\, \xi_h - \pi_h \theta_\omega \in Y_{0h}^k\}$$

to find the temperature θ, where $\pi_h : \mathscr{L}(Y, Y_h^k)$ is the usual Lagrange interpolation operator and $Y_{0h}^k \subset Y_0$. Let $\boldsymbol{V}_h^k = \boldsymbol{X}_h^k \bigcap \boldsymbol{H}_0^1(\Omega)$, and we use the space

$$\boldsymbol{V}_h^k(\boldsymbol{w}) \equiv \{\boldsymbol{v}_h \in \boldsymbol{X}_h^k;\, \boldsymbol{v}_h - \Pi_h \boldsymbol{u}_{\boldsymbol{w}} \in \boldsymbol{V}_h^k\},$$

where $\Pi_h = \pi_h^3$, to find the velocity $\boldsymbol{u}$. Further, the discrete kernel space of the divergence operator can be defined by

$$\boldsymbol{X}_{0h}^k = \{\boldsymbol{v}_h \in \boldsymbol{X}_h^k;\, b(\boldsymbol{v}_h, q_h) = 0 \quad \forall q_h \in Q_h^{k-1}\}.$$

We recall the following inverse estimate from Theorem 3.2.6 of [11]. On a quasi-uniform mesh there holds

$$\|\boldsymbol{v}_h\|_{m,q} \le C_{inv} h^{\imath - m + 3(1/q - 1/p)} \|\boldsymbol{v}_h\|_{\imath,p} \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{X}_h^k, \tag{3.3}$$

where $C_{inv} > 0$ is a generic constant independent of the mesh-size h, $\imath$ and m are two real numbers with $0 \le \imath \le m \le 1$, p and q are two integers with $1 \le p \le q \le \infty$.

From the Fortin criterion, the following discrete inf-sup condition (see Chap. 2 of [8] or [30]) is established,

$$\inf_{0 \ne q_h \in Q_h^{k-1}} \sup_{\boldsymbol{0} \ne \boldsymbol{v}_h \in \boldsymbol{V}_h^k} \frac{(q_h, \operatorname{div} \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{1,2} \|q_h\|_0} \ge \beta^*, \tag{3.4}$$

where β^* is a generic positive constant depending on the domain Ω.

Let N be a positive integer and $0 = t_0 < t_1 < \ldots < t_N = T$ be a uniform partition of $[0, T]$ with time step $\Delta t = T/N$ and $t_n = n\Delta t, 0 \le n \le N$. For any function $\boldsymbol{w}(t)$, we write $\boldsymbol{w}^n$ as the value of $\boldsymbol{w}$ at $t = n\Delta t$. The backward difference form is $d_t \boldsymbol{w}^n = (\boldsymbol{w}^n - \boldsymbol{w}^{n-1})/\Delta t$ for any sequence $\{\boldsymbol{w}^n\}$. We set $\nu^n(\theta) = \nu(x, t_n, \theta)$, and $\sigma^n(\theta)$, $\kappa^n(\theta)$ and $\boldsymbol{\beta}^n(\theta)$ are defined similarly.

Starting with the initial datum $\theta_h^0 \in Y_{0h}^k(\theta_D)$, $\boldsymbol{B}_h^0 \in \boldsymbol{W}_h^k$ and $\boldsymbol{u}_h^0 \in \boldsymbol{V}_h^k(\boldsymbol{u}_D)$, with $k \geq 1$, then our aim is to find $\left\{(\boldsymbol{u}_h^n, p_h^n, \theta_h^n, \boldsymbol{B}_h^n) \in \boldsymbol{V}_h^k(\boldsymbol{u}_D) \times Q_h^{k-1} \times Y_{0h}^k(\theta_D) \times \boldsymbol{W}_h^k; n = 1, \ldots, N\right\}$, for any $\left\{(\boldsymbol{v}_h, q_h, \varphi_h, \boldsymbol{C}_h) \in \boldsymbol{V}_h^k \times Q_h^{k-1} \times Y_{0h}^k \times \boldsymbol{W}_h^k\right\}$ such that

$$\begin{cases} (d_t \boldsymbol{u}_h^n, \boldsymbol{v}_h) + \mathcal{A}_1\big(\nu^n\big(\theta_h^{n-1}\big), \boldsymbol{u}_h^n, \boldsymbol{v}_h\big) + \mathcal{O}_1\big(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{v}_h\big) - \big(\boldsymbol{\beta}^n\big(\theta_h^{n-1}\big)\theta_h^n, \boldsymbol{v}_h\big) \\ \quad +\mu\big(\boldsymbol{B}_h^{n-1} \times \mathbf{curl}\, \boldsymbol{B}_h^n, \boldsymbol{v}_h\big) + b(\boldsymbol{v}_h, p_h^n) = (\boldsymbol{f}^n, \boldsymbol{v}_h), & (3.5) \\ b(\boldsymbol{u}_h^n, q_h) = 0, & (3.6) \\ (d_t \boldsymbol{B}_h^n, \boldsymbol{C}_h) + \big(\sigma^n\big(\theta_h^{n-1}\big)\mathbf{curl}\, \boldsymbol{B}_h^n, \mathbf{curl}\, \boldsymbol{C}_h\big) - \big(\boldsymbol{u}_h^n \times \boldsymbol{B}_h^{n-1}, \mathbf{curl}\, \boldsymbol{C}_h\big) = (\boldsymbol{g}^n, \boldsymbol{C}_h), & (3.7) \\ (d_t \theta_h^n, \varphi_h) + \mathcal{A}_2\big(\kappa^n\big(\theta_h^{n-1}\big), \theta_h^n, \varphi_h\big) + \mathcal{O}_2\big(\boldsymbol{u}_h^{n-1}, \theta_h^n, \varphi_h\big) = (\psi^n, \varphi_h), & (3.8) \end{cases}$$

with the initial values satisfy $\boldsymbol{u}_h^0 = P_{0h}\boldsymbol{u}^0$, $\boldsymbol{B}_h^0 = P_{1h}\boldsymbol{B}^0$ and $\theta_h^0 = P_{2h}\theta^0$, where $P_{0h}\boldsymbol{u}^0 \in \boldsymbol{V}_h^k(\boldsymbol{u}_D)$, $P_{1h}\boldsymbol{B}^0 \in \boldsymbol{W}_h^k$ and $P_{2h}\theta^0 \in Y_{0h}^k(\theta_D)$ are corresponding L^2 projection or interpolation functions which satisfy the following estimates [1, 11, 28]:

$$\begin{aligned} &\big\|\boldsymbol{u}^0 - P_{0h}\boldsymbol{u}^0\big\|_0 \leq Ch^{\ell+1}\big\|\boldsymbol{u}^0\big\|_{\ell+1,2}, \quad \big\|\boldsymbol{B}^0 - P_{1h}\boldsymbol{B}^0\big\|_0 \leq Ch^{\ell}\big\|\boldsymbol{B}^0\big\|_{\ell,2}, \\ &\big\|\theta^0 - P_{2h}\theta^0\big\|_0 \leq Ch^{\ell+1}\big\|\theta^0\big\|_{\ell+1,2}, \end{aligned} \tag{3.9}$$

with $\ell = \min\{k, s\}$, where $k \geq 1$ is the order index of the finite element spaces, $s > 1/2$ is the index of regularity of the initial values.

Remark 3.1. The coefficients depend on the temperature will increase the nonlinearity of the model and make the problem more intricate. The existence of solution to problem (3.5)–(3.8) will be proved in next section. Furthermore, we will prove that the discrete solution converges to a weak solution of the continuous model as Δt and h tend to zero.

Remark 3.2. When Ω is a non-convex polyhedron, the difficulty comes from the fact that the magnetic induction is in general not in $\boldsymbol{H}^1(\Omega)$ (see Rem. 3.3.1 of [21] for an explanation). The approximation of the magnetic induction with classical $\boldsymbol{H}^1$-conforming Lagrange finite elements can not capture the singularities and may converge to a wrong solution, refer to [14, 15, 21], etc. This is the reason why we choose Nédélec edge element to approximate the magnetic induction in (3.7).

Remark 3.3. In fact, the discrete solution for the magnetic induction still satisfies the weakly divergence free property. For any $s_h \in S_h$, by choosing $\boldsymbol{C}_h = \nabla s_h$ in (3.7), we can obtain $(d_t \boldsymbol{B}_h^n, \nabla s_h) = 0$, namely, $\big(\big(\boldsymbol{B}_h^n - \boldsymbol{B}_h^{n-1}\big)/\Delta t, \nabla s_h\big) = 0$. Due to $\big(\boldsymbol{B}_h^0, \nabla s_h\big) = 0$, it implies that $(\boldsymbol{B}_h^n, \nabla s_h) = 0$, where $n = 0, 1, \ldots, N$. Thus there is no need to add a Lagrange multiplier in the magnetic equation as in [45].

The following result gives a maximum principle for strong solution of (1.1)–(1.5). The detailed proof can follow the same line as Lemma 3.5 of [38] or Lemma 3.1 of [37].

Lemma 3.4. *Let $(\boldsymbol{u}, \boldsymbol{B}, \theta)$ be a strong solution of (1.1)–(1.5) for $t \in (0, T]$. Then*

$$\theta(x,t) \leq e^{t/2}\|\psi\|_{L^2(0,t;L^2(\Omega))} + \max\left\{\sup_{\Omega} \theta^0, \sup_{(0,T]\times\partial\Omega} \theta_D\right\}, \quad \forall (x,t) \in Q_T. \tag{3.10}$$

By the maximum principle (3.10), we know that temperature θ of (1.1)–(1.5) is uniformly bounded. Therefore, we can assume the given functions $\nu, \kappa, \sigma \in \mathscr{C}^{0,1}\big(\overline{\Omega} \times [0,T] \times \mathbb{R}; \mathbb{R}^+\big)$ and $\boldsymbol{\beta} \in \mathscr{C}^{0,1}\big(\overline{\Omega} \times [0,T] \times \mathbb{R}; \mathbb{R}^3\big)$ so as to satisfy

$$\begin{aligned}
&0<\sigma_0\le\sigma(x,t,\varepsilon)\le\sigma_1, \qquad 0<\kappa_0\le\kappa(x,t,\varepsilon)\le\kappa_1\\
&|\boldsymbol{\beta}(x,t,\varepsilon)|\le\boldsymbol{\beta}_1, \qquad\qquad 0<\nu_0\le\nu(x,t,\varepsilon)\le\nu_1 \qquad \text{for all}\, \varepsilon\in\mathbb{R},
\end{aligned}\tag{3.11}$$

with positive constants $\sigma_0,\sigma_1,\kappa_0,\kappa_1,\nu_0,\nu_1$ and $\boldsymbol{\beta}_1$.

We now show that the solution of (3.5)–(3.8) enjoys a stability property, which holds regardless of the sizes of h and Δt, and will play an important role in proving the convergence and well-posedness of the fully discrete solution.

Lemma 3.5. *Under the condition (3.11), for all $1\le m\le N$, the numerical solution $(\boldsymbol{u}_h^n,p_h^n,\theta_h^n,\boldsymbol{B}_h^n)$ of (3.5)–(3.8) satisfies the following stability estimate,*

$$\begin{aligned}
&\|\boldsymbol{u}_h^m\|_0^2+\mu\|\boldsymbol{B}_h^m\|_0^2+\|\theta_h^m\|_0^2+\sum_{n=1}^m\Big(\big\|\theta_h^n-\theta_h^{n-1}\big\|_0^2+\big\|\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}\big\|_0^2+\mu\big\|\boldsymbol{B}_h^n-\boldsymbol{B}_h^{n-1}\big\|_0^2\Big)\\
&\quad+\Delta t\sum_{n=1}^m\left(\left\|\sqrt{\nu^n\big(\theta_h^{n-1}\big)}\nabla\boldsymbol{u}_h^n\right\|_0^2+\mu\left\|\sqrt{\sigma^n\big(\theta_h^{n-1}\big)}\,\mathbf{curl}\,\boldsymbol{B}_h^n\right\|_0^2+\left\|\sqrt{\kappa^n\big(\theta_h^{n-1}\big)}\nabla\theta_h^n\right\|_0^2\right)\le C,
\end{aligned}$$

where C is a generic constant depending on $\boldsymbol{f}$, $\boldsymbol{g}$, ψ, μ, $\boldsymbol{u}^0$, $\boldsymbol{B}^0$, $\theta^0,\sigma,\nu,\boldsymbol{\beta}$, κ.

Proof. Choosing $\varphi_h=2\theta_h^n$ in (3.8) and using $2(a-b,a)=a^2-b^2+(a-b)^2$, there holds

$$\|\theta_h^n\|_0^2-\big\|\theta_h^{n-1}\big\|_0^2+\big\|\theta_h^n-\theta_h^{n-1}\big\|_0^2+2\Delta t\left\|\sqrt{\kappa^n\big(\theta_h^{n-1}\big)}\nabla\theta_h^n\right\|_0^2=2\Delta t(\psi^n,\theta_h^n).$$

By virtue of the Young inequality and (2.1), summing up from $n=1$ to m, we deduce

$$\begin{aligned}
&\|\theta_h^m\|_0^2+\sum_{n=1}^m\big\|\theta_h^n-\theta_h^{n-1}\big\|_0^2+\Delta t\sum_{n=1}^m\left\|\sqrt{\kappa^n\big(\theta_h^{n-1}\big)}\nabla\theta_h^n\right\|_0^2\\
&\qquad\le\big\|\theta_h^0\big\|_0^2+\Delta t\kappa_0^{-1}c_1^2\sum_{n=1}^m\|\psi^n\|_0^2\le C_{00}.
\end{aligned}\tag{3.12}$$

Taking $\boldsymbol{v}_h=2\boldsymbol{u}_h^n$ in (3.5), $q_h=-2p_h^n$ in (3.6), $\boldsymbol{C}_h=2\mu\boldsymbol{B}_h^n\in\boldsymbol{W}_{0h}^k$ in (3.7) and adding the three equations, there holds

$$\begin{aligned}
&\|\boldsymbol{u}_h^n\|_0^2-\big\|\boldsymbol{u}_h^{n-1}\big\|_0^2+\big\|\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}\big\|_0^2+\mu\|\boldsymbol{B}_h^n\|_0^2-\mu\big\|\boldsymbol{B}_h^{n-1}\big\|_0^2\\
&\quad+\mu\big\|\boldsymbol{B}_h^n-\boldsymbol{B}_h^{n-1}\big\|_0^2+2\Delta t\left\|\sqrt{\nu^n\big(\theta_h^{n-1}\big)}\nabla\boldsymbol{u}_h^n\right\|_0^2+2\Delta t\mu\left\|\sqrt{\sigma^n\big(\theta_h^{n-1}\big)}\,\mathbf{curl}\,\boldsymbol{B}_h^n\right\|_0^2\\
&\quad=2\Delta t\big(\boldsymbol{\beta}^n\big(\theta_h^{n-1}\big)\theta_h^n,\boldsymbol{u}_h^n\big)+2\Delta t(\boldsymbol{f}^n,\boldsymbol{u}_h^n)+2\mu\Delta t(\boldsymbol{g}^n,\boldsymbol{B}_h^n).
\end{aligned}\tag{3.13}$$

By virtue of Young inequality, (2.1), (3.1) and (3.12), we have

$$\begin{aligned}
2\big|\big(\boldsymbol{\beta}^n\big(\theta_h^{n-1}\big)\theta_h^n,\boldsymbol{u}_h^n\big)\big|&\le2\nu_0^{-1}c_1^2\|\boldsymbol{\beta}^n\|_{\mathscr{C}(\bar\Omega\times\mathbb{R};\mathbb{R}^3)}^2\|\theta_h^n\|_0^2+\frac12\left\|\sqrt{\nu^n\big(\theta_h^{n-1}\big)}\nabla\boldsymbol{u}_h^n\right\|_0^2\\
&\le2\nu_0^{-1}c_1^2\|\boldsymbol{\beta}^n\|_{\mathscr{C}(\bar\Omega\times\mathbb{R};\mathbb{R}^3)}^2C_{00}+\frac12\left\|\sqrt{\nu^n\big(\theta_h^{n-1}\big)}\nabla\boldsymbol{u}_h^n\right\|_0^2,\\
2|(\boldsymbol{f}^n,\boldsymbol{u}_h^n)+\mu(\boldsymbol{g}^n,\boldsymbol{B}_h^n)|&\le2\nu_0^{-1}c_1^2\|\boldsymbol{f}^n\|_0^2+\frac12\left\|\sqrt{\nu^n\big(\theta_h^{n-1}\big)}\nabla\boldsymbol{u}_h^n\right\|_0^2\\
&\quad+\mu\sigma_0^{-1}C_*^2\|\boldsymbol{g}^n\|_0^2+\mu\left\|\sqrt{\sigma^n\big(\theta_h^{n-1}\big)}\,\mathbf{curl}\,\boldsymbol{B}_h^n\right\|_0^2.
\end{aligned}$$

Inserting these inequalities into (3.13), and summing up from $n=1$ to m, there holds

$$
\begin{aligned}
&\|\boldsymbol{u}_h^m\|_0^2+\mu\|\boldsymbol{B}_h^m\|_0^2+\sum_{n=1}^m\Big(\big\|\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}\big\|_0^2+\mu\big\|\boldsymbol{B}_h^n-\boldsymbol{B}_h^{n-1}\big\|_0^2\Big)\\
&\quad+\Delta t\sum_{n=1}^m\left(\left\|\sqrt{\nu^n\big(\theta_h^{n-1}\big)}\nabla\boldsymbol{u}_h^n\right\|_0^2+\mu\left\|\sqrt{\sigma^n\big(\theta_h^{n-1}\big)}\,\mathbf{curl}\,\boldsymbol{B}_h^n\right\|_0^2\right)\\
&\quad\le\big\|\boldsymbol{u}_h^0\big\|_0^2+\mu\big\|\boldsymbol{B}_h^0\big\|_0^2+C\Delta t\sum_{n=1}^m\Big(1+\|\boldsymbol{f}^n\|_0^2+\|\boldsymbol{g}^n\|_0^2\Big).
\end{aligned}
\tag{3.14}
$$

A combination of (3.12) and (3.14) , then we can obtain the desired conclusion. □

The following estimates for the trilinear term in Navier–Stokes equations can be referred to Lemma 4.5 of [53].

Theorem 3.6. *For $i=1,2$, let $\boldsymbol{u}^i$ be functions in $\boldsymbol{L}^\infty(\Omega)\cap\boldsymbol{W}^{1,3}(\Omega)$, and $\boldsymbol{u}_h^i$ functions in $\boldsymbol{X}_h^k$. Then, for $\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)$, $\bar{\boldsymbol{u}}^i\in\big(\boldsymbol{L}^\infty(\Omega)\cap\boldsymbol{W}^{1,3}(\Omega)\big)$, $\bar{\boldsymbol{w}}^i\in\boldsymbol{X}_h^k$, and $(p,q)=(0,1)$ or $(p,q)=(1,0)$, we have*

$$
\begin{aligned}
&\big|\mathcal{O}_1\big(\boldsymbol{u}^1,\boldsymbol{u}^2,\boldsymbol{v}\big)-\mathcal{O}_1\big(\boldsymbol{u}_h^1,\boldsymbol{u}_h^2,\boldsymbol{v}\big)\big|\\
&\quad\le C\left[\min\Big\{\big\|\boldsymbol{u}^i\big\|_{0,\infty}+\big\|\boldsymbol{u}^i\big\|_{1,3};i=1,2\Big\}+\sum_{i=1}^2\Big(\big\|\bar{\boldsymbol{w}}_h^i\big\|_{0,\infty}+\big\|\bar{\boldsymbol{w}}_h^i\big\|_{1,3}\Big)\right]\\
&\quad\cdot\sum_{i=1}^2\Big(\big\|\boldsymbol{u}^i-\bar{\boldsymbol{u}}^i\big\|_p+\big\|\bar{\boldsymbol{u}}^i-\bar{\boldsymbol{w}}_h^i\big\|_p+\big\|\bar{\boldsymbol{w}}_h^i-\boldsymbol{u}_h^i\big\|_p\Big)\|\boldsymbol{v}\|_q+\big|\mathcal{O}_1\big(\bar{\boldsymbol{w}}_h^1-\boldsymbol{u}_h^1,\bar{\boldsymbol{w}}_h^2-\boldsymbol{u}_h^2,\boldsymbol{v}\big)\big|.
\end{aligned}
\tag{3.15}
$$

Moreover, if $\bar{\boldsymbol{w}}_h^2-\boldsymbol{u}_h^2\in\boldsymbol{X}_0$, then we readily see

$$
\begin{aligned}
&\big|\mathcal{O}_1\big(\boldsymbol{u}^1,\boldsymbol{u}^2,\bar{\boldsymbol{w}}_h^2-\boldsymbol{u}_h^2\big)-\mathcal{O}_1\big(\boldsymbol{u}_h^1,\boldsymbol{u}_h^2,\bar{\boldsymbol{w}}_h^2-\boldsymbol{u}_h^2\big)\big|\\
&\quad\le C\left[\min\Big\{\big\|\boldsymbol{u}^i\big\|_{0,\infty}+\big\|\boldsymbol{u}^i\big\|_{1,3};i=1,2\Big\}+\sum_{i=1}^2\Big(\big\|\bar{\boldsymbol{w}}_h^i\big\|_{0,\infty}+\big\|\bar{\boldsymbol{w}}_h^i\big\|_{1,3}\Big)\right]\\
&\quad\cdot\sum_{i=1}^2\Big(\big\|\boldsymbol{u}^i-\bar{\boldsymbol{u}}^i\big\|_p+\big\|\bar{\boldsymbol{u}}^i-\bar{\boldsymbol{w}}_h^i\big\|_p+\big\|\bar{\boldsymbol{w}}_h^i-\boldsymbol{u}_h^i\big\|_p\Big)\big\|\bar{\boldsymbol{w}}_h^2-\boldsymbol{u}_h^2\big\|_q.
\end{aligned}
\tag{3.16}
$$

4. Well-posedness and convergence of the fully discrete solution

In this section, we will prove the well-posedness of the numerical solution to the problem (3.5)–(3.8) by using the Lax–Milgram theorem (see [22] for more details). Utilizing the stability of the numerical scheme and the compactness method, the existence of weak solution to the thermally coupled MHD model in three dimensions is established. Furthermore, the uniqueness of weak solution and the convergence of the proposed numerical method are also derived.

We first prove the well-posedness result for the discrete solution in the following theorem.

Theorem 4.1. *Under the condition of Lemma 3.5, then there exists a unique solution $(\boldsymbol{u}_h^n,\theta_h^n,\boldsymbol{B}_h^n)$ to scheme (3.5)–(3.8).*

Proof. We divide this proof into two steps:

Step 1. For any $\varphi_h\in Y_{0h}^k$, given θ_h^{n-1} and $\boldsymbol{u}_h^{n-1}$, find $\theta_h^n\in Y_{0h}^k(\theta_D)$, we can rewrite (3.8) as

$$
\frac{1}{\Delta t}\big(\theta_h^n,\varphi_h\big)+\mathcal{A}_2\big(\kappa^n\big(\theta_h^{n-1}\big),\theta_h^n,\varphi_h\big)+\mathcal{O}_2\big(\boldsymbol{u}_h^{n-1},\theta_h^n,\varphi_h\big)=\big(\psi^n,\varphi_h\big)+\frac{1}{\Delta t}\big(\theta_h^{n-1},\varphi_h\big).
\tag{4.1}
$$

Let us define

$$G(\theta_h^n,\varphi_h)=\frac{1}{\Delta t}(\theta_h^n,\varphi_h)+\mathcal{A}_2\big(\kappa^n\big(\theta_h^{n-1}\big),\theta_h^n,\varphi_h\big)+\mathcal{O}_2\big(\boldsymbol{u}_h^{n-1},\theta_h^n,\varphi_h\big).$$

It is easy to verify that $G(\theta_h^n,\varphi_h)$ satisfies the ellipticity and boundedness. An application of the Lax–Milgram theorem shows that problem (3.8) attains a unique solution θ_h^n.

Step 2. We prove the uniqueness of the solution of (3.5)–(3.7) based on a fixed θ_h^n. According to (3.5)–(3.7), the solution $(\boldsymbol{u}_h^n,\boldsymbol{B}_h^n)\in\big(\boldsymbol{X}_{0h}^k\times\boldsymbol{W}_{0h}^k\big)$ satisfy that for any $(\boldsymbol{v}_h,\boldsymbol{C}_h)\in\big(\boldsymbol{X}_{0h}^k\times\boldsymbol{W}_{0h}^k\big)$, there holds

$$\begin{aligned}\frac{1}{\Delta t}(\boldsymbol{u}_h^n,\boldsymbol{v}_h)&+\frac{\mu}{\Delta t}(\boldsymbol{B}_h^n,\boldsymbol{C}_h)+\big(\nu^n\big(\theta_h^{n-1}\big)\nabla\boldsymbol{u}_h^n,\nabla\boldsymbol{v}_h\big)+\mu\big(\sigma^n\big(\theta_h^{n-1}\big)\mathbf{curl}\,\boldsymbol{B}_h^n,\mathbf{curl}\,\boldsymbol{C}_h\big)\\&+\mathcal{O}_1\big(\boldsymbol{u}_h^{n-1},\boldsymbol{u}_h^n,\boldsymbol{v}_h\big)+\mu\big(\boldsymbol{B}_h^{n-1}\times\mathbf{curl}\,\boldsymbol{B}_h^n,\boldsymbol{v}_h\big)-\mu\big(\boldsymbol{u}_h^n\times\boldsymbol{B}_h^{n-1},\mathbf{curl}\,\boldsymbol{C}_h\big)\\&=\big(\boldsymbol{f}^n+\boldsymbol{\beta}^n\big(\theta_h^{n-1}\big)\theta_h^n,\boldsymbol{v}_h\big)+\mu(\boldsymbol{g}^n,\boldsymbol{C}_h)+\left(\frac{1}{\Delta t}\boldsymbol{u}_h^{n-1},\boldsymbol{v}_h\right)+\left(\frac{\mu}{\Delta t}\boldsymbol{B}_h^{n-1},\boldsymbol{C}_h\right).\end{aligned}\tag{4.2}$$

Let $\mathbf{U}^{n-1}=\big(\boldsymbol{u}_h^{n-1},\boldsymbol{B}_h^{n-1}\big)$, $\mathbf{U}^n=(\boldsymbol{u}_h^n,\boldsymbol{B}_h^n)$, $\boldsymbol{\Phi}=(\boldsymbol{v}_h,\boldsymbol{C}_h)$, and it is easy to see that

$$|||\mathbf{U}^n|||^2=\frac{1}{\Delta t}\|\boldsymbol{u}_h^n\|_0^2+\frac{\mu}{\Delta t}\|\boldsymbol{B}_h^n\|_0^2+\left\|\sqrt{\nu^n\big(\theta_h^{n-1}\big)}\nabla\boldsymbol{u}_h^n\right\|_0^2+\mu\left\|\sqrt{\sigma^n\big(\theta_h^{n-1}\big)}\mathbf{curl}\,\boldsymbol{B}_h^n\right\|_0^2$$

provides a norm on $\boldsymbol{X}_{0h}^k\times\boldsymbol{W}_{0h}^k$. Define the left-hand side of (4.2) as $I\big(\mathbf{U}^{n-1},\mathbf{U}^n,\boldsymbol{\Phi}\big)$, thus we can obtain

$$I\big(\mathbf{U}^{n-1},\mathbf{U}^n,\mathbf{U}^n\big)\geq|||\mathbf{U}^n|||^2.$$

We choose $\delta_2>0$ such that $1/(3+\delta_1)+1/(6-\delta_2)=1/2$ and $\boldsymbol{H}^1(\Omega)\hookrightarrow\hookrightarrow\boldsymbol{L}^{6-\delta_2}(\Omega)$, together with (3.2) and (3.3), then we have the following estimate

$$\begin{aligned}&\mu\big|\big(\boldsymbol{B}_h^{n-1}\times\mathbf{curl}\,\boldsymbol{B}_h^n,\boldsymbol{v}_h\big)\big|\\&\quad=\mu\big|\big(\big[\boldsymbol{B}_h^{n-1}-Z\big(\boldsymbol{B}_h^{n-1}\big)\big]\times\mathbf{curl}\,\boldsymbol{B}_h^n,\boldsymbol{v}_h\big)+\big(Z\big(\boldsymbol{B}_h^{n-1}\big)\times\mathbf{curl}\,\boldsymbol{B}_h^n,\boldsymbol{v}_h\big)\big|\\&\quad\leq Ch^{l+1/2}\big\|\mathbf{curl}\,\boldsymbol{B}_h^{n-1}\big\|_0\|\mathbf{curl}\,\boldsymbol{B}_h^n\|_{0,3}\|\boldsymbol{v}_h\|_{0,6}+C\big\|Z\big(\boldsymbol{B}_h^{n-1}\big)\big\|_{0,3+\delta_1}\|\mathbf{curl}\,\boldsymbol{B}_h^n\|_0\|\boldsymbol{v}_h\|_{0,6-\delta_2}\\&\quad\leq Ch^{l}\big\|\mathbf{curl}\,\boldsymbol{B}_h^{n-1}\big\|_0\|\mathbf{curl}\,\boldsymbol{B}_h^n\|_0\|\nabla\boldsymbol{v}_h\|_0+C\big\|\mathbf{curl}\,\boldsymbol{B}_h^{n-1}\big\|_0\|\mathbf{curl}\,\boldsymbol{B}_h^n\|_0\|\nabla\boldsymbol{v}_h\|_0,\end{aligned}$$

where we have used $\big\|Z\big(\boldsymbol{B}_h^{n-1}\big)\big\|_{0,3+\delta_1}\leq C\big\|\mathbf{curl}\,Z\big(\boldsymbol{B}_h^{n-1}\big)\big\|_0=C\big\|\mathbf{curl}\,\boldsymbol{B}_h^{n-1}\big\|_0$ according to Lemma 2.1.

Similarly, we have

$$\mu\big|\big(\boldsymbol{u}_h^n\times\boldsymbol{B}_h^{n-1},\mathbf{curl}\,\boldsymbol{C}_h\big)\big|\leq Ch^l\big\|\mathbf{curl}\,\boldsymbol{B}_h^{n-1}\big\|_0\|\mathbf{curl}\,\boldsymbol{C}_h\|_0\|\nabla\boldsymbol{u}_h^n\|_0+C\big\|\mathbf{curl}\,\boldsymbol{B}_h^{n-1}\big\|_0\|\mathbf{curl}\,\boldsymbol{C}_h\|_0\|\nabla\boldsymbol{u}_h^n\|_0.$$

This implies the continuity of $I\big(\mathbf{U}^{n-1},\mathbf{U}^n,\boldsymbol{\Phi}\big)$, namely,

$$|I\big(\mathbf{U}^{n-1},\mathbf{U}^n,\boldsymbol{\Phi}\big)|\leq C\Big(1+\big\|\boldsymbol{u}_h^{n-1}\big\|_{1,2}+\big(h^l+1\big)\big\|\mathbf{curl}\,\boldsymbol{B}_h^{n-1}\big\|_0\Big)|||\mathbf{U}^n|||\,|||\boldsymbol{\Phi}|||.$$

According to the Lax–Milgram theorem, we know that (3.5) and (3.7) admit a unique solution $(\boldsymbol{u}_h^n,\boldsymbol{B}_h^n)$. Combining Step 1 and Step 2, we have completed the proof of Theorem 4.1. □

Next, we will present the convergence analysis for the fully discrete solution to problem (3.5)–(3.8). To this end, we introduce some interpolated functions over the temporal variable, which will be used below. Let $\boldsymbol{u}_{h\Delta t}(\cdot,t)$, $\theta_{h\Delta t}(\cdot,t)$, $\boldsymbol{B}_{h\Delta t}(\cdot,t)$ be the piecewise linear continuous interpolation of the fully discrete solution $(\boldsymbol{u}_h^n,\theta_h^n,\boldsymbol{B}_h^n)$, $n=1,2,\ldots,N$ on $(t_{n-1},t_n]$, that is to say

$$\phi_{h\Delta t}(\cdot,t) := \frac{t-t_{n-1}}{\Delta t}\phi_h^+(\cdot,t) + \frac{t_n - t}{\Delta t}\phi_h^-(\cdot,t),$$

where $\phi_h^+ = \phi_h^n, \phi_h^- = \phi_h^{n-1}$, with $\phi = \boldsymbol{u}, \theta, \boldsymbol{B}$.

Let $\tilde{\theta}_{h\Delta t}(x,t)$, $\tilde{\boldsymbol{u}}_{h\Delta t}(x,t)$, $\tilde{p}_{h\Delta t}(x,t)$, $\tilde{\boldsymbol{B}}_{h\Delta t}(x,t)$, $\hat{\boldsymbol{u}}_{h\Delta t}(x,t)$, $\hat{\theta}_{h\Delta t}(x,t)$, $\hat{\boldsymbol{B}}_{h\Delta t}(x,t)$, $\tilde{\boldsymbol{f}}_{\Delta t}(x,t)$, $\tilde{\boldsymbol{g}}_{\Delta t}(x,t)$, $\tilde{\psi}_{\Delta t}(x,t)$, $\tilde{\sigma}_{\Delta t}(x,t)$, $\tilde{\nu}_{\Delta t}(x,t)$, $\tilde{\boldsymbol{\beta}}_{\Delta t}(x,t)$ and $\tilde{\kappa}_{\Delta t}(x,t)$ be the piecewise constant extensions of $\{\theta_h^n\}$, $\{\boldsymbol{u}_h^n\}$, $\{p_h^n\}$, $\{\boldsymbol{B}_h^n\}$, $\{\boldsymbol{u}_h^{n-1}\}$, $\{\theta_h^{n-1}\}$, $\{\boldsymbol{B}_h^{n-1}\}$, $\{\boldsymbol{f}^n\}$, $\{\boldsymbol{g}^n\}$, $\{\psi^n\}$, $\{\sigma^n\}$, $\{\nu^n\}$, $\{\boldsymbol{\beta}^n\}$ and $\{\kappa^n\}$, $n = 1, 2, \ldots, N$, respectively, namely, for all $t \in (t_{n-1}, t_n]$,

$$\begin{aligned}
&\tilde{\theta}_{h\Delta t}(\cdot,t) = \theta_h^n, \quad \tilde{\boldsymbol{u}}_{h\Delta t}(\cdot,t) = \boldsymbol{u}_h^n, \quad \tilde{p}_{h\Delta t}(\cdot,t) = p_h^n, \quad \tilde{\boldsymbol{B}}_{h\Delta t}(\cdot,t) = \boldsymbol{B}_h^n,\\
&\hat{\boldsymbol{u}}_{h\Delta t}(\cdot,t) = \boldsymbol{u}_h^{n-1}, \quad \hat{\theta}_{h\Delta t}(\cdot,t) = \theta_h^{n-1}, \quad \hat{\boldsymbol{B}}_{h\Delta t}(\cdot,t) = \boldsymbol{B}_h^{n-1}, \quad \tilde{\boldsymbol{f}}_{\Delta t}(\cdot,t) = \boldsymbol{f}^n,\\
&\tilde{\boldsymbol{g}}_{\Delta t}(\cdot,t) = \boldsymbol{g}^n, \quad \tilde{\psi}_{\Delta t}(\cdot,t) = \psi^n, \quad \tilde{\sigma}_{\Delta t}(\cdot,t) = \sigma^n, \quad \tilde{\nu}_{\Delta t}(\cdot,t) = \nu^n,\\
&\tilde{\boldsymbol{\beta}}_{\Delta t}(\cdot,t) = \boldsymbol{\beta}^n, \quad \tilde{\kappa}_{\Delta t}(\cdot,t) = \kappa^n.
\end{aligned}$$

With the above notations, we will prove the following priori stability estimates without any restrictions on h and Δt.

Lemma 4.2. *For the sequences $\{(\boldsymbol{u}_{h\Delta t}, \tilde{p}_{h\Delta t}, \boldsymbol{B}_{h\Delta t}, \theta_{h\Delta t})\}$, there exists a constant C independent of the mesh-size h and the time-step Δt such that*

$$\begin{aligned}
&\|(\boldsymbol{u}_{h\Delta t})_t + \nabla\tilde{p}_{h\Delta t}\|_{L^r(0,T;(\boldsymbol{X}\cap\boldsymbol{H}^{1+s}(\Omega))')} \le C, \quad \|(\boldsymbol{B}_{h\Delta t})_t\|_{L^r(0,T;(\boldsymbol{W}\cap\boldsymbol{H}^{1+s}(\Omega))')} \le C,\\
&\|(\theta_{h\Delta t})_t\|_{L^{4/3}(0,T;(Y\cap H^{1+s}(\Omega))')} \le C,
\end{aligned}$$

with $r = \frac{2}{2-\zeta}$, $\zeta = \min\Big\{l,\ \big(\frac12 + l\big)\frac{2\delta_1}{3(1+\delta_1)},\ \frac{2\delta_1}{3(1+\delta_1)},\ \frac12\Big\}$ and $s > \frac12$, where δ_1 is defined in Lemma 2.1 and l is defined in (3.2).

Proof. To validate the first assertion, we define the $\boldsymbol{L}^2$-orthogonal projection to $\boldsymbol{V}_h^k$ *via* $\mathscr{P}_h$. Setting $\frac1r + \frac1{r'} = 1$ and $r' = \frac{2}{\zeta}$, with $\zeta = \min\Big\{l,\ \big(\frac12 + l\big)\frac{2\delta_1}{3(1+\delta_1)},\ \frac{2\delta_1}{3(1+\delta_1)},\ \frac12\Big\}$, for any $\boldsymbol{v} \in L^{r'}\big(0,T;\boldsymbol{X}\cap\boldsymbol{H}^{1+s}(\Omega)\big)$, by (3.5), we have

$$\begin{aligned}
&((\boldsymbol{u}_{h\Delta t})_t, \mathscr{P}_h\boldsymbol{v}) + \frac12[((\hat{\boldsymbol{u}}_{h\Delta t}\cdot\nabla)\tilde{\boldsymbol{u}}_{h\Delta t}, \mathscr{P}_h\boldsymbol{v}) - ((\hat{\boldsymbol{u}}_{h\Delta t}\cdot\nabla)\mathscr{P}_h\boldsymbol{v}, \tilde{\boldsymbol{u}}_{h\Delta t})] + (\nabla\tilde{p}_{h\Delta t}, \mathscr{P}_h\boldsymbol{v})\\
&\quad + \Big(\tilde{\nu}_{\Delta t}\Big(\hat{\theta}_{h\Delta t}\Big)\nabla\tilde{\boldsymbol{u}}_{h\Delta t}, \nabla\mathscr{P}_h\boldsymbol{v}\Big) + \mu\Big(\hat{\boldsymbol{B}}_{h\Delta t}\times\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}, \mathscr{P}_h\boldsymbol{v}\Big) - \Big(\tilde{\boldsymbol{\beta}}_{\Delta t}\Big(\hat{\theta}_{h\Delta t}\Big)\tilde{\theta}_{h\Delta t}, \mathscr{P}_h\boldsymbol{v}\Big)\\
&\quad = \Big(\tilde{\boldsymbol{f}}_{\Delta t}, \mathscr{P}_h\boldsymbol{v}\Big).
\end{aligned} \tag{4.3}$$

By virtue of Lemma 3.5, the Cauchy–Schwarz inequality and the interpolation inequality

$$\|\hat{\boldsymbol{u}}_{h\Delta t}\|_{0,3} \le C\|\hat{\boldsymbol{u}}_{h\Delta t}\|_0^{1/2}\|\nabla\hat{\boldsymbol{u}}_{h\Delta t}\|_0^{1/2}, \tag{4.4}$$

thanks to $r' = \frac{2}{\zeta}$, which implies that $r' > 4$, we can derive that

$$\begin{aligned}
&\int_0^T\left|\frac12[((\hat{\boldsymbol{u}}_{h\Delta t}\cdot\nabla)\tilde{\boldsymbol{u}}_{h\Delta t}, \mathscr{P}_h\boldsymbol{v}) - ((\hat{\boldsymbol{u}}_{h\Delta t}\cdot\nabla)\mathscr{P}_h\boldsymbol{v}, \tilde{\boldsymbol{u}}_{h\Delta t})]\right|\mathrm{d}t\\
&\quad\le C\|\hat{\boldsymbol{u}}_{h\Delta t}\|_{0,3}\|\nabla\tilde{\boldsymbol{u}}_{h\Delta t}\|_0\|\mathscr{P}_h\boldsymbol{v}\|_{0,6} + C\|\hat{\boldsymbol{u}}_{h\Delta t}\|_{0,3}\|\tilde{\boldsymbol{u}}_{h\Delta t}\|_{0,6}\|\nabla\mathscr{P}_h\boldsymbol{v}\|_0\\
&\quad\le C\int_0^T\Big[\|\hat{\boldsymbol{u}}_{h\Delta t}\|_0^{1/2}\|\nabla\hat{\boldsymbol{u}}_{h\Delta t}\|_0^{1/2}\|\nabla\tilde{\boldsymbol{u}}_{h\Delta t}\|_0\|\nabla\mathscr{P}_h\boldsymbol{v}\|_0\Big]\mathrm{d}t\\
&\quad\le C\|\nabla\hat{\boldsymbol{u}}_{h\Delta t}\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}^{1/2}\|\nabla\tilde{\boldsymbol{u}}_{h\Delta t}\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\|\nabla\mathscr{P}_h\boldsymbol{v}\|_{L^4(0,T;\boldsymbol{L}^2(\Omega))} \le C\|\nabla\mathscr{P}_h\boldsymbol{v}\|_{L^{r'}(0,T;\boldsymbol{L}^2(\Omega))}.
\end{aligned}$$

It can be checked that $1 < r < 2$, then we obtain

$$\int_0^T \left|\left(\tilde{\nu}_{\Delta t}\left(\hat{\theta}_{h\Delta t}\right)\nabla\tilde{\boldsymbol{u}}_{h\Delta t}, \nabla\mathscr{P}_h\boldsymbol{v}\right)\right| \mathrm{d}t$$
$$\le C\|\tilde{\nu}_{\Delta t}\|_{\mathscr{C}\left(\bar{\Omega}\times\mathbb{R};\mathbb{R}^+\right)}\|\nabla\tilde{\boldsymbol{u}}_{h\Delta t}\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\|\nabla\mathscr{P}_h\boldsymbol{v}\|_{L^{r'}(0,T;\boldsymbol{L}^2(\Omega))} \le C\|\nabla\mathscr{P}_h\boldsymbol{v}\|_{L^{r'}(0,T;\boldsymbol{L}^2(\Omega))}.$$

By using of Lemma 2.1, (3.2), (3.3), (2.1), there holds

$$\begin{aligned}
&\int_0^T \left|\mu\left(\hat{\boldsymbol{B}}_{h\Delta t}\times\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}, \mathscr{P}_h\boldsymbol{v}\right)\right| \mathrm{d}t \\
&= \int_0^T \left|\mu\left(\left[\hat{\boldsymbol{B}}_{h\Delta t} - Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right) + Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right)\right]\times\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}, \mathscr{P}_h\boldsymbol{v}\right)\right| \mathrm{d}t \\
&\overset{(3.2)}{\le} \int_0^T \Big|\mu C_{inv}h^l\left\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\right\|_0\left\|\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}\right\|_0\|\mathscr{P}_h\boldsymbol{v}\|_{0,6} \\
&\quad + \mu\left\|Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right)\right\|_{0,3}\left\|\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}\right\|_0\|\mathscr{P}_h\boldsymbol{v}\|_{0,6}\Big| \mathrm{d}t \\
&\overset{\text{Lemma 2.1}}{\le} C\int_0^T \Big|\mu C_{inv}h^l\left\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\right\|_0\left\|\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}\right\|_0\|\nabla\mathscr{P}_h\boldsymbol{v}\|_0 \\
&\quad + \mu\left\|Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right)\right\|_0^{\frac{2\delta_1}{3(1+\delta_1)}}\left\|Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right)\right\|_{0,3+\delta_1}^{\frac{\delta_1+3}{3(1+\delta_1)}}\left\|\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}\right\|_0\|\nabla\mathscr{P}_h\boldsymbol{v}\|_0\Big| \mathrm{d}t =: C\int_0^T |I_1+I_2|\,\mathrm{d}t,
\end{aligned}$$

where we have used the interpolation inequality

$$\left\|Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right)\right\|_{0,3} \le \left\|Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right)\right\|_0^{\frac{2\delta_1}{3(1+\delta_1)}}\left\|Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right)\right\|_{0,3+\delta_1}^{\frac{\delta_1+3}{3(1+\delta_1)}}. \tag{4.5}$$

By virtue of (3.3) and Lemma 3.5, we obtain

$$\begin{aligned}
\int_0^T |I_1|\,\mathrm{d}t &\le \int_0^T \left|\mu C_{inv}h^l\left\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\right\|_0^l\left\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\right\|_0^{1-l}\left\|\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}\right\|_0\|\nabla\mathscr{P}_h\boldsymbol{v}\|_0\right| \mathrm{d}t \\
&\overset{(3.3)}{\le} \int_0^T \left|\mu C_{inv}C_{inv}^l\left\|\hat{\boldsymbol{B}}_{h\Delta t}\right\|_0^l\left\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\right\|_0^{1-l}\left\|\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}\right\|_0\|\nabla\mathscr{P}_h\boldsymbol{v}\|_0\right| \mathrm{d}t \\
&\le C\left\|\hat{\boldsymbol{B}}_{h\Delta t}\right\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}^l\left\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\right\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}^{1-l}\left\|\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}\right\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\|\nabla\mathscr{P}_h\boldsymbol{v}\|_{L^{\frac{2}{l}}(0,T;\boldsymbol{L}^2(\Omega))} \\
&\overset{\text{Lemma 3.5}}{\le} C\|\nabla\mathscr{P}_h\boldsymbol{v}\|_{L^{\frac{2}{l}}(0,T;\boldsymbol{L}^2(\Omega))} \le C\|\nabla\mathscr{P}_h\boldsymbol{v}\|_{L^{r'}(0,T;\boldsymbol{L}^2(\Omega))}.
\end{aligned}$$

Concerning the other term, it can be decomposed as

$$\begin{aligned}
\int_0^T |I_2|\,\mathrm{d}t &\le \int_0^T \left|\mu\left\|Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right) - \hat{\boldsymbol{B}}_{h\Delta t} + \hat{\boldsymbol{B}}_{h\Delta t}\right\|_0^{\frac{2\delta_1}{3(1+\delta_1)}}\left\|Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right)\right\|_{0,3+\delta_1}^{\frac{\delta_1+3}{3(1+\delta_1)}}\left\|\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}\right\|_0\|\nabla\mathscr{P}_h\boldsymbol{v}\|_0\right| \mathrm{d}t \\
&\le \int_0^T \left|\mu\left\|Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right) - \hat{\boldsymbol{B}}_{h\Delta t}\right\|_0^{\frac{2\delta_1}{3(1+\delta_1)}}\left\|Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right)\right\|_{0,3+\delta_1}^{\frac{\delta_1+3}{3(1+\delta_1)}}\left\|\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}\right\|_0\|\nabla\mathscr{P}_h\boldsymbol{v}\|_0\right| \mathrm{d}t \\
&\quad + \int_0^T \left|\mu\left\|\hat{\boldsymbol{B}}_{h\Delta t}\right\|_0^{\frac{2\delta_1}{3(1+\delta_1)}}\left\|Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right)\right\|_{0,3+\delta_1}^{\frac{\delta_1+3}{3(1+\delta_1)}}\left\|\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}\right\|_0\|\nabla\mathscr{P}_h\boldsymbol{v}\|_0\right| \mathrm{d}t \\
&=: \int_0^T |I_{21}|\,\mathrm{d}t + \int_0^T |I_{22}|\,\mathrm{d}t.
\end{aligned}$$

With the help of (3.3), Lemmas 2.1, 3.5 and the interpolation inequality (4.5), we continue to deduce

$$
\begin{aligned}
&\int_0^T |I_{21}|\,\mathrm{d}t \\
&\le \int_0^T \left| Ch^{\left(\frac{1}{2}+l\right)\left(\frac{2\delta_1}{3(1+\delta_1)}\right)} \left\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\right\|_0^{\frac{2\delta_1}{3(1+\delta_1)}} \left\|Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right)\right\|_{0,3+\delta_1}^{\frac{\delta_1+3}{3(1+\delta_1)}} \left\|\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}\right\|_0 \|\nabla \mathscr{P}_h \boldsymbol{v}\|_0 \right| \mathrm{d}t \\
&\overset{\text{Lemma 2.1}}{\le} \int_0^T \left| Ch^{\left(\frac{1}{2}+l\right)\left(\frac{2\delta_1}{3(1+\delta_1)}\right)} \left\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\right\|_0 \left\|\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}\right\|_0 \|\nabla \mathscr{P}_h \boldsymbol{v}\|_0 \right| \mathrm{d}t \\
&\le \int_0^T \left| Ch^{\left(\frac{1}{2}+l\right)\left(\frac{2\delta_1}{3(1+\delta_1)}\right)} \left\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\right\|_0^{\left(\frac{1}{2}+l\right)\left(\frac{2\delta_1}{3(1+\delta_1)}\right)} \left\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\right\|_0^{1-\left(\frac{1}{2}+l\right)\left(\frac{2\delta_1}{3(1+\delta_1)}\right)} \left\|\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}\right\|_0 \|\nabla \mathscr{P}_h \boldsymbol{v}\|_0 \right| \mathrm{d}t \\
&\overset{(3.3)}{\le} \int_0^T \left| CC_{inv}^{\left(\frac{1}{2}+l\right)\left(\frac{2\delta_1}{3(1+\delta_1)}\right)} \left\|\hat{\boldsymbol{B}}_{h\Delta t}\right\|_0^{\left(\frac{1}{2}+l\right)\left(\frac{2\delta_1}{3(1+\delta_1)}\right)} \left\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\right\|_0^{1-\left(\frac{1}{2}+l\right)\left(\frac{2\delta_1}{3(1+\delta_1)}\right)} \left\|\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}\right\|_0 \|\nabla \mathscr{P}_h \boldsymbol{v}\|_0 \right| \mathrm{d}t \\
&\le C \left\|\hat{\boldsymbol{B}}_{h\Delta t}\right\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}^{\left(\frac{1}{2}+l\right)\left(\frac{2\delta_1}{3(1+\delta_1)}\right)} \left\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\right\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}^{1-\left(\frac{1}{2}+l\right)\left(\frac{2\delta_1}{3(1+\delta_1)}\right)} \left\|\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}\right\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \\
&\quad \times \|\nabla \mathscr{P}_h \boldsymbol{v}\|_{L^{\frac{2}{\left(\frac{1}{2}+l\right)\left(\frac{2\delta_1}{3(1+\delta_1)}\right)}}(0,T;\boldsymbol{L}^2(\Omega))} \\
&\overset{\text{Lemma 3.5}}{\le} C\|\nabla \mathscr{P}_h \boldsymbol{v}\|_{L^{\frac{2}{\left(\frac{1}{2}+l\right)\left(\frac{2\delta_1}{3(1+\delta_1)}\right)}}(0,T;\boldsymbol{L}^2(\Omega))} \le C\|\nabla \mathscr{P}_h \boldsymbol{v}\|_{L^{r'}(0,T;\boldsymbol{L}^2(\Omega))},
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^T |I_{22}|\,\mathrm{d}t &\le \int_0^T \left| \mu \left\|\hat{\boldsymbol{B}}_{h\Delta t}\right\|_0^{\frac{2\delta_1}{3(1+\delta_1)}} \left\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\right\|_0^{\frac{\delta_1+3}{3(1+\delta_1)}} \left\|\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}\right\|_0 \|\nabla \mathscr{P}_h \boldsymbol{v}\|_0 \right| \mathrm{d}t \\
&\le C \left\|\hat{\boldsymbol{B}}_{h\Delta t}\right\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}^{\frac{2\delta_1}{3(1+\delta_1)}} \left\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\right\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}^{\frac{\delta_1+3}{3(1+\delta_1)}} \\
&\quad \times \left\|\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}\right\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \|\nabla \mathscr{P}_h \boldsymbol{v}\|_{L^{\frac{2}{\left(\frac{2\delta_1}{3(1+\delta_1)}\right)}}(0,T;\boldsymbol{L}^2(\Omega))} \\
&\le C\|\nabla \mathscr{P}_h \boldsymbol{v}\|_{L^{\frac{2}{\left(\frac{2\delta_1}{3(1+\delta_1)}\right)}}(0,T;\boldsymbol{L}^2(\Omega))} \le C\|\nabla \mathscr{P}_h \boldsymbol{v}\|_{L^{r'}(0,T;\boldsymbol{L}^2(\Omega))}.
\end{aligned}
$$

Combined with these estimates, we get the following conclusion

$$
\int_0^T \left| \mu\left(\hat{\boldsymbol{B}}_{h\Delta t} \times \mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}, \mathscr{P}_h \boldsymbol{v}\right)\right| \mathrm{d}t \le C\|\nabla \mathscr{P}_h \boldsymbol{v}\|_{L^{r'}(0,T;\boldsymbol{L}^2(\Omega))}. \tag{4.6}
$$

Due to $1 < r < 2$, we have

$$
\int_0^T \left|\left(\tilde{\boldsymbol{\beta}}_{\Delta t}\left(\hat{\theta}_{h\Delta t}\right)\tilde{\theta}_{h\Delta t}, \mathscr{P}_h \boldsymbol{v}\right)\right| \mathrm{d}t \le C\|\nabla \mathscr{P}_h \boldsymbol{v}\|_{L^{r'}(0,T;\boldsymbol{L}^2(\Omega))}.
$$

Combining these inequalities with (4.3), we have

$$
\int_0^T |((\boldsymbol{u}_{h\Delta t})_t + \nabla \tilde{p}_{h\Delta t}, \boldsymbol{v})|\,\mathrm{d}t \le C\|\nabla \mathscr{P}_h \boldsymbol{v}\|_{L^{r'}(0,T;\boldsymbol{L}^2(\Omega))} \le C\|\nabla \boldsymbol{v}\|_{L^{r'}(0,T;\boldsymbol{L}^2(\Omega))} + Ch^s\|\nabla \boldsymbol{v}\|_{L^{r'}(0,T;\boldsymbol{H}^s(\Omega))}, \tag{4.7}
$$

where we have used the properties of $\boldsymbol{L}^2$-orthogonal projection $((\boldsymbol{u}_{h\Delta t})_t, \boldsymbol{v}) = ((\boldsymbol{u}_{h\Delta t})_t, \mathscr{P}_h \boldsymbol{v})$ and (see (2.7) of [45])

$$
\|\nabla \mathscr{P}_h \boldsymbol{v}\|_0 \le \|\nabla \boldsymbol{v}\|_0 + \|\nabla(\mathscr{P}_h \boldsymbol{v} - \boldsymbol{v})\|_0 \le C\|\nabla \boldsymbol{v}\|_0 + Ch^s\|\nabla \boldsymbol{v}\|_{s,2}.
$$

Thus it yields that

$$\|(\boldsymbol{u}_{h\Delta t})_t + \nabla \tilde{p}_{h\Delta t}\|_{L^r(0,T;(\boldsymbol{X}\cap \boldsymbol{H}^{1+s}(\Omega))')} \leq C. \tag{4.8}$$

To validate the second assertion, we define the $\boldsymbol{L}^2$-orthogonal projection to $\boldsymbol{W}_{0h}^k$ *via* $\mathscr{Q}_h$. From (3.7), we know that $(\boldsymbol{u}_{h\Delta t}, \boldsymbol{B}_{h\Delta t})$ satisfy:

$$((\boldsymbol{B}_{h\Delta t})_t, \mathscr{Q}_h\boldsymbol{C}) + \left(\tilde{\sigma}_{\Delta t}\left(\hat{\theta}_{h\Delta t}\right)\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}, \mathbf{curl}\,\mathscr{Q}_h\boldsymbol{C}\right) - \left(\tilde{\boldsymbol{u}}_{h\Delta t}\times\hat{\boldsymbol{B}}_{h\Delta t}, \mathbf{curl}\,\mathscr{Q}_h\boldsymbol{C}\right) = (\tilde{\boldsymbol{g}}_{\Delta t}, \mathscr{Q}_h\boldsymbol{C}). \tag{4.9}$$

For any $\boldsymbol{C} \in L^{r'}\left(0,T;\boldsymbol{W}\cap\boldsymbol{H}^{1+s}(\Omega)\right)$, by virtue of Lemma 3.5 and the Cauchy–Schwarz inequality, there holds

$$\int_0^T \left|\left(\tilde{\sigma}_{\Delta t}\left(\hat{\theta}_{h\Delta t}\right)\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}, \mathbf{curl}\,\mathscr{Q}_h\boldsymbol{C}\right)\right| \mathrm{d}t \leq C\|\mathbf{curl}\,\mathscr{Q}_h\boldsymbol{C}\|_{L^{r'}(0,T;\boldsymbol{L}^2(\Omega))}.$$

Adopting the same techniques as (4.6), applying Schwarz's inequality, equations (3.2), (3.3), Lemmas 3.5, 2.1 and the interpolation inequality (4.5), we readily see

$$\begin{aligned}
&\int_0^T \left|\left(\tilde{\boldsymbol{u}}_{h\Delta t}\times\hat{\boldsymbol{B}}_{h\Delta t}, \mathbf{curl}\,\mathscr{Q}_h\boldsymbol{C}\right)\right|\mathrm{d}t = \int_0^T \left|\left(\tilde{\boldsymbol{u}}_{h\Delta t}\times\left(\hat{\boldsymbol{B}}_{h\Delta t} - Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right) + Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right)\right), \mathbf{curl}\,\mathscr{Q}_h\boldsymbol{C}\right)\right|\mathrm{d}t \\
&\quad\leq \int_0^T \left|\|\tilde{\boldsymbol{u}}_{h\Delta t}\|_{0,\infty}\left\|\hat{\boldsymbol{B}}_{h\Delta t} - Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right)\right\|_0\|\mathbf{curl}\,\mathscr{Q}_h\boldsymbol{C}\|_0 + \|\tilde{\boldsymbol{u}}_{h\Delta t}\|_{0,6}\left\|Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right)\right\|_{0,3}\|\mathbf{curl}\,\mathscr{Q}_h\boldsymbol{C}\|_0\right|\mathrm{d}t \\
&\quad\leq \int_0^T \left|C_{inv}h^l\|\nabla\tilde{\boldsymbol{u}}_{h\Delta t}\|_0\left\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\right\|_0\|\mathbf{curl}\,\mathscr{Q}_h\boldsymbol{C}\|_0\right. \\
&\qquad\left. + C\|\nabla\tilde{\boldsymbol{u}}_{h\Delta t}\|_0\left\|Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right)\right\|_0^{\frac{2\delta_1}{3(1+\delta_1)}}\left\|Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right)\right\|_{0,3+\delta_1}^{\frac{\delta_1+3}{3(1+\delta_1)}}\|\mathbf{curl}\,\mathscr{Q}_h\boldsymbol{C}\|_0\right|\mathrm{d}t \\
&\quad\leq \int_0^T \left|C_{inv}C_{inv}^l\|\nabla\tilde{\boldsymbol{u}}_{h\Delta t}\|_0\left\|\hat{\boldsymbol{B}}_{h\Delta t}\right\|_0^l\left\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\right\|_0^{1-l}\|\mathbf{curl}\,\mathscr{Q}_h\boldsymbol{C}\|_0\right. \\
&\qquad + C\|\nabla\tilde{\boldsymbol{u}}_{h\Delta t}\|_0\left\|Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right) - \hat{\boldsymbol{B}}_{h\Delta t}\right\|_0^{\frac{2\delta_1}{3(1+\delta_1)}}\left\|Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right)\right\|_{0,3+\delta_1}^{\frac{\delta_1+3}{3(1+\delta_1)}}\|\mathbf{curl}\,\mathscr{Q}_h\boldsymbol{C}\|_0 \\
&\qquad\left. + C\|\nabla\tilde{\boldsymbol{u}}_{h\Delta t}\|_0\left\|\hat{\boldsymbol{B}}_{h\Delta t}\right\|_0^{\frac{2\delta_1}{3(1+\delta_1)}}\left\|Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right)\right\|_{0,3+\delta_1}^{\frac{\delta_1+3}{3(1+\delta_1)}}\|\mathbf{curl}\,\mathscr{Q}_h\boldsymbol{C}\|_0\right|\mathrm{d}t \\
&\quad\leq C\left\|\hat{\boldsymbol{B}}_{h\Delta t}\right\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}^l\left\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\right\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}^{1-l}\|\nabla\tilde{\boldsymbol{u}}_{h\Delta t}\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\|\mathbf{curl}\,\mathscr{Q}_h\boldsymbol{C}\|_{L^{\frac{2}{l}}(0,T;\boldsymbol{L}^2(\Omega))} \\
&\qquad + C\left\|\hat{\boldsymbol{B}}_{h\Delta t}\right\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}^{\left(\frac{1}{2}+l\right)\left(\frac{2\delta_1}{3(1+\delta_1)}\right)}\left\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\right\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}^{1-\left(\frac{1}{2}+l\right)\left(\frac{2\delta_1}{3(1+\delta_1)}\right)}\|\nabla\tilde{\boldsymbol{u}}_{h\Delta t}\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \\
&\qquad\times\|\mathbf{curl}\,\mathscr{Q}_h\boldsymbol{C}\|_{L^{\frac{2}{\left(\frac{1}{2}+l\right)\left(\frac{2\delta_1}{3(1+\delta_1)}\right)}}(0,T;\boldsymbol{L}^2(\Omega))} + C\left\|\hat{\boldsymbol{B}}_{h\Delta t}\right\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}^{\frac{2\delta_1}{3(1+\delta_1)}}\left\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\right\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}^{\frac{\delta_1+3}{3(1+\delta_1)}} \\
&\qquad\times\|\nabla\tilde{\boldsymbol{u}}_{h\Delta t}\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\|\mathbf{curl}\,\mathscr{Q}_h\boldsymbol{C}\|_{L^{\frac{2}{\left(\frac{2\delta_1}{3(1+\delta_1)}\right)}}(0,T;\boldsymbol{L}^2(\Omega))} \\
&\quad\leq C\|\mathbf{curl}\,\mathscr{Q}_h\boldsymbol{C}\|_{L^{r'}(0,T;\boldsymbol{L}^2(\Omega))}.
\end{aligned}$$

Combining these inequalities with (4.9), we obtain

$$\int_0^T |((\boldsymbol{B}_{h\Delta t})_t, \boldsymbol{C})|\,\mathrm{d}t \leq C\|\mathbf{curl}\,\mathscr{Q}_h\boldsymbol{C}\|_{L^{r'}(0,T;\boldsymbol{L}^2(\Omega))}. \tag{4.10}$$

Here, we need to introduce H(**curl**)-orthogonal projections to $\boldsymbol{W}_{0h}^k$ *via* $\mathscr{Q}_h^{curl}$ (see (2.12) of [45]), then we can deduce

$$\begin{aligned}\|\mathbf{curl}\,\mathscr{Q}_h\boldsymbol{C}\|_0 &\leq \left\|\mathbf{curl}\,\mathscr{Q}_h\boldsymbol{C}-\mathbf{curl}\,\mathscr{Q}_h^{curl}\boldsymbol{C}\right\|_0+\left\|\mathbf{curl}\,\mathscr{Q}_h^{curl}\boldsymbol{C}-\mathbf{curl}\,\boldsymbol{C}\right\|_0+\|\mathbf{curl}\,\boldsymbol{C}\|_0\\ &\leq C_{inv}h^{-1}\left\|\mathscr{Q}_h\boldsymbol{C}-\boldsymbol{C}+\boldsymbol{C}-\mathscr{Q}_h^{curl}\boldsymbol{C}\right\|_0+Ch^s\|\mathbf{curl}\,\boldsymbol{C}\|_{s,2}+\|\mathbf{curl}\,\boldsymbol{C}\|_0\\ &\leq Ch^s\|\boldsymbol{C}\|_{1+s,2}+C\|\mathbf{curl}\,\boldsymbol{C}\|_0.\end{aligned}$$

Thus it yields that

$$\|(\boldsymbol{B}_{h\Delta t})_t\|_{L^r(0,T;(\boldsymbol{W}\cap\boldsymbol{H}^{1+s}(\Omega))')}\leq C. \tag{4.11}$$

Similarly, we define the L^2 projection to Y_{0h}^k *via* $\mathscr{R}_h$. By (3.8), we know that $(\boldsymbol{u}_{h\Delta t},\theta_{h\Delta t})$ satisfy:

$$((\theta_{h\Delta t})_t,\mathscr{R}_h\varphi)+\left(\tilde{\kappa}_{\Delta t}\left(\hat{\theta}_{h\Delta t}\right)\nabla\tilde{\theta}_{h\Delta t},\nabla\mathscr{R}_h\varphi\right)+\mathcal{O}_2\left(\hat{\boldsymbol{u}}_{h\Delta t},\tilde{\theta}_{h\Delta t},\mathscr{R}_h\varphi\right)=\left(\tilde{\psi}_{\Delta t},\mathscr{R}_h\varphi\right). \tag{4.12}$$

For any $\varphi\in L^4\left(0,T;Y_0\cap H^{1+s}(\Omega)\right)$, by virtue of Lemma 3.5, the Cauchy–Schwarz inequality and the interpolation inequality (4.4), there holds

$$\begin{aligned}&\int_0^T\left|\frac{1}{2}\left[\left(\hat{\boldsymbol{u}}_{h\Delta t}\cdot\nabla\tilde{\theta}_{h\Delta t},\mathscr{R}_h\varphi\right)-\left(\hat{\boldsymbol{u}}_{h\Delta t}\cdot\nabla\mathscr{R}_h\varphi,\tilde{\theta}_{h\Delta t}\right)\right]\right|\mathrm{d}t\\ &\quad\leq C\int_0^T\left[\|\hat{\boldsymbol{u}}_{h\Delta t}\|_0^{1/2}\|\nabla\hat{\boldsymbol{u}}_{h\Delta t}\|_0^{1/2}\left\|\nabla\tilde{\theta}_{h\Delta t}\right\|_0\|\nabla\mathscr{R}_h\varphi\|_0\right]\mathrm{d}t\\ &\quad\leq C\|\nabla\hat{\boldsymbol{u}}_{h\Delta t}\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}^{1/2}\left\|\nabla\tilde{\theta}_{h\Delta t}\right\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\|\nabla\mathscr{R}_h\varphi\|_{L^4(0,T;\boldsymbol{L}^2(\Omega))}\leq C\|\nabla\mathscr{R}_h\varphi\|_{L^4(0,T;\boldsymbol{L}^2(\Omega))}\end{aligned}$$

and

$$\int_0^T\left|\left(\tilde{\kappa}_{\Delta t}\left(\hat{\theta}_{h\Delta t}\right)\nabla\tilde{\theta}_{h\Delta t},\nabla\mathscr{R}_h\varphi\right)\right|\mathrm{d}t\leq C\|\nabla\mathscr{R}_h\varphi\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}.$$

Combining these inequalities with (4.12), applying the properties of L^2-orthogonal projection, then we can arrive at

$$\int_0^T|((\theta_{h\Delta t})_t,\varphi)|\,\mathrm{d}t\leq C\|\nabla\mathscr{R}_h\varphi\|_{L^4(0,T;\boldsymbol{L}^2(\Omega))}, \tag{4.13}$$

using the same techniques as demonstrated in (4.8), which implies that

$$\|(\theta_{h\Delta t})_t\|_{L^{4/3}(0,T;(Y\cap H^{1+s}(\Omega))')}\leq C, \tag{4.14}$$

then the results now follows. □

Remark 4.3. *A priori* stability estimate of the above time derivatives for discrete finite element solution plays a key role in the subsequent strong convergence of the scheme. Such type of estimate for MHD model was first developed in [45]. The proof therein seems to be not complete and there are some minor gaps in the bounds on the important Lorentz force terms (*e.g.*, the estimates in line 19 and line 20 of page 1073 are not valid and similar problem arises in line 2-3 of page 1074). Here we give a rigorous proof with a different index r.

Concerning the discrete pressure solution, it enjoys the following stability estimate without any restrictions on h and Δt.

Lemma 4.4. *For the sequence* $\{\boldsymbol{u}_{h\Delta t}, \tilde{p}_{h\Delta t}\}$, *we have*

$$\|(\boldsymbol{u}_{h\Delta t})_t\|_{L^r(0,T;(\boldsymbol{X}\cap \boldsymbol{H}^{1+s}(\Omega))')} \leq C, \qquad \|\tilde{p}_{h\Delta t}\|_{L^r(0,T;L^2(\Omega)\cap H^{-s,2}(\Omega))} \leq C. \tag{4.15}$$

Proof. For any $\boldsymbol{v} \in L^{r'}\big(0,T;\boldsymbol{V}\cap \boldsymbol{H}^{1+s}(\Omega)\big)$, by (4.3), we have

$$\begin{aligned}((\boldsymbol{u}_{h\Delta t})_t, \mathscr{P}_h\boldsymbol{v}) + \frac{1}{2}[((\hat{\boldsymbol{u}}_{h\Delta t}\cdot\nabla)\tilde{\boldsymbol{u}}_{h\Delta t}, \mathscr{P}_h\boldsymbol{v}) - ((\hat{\boldsymbol{u}}_{h\Delta t}\cdot\nabla)\mathscr{P}_h\boldsymbol{v}, \tilde{\boldsymbol{u}}_{h\Delta t})] + \Big(\tilde{\nu}_{\Delta t}\Big(\hat{\theta}_{h\Delta t}\Big)\nabla\tilde{\boldsymbol{u}}_{h\Delta t}, \nabla\mathscr{P}_h\boldsymbol{v}\Big)\\ + \mu\Big(\hat{\boldsymbol{B}}_{h\Delta t}\times \mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}, \mathscr{P}_h\boldsymbol{v}\Big) - \Big(\tilde{\boldsymbol{\beta}}_{\Delta t}\Big(\hat{\theta}_{h\Delta t}\Big)\tilde{\theta}_{h\Delta t}, \mathscr{P}_h\boldsymbol{v}\Big) = \Big(\tilde{\boldsymbol{f}}_{\Delta t}, \mathscr{P}_h\boldsymbol{v}\Big).\end{aligned} \tag{4.16}$$

Using the similar method as in the proof of Lemma 4.2, we can obtain

$$\int_0^T |((\boldsymbol{u}_{h\Delta t})_t, \boldsymbol{v})|\,\mathrm{d}t \leq C\|\nabla\boldsymbol{v}\|_{L^{r'}(0,T;\boldsymbol{L}^2(\Omega))} + Ch^s\|\nabla\boldsymbol{v}\|_{L^{r'}(0,T;\boldsymbol{H}^s(\Omega))}. \tag{4.17}$$

For all $\boldsymbol{v} \in L^{r'}\big(0,T;\boldsymbol{X}\cap\boldsymbol{H}^{1+s}(\Omega)\big)$, by virtue of (4.7), (4.17) and Lemma 4.2, there holds

$$\begin{aligned}\int_0^T |(\nabla\tilde{p}_{h\Delta t}, \mathscr{P}_h\boldsymbol{v})|\,\mathrm{d}t = \int_0^T |(\nabla\tilde{p}_{h\Delta t}, \boldsymbol{v})|\,\mathrm{d}t \leq \int_0^T |(\nabla\tilde{p}_{h\Delta t} + (\boldsymbol{u}_{h\Delta t})_t, \boldsymbol{v})|\,\mathrm{d}t + \int_0^T |((\boldsymbol{u}_{h\Delta t})_t, \boldsymbol{v})|\,\mathrm{d}t\\ \leq C\|\nabla\boldsymbol{v}\|_{L^{r'}(0,T;\boldsymbol{L}^2(\Omega))} + Ch^s\|\nabla\boldsymbol{v}\|_{L^{r'}(0,T;\boldsymbol{H}^s(\Omega))},\end{aligned}$$

which implies that

$$\|\nabla\tilde{p}_{h\Delta t}\|_{L^r(0,T;\boldsymbol{H}^{-1}(\Omega)\cap\boldsymbol{H}^{-1-s,2}(\Omega))} \leq C. \tag{4.18}$$

Then we can obtain $\|(\boldsymbol{u}_{h\Delta t})_t\|_{L^r(0,T;(\boldsymbol{X}\cap\boldsymbol{H}^{1+s}(\Omega))')} \leq C$ by using Lemma 4.2. Thanks to the Poincaré type inequality (Cor. 2.1 of [23]), we find that

$$\|\tilde{p}_{h\Delta t}\|_{L^r(0,T;L^2(\Omega)\cap H^{-s,2}(\Omega))} \leq \|\nabla\tilde{p}_{h\Delta t}\|_{L^r(0,T;\boldsymbol{H}^{-1}(\Omega)\cap\boldsymbol{H}^{-1-s,2}(\Omega))} \leq C, \tag{4.19}$$

then the conclusion now follows. □

We also need to recall the Aubin-Lions' compactness result for Bochner spaces (refer to Lem. 2.8 of [21]).

Lemma 4.5. *Let* F *be a Banach space,* F_0 *and* F_1 *be two reflexive Banach spaces. Assume* $F_0 \Subset F$ *with compact injection,* $F \subset F_1$ *with continuous injection. Then the space*

$$\left\{w\middle| w\in L^{p_0}(0,T;F_0), \frac{\partial w}{\partial t}\in L^{p_1}(0,T;F_1)\right\} \Subset L^{p_0}(0,T;F)$$

with $1 < p_0 < +\infty$, $1 < p_1 < +\infty$.

Next, we present some basic convergence results for the fully discrete solution in the following theorem.

Theorem 4.6. *There exist functions* $\boldsymbol{u} \in L^\infty\big(0,T;\boldsymbol{L}^2(\Omega)\big)\cap L^2\big(0,T;\boldsymbol{H}_0^1(\Omega)\big), \boldsymbol{B} \in L^\infty\big(0,T;\boldsymbol{L}^2(\Omega)\big)\cap L^2(0,T;\boldsymbol{W}), p\in L^r\big(0,T;L^2(\Omega)\big), \theta\in L^\infty\big(0,T;L^2(\Omega)\big)\cap L^2\big(0,T;H_0^1(\Omega)\big)$, *such that, as* $h,\Delta t\to 0$,

$$\begin{aligned}
&\tilde{\boldsymbol{u}}_{h\Delta t},\, \hat{\boldsymbol{u}}_{h\Delta t},\, \boldsymbol{u}_{h\Delta t} \rightharpoonup \boldsymbol{u} && \text{weakly}^* && \text{in} && L^\infty\big(0,T;\boldsymbol{L}^2(\Omega)\big),\\
&\tilde{\boldsymbol{B}}_{h\Delta t},\, \hat{\boldsymbol{B}}_{h\Delta t},\, \boldsymbol{B}_{h\Delta t} \rightharpoonup \boldsymbol{B} && \text{weakly}^* && \text{in} && L^\infty\big(0,T;\boldsymbol{L}^2(\Omega)\big),\\
&\tilde{\theta}_{h\Delta t},\, \hat{\theta}_{h\Delta t},\, \theta_{h\Delta t} \rightharpoonup \theta && \text{weakly}^* && \text{in} && L^\infty\big(0,T;L^2(\Omega)\big),\\
&\tilde{\boldsymbol{u}}_{h\Delta t},\, \hat{\boldsymbol{u}}_{h\Delta t},\, \boldsymbol{u}_{h\Delta t} \rightharpoonup \boldsymbol{u} && \text{weakly} && \text{in} && L^2\big(0,T;\boldsymbol{H}_0^1(\Omega)\big),
\end{aligned}$$

$$
\begin{aligned}
&\tilde{\boldsymbol{B}}_{h\Delta t},\, \hat{\boldsymbol{B}}_{h\Delta t},\, \boldsymbol{B}_{h\Delta t} \rightharpoonup \boldsymbol{B} && \text{weakly} && \text{in} && L^2(0,T;\boldsymbol{W}),\\
&\tilde{\theta}_{h\Delta t},\, \hat{\theta}_{h\Delta t},\, \theta_{h\Delta t} \rightharpoonup \theta && \text{weakly} && \text{in} && L^2\big(0,T;H_0^1(\Omega)\big),\\
&\tilde{p}_{h\Delta t} \rightharpoonup p && \text{weakly} && \text{in} && L^r\big(0,T;L^2(\Omega)\cap H^{-s,2}(\Omega)\big),\\
&(\boldsymbol{u}_{h\Delta t})_t \rightharpoonup \boldsymbol{u}_t && \text{weakly} && \text{in} && L^r\Big(0,T;\big(\boldsymbol{X}\cap \boldsymbol{H}^{1+s}(\Omega)\big)'\Big),\\
&(\boldsymbol{B}_{h\Delta t})_t \rightharpoonup \boldsymbol{B}_t && \text{weakly} && \text{in} && L^r\Big(0,T;\big(\boldsymbol{W}\cap \boldsymbol{H}^{1+s}(\Omega)\big)'\Big),\\
&(\theta_{h\Delta t})_t \rightharpoonup \theta_t && \text{weakly} && \text{in} && L^{4/3}\Big(0,T;\big(Y\cap H^{1+s}(\Omega)\big)'\Big),\\
&\tilde{\boldsymbol{u}}_{h\Delta t},\, \hat{\boldsymbol{u}}_{h\Delta t},\, \boldsymbol{u}_{h\Delta t} \to \boldsymbol{u} && && \text{in} && L^2(0,T;\boldsymbol{L}^q(\Omega)),\\
&\tilde{\theta}_{h\Delta t},\, \hat{\theta}_{h\Delta t},\, \theta_{h\Delta t} \to \theta && && \text{in} && L^2(0,T;L^q(\Omega)),\\
&\tilde{\boldsymbol{B}}_{h\Delta t},\, \hat{\boldsymbol{B}}_{h\Delta t},\, \boldsymbol{B}_{h\Delta t} \to \boldsymbol{B} && && \text{in} && L^2\big(0,T;\boldsymbol{L}^2(\Omega)\big),
\end{aligned}
$$

with $1 \le q < 6$*, where* $\to$ *means strong convergence,* $\rightharpoonup$ *means weak convergence.*

Proof. The statements of Lemma 3.5 can imply that $\Big\{\Big(\tilde{\boldsymbol{u}}_{h\Delta t},\tilde{p}_{h\Delta t},\tilde{\boldsymbol{B}}_{h\Delta t},\tilde{\theta}_{h\Delta t}\Big)\Big\}$, $\Big\{\Big(\hat{\boldsymbol{u}}_{h\Delta t},\hat{\boldsymbol{B}}_{h\Delta t},\hat{\theta}_{h\Delta t}\Big)\Big\}$ and $\{(\boldsymbol{u}_{h\Delta t},\boldsymbol{B}_{h\Delta t},\theta_{h\Delta t})\}$ are all bounded sequences and thus have the corresponding weak convergent subsequence (see *e.g.* [51]). The other weak convergence results can be deduced by the statements of Lemmas 4.2 and 4.4. We say that the above three subsequences (still denoted by the same notations) enjoy the same accumulation function $(\boldsymbol{u},\boldsymbol{B},\theta)$. In fact, applying the interpolation inequality, Hölder inequality and Lemma 3.5, we have

$$
\begin{aligned}
\Big\|\theta_{h\Delta t}-\tilde{\theta}_{h\Delta t}\Big\|^2_{L^2(0,T;L^q(\Omega))} &= \frac{\Delta t}{3}\sum_{n=1}^{m}\big\|\theta_h^n-\theta_h^{n-1}\big\|^2_{L^q(\Omega)}\\
&\le \frac{\Delta t}{3}\sum_{n=1}^{m}\big\|\theta_h^n-\theta_h^{n-1}\big\|^{2\alpha}_{L^1(\Omega)}\big\|\theta_h^n-\theta_h^{n-1}\big\|^{2-2\alpha}_{L^6(\Omega)}\\
&\le C\left(\sum_{n=1}^{m}\Delta t\big\|\theta_h^n-\theta_h^{n-1}\big\|^2_{L^1(\Omega)}\right)^{\alpha}\left(\sum_{n=1}^{m}\Delta t\big\|\theta_h^n-\theta_h^{n-1}\big\|^2_{L^6(\Omega)}\right)^{1-\alpha}\\
&\le C(\Delta t)^{\alpha}\left(\sum_{n=1}^{m}\big\|\theta_h^n-\theta_h^{n-1}\big\|^2_{L^2(\Omega)}\right)^{\alpha}\left(\sum_{n=1}^{m}\Delta t\big\|\theta_h^n-\theta_h^{n-1}\big\|^2_{L^6(\Omega)}\right)^{1-\alpha}\xrightarrow{\Delta t\to 0} 0,
\end{aligned}
$$

with $\alpha=\frac{6-q}{5q}$, and we continue to obtain

$$
\Big\|\theta_{h\Delta t}-\hat{\theta}_{h\Delta t}\Big\|^2_{L^2(0,T;L^q(\Omega))}\xrightarrow{\Delta t\to 0} 0,
$$

which implies $\{\theta_{h\Delta t}\}$, $\Big\{\tilde{\theta}_{h\Delta t}\Big\}$ and $\Big\{\hat{\theta}_{h\Delta t}\Big\}$ converge to the same limit θ, as $h,\Delta t\to 0$. Furthermore, they converge strongly to θ in $L^2(0,T;L^q(\Omega))$ by a combination of Lemmas 4.2 and 4.5. Similarly, we can show that $\{\boldsymbol{u}_{h\Delta t}\}$, $\{\tilde{\boldsymbol{u}}_{h\Delta t}\}$ and $\{\hat{\boldsymbol{u}}_{h\Delta t}\}$ converge strongly to $\boldsymbol{u}$ in $L^2(0,T;\boldsymbol{L}^q(\Omega))$ and $\{\boldsymbol{B}_{h\Delta t}\}$, $\Big\{\tilde{\boldsymbol{B}}_{h\Delta t}\Big\}$ and $\Big\{\hat{\boldsymbol{B}}_{h\Delta t}\Big\}$ converge strongly to $\boldsymbol{B}$ in $L^2\big(0,T;\boldsymbol{L}^2(\Omega)\big)$ as $h,\Delta t\to 0$. The proof is completed. □

Considering that the viscosity coefficients $\nu(\cdot)$, $\sigma(\cdot)$, $\kappa(\cdot)$ and $\boldsymbol{\beta}(\cdot)$ are assumed to be in $\mathscr{C}^{0,1}\big(\bar{\Omega}\times\mathbb{R};\mathbb{R}\big)$, here we may define $|\cdot|_{\mathscr{C}^{0,1}(\bar{\Omega}\times\mathbb{R};\mathbb{R})}$ by

$$
|\lambda|_{\mathscr{C}^{0,1}(\bar{\Omega}\times\mathbb{R};\mathbb{R})}=\sup\left\{\frac{|\lambda(x,\theta)-\lambda(y,\chi)|}{|(x,\theta)-(y,\chi)|};(x,\theta),(y,\chi)\in\bar{\Omega}\times\mathbb{R}\right\}, \tag{4.20}
$$

where λ can be taken as ν, σ, κ and $\boldsymbol{\beta}$.

Now, we will prove that the accumulation function $(\boldsymbol{u}, \boldsymbol{B}, \theta)$ is indeed a weak solution to (2.5)–(2.7), which provides a numerical version of the existence analysis of the thermally coupled incompressible MHD problems with temperature-dependent coefficients.

Theorem 4.7. *Suppose that the initial values satisfy* $\boldsymbol{u}(0) = \boldsymbol{u}^0$, $\boldsymbol{B}(0) = \boldsymbol{B}^0$, $\theta(0) = \theta^0$, *and* $\lim_{\Delta t\to 0}\left\|\tilde{\boldsymbol{f}}_{\Delta t} - \boldsymbol{f}\right\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} = 0$, $\lim_{\Delta t\to 0}\|\tilde{\boldsymbol{g}}_{\Delta t} - \boldsymbol{g}\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} = 0$, $\lim_{\Delta t\to 0}\left\|\tilde{\psi}_{\Delta t} - \psi\right\|_{L^2(0,T;L^2(\Omega))} = 0$. *Then there exists a subsequence of* $\{\tilde{\boldsymbol{u}}_{h\Delta t}\}$, $\left\{\tilde{\boldsymbol{B}}_{h\Delta t}\right\}$, $\left\{\tilde{\theta}_{h\Delta t}\right\}$ *and* $\{\tilde{p}_{h\Delta t}\}$ *converges to* $(\boldsymbol{u}, \boldsymbol{B}, \theta, p)$, *which is a weak solution of (2.5)–(2.7) as* $h, \Delta t \to 0$.

Proof. According to the approximation properties of finite element space, for any $\boldsymbol{v} \in \mathscr{C}_0^\infty(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$, there exists a function $\boldsymbol{v}_h = \boldsymbol{H}_h\boldsymbol{v} \in \boldsymbol{V}_h^k$ such that

$$\boldsymbol{v}_h \xrightarrow{h\to 0} \boldsymbol{v} \qquad \text{in } \boldsymbol{H}_0^1(\Omega),$$

where $\boldsymbol{H}_h$ is the H^1-orthogonal projection operator to $\boldsymbol{V}_h^k$ (see [7,9]). For any $\delta(t) \in \mathscr{C}^\infty([0,T])$, by virtue of the Young inequality, Lemma 3.5, (4.20) and Theorem 4.6, we can verify the estimates one by one,

$$\begin{aligned}
&\left|\int_0^T \left[\left(\tilde{\nu}_{\Delta t}\left(\hat{\theta}_{h\Delta t}\right)\nabla\tilde{\boldsymbol{u}}_{h\Delta t}, \nabla\boldsymbol{v}_h\right) - (\nu(\theta)\nabla\boldsymbol{u}, \nabla\boldsymbol{v})\right]\delta(t)\,\mathrm{d}t\right| \\
&\quad\le C\left\|(\tilde{\nu}_{\Delta t} - \nu)\left(\hat{\theta}_{h\Delta t}\right)\right\|_{L^2(0,T;L^4(\Omega))}\|\nabla\tilde{\boldsymbol{u}}_{h\Delta t}\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\|\nabla\boldsymbol{v}_h\,\delta(t)\|_{L^\infty(0,T;\boldsymbol{L}^4(\Omega))} \\
&\qquad + C|\nu|_{\mathscr{C}^{0,1}\left(\bar{\Omega}\times\mathbb{R};\mathbb{R}\right)}\left\|\hat{\theta}_{h\Delta t} - \theta\right\|_{L^2(0,T;L^4(\Omega))}\|\nabla\tilde{\boldsymbol{u}}_{h\Delta t}\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\|\nabla\boldsymbol{v}_h\,\delta(t)\|_{L^\infty(0,T;\boldsymbol{L}^4(\Omega))} \\
&\qquad + C\|\nu\|_{\mathscr{C}\left(\bar{\Omega}\times\mathbb{R};\mathbb{R}^+\right)}\|\nabla\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\|[\nabla\boldsymbol{v}_h - \nabla\boldsymbol{v}]\delta(t)\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \\
&\qquad + C\|\nu\|_{\mathscr{C}\left(\bar{\Omega}\times\mathbb{R};\mathbb{R}^+\right)}\left|\int_0^T (\nabla\tilde{\boldsymbol{u}}_{h\Delta t} - \nabla\boldsymbol{u}, \nabla\boldsymbol{v}_h)\,\delta(t)\,\mathrm{d}t\right| \xrightarrow{h,\Delta t\to 0} 0,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_0^T [\mathcal{O}_1(\hat{\boldsymbol{u}}_{h\Delta t}, \tilde{\boldsymbol{u}}_{h\Delta t}, \boldsymbol{v}_h) - \mathcal{O}_1(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})]\delta(t)\,\mathrm{d}t\right| \\
&\quad\le C\|\hat{\boldsymbol{u}}_{h\Delta t} - \boldsymbol{u}\|_{L^2(0,T;\boldsymbol{L}^4(\Omega))}\|\nabla\tilde{\boldsymbol{u}}_{h\Delta t}\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\|\boldsymbol{v}_h\delta(t)\|_{L^\infty(0,T;\boldsymbol{L}^4(\Omega))} \\
&\qquad + C\|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{L}^4(\Omega))}\|\nabla\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\|(\boldsymbol{v}_h - \boldsymbol{v})\delta(t)\|_{L^\infty(0,T;\boldsymbol{L}^4(\Omega))} \\
&\qquad + C\|\hat{\boldsymbol{u}}_{h\Delta t} - \boldsymbol{u}\|_{L^2(0,T;\boldsymbol{L}^4(\Omega))}\|\nabla\boldsymbol{v}_h\delta(t)\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\|\tilde{\boldsymbol{u}}_{h\Delta t}\|_{L^2(0,T;\boldsymbol{L}^4(\Omega))} \\
&\qquad + C\|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{L}^4(\Omega))}\|\nabla\boldsymbol{v}_h\delta(t)\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\|\tilde{\boldsymbol{u}}_{h\Delta t} - \boldsymbol{u}\|_{L^2(0,T;\boldsymbol{L}^4(\Omega))} \\
&\qquad + C\|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{L}^4(\Omega))}\|\nabla(\boldsymbol{v}_h - \boldsymbol{v})\delta(t)\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{L}^4(\Omega))} \\
&\qquad + C\left|\int_0^T ((\boldsymbol{u}\cdot\nabla)(\tilde{\boldsymbol{u}}_{h\Delta t} - \boldsymbol{u}), \boldsymbol{v}_h)\,\delta(t)\,\mathrm{d}t\right| \xrightarrow{h,\Delta t\to 0} 0.
\end{aligned}$$

On the other hand, by applying (3.2), inverse inequality (3.3) and Lemma 3.5, we derive

$$\left|\int_0^T \left[\mu\left(\hat{\boldsymbol{B}}_{h\Delta t} \times \mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}, \boldsymbol{v}_h\right) - \mu(\boldsymbol{B} \times \mathbf{curl}\,\boldsymbol{B}, \boldsymbol{v})\right]\delta(t)\,\mathrm{d}t\right|$$

$$
\begin{aligned}
&= \left|\int_0^T \left[\mu\Big(\Big[\hat{\boldsymbol{B}}_{h\Delta t} - Z\Big(\hat{\boldsymbol{B}}_{h\Delta t}\Big)\Big] \times \mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}, \boldsymbol{v}_h\Big)\right.\right.\\
&\quad \left.\left. +\ \mu\Big(Z\Big(\hat{\boldsymbol{B}}_{h\Delta t}\Big) \times \mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}, \boldsymbol{v}_h\Big) - \mu(\boldsymbol{B} \times \mathbf{curl}\,\boldsymbol{B}, \boldsymbol{v})\right]\delta(t)\,\mathrm{d}t\right|\\
&\le Ch^{l+\frac{1}{2}}\Big\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\Big\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\Big\|\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}\Big\|_{L^2(0,T;\boldsymbol{L}^3(\Omega))}\|\boldsymbol{v}_h\delta(t)\|_{L^\infty(0,T;\boldsymbol{L}^6(\Omega))}\\
&\quad + \left|\int_0^T \left[\mu\Big(Z\Big(\hat{\boldsymbol{B}}_{h\Delta t}\Big) \times \mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}, \boldsymbol{v}_h\Big) - \mu(\boldsymbol{B} \times \mathbf{curl}\,\boldsymbol{B}, \boldsymbol{v})\right]\delta(t)\,\mathrm{d}t\right|\\
&\le Ch^{l}\Big\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\Big\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\Big\|\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}\Big\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\|\boldsymbol{v}_h\,\delta(t)\|_{L^\infty(0,T;\boldsymbol{L}^6(\Omega))}\\
&\quad + \left|\int_0^T \left[\mu\Big(Z\Big(\hat{\boldsymbol{B}}_{h\Delta t}\Big) \times \mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}, \boldsymbol{v}_h\Big) - \mu(\boldsymbol{B} \times \mathbf{curl}\,\boldsymbol{B}, \boldsymbol{v})\right]\delta(t)\,\mathrm{d}t\right|\\
&=: F_1 + |F_2|
\end{aligned}
$$

as $h \to 0$, it can be clearly deduced $F_1 = Ch^l\Big\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\Big\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\Big\|\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}\Big\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}$ $\|\boldsymbol{v}_h\,\delta(t)\|_{L^\infty(0,T;\boldsymbol{L}^6(\Omega))} \to 0$. Next, we just need to show $|F_2| \to 0$ when $h, \Delta t \to 0$.

From Lemma 3.5, we know that $\Big\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\Big\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \le C$. Thanks to $\operatorname{div} Z\Big(\hat{\boldsymbol{B}}_{h\Delta t}\Big) = 0$, we can derive

$$
\begin{aligned}
\Big\|Z\Big(\hat{\boldsymbol{B}}_{h\Delta t}\Big)\Big\|_{L^2(0,T;\boldsymbol{\mathcal{H}}(\Omega))} &= \Big\|\mathbf{curl}\,Z\Big(\hat{\boldsymbol{B}}_{h\Delta t}\Big)\Big\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\\
&= \Big\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\Big\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \le C.
\end{aligned} \tag{4.21}
$$

Noticing that $\boldsymbol{\mathcal{H}}(\Omega) \hookrightarrow\hookrightarrow \boldsymbol{H}^s(\Omega) \hookrightarrow\hookrightarrow \boldsymbol{L}^3(\Omega)$, where $\hookrightarrow\hookrightarrow$ means the compact imbedding, $s > 1/2$ is a constant depending on Ω (*cf.* [48]). We can choose $\delta_2 > 0$ such that $1/(3+\delta_1) + 1/(6-\delta_2) = 1/2$ and $\boldsymbol{H}^1(\Omega) \hookrightarrow\hookrightarrow \boldsymbol{L}^{6-\delta_2}(\Omega)$, together with the fact that $\tilde{\boldsymbol{B}}_{h\Delta t}$ converges to $\boldsymbol{B}$ in the sense of weak convergence in $L^2(0,T;\boldsymbol{H}(\mathbf{curl};\Omega))$ according to Theorem 4.6, then we have

$$
\begin{aligned}
|F_2| &= \left|\int_0^T \left[\mu\Big(Z\Big(\hat{\boldsymbol{B}}_{h\Delta t}\Big) \times \mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t}, \boldsymbol{v}_h\Big) - \mu(\boldsymbol{B} \times \mathbf{curl}\,\boldsymbol{B}, \boldsymbol{v})\right]\delta(t)\,\mathrm{d}t\right|\\
&\le C\mu\Big\|Z\Big(\hat{\boldsymbol{B}}_{h\Delta t}\Big) - \boldsymbol{B}\Big\|_{L^2(0,T;\boldsymbol{L}^3(\Omega))}\|\mathbf{curl}\,\boldsymbol{B}\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\|\boldsymbol{v}_h\delta(t)\|_{L^\infty(0,T;\boldsymbol{L}^6(\Omega))}\\
&\quad + C\mu\|\boldsymbol{B}\|_{L^2(0,T;\boldsymbol{\mathcal{H}}(\Omega))}\|\mathbf{curl}\,\boldsymbol{B}\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\|(\boldsymbol{v}_h - \boldsymbol{v})\delta(t)\|_{L^\infty\left(0,T;\boldsymbol{L}^{6-\delta_2}(\Omega)\right)}\\
&\quad + C\mu\left|\int_0^T \Big(Z\Big(\hat{\boldsymbol{B}}_{h\Delta t}\Big) \times \Big(\mathbf{curl}\,\tilde{\boldsymbol{B}}_{h\Delta t} - \mathbf{curl}\,\boldsymbol{B}\Big), \boldsymbol{v}_h\Big)\,\delta(t)\,\mathrm{d}t\right| \xrightarrow{h,\Delta t \to 0} 0,
\end{aligned}
$$

where we have used the fact that

$$
\begin{aligned}
\Big\|Z\Big(\hat{\boldsymbol{B}}_{h\Delta t}\Big) - \boldsymbol{B}\Big\|_{L^2(0,T;\boldsymbol{L}^3(\Omega))} &\le \Big\|Z\Big(\hat{\boldsymbol{B}}_{h\Delta t}\Big) - \boldsymbol{B}\Big\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}^{\frac{2\delta_1}{3(1+\delta_1)}}\Big\|Z\Big(\hat{\boldsymbol{B}}_{h\Delta t}\Big) - \boldsymbol{B}\Big\|_{L^2\left(0,T;\boldsymbol{L}^{3+\delta_1}(\Omega)\right)}^{\frac{3+\delta_1}{3(1+\delta_1)}}\\
&\le C\Big\|Z\Big(\hat{\boldsymbol{B}}_{h\Delta t}\Big) - \boldsymbol{B}\Big\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}^{\frac{2\delta_1}{3(1+\delta_1)}}\Big\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t} - \mathbf{curl}\,\boldsymbol{B}\Big\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}^{\frac{3+\delta_1}{3(1+\delta_1)}}
\end{aligned}
$$

and

$$
\Big\|Z\Big(\hat{\boldsymbol{B}}_{h\Delta t}\Big) - \boldsymbol{B}\Big\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}^{\frac{2\delta_1}{3(1+\delta_1)}} \le \left(\Big\|Z\Big(\hat{\boldsymbol{B}}_{h\Delta t}\Big) - \hat{\boldsymbol{B}}_{h\Delta t}\Big\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} + \Big\|\hat{\boldsymbol{B}}_{h\Delta t} - \boldsymbol{B}\Big\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\right)^{\frac{2\delta_1}{3(1+\delta_1)}}
$$

$$\leq \left(Ch^{(l+\frac{1}{2})} \left\| \mathbf{curl}\, \hat{\boldsymbol{B}}_{h\Delta t} \right\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} + \left\| \hat{\boldsymbol{B}}_{h\Delta t} - \boldsymbol{B} \right\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \right)^{\frac{2\delta_1}{3(1+\delta_1)}}$$
$$\xrightarrow{h,\Delta t\to 0} 0.$$

Based on Theorem 4.6 and the definition of weak convergence, there hold

$$\int_0^T (\operatorname{div} \boldsymbol{v}_h, \tilde{p}_{h\Delta t})\, \delta(t)\, \mathrm{d}t \xrightarrow{h,\Delta t\to 0} \int_0^T (\operatorname{div} \boldsymbol{v}, p)\, \delta(t)\, \mathrm{d}t, \quad \int_0^T (\operatorname{div} \tilde{\boldsymbol{u}}_{h\Delta t}, q_h)\, \delta(t)\, \mathrm{d}t \xrightarrow{h,\Delta t\to 0} \int_0^T (\operatorname{div} \boldsymbol{u}, q)\, \delta(t)\, \mathrm{d}t.$$

By using Lemma 4.4, so we can extract a subsequence of $\{(\boldsymbol{u}_{h\Delta t})_t\}$, which has common subscript and is denoted by the same notation such that

$$\left| \int_0^T [((\boldsymbol{u}_{h\Delta t})_t, \boldsymbol{v}_h) - (\partial_t \boldsymbol{u}, \boldsymbol{v})]\delta(t)\, \mathrm{d}t \right| \xrightarrow{h,\Delta t\to 0} 0.$$

Hence it yields that

$$\begin{aligned} \int_0^T [(\boldsymbol{u}_t, \boldsymbol{v}) + (\nu(\theta)\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + \mathcal{O}_1(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + \mu(\boldsymbol{B} \times \mathbf{curl}\, \boldsymbol{B}, \boldsymbol{v}) - (\boldsymbol{\beta}(\theta)\theta, \boldsymbol{v}) \\ - (\operatorname{div} \boldsymbol{v}, p) + (\operatorname{div} \boldsymbol{u}, q)]\delta(t)\, \mathrm{d}t = \int_0^T (\boldsymbol{f}, \boldsymbol{v})\delta(t)\, \mathrm{d}t. \end{aligned} \tag{4.22}$$

In the next step, for any $\boldsymbol{C} \in \mathscr{C}_0^\infty(\Omega) \cap \boldsymbol{W}_0$, there exists $\boldsymbol{C}_h = \boldsymbol{O}_h \boldsymbol{C} \in \boldsymbol{W}_{0h}^k$ such that

$$\boldsymbol{C}_h \xrightarrow{h\to 0} \boldsymbol{C} \qquad \text{in } \boldsymbol{W}_0,$$

where $\boldsymbol{O}_h$ is the $H(\mathrm{curl})$-orthogonal projection operator to $\boldsymbol{W}_{0h}^k$. Making use of Young's inequality, Lemma 3.5, (4.20) and Theorem 4.6, there holds

$$\begin{aligned} &\left| \int_0^T \left[\left(\tilde{\sigma}_{\Delta t}\left(\hat{\theta}_{h\Delta t}\right) \mathbf{curl}\, \tilde{\boldsymbol{B}}_{h\Delta t}, \mathbf{curl}\, \boldsymbol{C}_h \right) - (\sigma(\theta)\mathbf{curl}\, \boldsymbol{B}, \mathbf{curl}\, \boldsymbol{C}) \right] \delta(t)\, \mathrm{d}t \right| \\ &\quad \leq C \left\| (\tilde{\sigma}_{\Delta t} - \sigma)\left(\hat{\theta}_{h\Delta t}\right) \right\|_{L^2(0,T;L^6(\Omega))} \left\| \mathbf{curl}\, \tilde{\boldsymbol{B}}_{h\Delta t} \right\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \| \mathbf{curl}\, \boldsymbol{C}_h \delta(t) \|_{L^\infty(0,T;\boldsymbol{L}^3(\Omega))} \\ &\qquad + C|\sigma|_{\mathscr{C}^{0,1}\left(\bar{\Omega}\times\mathbb{R};\mathbb{R}\right)} \left\| \hat{\theta}_{h\Delta t} - \theta \right\|_{L^2\left(0,T;L^{6-\delta_2}(\Omega)\right)} \left\| \mathbf{curl}\, \tilde{\boldsymbol{B}}_{h\Delta t} \right\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \| \mathbf{curl}\, \boldsymbol{C}_h \delta(t) \|_{L^\infty\left(0,T;\boldsymbol{L}^{3+\delta_1}(\Omega)\right)} \\ &\qquad + C\|\sigma\|_{\mathscr{C}\left(\bar{\Omega}\times\mathbb{R};\mathbb{R}^+\right)} \| \mathbf{curl}\, \boldsymbol{B} \|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \| [\mathbf{curl}\, \boldsymbol{C}_h - \mathbf{curl}\, \boldsymbol{C}]\delta(t) \|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \\ &\qquad + C\|\sigma\|_{\mathscr{C}\left(\bar{\Omega}\times\mathbb{R};\mathbb{R}^+\right)} \left| \int_0^T \left(\mathbf{curl}\, \tilde{\boldsymbol{B}}_{h\Delta t} - \mathbf{curl}\, \boldsymbol{B}, \mathbf{curl}\, \boldsymbol{C}_h \right) \delta(t)\, \mathrm{d}t \right| \xrightarrow{h,\Delta t\to 0} 0. \end{aligned}$$

In addition, by applying Lemma 3.5, we deduce that

$$\begin{aligned} &\left| \int_0^T \left[\left(\tilde{\boldsymbol{u}}_{h\Delta t} \times \hat{\boldsymbol{B}}_{h\Delta t}, \mathbf{curl}\, \boldsymbol{C}_h \right) - (\boldsymbol{u} \times \boldsymbol{B}, \mathbf{curl}\, \boldsymbol{C}) \right] \delta(t)\, \mathrm{d}t \right| \\ &\quad \leq Ch^l \left\| \mathbf{curl}\, \hat{\boldsymbol{B}}_{h\Delta t} \right\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \| \mathbf{curl}\, \boldsymbol{C}_h\, \delta(t) \|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \| \tilde{\boldsymbol{u}}_{h\Delta t} \|_{L^2(0,T;\boldsymbol{L}^6(\Omega))} \\ &\qquad + \left| \int_0^T \left[\left(\tilde{\boldsymbol{u}}_{h\Delta t} \times Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right), \mathbf{curl}\, \boldsymbol{C}_h \right) - (\boldsymbol{u} \times \boldsymbol{B}, \mathbf{curl}\, \boldsymbol{C}) \right] \delta(t)\, \mathrm{d}t \right| \end{aligned}$$

$$=: E_1 + |E_2|,$$

as $h \to 0$, it can be clearly derived $E_1 = Ch^l \left\|\mathbf{curl}\,\hat{\boldsymbol{B}}_{h\Delta t}\right\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \|\mathbf{curl}\,\boldsymbol{C}_h\,\delta(t)\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}$ $\|\tilde{\boldsymbol{u}}_{h\Delta t}\|_{L^2(0,T;\boldsymbol{L}^6(\Omega))} \to 0$. Next, we just need to show $|E_2| \to 0$ when $h, \Delta t \to 0$. By using of Young's inequality, Theorem 4.6, (4.21), Lemmas 2.1 and 3.5, we continue to derive

$$\begin{aligned}
|E_2| &= \left|\int_0^T \left[\left(\tilde{\boldsymbol{u}}_{h\Delta t} \times Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right), \mathbf{curl}\,\boldsymbol{C}_h\right) - (\boldsymbol{u} \times \boldsymbol{B}, \mathbf{curl}\,\boldsymbol{C})\right]\delta(t)\,\mathrm{d}t\right| \\
&\le C\|\tilde{\boldsymbol{u}}_{h\Delta t} - \boldsymbol{u}\|_{L^2\left(0,T;\boldsymbol{L}^{6-\delta_2}(\Omega)\right)} \left\|Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right)\right\|_{L^2(0,T;\boldsymbol{\mathcal{H}}(\Omega))} \|\mathbf{curl}\,\boldsymbol{C}_h\,\delta(t)\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \\
&\quad + C\|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{L}^6(\Omega))} \left\|Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right) - \boldsymbol{B}\right\|_{L^2(0,T;\boldsymbol{L}^3(\Omega))} \|\mathbf{curl}\,\boldsymbol{C}_h\,\delta(t)\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \\
&\quad + C\|\boldsymbol{u}\|_{L^\infty\left(0,T;\boldsymbol{L}^{6-\delta_2}(\Omega)\right)} \|\boldsymbol{B}\|_{L^2(0,T;\boldsymbol{\mathcal{H}}(\Omega))} \|(\mathbf{curl}\,\boldsymbol{C}_h - \mathbf{curl}\,\boldsymbol{C})\delta(t)\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \xrightarrow{h,\Delta t \to 0} 0,
\end{aligned}$$

where we have used the fact that $1/(3+\delta_1) + 1/(6-\delta_2) + 1/2 = 1$ and $\left\|Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right)\right\|_{L^2\left(0,T;\boldsymbol{L}^{3+\delta_1}(\Omega)\right)} \le C\left\|Z\left(\hat{\boldsymbol{B}}_{h\Delta t}\right)\right\|_{L^2(0,T;\boldsymbol{\mathcal{H}}(\Omega))}$. Collecting these results yield that

$$\int_0^T [(\boldsymbol{B}_t, \boldsymbol{C}) + (\sigma(\theta)\mathbf{curl}\,\boldsymbol{B}, \mathbf{curl}\,\boldsymbol{C}) + (\boldsymbol{u} \times \boldsymbol{B}, \mathbf{curl}\,\boldsymbol{C})]\delta(t)\,\mathrm{d}t = \int_0^T (\boldsymbol{g}, \boldsymbol{C})\delta(t)\,\mathrm{d}t. \tag{4.23}$$

Finally, for any $\varphi \in \mathscr{C}_0^\infty(\Omega) \cap Y_0$, there exists $\varphi_h = A_h\varphi \in Y_{0h}^k$ such that

$$\varphi_h \xrightarrow{h \to 0} \varphi \qquad \text{in } Y_0,$$

where A_h is the H^1-orthogonal projection operator to Y_{0h}^k. In a similar argument, there holds

$$\begin{aligned}
&\left|\int_0^T \left[\left(\tilde{\kappa}_{\Delta t}\left(\hat{\theta}_{h\Delta t}\right)\nabla\tilde{\theta}_{h\Delta t}, \nabla\varphi_h\right) - (\kappa(\theta)\nabla\theta, \nabla\varphi)\right]\delta(t)\,\mathrm{d}t\right| \\
&\le C\left\|(\tilde{\kappa}_{\Delta t} - \kappa)\left(\hat{\theta}_{h\Delta t}\right)\right\|_{L^2(0,T;L^4(\Omega))} \left\|\nabla\tilde{\theta}_{h\Delta t}\right\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \|\nabla\varphi_h\,\delta(t)\|_{L^\infty(0,T;\boldsymbol{L}^4(\Omega))} \\
&\quad + C|\kappa|_{\mathscr{C}^{0,1}\left(\bar{\Omega}\times\mathbb{R};\mathbb{R}\right)} \left\|\hat{\theta}_{h\Delta t} - \theta\right\|_{L^2(0,T;L^4(\Omega))} \left\|\nabla\tilde{\theta}_{h\Delta t}\right\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \|\nabla\varphi_h\,\delta(t)\|_{L^\infty(0,T;\boldsymbol{L}^4(\Omega))} \\
&\quad + C\|\kappa\|_{\mathscr{C}\left(\bar{\Omega}\times\mathbb{R};\mathbb{R}^+\right)} \|\nabla\theta\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \|(\nabla\varphi_h - \nabla\varphi)\,\delta(t)\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \\
&\quad + C\|\kappa\|_{\mathscr{C}\left(\bar{\Omega}\times\mathbb{R};\mathbb{R}^+\right)} \left|\int_0^T \left(\nabla\tilde{\theta}_{h\Delta t} - \nabla\theta, \nabla\varphi_h\right)\delta(t)\,\mathrm{d}t\right| \xrightarrow{h,\Delta t \to 0} 0,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_0^T \left[\mathcal{O}_2\left(\hat{\boldsymbol{u}}_{h\Delta t}, \tilde{\theta}_{h\Delta t}, \varphi_h\right) - \mathcal{O}_2(\boldsymbol{u}, \theta, \varphi)\right]\delta(t)\,\mathrm{d}t\right| \\
&\le C\|\hat{\boldsymbol{u}}_{h\Delta t} - \boldsymbol{u}\|_{L^2(0,T;\boldsymbol{L}^4(\Omega))} \left\|\nabla\tilde{\theta}_{h\Delta t}\right\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \|\varphi_h\delta(t)\|_{L^\infty(0,T;L^4(\Omega))} \\
&\quad + C\|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{L}^4(\Omega))} \|\nabla\theta\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \|(\varphi_h - \varphi)\delta(t)\|_{L^\infty(0,T;L^4(\Omega))} \\
&\quad + C\|\hat{\boldsymbol{u}}_{h\Delta t} - \boldsymbol{u}\|_{L^2(0,T;\boldsymbol{L}^4(\Omega))} \|\nabla\varphi_h\delta(t)\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \left\|\tilde{\theta}_{h\Delta t}\right\|_{L^2(0,T;L^4(\Omega))}
\end{aligned}$$

$$
\begin{aligned}
&+C\|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{L}^4(\Omega))}\|\nabla\varphi_h\,\delta(t)\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\left\|\tilde{\theta}_{h\Delta t}-\theta\right\|_{L^2(0,T;L^4(\Omega))}\\
&+C\|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{L}^4(\Omega))}\|\nabla(\varphi_h-\varphi)\delta(t)\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\|\theta\|_{L^2(0,T;L^4(\Omega))}\\
&+C\left|\int_0^T\left(\boldsymbol{u}\cdot\nabla\left(\tilde{\theta}_{h\Delta t}-\nabla\theta\right),\varphi_h\right)\delta(t)\,\mathrm{d}t\right|\xrightarrow{h,\Delta t\to 0}0.
\end{aligned}
$$

Hence it yields that

$$
\int_0^T[(\theta_t,\varphi)+(\kappa(\theta)\nabla\theta,\nabla\varphi)+\mathcal{O}_2(\boldsymbol{u},\theta,\varphi)]\delta(t)\,\mathrm{d}t=\int_0^T(\psi,\varphi)\delta(t)\,\mathrm{d}t. \tag{4.24}
$$

Since $\mathscr{C}_0^\infty(\Omega)\cap\boldsymbol{H}_0^1(\Omega)$ is dense in $\boldsymbol{H}_0^1(\Omega)$, $\mathscr{C}_0^\infty(\Omega)\cap\boldsymbol{W}_0$ is dense in $\boldsymbol{W}_0$, $\mathscr{C}_0^\infty(\Omega)\cap Y_0$ is dense in Y_0 and $\mathscr{C}^\infty([0,T])$ is dense in $L^q([0,T])$ with $1\le q<\infty$, the proof of Theorem 4.7 can be completed by combining with (4.22), (4.23) and (4.24) in the distribute sense, as $h,\Delta t\to 0$. ☐

Remark 4.8. The first convergence result of finite element discretization for MHD with constant coefficients is given in Theorem 3.1 of [45], which gives the proof idea without going through all the details. Here we extend to the temperature dependent coefficients case with the details to show the techniques to treat these nonlinear terms.

We will also show the continuous system (2.5)–(2.7) has a unique solution. To this end, let us recall the Gronwall lemma in differential form.

Lemma 4.9 (Gronwall lemma)**.** *If $\eta(\cdot)$ is continuous differentiable function, and it is non-negative such that*

$$
\eta'(t)\le\phi(t)\eta(t)+\varphi_0(t)\qquad t\in[0,T], \tag{4.25}
$$

where $\phi(t)$ and $\varphi_0(t)$ are non-negative integrable functions, then there holds

$$
\eta(t)\le e^{\int_0^t\phi(s)\,\mathrm{d}s}\left[\eta(0)+\int_0^t\varphi_0(s)\,\mathrm{d}s\right]\qquad\forall\,t\in[0,T]. \tag{4.26}
$$

We will prove the uniqueness of the continuous system (2.5)–(2.7), provided that the exact solution is under a slight smooth assumption. More precisely, we need to make a smoother assumption on the weak solution for the magneto-thermal coupling model with temperature-dependent coefficients.

Theorem 4.10. *Let $(\boldsymbol{u},\boldsymbol{B},\theta,p)$ be the weak solution of the continuous system (2.5)–(2.7) and assume that $\boldsymbol{u}\in L^2\big(0,T;\boldsymbol{H}^{1+s}(\Omega)\big)\cap L^4\big(0,T;\boldsymbol{W}^{1,6}(\Omega)\big)$, $\mathbf{curl}\,\boldsymbol{B}\in L^2(0,T;\boldsymbol{H}^s(\Omega))\cap\boldsymbol{L}^4\big(0,T;\boldsymbol{L}^6(\Omega)\big)$, $\theta\in L^2\big(0,T;H^{1+s}(\Omega)\big)\cap L^4\big(0,T;W^{1,6}(\Omega)\big)$ with $s>1/2$. Then $(\boldsymbol{u},\boldsymbol{B},\theta,p)$ is the unique weak solution for system (2.5)–(2.7).*

Proof. Assume that problem (2.5)–(2.7) has two different weak solutions $(\boldsymbol{u}_1,\boldsymbol{B}_1,\theta_1,p_1)$ and $(\boldsymbol{u}_2,\boldsymbol{B}_2,\theta_2,p_2)$. Let $\boldsymbol{u}=\boldsymbol{u}_1-\boldsymbol{u}_2$, $\boldsymbol{B}=\boldsymbol{B}_1-\boldsymbol{B}_2$, $\theta=\theta_1-\theta_2$ and $p=p_1-p_2$ in (2.5)–(2.7), for any $(\boldsymbol{v},\boldsymbol{C},\varphi,q)\in(\boldsymbol{X}_0\times\boldsymbol{W}_0\times Y_0\times Q)$, there holds

$$
\left\{
\begin{aligned}
&\langle\boldsymbol{u}_t,\boldsymbol{v}\rangle+(\nu(\theta_1)\nabla\boldsymbol{u},\nabla\boldsymbol{v})+([\nu(\theta_1)-\nu(\theta_2)]\nabla\boldsymbol{u}_2,\nabla\boldsymbol{v})+((\boldsymbol{u}_1\cdot\nabla)\boldsymbol{u},\boldsymbol{v})+((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_2,\boldsymbol{v})-(\operatorname{div}\boldsymbol{v},p)\\
&+(\operatorname{div}\boldsymbol{u},q)+\mu(\boldsymbol{B}\times\mathbf{curl}\,\boldsymbol{B}_1,\boldsymbol{v})+\mu(\boldsymbol{B}_2\times\mathbf{curl}\,\boldsymbol{B},\boldsymbol{v})-([\boldsymbol{\beta}(\theta_1)-\boldsymbol{\beta}(\theta_2)]\theta_1,\boldsymbol{v})-(\boldsymbol{\beta}(\theta_2)\theta,\boldsymbol{v})=0, &(4.27)\\
&\langle\boldsymbol{B}_t,\boldsymbol{C}\rangle+([\sigma(\theta_1)-\sigma(\theta_2)]\mathbf{curl}\,\boldsymbol{B}_1,\mathbf{curl}\,\boldsymbol{C})+(\sigma(\theta_2)\mathbf{curl}\,\boldsymbol{B},\mathbf{curl}\,\boldsymbol{C})\\
&-(\boldsymbol{u}\times\boldsymbol{B}_1,\mathbf{curl}\,\boldsymbol{C})-(\boldsymbol{u}_2\times\boldsymbol{B},\mathbf{curl}\,\boldsymbol{C})=0, &(4.28)\\
&\langle\theta_t,\varphi\rangle+([\kappa(\theta_1)-\kappa(\theta_2)]\nabla\theta_1,\nabla\varphi)+(\kappa(\theta_2)\nabla\theta,\nabla\varphi)+(\boldsymbol{u}\cdot\nabla\theta_1,\varphi)+(\boldsymbol{u}_2\cdot\nabla\theta,\varphi)=0. &(4.29)
\end{aligned}
\right.
$$

Choosing $\boldsymbol{v}=\boldsymbol{u}, q=p$ in (4.27), $\boldsymbol{C}=\mu\boldsymbol{B}$ in (4.28), $\varphi=\theta$ in (4.29) and adding the three equations, we have

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\Big(\|\boldsymbol{u}\|_0^2+\mu\|\boldsymbol{B}\|_0^2+\|\theta\|_0^2\Big)+\Big\|\sqrt{\nu(\theta_1)}\nabla\boldsymbol{u}\Big\|_0^2+\mu\Big\|\sqrt{\sigma(\theta_2)}\mathbf{curl}\,\boldsymbol{B}\Big\|_0^2+\Big\|\sqrt{\kappa(\theta_2)}\nabla\theta\Big\|_0^2\\ =-([\nu(\theta_1)-\nu(\theta_2)]\nabla\boldsymbol{u}_2,\nabla\boldsymbol{u})-((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_2,\boldsymbol{u})+(\boldsymbol{\beta}(\theta_2)\theta,\boldsymbol{u})\\ +([\boldsymbol{\beta}(\theta_1)-\boldsymbol{\beta}(\theta_2)]\theta_1,\boldsymbol{u})-\mu(\boldsymbol{B}\times\mathbf{curl}\,\boldsymbol{B}_1,\boldsymbol{u})+\mu(\boldsymbol{u}_1\times\boldsymbol{B},\mathbf{curl}\,\boldsymbol{B})\\ -\mu([\sigma(\theta_1)-\sigma(\theta_2)]\mathbf{curl}\,\boldsymbol{B}_1,\mathbf{curl}\,\boldsymbol{B})-([\kappa(\theta_1)-\kappa(\theta_2)]\nabla\theta_1,\nabla\theta)-(\boldsymbol{u}\cdot\nabla\theta_1,\theta).\end{aligned}\tag{4.30}$$

By using the Young's inequalities, Sobolev inequalities (2.1)–(2.4), (4.20) and the interpolation inequalities, we find

$$\begin{aligned}|((\nu(\theta_1)-\nu(\theta_2))\nabla\boldsymbol{u}_2,\nabla\boldsymbol{u})|&\le|\nu|_{\mathscr{C}^{0,1}(\bar{\Omega}\times\mathbb{R};\mathbb{R})}\|\theta\|_{0,3}\|\nabla\boldsymbol{u}_2\|_{0,6}\|\nabla\boldsymbol{u}\|_0\\ &\le\frac{1}{6}\Big\|\sqrt{\nu(\theta_1)}\nabla\boldsymbol{u}\Big\|_0^2+6\nu_0^{-1}|\nu|^2_{\mathscr{C}^{0,1}(\bar{\Omega}\times\mathbb{R};\mathbb{R})}\|\nabla\boldsymbol{u}_2\|_{0,6}^2\|\theta\|_{0,3}^2\\ &\le\frac{1}{6}\Big\|\sqrt{\nu(\theta_1)}\nabla\boldsymbol{u}\Big\|_0^2+6\nu_0^{-1}|\nu|^2_{\mathscr{C}^{0,1}(\bar{\Omega}\times\mathbb{R};\mathbb{R})}\|\nabla\boldsymbol{u}_2\|_{0,6}^2\|\theta\|_0\|\theta\|_{0,6}\\ &\le\frac{1}{6}\Big\|\sqrt{\nu(\theta_1)}\nabla\boldsymbol{u}\Big\|_0^2+4\kappa_0^{-1}\Big(c_1\,6\nu_0^{-1}|\nu|^2_{\mathscr{C}^{0,1}(\bar{\Omega}\times\mathbb{R};\mathbb{R})}\|\nabla\boldsymbol{u}_2\|_{0,6}^2\Big)^2\|\theta\|_0^2+\frac{1}{4}\Big\|\sqrt{\kappa(\theta_2)}\nabla\theta\Big\|_0^2,\end{aligned}$$

and

$$\begin{aligned}|((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_2,\boldsymbol{u})|\le\frac{1}{6}\Big\|\sqrt{\nu(\theta_1)}\nabla\boldsymbol{u}\Big\|_0^2+6\nu_0^{-1}c_1^2\|\nabla\boldsymbol{u}_2\|_{0,3}^2\|\boldsymbol{u}\|_0^2,\ \mu|(\boldsymbol{B}\times\mathbf{curl}\,\boldsymbol{B}_1,\boldsymbol{u})-(\boldsymbol{u}_1\times\boldsymbol{B},\mathbf{curl}\,\boldsymbol{B})|\\ \le\mu\|\boldsymbol{B}\|_0\|\mathbf{curl}\,\boldsymbol{B}_1\|_{0,3}\|\boldsymbol{u}\|_{0,6}+\mu\|\boldsymbol{u}_1\|_{0,\infty}\|\boldsymbol{B}\|_0\|\mathbf{curl}\,\boldsymbol{B}\|_0\\ \le\frac{1}{6}\Big\|\sqrt{\nu(\theta_1)}\nabla\boldsymbol{u}\Big\|_0^2+6\nu_0^{-1}c_1^2\mu^2\|\mathbf{curl}\,\boldsymbol{B}_1\|_{0,3}^2\|\boldsymbol{B}\|_0^2+\frac{\mu}{2}\Big\|\sqrt{\sigma(\theta_2)}\mathbf{curl}\,\boldsymbol{B}\Big\|_0^2+2\sigma_0^{-1}\mu\|\boldsymbol{u}_1\|_{0,\infty}^2\|\boldsymbol{B}\|_0^2.\end{aligned}$$

Similarly, we can deduce

$$\begin{aligned}&|([\boldsymbol{\beta}(\theta_1)-\boldsymbol{\beta}(\theta_2)]\theta_1,\boldsymbol{u})+(\boldsymbol{\beta}(\theta_2)\theta,\boldsymbol{u})-\mu([\sigma(\theta_1)-\sigma(\theta_2)]\mathbf{curl}\,\boldsymbol{B}_1,\mathbf{curl}\,\boldsymbol{B})|\\ &\quad\le|\boldsymbol{\beta}|_{\mathscr{C}^{0,1}(\bar{\Omega}\times\mathbb{R};\mathbb{R})}\|\theta\|_0\|\theta_1\|_{0,6}\|\boldsymbol{u}\|_{0,3}+\|\boldsymbol{\beta}\|_{\mathscr{C}(\bar{\Omega}\times\mathbb{R};\mathbb{R}^3)}\|\theta\|_0\|\boldsymbol{u}\|_0\\ &\qquad+|\sigma|_{\mathscr{C}^{0,1}(\bar{\Omega}\times\mathbb{R};\mathbb{R})}\mu\|\theta\|_{0,3}\|\mathbf{curl}\,\boldsymbol{B}_1\|_{0,6}\|\mathbf{curl}\,\boldsymbol{B}\|_0\\ &\quad\le\frac{2}{6}\Big\|\sqrt{\nu(\theta_1)}\nabla\boldsymbol{u}\Big\|_0^2+\Big(6\nu_0^{-1}|\boldsymbol{\beta}|^2_{\mathscr{C}^{0,1}(\bar{\Omega}\times\mathbb{R};\mathbb{R})}c_1^2\|\theta_1\|_{0,6}^2+6\nu_0^{-1}\|\boldsymbol{\beta}\|^2_{\mathscr{C}(\bar{\Omega}\times\mathbb{R};\mathbb{R}^3)}c_1^2\Big)\|\theta\|_0^2\\ &\qquad+4\kappa_0^{-1}\Big(c_1\,2\sigma_0^{-1}|\sigma|^2_{\mathscr{C}^{0,1}(\bar{\Omega}\times\mathbb{R};\mathbb{R})}\mu\|\mathbf{curl}\,\boldsymbol{B}_1\|_{0,6}^2\Big)^2\|\theta\|_0^2+\frac{1}{4}\Big\|\sqrt{\kappa(\theta_2)}\nabla\theta\Big\|_0^2+\frac{\mu}{2}\Big\|\sqrt{\sigma(\theta_2)}\mathbf{curl}\,\boldsymbol{B}\Big\|_0^2,\end{aligned}$$

and

$$\begin{aligned}&|([\kappa(\theta_1)-\kappa(\theta_2)]\nabla\theta_1,\nabla\theta)+(\boldsymbol{u}\cdot\nabla\theta_1,\theta)|\le|\kappa|_{\mathscr{C}^{0,1}(\bar{\Omega}\times\mathbb{R};\mathbb{R})}\|\theta\|_{0,3}\|\nabla\theta_1\|_{0,6}\|\nabla\theta\|_0+\|\theta\|_0\|\nabla\theta_1\|_{0,3}\|\boldsymbol{u}\|_{0,6}\\ &\quad\le\frac{2}{4}\Big\|\sqrt{\kappa(\theta_2)}\nabla\theta\Big\|_0^2+4\kappa_0^{-1}\Big(c_1\,4\kappa_0^{-1}|\kappa|^2_{\mathscr{C}^{0,1}(\bar{\Omega}\times\mathbb{R};\mathbb{R})}\|\nabla\theta_1\|_{0,6}^2\Big)^2\|\theta\|_0^2+\frac{1}{6}\Big\|\sqrt{\nu(\theta_1)}\nabla\boldsymbol{u}\Big\|_0^2+6\nu_0^{-1}c_1^2\|\nabla\theta_1\|_{0,3}^2\|\theta\|_0^2.\end{aligned}$$

Combining these inequalities with (4.30), then we come to

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\|\boldsymbol{u}\|_0^2+\mu\|\boldsymbol{B}\|_0^2+\|\theta\|_0^2\Big)\le C\Big[1+\|\nabla\boldsymbol{u}_2\|_{0,6}^4+\|\nabla\boldsymbol{u}_2\|_{0,3}^2+\|\nabla\theta_1\|_{0,3}^2+\|\mathbf{curl}\,\boldsymbol{B}_1\|_{0,3}^2+\|\boldsymbol{u}_1\|_{0,\infty}^2\\ +\|\mathbf{curl}\,\boldsymbol{B}_1\|_{0,6}^4+\|\nabla\theta_1\|_{0,6}^4\Big]\Big(\|\boldsymbol{u}\|_0^2+\mu\|\boldsymbol{B}\|_0^2+\|\theta\|_0^2\Big).\end{aligned}\tag{4.31}$$

Applying Lemma 4.9, by using $\boldsymbol{u}(0)=0$, $\boldsymbol{B}(0)=0$, $\theta(0)=0$, we deduce that $\boldsymbol{u}=0$, $\boldsymbol{B}=0$ and $\theta=0$. Consequently, equation (4.27) is simplified as $(p,\operatorname{div}\boldsymbol{v})=0$, for any $\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)$. Combining with the inf-sup condition with continuous form, then we deduce $p=0$, which brings the proof to an end. □

Theorem 4.11. *Under the same conditions of Theorems 4.7 and 4.10, then the whole sequence of* $\{\tilde{\boldsymbol{u}}_{h\Delta t}\}$, $\left\{\tilde{\boldsymbol{B}}_{h\Delta t}\right\}$, $\left\{\tilde{\theta}_{h\Delta t}\right\}$ *and* $\{\tilde{p}_{h\Delta t}\}$ *converges to the unique weak solution* $(\boldsymbol{u}, \boldsymbol{B}, \theta, p)$.

Proof. Based on Theorems 4.7 and 4.10, we know that each subsequence of $\{\tilde{\boldsymbol{u}}_{h\Delta t}\}$, $\left\{\tilde{\boldsymbol{B}}_{h\Delta t}\right\}$, $\left\{\tilde{\theta}_{h\Delta t}\right\}$ and $\{\tilde{p}_{h\Delta t}\}$ has the same limit $(\boldsymbol{u}, \boldsymbol{B}, \theta, p)$, which is the unique weak solution to the system (2.5)–(2.7). Thus the whole sequence of $\{\tilde{\boldsymbol{u}}_{h\Delta t}\}$, $\left\{\tilde{\boldsymbol{B}}_{h\Delta t}\right\}$, $\left\{\tilde{\theta}_{h\Delta t}\right\}$ and $\{\tilde{p}_{h\Delta t}\}$ converges to the unique weak solution $(\boldsymbol{u}, \boldsymbol{B}, \theta, p)$. □

5. Error estimates for the magneto-heat coupling model with temperature-dependent coefficients

In this section, we mainly consider the error estimates of the fully discrete finite element method for the MHD system coupled the thermal equation with temperature-dependent coefficients. Under the hypothesis of a low regularity for the exact solution, we rigorously establish the error estimates for the velocity, temperature and magnetic induction unconditionally in the sense that the time step is restricted but is independent of the spacial mesh size. We also prove a sub-optimal error estimate for the pressure as a supplementary result, which is consistent with the pioneering work [54].

We first recall a discrete version of the Gronwall inequality in a slightly more general form than usually found in the literature, and this detailed proof can be found in [29].

Lemma 5.1. *Let* $C_*, \Delta t, a_n, b_n, c_n$ *and* d_n *be non-negative numbers with* $n \geq 0$ *such that*

$$a_m + \Delta t \sum_{n=0}^{m} b_n \leq \Delta t \sum_{n=0}^{m} d_n a_n + \Delta t \sum_{n=0}^{m} c_n + C_* \qquad \forall\, m \geq 0.$$

Suppose that $\Delta t d_n < 1$, *for all* n, *and set* $\lambda_n = (1 - \Delta t d_n)^{-1}$. *Then*

$$a_m + \Delta t \sum_{n=0}^{m} b_n \leq \exp\left(\Delta t \sum_{n=0}^{m} \lambda_n d_n\right)\left\{\Delta t \sum_{n=0}^{m} c_n + C_*\right\} \qquad \forall\, m \geq 0.$$

Before proceeding, we need to make a regularity assumption for the weak solution of (2.5)–(2.7), which will be helpful for the error analysis of the numerical solution.

Assumption 5.2. *Suppose that the weak solution* $(\boldsymbol{u}, p, \boldsymbol{B}, \theta)$ *satisfies the following regularity,*

$$\begin{aligned}
&\boldsymbol{u} \in L^\infty\big(0, T; \boldsymbol{H}^{s+1}(\Omega)\big), \quad p \in L^\infty(0, T; H^s(\Omega)), \quad \theta \in L^\infty\big(0, T; H^{1+s}(\Omega)\big), \quad \boldsymbol{B} \in L^\infty(0, T; \boldsymbol{H}^s(\Omega)),\\
&\mathbf{curl}\,\boldsymbol{B} \in L^\infty(0, T; \boldsymbol{H}^s(\Omega)), \quad \boldsymbol{u}_t \in L^2\big(0, T; \boldsymbol{L}^2(\Omega)\big), \quad \boldsymbol{B}_t \in L^2\big(0, T; \boldsymbol{L}^2(\Omega)\big), \quad \theta_t \in L^2\big(0, T; H^1(\Omega)\big),\\
&\boldsymbol{u}_{tt} \in L^2\big(0, T; \boldsymbol{L}^2(\Omega)\big), \quad \boldsymbol{B}_{tt} \in L^2\big(0, T; \boldsymbol{L}^2(\Omega)\big), \quad \theta_{tt} \in L^2\big(0, T; L^2(\Omega)\big),
\end{aligned}$$

where the exponent $s > 1/2$ *depends on* Ω.

Remark 5.3. The above regularity assumption for the solution $(\boldsymbol{u}, p, \boldsymbol{B}, \theta)$ is even weaker than most of the hypotheses in the literature, see, *e.g.* [27,29,38,49,54]. We hope it is a reasonable assumption and may be valid for a general polyhedral with Lipschitz boundary. In fact, it is enough to facilitate the subsequent error analysis of the magneto-thermal coupling model with temperature-dependent coefficients.

We next define some useful Galerkin and Ritz projections: given $(\boldsymbol{u}, p, \theta) \in (\boldsymbol{X}_0 \times Q \times Y_0)$, find $\mathcal{P}_h \boldsymbol{u} \in \boldsymbol{V}_h^k(\boldsymbol{u}_D)$, $\mathcal{Q}_h p \in Q_h^{k-1}$ and $\mathcal{R}_h \theta \in Y_{0h}^k(\theta_D)$, for any $(\boldsymbol{v}_h, q_h, \varphi_h) \in \big(\boldsymbol{V}_h^k \times Q_h^{k-1} \times Y_{0h}^k\big)$, such that

$$\begin{aligned}
(\nu(\theta)\nabla \mathcal{P}_h \boldsymbol{u}, \nabla \boldsymbol{v}_h) + b(\boldsymbol{v}_h, \mathcal{Q}_h p) - b(\mathcal{P}_h \boldsymbol{u}, q_h) &= (\nu(\theta)\nabla \boldsymbol{u}, \nabla \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p) - b(\boldsymbol{u}, q_h),\\
(\kappa(\theta)\nabla \mathcal{R}_h \theta, \nabla \varphi_h) &= (\kappa(\theta)\nabla \theta, \nabla \varphi_h).
\end{aligned} \tag{5.1}$$

We define the Fortin operator $\mathcal{F}_h$ from $\boldsymbol{W}_0$ to $\boldsymbol{W}_h^k$: given $\boldsymbol{B} \in \boldsymbol{W}_0$, find $\mathcal{F}_h\boldsymbol{B} \in \boldsymbol{W}_h^k$, for any $\boldsymbol{C}_h \in \boldsymbol{W}_h^k$ and $\psi_h \in S_h$ such that

$$(\sigma(\theta)\mathbf{curl}\,\mathcal{F}_h\boldsymbol{B}, \mathbf{curl}\,\boldsymbol{C}_h) = (\sigma(\theta)\mathbf{curl}\,\boldsymbol{B}, \mathbf{curl}\,\boldsymbol{C}_h), \quad (\mathcal{F}_h\boldsymbol{B}, \nabla\psi_h) = (\boldsymbol{B}, \nabla\psi_h). \tag{5.2}$$

By a similar argument to the constant coefficients case as [5,23,28] and the temperature-dependent coefficients case [54] (refer to Lem. 1 for more details), we can prove the following approximation properties

$$\begin{aligned} \|\boldsymbol{u} - \mathcal{P}_h\boldsymbol{u}\|_0 + h\|\nabla(\boldsymbol{u} - \mathcal{P}_h\boldsymbol{u})\|_0 + h\|p - \mathcal{Q}_h p\|_0 &\le C_e h^{1+\ell}\|\boldsymbol{u}\|_{1+\ell,2} + C_e h^{1+\ell}\|p\|_{\ell,2}, \\ \|\boldsymbol{B} - \mathcal{F}_h\boldsymbol{B}\|_0 + \|\mathbf{curl}\,(\boldsymbol{B} - \mathcal{F}_h\boldsymbol{B})\|_0 &\le C_e h^{\ell}\Big(\|\boldsymbol{B}\|_{\ell,2} + \|\mathbf{curl}\,\boldsymbol{B}\|_{\ell,2}\Big), \\ \|\theta - \mathcal{R}_h\theta\|_{1,2} &\le C_e h^{\ell}\|\theta\|_{\ell+1,2} \end{aligned} \tag{5.3}$$

with $\ell = \min\{k, s\}$, where $k \ge 1$ is the order index of the finite element spaces, $s > 1/2$ is the index of regularity of the exact solution.

By virtue of the properties of projection, we present that the solution to (2.5)–(2.7) has the following estimates, which hold regardless of the sizes h and Δt. Moreover, the stability property is beneficial to the subsequent error analysis in full discretization.

Lemma 5.4. *Suppose Assumption 5.2 holds. Let $(\boldsymbol{u}, \theta, \boldsymbol{B})$ be the unique solution of (2.5)–(2.7), then the following estimates are established,*

$$\|\boldsymbol{u} - \mathcal{P}_h\boldsymbol{u}\|_{0,\infty} + \|\nabla(\boldsymbol{u} - \mathcal{P}_h\boldsymbol{u})\|_{0,3} + \|\mathbf{curl}\,(\boldsymbol{B} - \mathcal{F}_h\boldsymbol{B})\|_{0,3} + \|\theta - \mathcal{R}_h\theta\|_{0,\infty} + \|\nabla(\theta - \mathcal{R}_h\theta)\|_{0,3} \le C_r, \tag{5.4}$$

where C_r is a generic constant depending on the regularity of the domain Ω.

Proof. Let $\boldsymbol{U} = I_h\boldsymbol{u} \in \boldsymbol{X}_h^k$, with $\boldsymbol{U}$ is the standard Lagrange nodal interpolant. By using the finite element approximations, including (5.3), inverse inequality (3.3), Sobolev's embedding theorem and Assumption 5.2, we directly see

$$\begin{aligned} \|\nabla\mathcal{P}_h\boldsymbol{u}\|_{0,3} &= \|\nabla(\mathcal{P}_h\boldsymbol{u} - \boldsymbol{U} + \boldsymbol{U} - \boldsymbol{u} + \boldsymbol{u})\|_{0,3} \le \|\nabla(\mathcal{P}_h\boldsymbol{u} - \boldsymbol{U})\|_{0,3} + \|\nabla(\boldsymbol{U} - \boldsymbol{u})\|_{0,3} + \|\nabla\boldsymbol{u}\|_{0,3} \\ &\le C_{inv}h^{-1/2}[\|\nabla(\mathcal{P}_h\boldsymbol{u} - \boldsymbol{u})\|_0 + \|\nabla(\boldsymbol{u} - \boldsymbol{U})\|_0] + \|\nabla(\boldsymbol{U} - \boldsymbol{u})\|_{0,3} + \|\nabla\boldsymbol{u}\|_{0,3} \\ &\le \|\nabla\boldsymbol{u}\|_{0,3} + Ch^{\ell-1/2}\Big[\|\boldsymbol{u}\|_{1+\ell,2} + \|p\|_{\ell,2}\Big] \le C_r. \end{aligned}$$

We can verify that other terms are bounded in a similar way, this bring the proof to an end. □

Let $(e_{1h}^n, e_{2h}^n, e_{3h}^n, e_{4h}^n) = (\boldsymbol{u}_h^n - \mathcal{P}_h\boldsymbol{u}^n, p_h^n - \mathcal{Q}_h p^n, \theta_h^n - \mathcal{R}_h\theta^n, \boldsymbol{B}_h^n - \mathcal{F}_h\boldsymbol{B}^n)$. A combination of (2.5)–(2.7), (3.5)–(3.8) and (5.1), (5.2) yields the following truncation error equations:

$$\begin{cases} (d_t e_{1h}^n, \boldsymbol{v}_h) + \mathcal{A}_1\big(\nu^n(\theta_h^{n-1}), e_{1h}^n, \boldsymbol{v}_h\big) + b(\boldsymbol{v}_h, e_{2h}^n) = \langle R_{L1h}^n + R_{N1h}^n, \boldsymbol{v}_h\rangle, & (5.5) \\ b(e_{1h}^n, q_h) = 0, & (5.6) \\ (d_t e_{3h}^n, \varphi_h) + \mathcal{A}_2\big(\kappa^n(\theta_h^{n-1}), e_{3h}^n, \varphi_h\big) = \langle R_{L2h}^n, \varphi_h\rangle + \langle R_{N2h}^n, \varphi_h\rangle, & (5.7) \\ (d_t e_{4h}^n, \boldsymbol{C}_h) + \big(\sigma^n(\theta_h^{n-1})\mathbf{curl}\,e_{4h}^n, \mathbf{curl}\,\boldsymbol{C}_h\big) = \langle R_{L3h}^n + R_{N3h}^n, \boldsymbol{C}_h\rangle, & (5.8) \end{cases}$$

where

$$\begin{aligned} &\langle R_{L1h}^n, \boldsymbol{v}_h\rangle = \langle \partial_t\boldsymbol{u}^n - d_t\mathcal{P}_h\boldsymbol{u}^n, \boldsymbol{v}_h\rangle, \\ &\langle R_{N1h}^n, \boldsymbol{v}_h\rangle = \langle R_{D1h}^n + R_{C1h}^n + R_{Bh}^n + R_{A1h}^n, \boldsymbol{v}_h\rangle \end{aligned}$$

$$= \left\{ \mathcal{A}_1(\nu^n(\theta^n), \boldsymbol{u}^n, \boldsymbol{v}_h) - \mathcal{A}_1\left(\nu^n\left(\theta_h^{n-1}\right), \boldsymbol{u}^n, \boldsymbol{v}_h\right) \right\} + \left\{ \mathcal{O}_1(\boldsymbol{u}^n, \boldsymbol{u}^n, \boldsymbol{v}_h) - \mathcal{O}_1\left(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{v}_h\right) \right\}$$
$$+ \left\{ \left(\boldsymbol{\beta}^n\left(\theta_h^{n-1}\right)\theta_h^n, \boldsymbol{v}_h\right) - (\boldsymbol{\beta}^n(\theta^n)\theta^n, \boldsymbol{v}_h) \right\} + \left\{ \mu(\boldsymbol{B}^n \times \mathbf{curl}\, \boldsymbol{B}^n, \boldsymbol{v}_h) - \mu\left(\boldsymbol{B}_h^{n-1} \times \mathbf{curl}\, \boldsymbol{B}_h^n, \boldsymbol{v}_h\right) \right\},$$

and

$$\begin{aligned}
\langle R_{L2h}^n, \varphi_h \rangle &= \langle \partial_t \theta^n - d_t \mathcal{R}_h \theta^n, \varphi_h \rangle, \\
\langle R_{N2h}^n, \varphi_h \rangle &= \langle R_{D2h}^n + R_{C2h}^n, \varphi_h \rangle \\
&= \left\{ \mathcal{A}_2(\kappa^n(\theta^n), \theta^n, \varphi_h) - \mathcal{A}_2\left(\kappa^n\left(\theta_h^{n-1}\right), \theta^n, \varphi_h\right) \right\} + \left\{ \mathcal{O}_2(\boldsymbol{u}^n, \theta^n, \varphi_h) - \mathcal{O}_2\left(\boldsymbol{u}_h^{n-1}, \theta_h^n, \varphi_h\right) \right\}, \\
\langle R_{L3h}^n, \boldsymbol{C}_h \rangle &= \langle \partial_t \boldsymbol{B}^n - d_t \mathcal{F}_h \boldsymbol{B}^n, \boldsymbol{C}_h \rangle, \\
\langle R_{N3h}^n, \boldsymbol{C}_h \rangle &= \langle R_{A2h}^n + R_{E1h}^n, \boldsymbol{C}_h \rangle \\
&= \left\{ \left(\boldsymbol{u}_h^n \times \boldsymbol{B}_h^{n-1}, \mathbf{curl}\, \boldsymbol{C}_h\right) - (\boldsymbol{u}^n \times \boldsymbol{B}^n, \mathbf{curl}\, \boldsymbol{C}_h) \right\} \\
&\quad + \left\{ (\sigma^n(\theta^n)\mathbf{curl}\, \boldsymbol{B}^n, \mathbf{curl}\, \boldsymbol{C}_h) - \left(\sigma^n\left(\theta_h^{n-1}\right)\mathbf{curl}\, \boldsymbol{B}^n, \mathbf{curl}\, \boldsymbol{C}_h\right) \right\}.
\end{aligned}$$

The following lemma can be referred to Lemma 2 of [54].

Lemma 5.5. *Let ϕ_i, θ_i, $i = 1, 2$, and χ be functions in $L^2(\Omega)$, and λ a function in $\mathscr{C}^{0,1}\left(\bar{\Omega} \times \mathbb{R}; \mathbb{R}\right)$. Then for any $\bar{\phi} \in L^{p_0}(\Omega)$, where $p_0 \le \infty$, it holds that*

$$\begin{aligned}
&\left| \int_\Omega \lambda(\cdot, \theta_1)\phi_1\, \chi\, \mathrm{d}x - \int_\Omega \lambda(\cdot, \theta_2)\phi_2\, \chi\, \mathrm{d}x \right| \\
&\quad \le \max\left\{ \|\lambda\|_{\mathscr{C}(\bar{\Omega} \times \mathbb{R}; \mathbb{R})}, \left\|\bar{\phi}\right\|_{0,p_0} |\lambda|_{\mathscr{C}^{0,1}(\bar{\Omega} \times \mathbb{R}; \mathbb{R})} \right\} \left(\|\theta_1 - \theta_2\|_{0,q_0} + \left\|\phi_1 - \bar{\phi}\right\|_{0,q_0} + \left\|\bar{\phi} - \phi_2\right\|_{0,q_0} \right) \|\chi\|_{0,r_0},
\end{aligned} \tag{5.9}$$

where $1/p_0 + 1/q_0 + 1/r_0 = 1$, and $|\lambda|_{\mathscr{C}^{0,1}(\bar{\Omega} \times \mathbb{R}; \mathbb{R})}$ is defined in (4.20).

With the above preparations, we can now establish the following error estimates for the velocity, temperature and magnetic induction.

Theorem 5.6. *Suppose Assumption 5.2 holds, and the initial approximations $\boldsymbol{u}_h^0$, θ_h^0, $\boldsymbol{B}_h^0$ satisfy*

$$\left\|\boldsymbol{u}^0 - \boldsymbol{u}_h^0\right\|_0,\ \left\|\theta^0 - \theta_h^0\right\|_0,\ \left\|\boldsymbol{B}^0 - \boldsymbol{B}_h^0\right\|_0 \le C^* h^\ell$$

with $\ell = \min\{k, s\}$. Then there exists a positive constant Δt_0 such that when $\Delta t \le \Delta t_0$, the continuous problem (2.5)–(2.7) and the finite element system (3.5)–(3.8) admits a unique solution $(\boldsymbol{u}^n, \theta^n, \boldsymbol{B}^n)$ and $(\boldsymbol{u}_h^n, \theta_h^n, \boldsymbol{B}_h^n)$, respectively, which satisfies

$$\begin{aligned}
&\|\boldsymbol{u}^m - \boldsymbol{u}_h^m\|_0^2 + \|\theta^m - \theta_h^m\|_0^2 + \mu\|\boldsymbol{B}^m - \boldsymbol{B}_h^m\|_0^2 + \Delta t \sum_{n=1}^m \Big[\nu_0 \|\nabla(\boldsymbol{u}^n - \boldsymbol{u}_h^n)\|_0^2 \\
&\quad + \kappa_0 \|\nabla(\theta^n - \theta_h^n)\|_0^2 + \mu\sigma_0 \|\mathbf{curl}\,(\boldsymbol{B}^n - \boldsymbol{B}_h^n)\|_0^2 \Big] \le C^* \left((\Delta t)^2 + h^{2\ell} \right).
\end{aligned}$$

Proof. Substituting $(e_{1h}^n, e_{2h}^n, e_{3h}^n, \mu e_{4h}^n)$ into $(\boldsymbol{v}_h, q_h, \varphi_h, \boldsymbol{C}_h)$ in (5.5)–(5.8), there holds

$$\left\{ \begin{aligned}
&(d_t e_{1h}^n, e_{1h}^n) + \mathcal{A}_1\left(\nu^n\left(\theta_h^{n-1}\right), e_{1h}^n, e_{1h}^n\right) = \langle R_{L1h}^n + R_{N1h}^n, e_{1h}^n \rangle, && (5.10)\\
&(d_t e_{3h}^n, e_{3h}^n) + \mathcal{A}_2\left(\kappa^n\left(\theta_h^{n-1}\right), e_{3h}^n, e_{3h}^n\right) = \langle R_{L2h}^n + R_{N2h}^n, e_{3h}^n \rangle, && (5.11)\\
&\mu(d_t e_{4h}^n, e_{4h}^n) + \mu\left(\sigma^n\left(\theta_h^{n-1}\right)\mathbf{curl}\, e_{4h}^n, \mathbf{curl}\, e_{4h}^n\right) = \mu\langle R_{L3h}^n + R_{N3h}^n, e_{4h}^n \rangle. && (5.12)
\end{aligned} \right.$$

Due to $2b(b-a) = b^2 - a^2 + (b-a)^2$, we conclude that

$$
\begin{aligned}
(d_t e_{1h}^n, e_{1h}^n) + \mathcal{A}_1\big(\nu^n\big(\theta_h^{n-1}\big), e_{1h}^n, e_{1h}^n\big) &\geq \frac{1}{2}\Big[d_t\|e_{1h}^n\|_0^2 + \Delta t\|d_t e_{1h}^n\|_0^2\Big] + \nu_0\|\nabla e_{1h}^n\|_0^2, &(5.13)\\
(d_t e_{3h}^n, e_{3h}^n) + \mathcal{A}_2\big(\kappa^n\big(\theta_h^{n-1}\big), e_{3h}^n, e_{3h}^n\big) &\geq \frac{1}{2}\Big[d_t\|e_{3h}^n\|_0^2 + \Delta t\|d_t e_{3h}^n\|_0^2\Big] + \kappa_0\|\nabla e_{3h}^n\|_0^2 &(5.14)
\end{aligned}
$$

and

$$
\mu(d_t e_{4h}^n, e_{4h}^n) + \mu\big(\sigma^n\big(\theta_h^{n-1}\big)\mathbf{curl}\, e_{4h}^n, \mathbf{curl}\, e_{4h}^n\big) \geq \frac{\mu}{2}\Big[d_t\|e_{4h}^n\|_0^2 + \Delta t\|d_t e_{4h}^n\|_0^2\Big] + \mu\sigma_0\|\mathbf{curl}\, e_{4h}^n\|_0^2. \tag{5.15}
$$

Next we estimate the right-hand side of (5.10). Concerning the first term, we have

$$
\begin{aligned}
\langle R_{L1h}^n, e_{1h}^n\rangle &= \langle \partial_t \boldsymbol{u}^n - d_t\mathcal{P}_h\boldsymbol{u}^n, e_{1h}^n\rangle\\
&= (\partial_t\boldsymbol{u}^n - d_t\boldsymbol{u}^n, e_{1h}^n) + (d_t\boldsymbol{u}^n - d_t\mathcal{P}_h\boldsymbol{u}^n, e_{1h}^n).
\end{aligned}\tag{5.16}
$$

By integration formula and Cauchy–Schwarz inequality,

$$
\begin{aligned}
|\partial_t\boldsymbol{u}^n - d_t\boldsymbol{u}^n| &= \left|\partial_t\boldsymbol{u}^n - \frac{\boldsymbol{u}^n - \boldsymbol{u}^{n-1}}{\Delta t}\right| = \left|\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}(\partial_t\boldsymbol{u}^n(t) - \partial_t\boldsymbol{u}(\xi))\,\mathrm{d}\xi\right|\\
&\leq \left|\int_{t_{n-1}}^{t_n}\partial_{tt}\boldsymbol{u}(t)\,\mathrm{d}t\right| \leq C\sqrt{\Delta t}\|\partial_{tt}\boldsymbol{u}\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))}.
\end{aligned}
$$

Thanks to the approximation error estimate (5.3),

$$
\begin{aligned}
|d_t\boldsymbol{u}^n - d_t\mathcal{P}_h\boldsymbol{u}^n| &= \frac{1}{\Delta t}\big|(\boldsymbol{u}^n - \mathcal{P}_h\boldsymbol{u}^n) - \big(\boldsymbol{u}^{n-1} - \mathcal{P}_h\boldsymbol{u}^{n-1}\big)\big|\\
&\leq \frac{1}{\Delta t}C_e h^{\ell+1}\Big(\|\boldsymbol{u}^n\|_{1+\ell,2} + \big\|\boldsymbol{u}^{n-1}\big\|_{1+\ell,2}\Big).
\end{aligned}\tag{5.17}
$$

Then we have

$$
\langle R_{L1h}^n, e_{1h}^n\rangle \leq C\bigg\{\sqrt{\Delta t}\|\partial_{tt}\boldsymbol{u}\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))} + C_e\frac{h^{\ell+1}}{\Delta t}\Big(\|\boldsymbol{u}^n\|_{1+\ell,2} + \big\|\boldsymbol{u}^{n-1}\big\|_{1+\ell,2}\Big)\bigg\}\|e_{1h}^n\|_0. \tag{5.18}
$$

Similarly, we can prove

$$
\begin{aligned}
\langle R_{L2h}^n, e_{3h}^n\rangle &= \langle\partial_t\theta^n - d_t\mathcal{R}_h\theta^n, e_{3h}^n\rangle\\
&\leq C\bigg\{\sqrt{\Delta t}\|\partial_{tt}\theta\|_{L^2(t_{n-1},t_n;L^2(\Omega))} + C_e\frac{h^{\ell+1}}{\Delta t}\Big(\|\theta^n\|_{1+\ell,2} + \big\|\theta^{n-1}\big\|_{1+\ell,2}\Big)\bigg\}\|e_{3h}^n\|_0
\end{aligned}\tag{5.19}
$$

and

$$
\begin{aligned}
\mu\langle R_{L3h}^n, e_{4h}^n\rangle &= \mu\langle\partial_t\boldsymbol{B}^n - d_t\mathcal{F}_h\boldsymbol{B}^n, e_{4h}^n\rangle\\
&\leq C\bigg\{\sqrt{\Delta t}\|\partial_{tt}\boldsymbol{B}\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))} + \frac{C_e h^\ell}{\Delta t}\Big(\|\boldsymbol{B}^n\|_{\ell,2} + \big\|\boldsymbol{B}^{n-1}\big\|_{\ell,2}\Big)\bigg\}\|e_{4h}^n\|_0.
\end{aligned}\tag{5.20}
$$

Making use of Lemma 5.5, by choosing $\big(\phi_1, \bar{\phi}, \phi_2\big) = (\nabla\boldsymbol{u}^n, \nabla\boldsymbol{u}^n, \nabla\boldsymbol{u}^n)$, $(\theta_1, \theta_2) = \big(\theta^n, \theta_h^{n-1}\big)$ and $(\lambda, \chi) = (\nu^n, \nabla e_{1h}^n)$, then we arrive at

$$
\begin{aligned}
\langle R_{D1h}^n, e_{1h}^n\rangle &= \mathcal{A}_1\big(\nu^n(\theta^n) - \nu^n\big(\theta_h^{n-1}\big), \boldsymbol{u}^n, e_{1h}^n\big)\\
&\leq \max\Big\{\|\nu^n\|_{\mathscr{C}(\bar{\Omega}\times\mathbb{R};\mathbb{R}^+)}, \|\nabla\boldsymbol{u}^n\|_{0,3}|\nu^n|_{\mathscr{C}^{0,1}(\bar{\Omega}\times\mathbb{R};\mathbb{R})}\Big\}\big\|\nabla\big(\theta^n - \theta_h^{n-1}\big)\big\|_0\|\nabla e_{1h}^n\|_0,
\end{aligned}\tag{5.21}
$$

which, together with

$$\theta^n - \theta_h^{n-1} = \theta^n - \theta^{n-1} + \theta^{n-1} - \mathcal{R}_h\theta^{n-1} - e_{3h}^{n-1},$$
$$\big\|\nabla\big(\theta^{n-1} - \mathcal{R}_h\theta^{n-1}\big)\big\|_0 \le C_e h^\ell \big\|\theta^{n-1}\big\|_{1+\ell,2}$$

and (5.4), we derive

$$\langle R_{D1h}^n, e_{1h}^n\rangle \le C^*\Big\{\sqrt{\Delta t}\|\partial_t\theta\|_{L^2(t_{n-1},t_n;H^1(\Omega))} + C_e h^\ell\big\|\theta^{n-1}\big\|_{1+\ell,2} + \big\|\nabla e_{3h}^{n-1}\big\|_0\Big\}\|\nabla e_{1h}^n\|_0.$$

Concerning the term $\langle R_{D2h}^n, e_{3h}^n\rangle$, by using of Lemma 5.5 again, we can choose $\big(\phi_1,\, \bar{\phi},\, \phi_2\big) = (\nabla\theta^n, \nabla\theta^n, \nabla\mathcal{R}_h\theta^n)$, $(\theta_1,\theta_2) = \big(\theta^n, \theta_h^{n-1}\big)$ and $(\lambda,\chi) = (\kappa^n, \nabla e_{3h}^n)$, there holds

$$\begin{aligned}\langle R_{D2h}^n, e_{3h}^n\rangle &= \mathcal{A}_2(\kappa^n(\theta^n), \theta^n, e_{3h}^n) - \mathcal{A}_2\big(\kappa^n\big(\theta_h^{n-1}\big), \mathcal{R}_h\theta^n, e_{3h}^n\big)\\ &\le \max\Big\{\|\kappa^n\|_{\mathscr{C}(\bar\Omega\times\mathbb{R};\mathbb{R}^+)}, \|\nabla\theta^n\|_{0,3}|\kappa^n|_{\mathscr{C}^{0,1}(\bar\Omega\times\mathbb{R};\mathbb{R})}, \|\mathcal{R}_h\theta^n\|_{0,\infty}|\kappa^n|_{\mathscr{C}^{0,1}(\bar\Omega\times\mathbb{R};\mathbb{R})}\Big\}\\ &\quad\times \big\|\nabla\big(\theta^n - \theta_h^{n-1}\big)\big\|_0\|\nabla e_{3h}^n\|_0\\ &\le C^*\Big\{\sqrt{\Delta t}\|\partial_t\theta\|_{L^2(t_{n-1},t_n;H^1(\Omega))} + C_e h^\ell\big\|\theta^{n-1}\big\|_{1+\ell,2} + \big\|\nabla e_{3h}^{n-1}\big\|_0\Big\}\|\nabla e_{3h}^n\|_0.\end{aligned}$$

Similarly, by choose $\big(\phi_1,\, \bar{\phi},\, \phi_2\big) = (\theta^n, \theta^n, \theta_h^n)$, and $(\theta_1,\theta_2) = \big(\theta^n, \theta_h^{n-1}\big)$, $(\lambda,\chi) = (\boldsymbol{\beta}^n, e_{1h}^n)$, we deduce

$$\begin{aligned}\langle R_{Bh}^n, e_{1h}^n\rangle &= \big(\boldsymbol{\beta}^n\big(\theta_h^{n-1}\big)\theta_h^n, e_{1h}^n\big) - (\boldsymbol{\beta}^n(\theta^n)\theta^n, e_{1h}^n)\\ &= -\big(\boldsymbol{\beta}^n\big(\theta_h^{n-1}\big)[\theta^n - \theta_h^n], e_{1h}^n\big) - \big(\big[\boldsymbol{\beta}^n(\theta^n) - \boldsymbol{\beta}^n\big(\theta_h^{n-1}\big)\big]\theta^n, e_{1h}^n\big)\\ &\le C^*\Big\{\sqrt{\Delta t}\|\partial_t\theta\|_{L^2(t_{n-1},t_n;H^1(\Omega))} + C_e h^\ell\Big(\|\theta^n\|_{1+\ell,2} + \big\|\theta^{n-1}\big\|_{1+\ell,2}\Big) + \|e_{3h}^n\|_0 + \big\|e_{3h}^{n-1}\big\|_0\Big\}\|\nabla e_{1h}^n\|_0.\end{aligned}$$

Now we are in a position to bound $R_{Cih}^n, i = 1,2$. By (3.16), we can choose $\big(\boldsymbol{u}^1, \boldsymbol{u}^2, \boldsymbol{u}_h^1, \boldsymbol{u}_h^2\big) = \big(\boldsymbol{u}^n, \boldsymbol{u}^n, \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n\big)$ and $\big(\bar{\boldsymbol{u}}^1, \bar{\boldsymbol{u}}^2, \bar{\boldsymbol{w}}_h^1, \bar{\boldsymbol{w}}_h^2\big) = \big(\mathcal{P}_h\boldsymbol{u}^{n-1}, \mathcal{P}_h\boldsymbol{u}^{n-1}, \mathcal{P}_h\boldsymbol{u}^{n-1}, \mathcal{P}_h\boldsymbol{u}^n\big)$, then we have

$$\begin{aligned}\langle R_{C1h}^n, e_{1h}^n\rangle &= \mathcal{O}_1(\boldsymbol{u}^n, \boldsymbol{u}^n, e_{1h}^n) - \mathcal{O}_1\big(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, e_{1h}^n\big)\\ &\le C\left[\|\boldsymbol{u}^n\|_{0,\infty} + \|\boldsymbol{u}^n\|_{1,3} + \sum_{i=n-1}^{n}\Big(\big\|\mathcal{P}_h\boldsymbol{u}^i\big\|_{0,\infty} + \big\|\mathcal{P}_h\boldsymbol{u}^i\big\|_{1,3}\Big)\right]\\ &\quad\cdot\big(2\big\|\boldsymbol{u}^n - \mathcal{P}_h\boldsymbol{u}^{n-1}\big\|_0 + \big\|\mathcal{P}_h\boldsymbol{u}^{n-1} - \mathcal{P}_h\boldsymbol{u}^n\big\|_0 + \big\|e_{1h}^{n-1}\big\|_0 + \|e_{1h}^n\|_0\big)\|\nabla e_{1h}^n\|_0.\end{aligned}$$

Since

$$\begin{aligned}\big\|\boldsymbol{u}^n - \mathcal{P}_h\boldsymbol{u}^{n-1}\big\|_0 &\le \big\|\boldsymbol{u}^n - \boldsymbol{u}^{n-1}\big\|_0 + \big\|\boldsymbol{u}^{n-1} - \mathcal{P}_h\boldsymbol{u}^{n-1}\big\|_0\\ &\le C\sqrt{\Delta t}\|\partial_t\boldsymbol{u}\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))} + C_e h^{\ell+1}\big\|\boldsymbol{u}^{n-1}\big\|_{1+\ell,2}\end{aligned}$$

and

$$\begin{aligned}\big\|\mathcal{P}_h\boldsymbol{u}^{n-1} - \mathcal{P}_h\boldsymbol{u}^n\big\|_0 &\le \big\|\boldsymbol{u}^n - \boldsymbol{u}^{n-1}\big\|_0 + \big\|\boldsymbol{u}^{n-1} - \mathcal{P}_h\boldsymbol{u}^{n-1}\big\|_0 + \|\boldsymbol{u}^n - \mathcal{P}_h\boldsymbol{u}^n\|_0\\ &\le C\sqrt{\Delta t}\|\partial_t\boldsymbol{u}\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))} + C_e h^{\ell+1}\Big[\big\|\boldsymbol{u}^{n-1}\big\|_{1+\ell,2} + \|\boldsymbol{u}^n\|_{1+\ell,2}\Big],\end{aligned}$$

we have

$$\begin{aligned}\langle R_{C1h}^n, e_{1h}^n\rangle \le C\Big(&\sqrt{\Delta t}\|\partial_t\boldsymbol{u}\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))} + C_e h^{\ell+1}\big\|\boldsymbol{u}^{n-1}\big\|_{1+\ell,2}\\ &+ C_e h^{\ell+1}\|\boldsymbol{u}^n\|_{1+\ell,2} + \|e_{1h}^n\|_0 + \big\|e_{1h}^{n-1}\big\|_0\Big)\|\nabla e_{1h}^n\|_0.\end{aligned}$$

Similarly, by choosing $\big(\boldsymbol{u}^1,\boldsymbol{u}^2,\boldsymbol{u}_h^1,\boldsymbol{u}_h^2\big)=\big(\boldsymbol{u}^n,\theta^n,\boldsymbol{u}_h^{n-1},\theta_h^n\big)$ and $\big(\bar{\boldsymbol{u}}^1,\bar{\boldsymbol{u}}^2,\bar{\boldsymbol{w}}_h^1,\bar{\boldsymbol{w}}_h^2\big)=\big(\mathcal{P}_h\boldsymbol{u}^{n-1},\mathcal{R}_h\theta^n,\mathcal{P}_h\boldsymbol{u}^{n-1},\mathcal{R}_h\theta^n\big)$, we can obtain

$$\begin{aligned}\langle R_{C2h}^n,e_{3h}^n\rangle&=\mathcal{O}_2(\boldsymbol{u}^n,\theta^n,e_{3h}^n)-\mathcal{O}_2\big(\boldsymbol{u}_h^{n-1},\theta_h^n,e_{3h}^n\big)\\&\le C\Big(\sqrt{\Delta t}\|\partial_t\boldsymbol{u}\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))}+C_eh^\ell\big\|\boldsymbol{u}^{n-1}\big\|_{1+\ell,2}+C_eh^\ell\|\theta^n\|_{1+\ell,2}+\|e_{3h}^n\|_0+\big\|e_{1h}^{n-1}\big\|_0\Big)\|\nabla e_{3h}^n\|_0.\end{aligned}$$

To estimate the term $\mu\langle R_{E1h}^n,e_{4h}^n\rangle$, we employ Lemma 5.5 by choosing $\big(\phi_1,\bar{\phi},\phi_2\big)=(\mathbf{curl}\,\boldsymbol{B}^n,\mathbf{curl}\,\boldsymbol{B}^n,\mathbf{curl}\,\boldsymbol{B}^n)$, $(\theta_1,\theta_2)=\big(\theta^n,\theta_h^{n-1}\big)$, $(\lambda,\chi)=(\sigma^n,\mathbf{curl}\,e_{4h}^n)$ to get

$$\begin{aligned}\mu\langle R_{E1h}^n,e_{4h}^n\rangle&=\mu\big(\big[\sigma^n(\theta^n)-\sigma^n\big(\theta_h^{n-1}\big)\big]\mathbf{curl}\,\boldsymbol{B}^n,\mathbf{curl}\,e_{4h}^n\big)\\&\le\mu\max\Big\{\|\sigma^n\|_{\mathscr{C}(\bar{\Omega}\times\mathbb{R};\mathbb{R}^+)},\|\mathbf{curl}\,\boldsymbol{B}^n\|_{0,3}|\sigma^n|_{\mathscr{C}^{0,1}(\bar{\Omega}\times\mathbb{R};\mathbb{R})}\Big\}\big\|\nabla\big(\theta^n-\theta_h^{n-1}\big)\big\|_0\|\mathbf{curl}\,e_{4h}^n\|_0\\&\le C\Big(\sqrt{\Delta t}\|\partial_t\theta\|_{L^2(t_{n-1},t_n;H^1(\Omega))}+C_eh^\ell\big\|\theta^{n-1}\big\|_{1+\ell,2}+\big\|\nabla e_{3h}^{n-1}\big\|_0\Big)\|\mathbf{curl}\,e_{4h}^n\|_0.\end{aligned}$$

By using of (5.4) and Assumption 5.2, we can derive

$$\begin{aligned}&\mu\big|(\boldsymbol{B}^n\times\mathbf{curl}\,\boldsymbol{B}^n,e_{1h}^n)-\big(\boldsymbol{B}_h^{n-1}\times\mathbf{curl}\,\boldsymbol{B}_h^n,e_{1h}^n\big)+\big(\boldsymbol{u}_h^n\times\boldsymbol{B}_h^{n-1},\mathbf{curl}\,e_{4h}^n\big)-(\boldsymbol{u}^n\times\boldsymbol{B}^n,\mathbf{curl}\,e_{4h}^n)\big|\\&\quad=\mu\big|\big(\big[\boldsymbol{B}^n-\boldsymbol{B}_h^{n-1}\big]\times\mathbf{curl}\,\boldsymbol{B}^n,e_{1h}^n\big)+\big(\boldsymbol{u}^n\times\big[\boldsymbol{B}_h^{n-1}-\boldsymbol{B}^n\big],\mathbf{curl}\,e_{4h}^n\big)\\&\qquad+\big(\big[\mathcal{F}_h\boldsymbol{B}^{n-1}+e_{4h}^{n-1}\big]\times\mathbf{curl}\,[\boldsymbol{B}^n-\mathcal{F}_h\boldsymbol{B}^n],e_{1h}^n\big)-\big([\boldsymbol{u}^n-\mathcal{P}_h\boldsymbol{u}^n]\times\big[\mathcal{F}_h\boldsymbol{B}^{n-1}+e_{4h}^{n-1}\big],\mathbf{curl}\,e_{4h}^n\big)\big|\\&\quad\le C\mu\Big[\|\mathbf{curl}\,\boldsymbol{B}^n\|_{0,3}\big\|\boldsymbol{B}^n-\boldsymbol{B}_h^{n-1}\big\|_0\|\nabla e_{1h}^n\|_0+\|\boldsymbol{u}^n\|_{0,\infty}\big\|\boldsymbol{B}^n-\boldsymbol{B}_h^{n-1}\big\|_0\|\mathbf{curl}\,e_{4h}^n\|_0\\&\qquad+\big\|\mathcal{F}_h\boldsymbol{B}^{n-1}\big\|_{0,3}\|\mathbf{curl}\,(\boldsymbol{B}^n-\mathcal{F}_h\boldsymbol{B}^n)\|_0\|\nabla e_{1h}^n\|_0+\|\mathbf{curl}\,(\boldsymbol{B}^n-\mathcal{F}_h\boldsymbol{B}^n)\|_{0,3}\big\|e_{4h}^{n-1}\big\|_0\|\nabla e_{1h}^n\|_0\\&\qquad+\|\nabla(\boldsymbol{u}^n-\mathcal{P}_h\boldsymbol{u}^n)\|_0\big\|\mathcal{F}_h\boldsymbol{B}^{n-1}\big\|_{0,3}\|\mathbf{curl}\,e_{4h}^n\|_0+\|\boldsymbol{u}^n-\mathcal{P}_h\boldsymbol{u}^n\|_{0,\infty}\big\|e_{4h}^{n-1}\big\|_0\|\mathbf{curl}\,e_{4h}^n\|_0\Big]\\&\quad\le C\Big(C_r\sqrt{\Delta t}\|\partial_t\boldsymbol{B}\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))}+C_eC_rh^\ell\Big[\big\|\boldsymbol{B}^{n-1}\big\|_{\ell,2}+\|\mathbf{curl}\,\boldsymbol{B}^n\|_{\ell,2}\Big]+C_r\big\|e_{4h}^{n-1}\big\|_0\Big)\|\nabla e_{1h}^n\|_0\\&\qquad+C\Big(C_r\sqrt{\Delta t}\|\partial_t\boldsymbol{B}\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))}+C_eC_rh^\ell\Big[\big\|\boldsymbol{B}^{n-1}\big\|_{\ell,2}+\|\boldsymbol{u}^n\|_{\ell+1,2}\Big]+C_r\big\|e_{4h}^{n-1}\big\|_0\Big)\|\mathbf{curl}\,e_{4h}^n\|_0.\end{aligned}$$

Combining the above estimates and by Young's inequality, we can deduce that

$$\begin{aligned}&d_t\|e_{1h}^n\|_0^2+d_t\|e_{3h}^n\|_0^2+\mu\,d_t\|e_{4h}^n\|_0^2+\Delta t\Big[\|d_te_{1h}^n\|_0^2+\|d_te_{3h}^n\|_0^2+\mu\|d_te_{4h}^n\|_0^2\Big]+\nu_0\|\nabla e_{1h}^n\|_0^2+\frac{\kappa_0}{2}\|\nabla e_{3h}^n\|_0^2\\&\quad+\sigma_0\mu\|\mathbf{curl}\,e_{4h}^n\|_0^2+\frac{\kappa_0}{2}\|\nabla e_{3h}^n\|_0^2-\frac{\kappa_0}{2}\big\|\nabla e_{3h}^{n-1}\big\|_0^2\le c_\varepsilon^*\Big(\big\|e_{1h}^{n-1}\big\|_0^2+\big\|e_{3h}^{n-1}\big\|_0^2+\mu\big\|e_{4h}^{n-1}\big\|_0^2+\delta_n\Big),\end{aligned}\tag{5.22}$$

where

$$\begin{aligned}\delta_n&=C\Delta t\Big\{\|\partial_{tt}\boldsymbol{u}\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))}^2+\|\partial_{tt}\theta\|_{L^2(t_{n-1},t_n;L^2(\Omega))}^2+\|\partial_{tt}\boldsymbol{B}\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))}^2\\&\qquad+\|\partial_t\boldsymbol{u}\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))}^2+\|\partial_t\theta\|_{L^2(t_{n-1},t_n;H^1(\Omega))}^2+\|\partial_t\boldsymbol{B}\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))}^2\Big\}\\&\quad+C\frac{h^{2\ell}}{\Delta t}\Big\{\|\boldsymbol{u}^n\|_{\ell+1,2}^2+\big\|\boldsymbol{u}^{n-1}\big\|_{\ell+1,2}^2+\|\theta^n\|_{1+\ell,2}^2+\|\boldsymbol{B}^n\|_{\ell,2}^2+\big\|\theta^{n-1}\big\|_{1+\ell,2}^2+\big\|\boldsymbol{B}^{n-1}\big\|_{\ell,2}^2\Big\}\\&\quad+Ch^{2\ell}\Big\{\|\theta^n\|_{1+\ell,2}^2+\big\|\theta^{n-1}\big\|_{1+\ell,2}^2+\big\|\boldsymbol{B}^{n-1}\big\|_{\ell,2}^2+\|\mathbf{curl}\,\boldsymbol{B}^n\|_{\ell,2}^2+\|\boldsymbol{u}^n\|_{1+\ell,2}^2+\big\|\boldsymbol{u}^{n-1}\big\|_{1+\ell,2}^2\Big\},\end{aligned}$$

and

$$\Delta t\sum_{n=1}^{m}\delta_n\le C^*\Big((\Delta t)^2+h^{2\ell}\Big).\tag{5.23}$$

Summing up (5.22) from $n = 1$ to m, we conclude that

$$\begin{aligned}&\|e_{1h}^m\|_0^2 + \|e_{3h}^m\|_0^2 + \mu\|e_{4h}^m\|_0^2 + \Delta t\frac{\kappa_0}{2}\|\nabla e_{3h}^m\|_0^2 + \Delta t\sum_{n=1}^{m}\Big[\nu_0\|\nabla e_{1h}^n\|_0^2 \\ &\quad + \frac{\kappa_0}{2}\|\nabla e_{3h}^n\|_0^2 + \mu\sigma_0\|\mathbf{curl}\, e_{4h}^n\|_0^2 + \Delta t\|d_t e_{1h}^n\|_0^2 + \Delta t\|d_t e_{3h}^n\|_0^2 + \Delta t\|d_t e_{4h}^n\|_0^2\Big] \\ &\quad \le \Delta t\sum_{n=0}^{m-1} c_\varepsilon^*\Big(\|e_{1h}^n\|_0^2 + \|e_{3h}^n\|_0^2 + \|e_{4h}^n\|_0^2 + \Delta t\|\nabla e_{3h}^n\|_0^2\Big) + \Delta t\sum_{n=1}^{m}\delta_n.\end{aligned} \tag{5.24}$$

Applying the discrete version of the Gronwall inequality (see Lem. 5.1) to (5.24), and by using (5.23), then we arrive at

$$\begin{aligned}&\|e_{1h}^m\|_0^2 + \|e_{3h}^m\|_0^2 + \mu\|e_{4h}^m\|_0^2 + \Delta t\sum_{n=1}^{m}\Big[\nu_0\|\nabla e_{1h}^n\|_0^2 + \frac{\kappa_0}{2}\|\nabla e_{3h}^n\|_0^2 \\ &\quad + \mu\sigma_0\|\mathbf{curl}\, e_{4h}^n\|_0^2 + \Delta t\|d_t e_{1h}^n\|_0^2 + \Delta t\|d_t e_{3h}^n\|_0^2 + \Delta t\|d_t e_{4h}^n\|_0^2\Big] \le C^*\Big((\Delta t)^2 + h^{2\ell}\Big).\end{aligned} \tag{5.25}$$

By virtue of the identity $\boldsymbol{u}^n - \boldsymbol{u}_h^n = \boldsymbol{u}^n - \mathcal{P}_h\boldsymbol{u}^n - e_{1h}^n$, $\theta^n - \theta_h^n = \theta^n - \mathcal{R}_h\theta^n - e_{3h}^n$, $\boldsymbol{B}^n - \boldsymbol{B}_h^n = \boldsymbol{B}^n - \mathcal{F}_h\boldsymbol{B}^n - e_{4h}^n$, and the error bounds (5.3), we can obtain the desired result and the proof is completed. ☐

As far as error estimate for the pressure, we will prove the following sub-optimal result by following a similar argument developed in [54].

Theorem 5.7. *Under the same assumptions as Theorem 5.6, we have*

$$\Delta t\sum_{n=1}^{m}\|p^n - p_h^n\|_0^2 \le C^*(\Delta t)^{-1}\Big((\Delta t)^2 + h^{2\ell}\Big) \tag{5.26}$$

with $\ell = \min\{k, s\}$.

Proof. From (5.5), we know that

$$b(\boldsymbol{v}_h, e_{2h}^n) = \langle R_{L1h}^n, \boldsymbol{v}_h\rangle + \langle R_{N1h}^n, \boldsymbol{v}_h\rangle - (d_t e_{1h}^n, \boldsymbol{v}_h) - \mathcal{A}_1\big(\nu^n\big(\theta_h^{n-1}\big), e_{1h}^n, \boldsymbol{v}_h\big).$$

Making use of the inf-sup condition (3.4), we derive

$$\begin{aligned}\|e_{2h}^n\|_0 &\le \frac{1}{\beta^*}\sup_{\mathbf{0}\neq\boldsymbol{v}_h\in\boldsymbol{V}_h^k}\frac{b(\boldsymbol{v}_h, e_{2h}^n)}{\|\boldsymbol{v}_h\|_{1,2}} \\ &\le C^*\bigg\{\|R_{L1h}^n\|_{(\boldsymbol{V}_h^k)'} + \|R_{N1h}^n\|_{(\boldsymbol{V}_h^k)'} + \frac{1}{\sqrt{\Delta t}}\Big(\sqrt{\Delta t}\|d_t e_{1h}^n\|_0\Big) + \|\nabla e_{1h}^n\|_0\bigg\},\end{aligned} \tag{5.27}$$

where $R_{N1h}^n = R_{D1h}^n + R_{C1h}^n + R_{Bh}^n + R_{A1h}^n$.

We will estimate the terms in the last line of (5.27) one by one. According to the estimate (5.25), there holds

$$\Delta t\sum_{n=1}^{m}\Big(\Delta t\|d_t e_{1h}^n\|_0^2\Big) \le C^*\Big((\Delta t)^2 + h^{2\ell}\Big).$$

Concerning the term $\langle R_{C1h}^n, \boldsymbol{v}_h\rangle$, by (3.15), similar to the previous theorem, we can deduce

$$\langle R_{C1h}^n, \boldsymbol{v}_h\rangle \le C^*\Big(\sqrt{\Delta t}\|\partial_t \boldsymbol{u}\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))} + C_e h^{\ell+1}\big\|\boldsymbol{u}^{n-1}\big\|_{1+\ell,2} + C_e h^{\ell+1}\|\boldsymbol{u}^n\|_{1+\ell,2} + \|e_{1h}^n\|_0 + \big\|e_{1h}^{n-1}\big\|_0\Big)\|\boldsymbol{v}_h\|_{1,2} + \mathcal{O}_1\big(e_{1h}^{n-1}, e_{1h}^n, \boldsymbol{v}_h\big).$$

A combination of the inverse inequality, Poincaré type inequality and (5.25) yields

$$\begin{aligned}\big\|e_{1h}^{n-1}\big\|_{0,3} &\le C\min\Big\{h^{-1/2}\big\|e_{1h}^{n-1}\big\|_0,\ \big\|\nabla e_{1h}^{n-1}\big\|_0\Big\}\\ &\le C^*\min\Big\{h^{-1/2}\big(\Delta t + h^\ell\big), (\Delta t)^{-1/2}\big(\Delta t + h^\ell\big)\Big\} \le C^*,\end{aligned}$$

which implies that

$$|\mathcal{O}_1\big(e_{1h}^{n-1}, e_{1h}^n, \boldsymbol{v}_h\big)| \le C^*\|\nabla e_{1h}^n\|_0\|\boldsymbol{v}_h\|_{0,6} + C^*\|\nabla \boldsymbol{v}_h\|_0\|e_{1h}^n\|_{0,6}.$$

Hence it holds that

$$\|R_{C1h}^n\|_{(\boldsymbol{V}_h^k)'} \le C^*\Big(\sqrt{\Delta t}\|\partial_t \boldsymbol{u}\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))} + C_e h^{\ell+1}\big\|\boldsymbol{u}^{n-1}\big\|_{1+\ell,2} + C_e h^{\ell+1}\|\boldsymbol{u}^n\|_{1+\ell,2} + \|e_{1h}^n\|_0 + \big\|e_{1h}^{n-1}\big\|_0 + \|\nabla e_{1h}^n\|_0\Big).$$

In a similar way, we can prove

$$\|R_{D1h}^n\|_{(\boldsymbol{V}_h^k)'} \le C^*\Big\{\sqrt{\Delta t}\|\partial_t \theta\|_{L^2(t_{n-1};t_n;H^1(\Omega))} + C_e h^\ell\big\|\theta^{n-1}\big\|_{1+\ell,2} + \big\|\nabla e_{3h}^{n-1}\big\|_0\Big\},$$

$$\|R_{Bh}^n\|_{(\boldsymbol{V}_h^k)'} \le C^*\Big\{\sqrt{\Delta t}\|\partial_t \theta\|_{L^2(t_{n-1};t_n;H^1(\Omega))} + C_e h^\ell\Big(\|\theta^n\|_{1+\ell,2} + \big\|\theta^{n-1}\big\|_{1+\ell,2}\Big) + \|e_{3h}^n\|_0 + \big\|e_{3h}^{n-1}\big\|_0\Big\},$$

and

$$\|R_{A1h}^n\|_{(\boldsymbol{V}_h^k)'} \le C\Big(C_r\sqrt{\Delta t}\|\partial_t \boldsymbol{B}\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))} + C_e C_r h^\ell\big\|\boldsymbol{B}^{n-1}\big\|_{\ell,2} + C_r\big\|e_{4h}^{n-1}\big\|_0 + C_e C_r h^\ell\|\mathbf{curl}\,\boldsymbol{B}^n\|_{\ell,2} + C_r\|\mathbf{curl}\, e_{4h}^n\|_0 + \big\|e_{4h}^{n-1}\times \mathbf{curl}\, e_{4h}^n\big\|_{0,6/5}\Big).$$

To bound the last term in the last line, we continue to derive

$$\begin{aligned}\big\|e_{4h}^{n-1}\times \mathbf{curl}\, e_{4h}^n\big\|_{0,6/5} &= \big\|\big[e_{4h}^{n-1} - Z\big(e_{4h}^{n-1}\big) + Z\big(e_{4h}^{n-1}\big)\big]\times \mathbf{curl}\, e_{4h}^n\big\|_{0,6/5}\\ &\le \big\|e_{4h}^{n-1} - Z\big(e_{4h}^{n-1}\big)\big\|_0\|\mathbf{curl}\, e_{4h}^n\|_{0,3} + \big\|Z\big(e_{4h}^{n-1}\big)\big\|_{0,3}\|\mathbf{curl}\, e_{4h}^n\|_0\\ &\le C_{inv}h^l\big\|\mathbf{curl}\, e_{4h}^{n-1}\big\|_0\|\mathbf{curl}\, e_{4h}^n\|_0 + \big\|Z\big(e_{4h}^{n-1}\big)\big\|_{0,3+\delta_1}\|\mathbf{curl}\, e_{4h}^n\|_0\\ &\le C_{inv}h^l\big\|\mathbf{curl}\, e_{4h}^{n-1}\big\|_0\|\mathbf{curl}\, e_{4h}^n\|_0 + \big\|\mathbf{curl}\, e_{4h}^{n-1}\big\|_0\|\mathbf{curl}\, e_{4h}^n\|_0\\ &\le C\big\|\mathbf{curl}\, e_{4h}^{n-1}\big\|_0\|\mathbf{curl}\, e_{4h}^n\|_0.\end{aligned}$$

By virtue of (5.18), we have

$$\|R_{L1h}^n\|_{(\boldsymbol{V}_h^k)'} \le C\Big\{\sqrt{\Delta t}\|\partial_{tt} \boldsymbol{u}\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))} + C_e\frac{h^{\ell+1}}{\Delta t}\Big(\|\boldsymbol{u}^n\|_{1+\ell,2} + \big\|\boldsymbol{u}^{n-1}\big\|_{1+\ell,2}\Big)\Big\}.$$

A combination of the above estimates, we deduce that

$$\|e_{2h}^n\|_0 \le C^*\Big\{\sqrt{\Delta t}\Big[\|\partial_t \boldsymbol{u}\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))} + \|\partial_{tt} \boldsymbol{u}\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))} + \|\partial_t \theta\|_{L^2(t_{n-1};t_n;H^1(\Omega))} + \|\partial_t \boldsymbol{B}\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))}\Big] + h^\ell\Big[\big\|\boldsymbol{u}^{n-1}\big\|_{1+\ell,2} + \|\boldsymbol{u}^n\|_{1+\ell,2} + \|\theta^n\|_{1+\ell,2}$$

$$+ \left\|\theta^{n-1}\right\|_{1+\ell,2} + \left\|\boldsymbol{B}^{n-1}\right\|_{\ell,2} + \|\mathbf{curl}\,\boldsymbol{B}^n\|_{\ell,2}\Big] + \frac{h^{\ell+1}}{\Delta t}\Big(\|\boldsymbol{u}^n\|_{1+\ell,2} + \left\|\boldsymbol{u}^{n-1}\right\|_{1+\ell,2}\Big)$$
$$+ \|e_{1h}^n\|_0 + \left\|e_{1h}^{n-1}\right\|_0 + \left\|e_{3h}^{n-1}\right\|_0 + \|\nabla e_{1h}^n\|_0 + \left\|\mathbf{curl}\,e_{4h}^{n-1}\right\|_0\|\mathbf{curl}\,e_{4h}^n\|_0 + \left\|\nabla e_{3h}^{n-1}\right\|_0$$
$$+ \|e_{3h}^n\|_0 + \|\nabla e_{3h}^n\|_0 + \left\|e_{4h}^{n-1}\right\|_0 + \|\mathbf{curl}\,e_{4h}^n\|_0 + \frac{1}{\sqrt{\Delta t}}\Big(\sqrt{\Delta t}\|d_t e_{1h}^n\|_0\Big)\Big\},$$

which, together with (5.25), we derive

$$\Delta t\sum_{n=1}^{m}\|e_{2h}^n\|_0^2 \le C^*\Bigg\{(\Delta t)^2 + h^{2\ell} + (\Delta t)^{-1}\Delta t\sum_{n=1}^{m}\Big(\Delta t\|d_t e_{1h}^n\|_0^2\Big)$$
$$+ (\Delta t)^{-1}\Big[(\Delta t)^2 + h^{2\ell}\Big]\sum_{n=1}^{m}\Delta t\|\mathbf{curl}\,e_{4h}^n\|_0^2\Bigg\} \le C^*(\Delta t)^{-1}\Big((\Delta t)^2 + h^{2\ell}\Big).$$

Then the proof can be completed by applying $p^n - p_h^n = p^n - \mathcal{Q}_h p^n - e_{2h}^n$ and (5.3). □

6. Numerical experiments

In this section, we consider two numerical experiments to verify the convergence properties of the fully finite element discretization for the MHD coupled thermal equation with temperature-dependent coefficients (1.1)–(1.7). The parallel code is developed based on the finite element package-Parallel Hierarchical Grids (PHG), *cf.* [56, 57]. The computations were (partly) done on the high performance computers of State Key Laboratory of Scientific and Engineering Computing, Chinese Academy of Sciences.

The two examples are used to verify the optimal error estimates of the fully discrete finite element method proposed in Section 3. In all examples, the domain under consideration is $\Omega = (0,1)^3$ and the finite element mesh is obtained by a uniform tetrahedral partition. We employ the continuous P_2 finite element for discretizing the velocity $\boldsymbol{u}$ and the temperature θ, the continuous P_1 element for discretizing the pressure p, and the second order edge element method for discretizing magnetic induction $\boldsymbol{B}$.

Example 6.1. This example is to show the temporal error of the Euler semi-implicit scheme, when $\Delta t \to 0$. Setting the parameters $\nu(\theta) = 1, \sigma(\theta) = 1, \kappa(\theta) = \theta$, $\mu = 1$ and $\boldsymbol{\beta}(\theta) = (0,0,1)$, the time interval $[0,1]$. The analytic solution is chosen as

$$\boldsymbol{u} = \big(ye^{-t}, z\cos(t), x\big),\quad p = x - y,\quad \boldsymbol{B} = \big(ye^{-t}, 0, 0\big),\quad \theta = ye^{-t}.$$

Since the exact solution is linear in spatial variables, the major error comes from the discretization of the time variable. We fix a tetrahedral mesh with $h_0 = 0.433$, the terminal time $T = 1$, and test the convergence rate at each time step. We list the discretization error for all unknowns at the last moment $T = 1$ in Table 1, from which it shows perfectly that the temporal convergence rate of the Euler discrete scheme is first-order.

Example 6.2. This example is to test the convergence rate for our numerical scheme when both the timestep and the meshwidth are refined at the same time. Setting the parameters $\nu(\theta) = \theta, \sigma(\theta) = 1, \kappa(\theta) = \theta$, $\mu = 1$ and $\boldsymbol{\beta}(\theta) = (0,0,-1)$. The initial mesh $h_0 = 0.866$ and the time interval is $[0,1]$. The analytic solution is chosen as

$$\boldsymbol{u} = (\sin(y)\sin(t), 0, 0),\quad p = 0,\quad \boldsymbol{B} = (0, \sin(x)\sin(t), 0),\quad \theta = 1 + \sin(y)\sin(t).$$

To test the validity of Theorem 5.6, the following error bounds are denoted by

$$E(\boldsymbol{u}) := \left(\Delta t\sum_{n=1}^{m}\|\nabla(\boldsymbol{u}^n - \boldsymbol{u}_h^n)\|_0^2\right)^{1/2},\quad E(\theta) := \left(\Delta t\sum_{n=1}^{m}\|\nabla(\theta^n - \theta_h^n)\|_0^2\right)^{1/2},$$

TABLE 1. The convergence rate of Euler scheme at terminal time $T = 1$ (Example 6.1).

Δt	$\left\|\boldsymbol{u}(T) - \boldsymbol{u}_h^N\right\|_0$	Order	$\left\|\boldsymbol{u}(T) - \boldsymbol{u}_h^N\right\|_{1,2}$	Order	$\left\|p(T) - p_h^N\right\|_0$	Order
0.1000	7.167e-05	–	6.332e-04	–	8.947e-03	–
0.0500	3.612e-05	0.9884	3.189e-04	0.9898	4.374e-03	1.0323
0.0250	1.816e-05	0.9922	1.606e-04	0.9896	2.157e-03	1.0198
0.0125	9.179e-06	0.9843	8.230e-05	0.9644	1.069e-03	1.0130
Δt	$\left\|\boldsymbol{B}(T) - \boldsymbol{B}_h^N\right\|_0$	Order	$\left\|\boldsymbol{B}(T) - \boldsymbol{B}_h^N\right\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$	Order		
0.1000	8.568e-04	–	4.181e-03	–		
0.0500	4.181e-04	1.0351	2.039e-03	1.0357		
0.0250	2.065e-04	1.0175	1.007e-03	1.0178		
0.0125	1.026e-04	1.0087	5.005e-04	1.0089		
Δt	$\left\|\theta(T) - \theta_h^N\right\|_0$	Order	$\left\|\theta(T) - \theta_h^N\right\|_{1,2}$	Order		
0.1000	1.338e-03	–	1.371e-02	–		
0.0500	7.032e-04	0.9278	7.118e-03	0.9460		
0.0250	3.609e-04	0.9623	3.629e-03	0.9721		
0.0125	1.826e-04	0.9830	1.831e-03	0.9871		

TABLE 2. The convergence rate of Euler scheme at terminal time $T = 1$ (Example 6.2).

$(\Delta t, h)$	$E(\boldsymbol{u})$	Order	$E(\boldsymbol{B})$	Order	$E(\theta)$	Order	$\left\|p(T) - p_h^N\right\|_0$	Order
$(\Delta t_0, h_0)$	5.827e-03	–	9.387e-03	–	3.749e-02	–	3.883e-03	–
$(\Delta t_0/4, h_0/2)$	1.148e-03	2.3437	2.560e-03	1.8746	7.187e-03	2.3831	1.014e-03	1.9371
$(\Delta t_0/16, h_0/4)$	2.636e-04	2.1226	6.528e-04	1.9715	1.662e-03	2.1125	2.418e-04	2.0680
$(\Delta t_0/64, h_0/8)$	6.400e-05	2.0422	1.627e-04	2.0042	4.062e-04	2.0327	6.316e-05	1.9368
$(\Delta t, h)$	$\left\|\boldsymbol{u}(T) - \boldsymbol{u}_h^N\right\|_0$	Order	$\left\|\boldsymbol{B}(T) - \boldsymbol{B}_h^N\right\|_0$	Order	$\left\|\theta(T) - \theta_h^N\right\|_0$	Order		
$(\Delta t_0, h_0)$	4.979e-04	–	3.175e-03	–	9.892e-04	–		
$(\Delta t_0/4, h_0/2)$	6.824e-05	2.8671	1.075e-03	1.5626	3.233e-04	1.6136		
$(\Delta t_0/16, h_0/4)$	1.326e-05	2.3641	3.439e-04	1.6439	1.170e-04	1.4661		
$(\Delta t_0/64, h_0/8)$	4.071e-06	1.7031	9.665e-05	1.8311	3.628e-05	1.6893		

$$E(\boldsymbol{B}) := \left(\Delta t \sum_{n=1}^{m} \|\mathbf{curl}\,(\boldsymbol{B}^n - \boldsymbol{B}_h^n)\|_0^2\right)^{1/2}.$$

Setting $\Delta t_0 = h_0^2$, Table 2 shows that the convergence rate for the backward Euler scheme at the terminal time. The initial conditions, boundary conditions and source terms are determined by the analytical solution. Both the timestep and the meshwidth are refined at the same time such that $\Delta t = \mathcal{O}(h^2)$. The corresponding convergent results are displayed in Table 2 and an $\mathcal{O}(h^2)$ convergence of the proposed numerical scheme can be observed, which agrees with the theoretical results developed in this paper.

7. SUMMARY

We have studied a fully discrete finite element scheme for the 3D thermally coupled incompressible MHD problems with variable coefficients problems. The proposed scheme has the nice features that it only needs to solve one linear system at each time step and the magnetic induction is approximated by $\boldsymbol{H}(\mathbf{curl})$-conforming Nédélec edge element, which make it quite attractive to solve these highly nonlinear MHD models with possibly non-smooth magnetic induction solution. We prove that the fully discrete solution converges to a weak solution of the continuous problem as both meshwidth and timestep size tend to zero, and it is unique under a further smooth assumption. Thus we have given a numerical verification of the existence of weak solution to this model, which is still missing in the literature. Under a quite low regularity assumption for the exact solution, we rigorously establish the error estimates for the velocity, temperature and magnetic induction unconditionally in the sense that the time step is restricted but is independent of the spacial mesh size. Whether the plain convergence or error estimate, the technique and results of this paper have some improvements on that of relevant papers.

Acknowledgements. The authors thank the anonymous referees very much for their helpful comments and suggestions which helped to improve the presentation of this paper. Shipeng Mao is supported by National Natural Science Foundation of China (Nos 11871467, 11471329), Qianqian Ding is supported by Shandong Province Natural Science Foundation ZR2021QA054 and China Postdoctoral Science Foundation 2021M691951.

Funding: The research is supported by National Natural Science Foundation of China (Nos 11871467, 11471329), Shandong Province Natural Science Foundation ZR2021QA054 and China Postdoctoral Science Foundation 2021M691951.

References

[1] A. Ait Ou Ammi and M. Marion, Nonlinear Galerkin methods and mixed finite elements: two-grid algorithms for the Navier–Stokes equations. *Numer. Math.* **68** (1994) 189–213.

[2] C. Amrouche, C. Bernardi, M. Dauge and V. Girault, Vector potentials in three-dimensional non-smooth domains. *Math. Methods Appl. Sci.* **21** (1998) 823–864.

[3] D.A. Anderson, J.C. Tannehill and R.H. Pletcher, Computational Fluid Mechanics and Heat Transfer. Series in *Computational Methods in Mechanics and Thermal Sciences*. Hemisphere Publishing Corp., Washington, DC; McGraw-Hill Book Co., New York (1984).

[4] F. Armero and J.C. Simo, Long-term dissipativity of time-stepping algorithms for an abstract evolution equation with applications to the incompressible MHD and Navier–Stokes equations. *Comput. Methods Appl. Mech. Eng.* **131** (1996) 41–90.

[5] D. Boffi, Fortin operator and discrete compactness for edge elements. *Numer. Math.* **87** (2000) 229–246.

[6] D. Boffi, F. Brezzi and M. Fortin, Mixed Finite Element Methods and Applications. Vol. 44 of *Springer Series in Computational Mathematics*. Springer, Heidelberg (2013).

[7] J.H. Bramble, J.E. Pasciak and O. Steinbach, On the stability of the L^2 projection in H^1. *Math. Comput.* **71** (2002) 147–156.

[8] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods. In: Vol. 15 of *Springer Series in Computational Mathematics*. Springer-Verlag, New York (1991).

[9] C. Carstensen, Merging the Bramble–Pasciak–Steinbach and the Crouzeix–Thomée criterion for H^1-stability of the L^2-projection onto finite element spaces. *Math. Comput.* **71** (2002) 157–163.

[10] Z. Cheng, N. Takahashi and B. Forghani, Electromagnetic and Thermal Field Modeling and Application in Electrical Engineering. Science Press, Beijing (2009).

[11] P.G. Ciarlet, The Finite Element Method for Elliptic Problems. *Studies in Mathematics and its Applications*. Vol. 4. North-Holland Publishing Co., Amsterdam-New York-Oxford (1978).

[12] M. Costabel and M. Dauge, Singularities of Maxwell's equations on polyhedral domains. In: Analysis, Numerics and Applications of Differential and Integral Equations (Stuttgart, 1996). Vol. 379 of *Pitman Res. Notes Math. Ser.* Longman, Harlow (1998) 69–76.

[13] M. Costabel and M. Dauge, Singularities of electromagnetic fields in polyhedral domains. *Arch. Ration. Mech. Anal.* **151** (2000) 221–276.

[14] M. Costabel and M. Dauge, Weighted regularization of Maxwell equations in polyhedral domains. A rehabilitation of nodal finite elements. *Numer. Math.*, **93** (2002) 239–277.

[15] M. Dauge, Singularities of corner problems and problems of corner singularities. In: Actes du 30ème Congrès d'Analyse Numérique: CANum '98 (Arles, 1998). Vol. 6 of *ESAIM Proc. Soc. Math. Appl. Indust.* Paris (1999) 19–40.

[16] P.A. Davidson, An Introduction to Magnetohydrodynamics. *Cambridge Texts in Applied Mathematics*. Cambridge University Press, Cambridge (2001).

[17] X. Dong and Y. He, Optimal convergence analysis of Crank-Nicolson extrapolation scheme for the three-dimensional incompressible magnetohydrodynamics. *Comput. Math. Appl.* **76** (2018) 2678–2700.

[18] P. Fernandes and G. Gilardi, Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions. *Math. Models Methods Appl. Sci.* **7** (1997) 957–991.

[19] H. Gao and W. Qiu, A semi-implicit energy conserving finite element method for the dynamical incompressible magnetohydrodynamics equations. *Comput. Methods Appl. Mech. Eng.* **346** (2019) 982–1001.

[20] J.-F. Gerbeau and C. Le Bris, Comparison between two numerical methods for a magnetostatic problem. *Calcolo* **37** (2000) 1–20.

[21] J.-F. Gerbeau, C. Le Bris and T. Lelièvre, Mathematical Methods for the Magnetohydrodynamics of Liquid Metals. *Numerical Mathematics and Scientific Computation*. Oxford University Press, Oxford (2006).

[22] D. Gilbarg and N.S. Trudinger, Elliptic Partial Differential Equations of Second Order. Reprint of the 1998 edition. *Classics in Mathematics*. Springer-Verlag, Berlin (2001).

[23] V. Girault and P.-A. Raviart, Finite Element Methods for Navier–Stokes Equations: Theory and Algorithms. Vol. 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin (1986).

[24] C. Greif, D. Li, D. Schötzau and X. Wei, A mixed finite element method with exactly divergence-free velocities for incompressible magnetohydrodynamics. *Comput. Methods Appl. Mech. Eng.* **199** (2010) 2840–2855.

[25] J.L. Guermond and P.D. Minev, Mixed finite element approximation of an MHD problem involving conducting and insulating regions: the 3D case. *Numer. Methods Part. Differ. Equ.* **19** (2003) 709–731.

[26] M.D. Gunzburger, A.J. Meir and J.S. Peterson, On the existence, uniqueness, and finite element approximation of solutions of the equations of stationary, incompressible magnetohydrodynamics. *Math. Comp.* **56** (1991) 523–563.

[27] Y. He, Unconditional convergence of the Euler semi-implicit scheme for the three-dimensional incompressible MHD equations. *IMA J. Numer. Anal.* **35** (2015) 767–801.

[28] J.G. Heywood and R. Rannacher, Finite element approximation of the nonstationary Navier-Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization. *SIAM J. Numer. Anal.* **19** (1982) 275–311.

[29] J.G. Heywood and R. Rannacher, Finite-element approximation of the nonstationary Navier-Stokes problem. IV. Error analysis for second-order time discretization. *SIAM J. Numer. Anal.* **27** (1990) 353–384.

[30] R. Hiptmair, Finite elements in computational electromagnetism. *Acta Numer.* **11** (2002) 237–339.

[31] R. Hiptmair, L. Li, S. Mao and W. Zheng, A fully divergence-free finite element method for magnetohydrodynamic equations. *Math. Models Methods Appl. Sci.* **28** (2018) 659–695.

[32] K. Hu, Y. Ma and J. Xu, Stable finite element methods preserving $\nabla \cdot B = 0$ exactly for MHD models. *Numer. Math.* **135** (2017) 371–396.

[33] B. Huang and Y. Liao, Global stability of combination of viscous contact wave with rarefaction wave for compressible Navier–Stokes equations with temperature-dependent viscosity. *Math. Models Methods Appl. Sci.* **27** (2017) 2321–2379.

[34] H. Kurose, D. Miyagi, N. Takahashi, N. Uchida and K. Kawanaka, 3-d eddy current analysis of induction heating apparatus considering heat emission, heat conduction, and temperature dependence of magnetic characteristics. *IEEE Trans. Magnetics* **45** (2009) 1847–1850.

[35] W. Layton, H. Tran and C. Trenchea, Stability of partitioned methods for magnetohydrodynamics flows at small magnetic Reynolds number. In: Recent Advances in Scientific Computing and Applications. Vol. 586 of *Contemp. Math.* Amer. Math. Soc., Providence, RI (2013) 231–238.

[36] A.E. Lifschitz, Magnetohydrodynamics and Spectral Theory. Vol. 4 of *Developments in Electromagnetic Theory and Applications*. Kluwer Academic Publishers Group, Dordrecht (1989).

[37] S.A. Lorca and J.L. Boldrini, Stationary solutions for generalized Boussinesq models. *J. Differ. Equ.* **124** (1996) 389–406.

[38] S.A. Lorca and J.L. Boldrini, The initial value problem for a generalized Boussinesq model. *Nonlinear Anal.* **36** (1999) 457–480.

[39] A.J. Meir, Thermally coupled, stationary, incompressible MHD flow; existence, uniqueness, and finite element approximation. *Numer. Methods Part. Differ. Equ.* **11** (1995) 311–337.

[40] A.J. Meir and P.G. Schmidt, On electromagnetically and thermally driven liquid-metal flows. In: Proceedings of the Third World Congress of Nonlinear Analysts, Part 5 (Catania, 2000). Vol. 47 (2001) 3281–3294.

[41] P. Monk, Finite Element Methods for Maxwell's Equations. *Numerical Mathematics and Scientific Computation*. Oxford University Press, New York (2003).

[42] R. Moreau, Magnetohydrodynamics. Vol. 3 of Fluid Mechanics and its Applications. Translated from the French by A.F. Wright. Kluwer Academic Publishers Group, Dordrecht (1990).

[43] S. Nadeem and N.S. Akbar, Effects of temperature dependent viscosity on peristaltic flow of a Jeffrey-six constant fluid in a non-uniform vertical tube. *Commun. Nonlinear Sci. Numer. Simul.* **15** (2010) 3950–3964.

[44] J.-C. Nédélec, Mixed finite elements in $\mathbf{R}^3$. *Numer. Math.* **35** (1980) 315–341.

[45] A. Prohl, Convergent finite element discretizations of the nonstationary incompressible magnetohydrodynamics system. *ESAIM: M2AN* **42** (2008) 1065–1087.

[46] H. Qiu, Error analysis of euler semi-implicit scheme for the nonstationary magneto-hydrodynamics problem with temperature dependent parameters. *J. Sci. Comput.* **85** (2020) 1–26.

[47] S.S. Ravindran, Partitioned time-stepping scheme for an MHD system with temperature-dependent coefficients. *IMA J. Numer. Anal.* **39** (2019) 1860–1887.

[48] D. Schötzau, Mixed finite element methods for stationary incompressible magneto-hydrodynamics. *Numer. Math.* **96** (2004) 771–800.

[49] M. Sermange and R. Temam, Some mathematical questions related to the MHD equations. *Comm. Pure Appl. Math.* **36** (1983) 635–664.

[50] D. Shi and Z. Yu, Nonconforming mixed finite element methods for stationary incompressible magnetohydrodynamics. *Int. J. Numer. Anal. Model.* **10** (2013) 904–919.

[51] J. Simon, Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl.* **146** (1987) 65–96.

[52] M. Tabata, Finite element approximation to infinite prandtl number boussinesq equations with temperature-dependent coefficients – thermal convection problems in a spherical shell. *Future Generation Comput. Syst.* **22** (2006) 521–531.

[53] M. Tabata and D. Tagami, Error estimates for finite element approximations of drag and lift in nonstationary Navier–Stokes flows. *Japan J. Indust. Appl. Math.* **17** (2000) 371–389.

[54] M. Tabata and D. Tagami, Error estimates of finite element methods for nonstationary thermal convection problems with temperature-dependent coefficients. *Numer. Math.* **100** (2005) 351–372.

[55] M. Wiedmer, Finite element approximation for equations of magnetohydrodynamics. *Math. Comp.* **69** (2000) 83–101.

[56] L.-B. Zhang, A parallel algorithm for adaptive local refinement of tetrahedral meshes using bisection. *Numer. Math. Theory Methods Appl.* **2** (2009) 65–89.

[57] L. Zhang, T. Cui and H. Liu, A set of symmetric quadrature rules on triangles and tetrahedra. *J. Comput. Math.* **27** (2009) 89–96.

[58] G.-D. Zhang, J. Yang and C. Bi, Second order unconditionally convergent and energy stable linearized scheme for MHD equations. *Adv. Comput. Math.* **44** (2018) 505–540.