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A PRIORI AND *A POSTERIORI* ERROR ESTIMATES FOR THE QUAD-CURL EIGENVALUE PROBLEM

Lixiu Wang[1,2], Qian Zhang[3,*], Jiguang Sun[3] and Zhimin Zhang[2,4]

Abstract. In this paper, we consider *a priori* and *a posteriori* error estimates of the $H(\mathrm{curl}^2)$-conforming finite element when solving the quad-curl eigenvalue problem. An *a priori* estimate of eigenvalues with convergence order $2(s-1)$ is obtained if the corresponding eigenvector $\boldsymbol{u} \in \boldsymbol{H}^{s-1}(\Omega)$ and $\nabla \times \boldsymbol{u} \in \boldsymbol{H}^{s}(\Omega)$. For the *a posteriori* estimate, by analyzing the associated source problem, we obtain lower and upper bounds for the errors of eigenvectors in the energy norm and upper bounds for the errors of eigenvalues. Numerical examples are presented for validation.

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1. Introduction

The quad-curl operator has important applications in the inverse electromagnetic scattering theory [8,9]. The corresponding quad-curl eigenvalue problem plays a fundamental role in the analysis and computation of the electromagnetic interior transmission eigenvalues [27,32,33]. Various numerical methods have been proposed for the quad-curl source problem, see, *e.g.*, [6,7,13,22,35,36,38,41,42,45]. However, there exist only a few results on the numerical methods for the quad-curl eigenvalue problem. The quad-curl eigenvalue problem was first proposed in [33], where J. Sun developed a mixed finite element method by introducing an auxiliary variable $\boldsymbol{w} = \nabla \times \nabla \times \boldsymbol{u}$ and proved an *a priori* error estimate. In [12], H. Chen *et al.* designed a different mixed scheme by introducing $\boldsymbol{\delta} = \nabla \times \boldsymbol{u}$. Two multigrid methods based on the Rayleigh quotient iteration and the inverse iteration with fixed shift were proposed and analyzed in [21].

Very recently, three of the authors and their collaborators constructed $H(\mathrm{curl}^2)$-conforming finite elements in both two and three dimensions (2D and 3D) to solve the quad-curl source problem [23,24,39,40,44]. Based on the conforming elements, in this paper, we consider the conforming finite element method for the eigenvalue problem and derive *a priori* and *a posteriori* error estimates.

Keywords and phrases. The quad-curl problem, eigenvalue problem, *a priori* error estimation, *a posteriori* error estimation, curl-curl conforming elements.

[1] School of Mathematics and Physics, University of Science and Technology Beijing, Beijing 100083, P.R. China.

[2] Beijing Computational Science Research Center, Beijing 100193, P.R. China.

[3] Department of Mathematical Sciences, Michigan Technological University, Houghton, MI 49931, USA.

[4] Department of Mathematics, Wayne State University, Detroit, MI 48202, USA.

*Corresponding author: qzhang15@mtu.edu

In the first part of the paper, we apply the classical framework of Babuška and Osborn [2, 29] to derive an *a priori* estimate. To this end, we prove the discrete compactness of the $H(\mathrm{curl}^2)$-conforming finite elements with div-free condition. We can show that the conforming method is convergent and it has a convergence order of $2(s-1)$ when the eigenvector $\boldsymbol{u} \in \boldsymbol{H}^{s-1}(\Omega)$ and $\nabla \times \boldsymbol{u} \in \boldsymbol{H}^{s}(\Omega)$.

At reentrant corners or material interfaces, the eigenvectors feature strong singularities [28]. For more efficient computation, adaptive local refinements are considered. *A posteriori* error estimators are essential for adaptive finite element methods. In addition, an inappropriate scheme for the quad-curl problem might lead to spurious eigenvalues. In this situation, *a posteriori* error estimators can be applied to test whether an eigenvalue is spurious. We refer to [3–5, 15, 17, 25, 30] for the *a posteriori* error estimates of electromagnetic problems and elliptic problems. In [10], S. Cao *et al.* developed an *a posteriori* error estimator for a decoupled finite element method for the quad-curl source problem. In terms of the quad-curl eigenvalue problem, to the authors' knowledge, no work on *a posteriori* error estimations has been done so far. Therefore, in the second part of the paper, we consider an *a posteriori* error estimate for the conforming finite element method.

Due to the large kernel space of the curl operator, the Helmholtz decomposition of splitting a vector field in $H(\mathrm{curl}^2;\Omega)$ into the irrotational and solenoidal components plays an important role in the analysis. However, in general, the irrotational component is not H^2-regular when Ω is non-convex. Therefore, we propose a new decomposition for $H_0(\mathrm{curl}^2;\Omega)$, which further splits the irrotational component into a function in $H^2(\Omega)$ and a function in the kernel of curl operator. To obtain an *a posteriori* error estimator for the eigenvalue problem, we apply the idea of [17] to relate the eigenvalue problem to a source problem. An *a posteriori* error estimator for the source problem is constructed by analyzing irrotational and solenoidal components, respectively. Then an *a posteriori* error estimator for simple eigenvalues is obtained. The proof uses the new decomposition and makes no additional regularity assumption.

For ease of presentation, we will focus on only 3D case, the similar arguments can be used to the 2D case. The rest of this paper is organized as follows. In Section 2, we present some notation, the $H(\mathrm{curl}^2)$-conforming elements, the new decomposition, and an $H(\mathrm{curl}^2)$-type Clément interpolation, which will be used in the *a posteriori* error analysis. In Section 3, we derive the *a priori* error estimate for the quad-curl eigenvalue problem. In Section 4, we prove the *a posteriori* error estimate. Finally, in Section 5, we show some numerical experiments.

2. Notation and basis tools

2.1. Notation

Let $\Omega \subset \mathbb{R}^3$ be a contractible Lipschitz domain. For a Lipschitz domain $D \subset \mathbb{R}^3$, $L^2(D)$ denotes the space of square integrable functions on D with norm $\|\cdot\|_D$. For a positive integer s, $H^s(D)$ denotes the space of scalar functions in $L^2(D)$ whose derivatives up to order s are also in $L^2(D)$. If $s = 0$, $H^0(D) = L^2(D)$. For vector functions, denote $\boldsymbol{L}^2(D) = (L^2(D))^3$ and $\boldsymbol{H}^s(D) = (H^s(D))^3$. We use $(\cdot,\cdot)_D$ and $\langle\cdot,\cdot\rangle_{\partial D}$ to stand for the L^2 inner products on D and ∂D. When $D = \Omega$, we omit the subscript Ω in the notation of norms and L^2 inner products.

For simplicity, denote $(\nabla\times)^2\boldsymbol{u} = \nabla \times \nabla \times \boldsymbol{u}$. We now define a space concerning the $\nabla\times$ operator

$$H(\mathrm{curl}^2;D) := \{\boldsymbol{u} \in \boldsymbol{L}^2(D) :\ (\nabla\times)^i\boldsymbol{u} \in \boldsymbol{L}^2(D),\ i = 1, 2\},$$

whose norm is given by

$$\|\boldsymbol{u}\|_{H(\mathrm{curl}^2;D)} = \sqrt{\sum_{i=0}^{2} \left((\nabla\times)^i\boldsymbol{u}, (\nabla\times)^i\boldsymbol{u}\right)_D}.$$

We also equip the space $H(\mathrm{curl}^2;D)$ with the following norm:

$$|\!|\!|\boldsymbol{u}|\!|\!|_D^2 = \|\boldsymbol{u}\|_D^2 + \|(\nabla\times)^2\boldsymbol{u}\|_D.$$

We drop the subscript D in $\|\cdot\|_D$ when $D=\Omega$. The spaces $H_0(\mathrm{curl}^2;D)$, $H_0^1(D)$, and $H(\mathrm{div}^0;D)$ are defined, respectively, as

$$\begin{aligned} H_0(\mathrm{curl}^2;D) &:= \{\boldsymbol{u}\in H(\mathrm{curl}^2;D):\ \boldsymbol{u}\times\boldsymbol{n}_D=0 \text{ and } \nabla\times\boldsymbol{u}\times\boldsymbol{n}_D=0 \text{ on } \partial D\},\\ H_0^1(D) &:= \{u\in H^1(D): u=0 \text{ on } \partial D\},\\ H(\mathrm{div}^0;D) &:= \{\boldsymbol{u}\in \boldsymbol{L}^2(D):\ \nabla\cdot\boldsymbol{u}=0\}, \end{aligned}$$

where $\boldsymbol{n}_D$ is the unit outward normal vector to ∂D.

Let $\mathcal{T}_h$ be a shape regular simplicial triangulation of Ω. Denote by $\mathcal{N}_h$, $\mathcal{E}_h$, and $\mathcal{F}_h$ the sets of vertices, edges, and faces. Let $\boldsymbol{\tau}_e$ and $\boldsymbol{n}_f$ be the unit tangent vector of an edge $e\in\mathcal{E}_h$ and the unit normal vector of a face $f\in\mathcal{F}_h$, respectively. We denote $\mathcal{N}_h^{\mathrm{int}}$ and $\mathcal{F}_h^{\mathrm{int}}$ as the sets of vertices and faces in the interior of Ω, respectively. Let $\mathcal{N}_h(T)$, $\mathcal{E}_h(T)$ and $\mathcal{F}_h(T)$ be the sets of vertices, edges, and faces on the element T. Denote by h_T the diameter of $T\in\mathcal{T}_h$. Denote $h=\max\limits_{T\in\mathcal{T}_h} h_T$. We use P_k to represent the space of polynomials with degrees at most k. Denote $\boldsymbol{P}_k=(P_k)^3$.

2.2. $\boldsymbol{H}(\mathbf{curl}^2)$-conforming elements in 3D

We apply the $H(\mathrm{curl}^2)$-conforming elements constructed in [23,24]. The shape function space for an element $T\in\mathcal{T}_h$ is

$$V_h^k(T)=\nabla P_{k+1}(T)\oplus \mathfrak{p}\boldsymbol{P}_{k-1}^+(T),$$

where

$$\mathfrak{p}\boldsymbol{u}=\int_0^1 \boldsymbol{u}(W+t(\boldsymbol{x}-W))\times t(\boldsymbol{x}-W)\mathrm{d}t,$$

and

$$\boldsymbol{P}_k^+(T)=\begin{cases}\boldsymbol{P}_1(T)\oplus B^1, & k=1,\\ \boldsymbol{P}_2(T)\oplus B^1\oplus \mathring{B}^2, & k=2,\\ \boldsymbol{P}_k(T)\oplus \mathring{B}^k, & k\geqslant 3.\end{cases}$$

Here B^1 and $\mathring{B}^k$ are spaces of modified bubbles defined on the Alfeld split of T, see [24] for more information.

Remark 2.1. For the bubble functions in $\boldsymbol{P}_k^+(T)$, we choose the barycenter of T as the base point W in the Poincaré operator $\mathfrak{p}$, see [14]. For other functions, we choose W to be the origin.

For $k\geqslant 2$, the $H(\mathrm{curl}^2)$-conforming element with the shape function space $V_h^k(T)$ is defined by the following degrees of freedom (DOFs).

– Vertex DOFs $\boldsymbol{M}_v(\boldsymbol{u})$ at all vertices $v_i\in\mathcal{N}_h$:

$$\boldsymbol{M}_v(\boldsymbol{u})=\{(\nabla\times\boldsymbol{u})(v_i)\}. \tag{2.1}$$

– Edge DOFs $\boldsymbol{M}_e(\boldsymbol{u})$ on all edges $e_i\in\mathcal{E}_h$:

$$\begin{aligned}\boldsymbol{M}_e(\boldsymbol{u})=&\left\{\int_{e_i}\boldsymbol{u}\cdot\boldsymbol{\tau}_{e_i}q\mathrm{d}S,\ \forall q\in P_k(e_i)\right\}\\ &\cup\left\{\frac{1}{\mathrm{length}(e_i)}\int_{e_i}\nabla\times\boldsymbol{u}\cdot\boldsymbol{q}\mathrm{d}S,\ \forall\boldsymbol{q}\circ F_T\in\boldsymbol{P}_{k-3}(\hat{e}_i)\right\}.\end{aligned} \tag{2.2}$$

– Face DOFs $\boldsymbol{M}_f(\boldsymbol{u})$ at all faces $f_i \in \mathcal{F}_h$ (with two mutually orthogonal unit vector $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ in the face f_i and the unit normal vector $\boldsymbol{n}_i$):

$$\begin{aligned}\boldsymbol{M}_f(\boldsymbol{u}) = &\left\{\int_{f_i} \nabla\times\boldsymbol{u}\cdot\boldsymbol{n}_i q\mathrm{d}A,\ \forall q\in P_{k-4}(f_i)/\mathbb{R}\right\}\\ &\cup\left\{\frac{1}{\mathrm{area}(f_i)}\int_{f_i}\nabla\times\boldsymbol{u}\cdot\boldsymbol{\tau}_1 q\mathrm{d}A,\ \forall q\in P_{k-4}(f_i)\right\}\\ &\cup\left\{\frac{1}{\mathrm{area}(f_i)}\int_{f_i}\nabla\times\boldsymbol{u}\cdot\boldsymbol{\tau}_2 q\mathrm{d}A,\ \forall q\in P_{k-4}(f_i)\right\}\\ &\cup\left\{\frac{1}{\mathrm{area}\,(f_i)}\int_{f_i}\boldsymbol{u}\cdot\boldsymbol{q}\mathrm{d}A, \forall\boldsymbol{q}=B_T\hat{\boldsymbol{q}}, \hat{\boldsymbol{q}}\in P_{k-2}(\hat{f}_i)\left[\hat{\boldsymbol{x}}-(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{n}}_i)\,\hat{\boldsymbol{n}}_i\right]|_{\hat{f}_i}\right\}.\end{aligned}\tag{2.3}$$

– Interior DOFs $\boldsymbol{M}_T(\boldsymbol{u})$ for all the elements $T_i \in \mathcal{T}_h$:

$$\begin{aligned}\boldsymbol{M}_T(\boldsymbol{u}) = &\left\{\int_{T_i}\nabla\times\boldsymbol{u}\cdot\boldsymbol{q}\mathrm{d}V,\ \forall\boldsymbol{q}\circ F_{T_i} = B_{T_i}^{-\mathrm{T}}\hat{\boldsymbol{q}}, \hat{\boldsymbol{q}}\in\boldsymbol{P}_{k-6}(\hat{T}_i)\times\hat{\boldsymbol{x}}\right\}\\ &\cup\left\{\int_{T_i}\boldsymbol{u}\cdot\boldsymbol{q}\mathrm{d}V,\ \forall\boldsymbol{q}\circ F_{T_i} = 1/\det(B_{T_i})B_{T_i}\hat{\boldsymbol{q}}\in P_{k-3}(\hat{T}_i)\hat{\boldsymbol{x}}\right\}.\end{aligned}\tag{2.4}$$

Here the notation $\hat{}$ denotes the corresponding variable on the reference element (the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$), and B_T is the Jacobian matrix of the transformation from $\hat{T}$ to T.

We can define an $H(\mathrm{curl}^2;\Omega)$ interpolation $\Pi_h^k\boldsymbol{u}$ by the DOFs (2.1)–(2.4). Applying the similar arguments of the proof of Theorem 5.41 in [26], we can obtain the following error estimates.

Theorem 2.2. *The interpolation Π_h^k satisfies the following approximation properties.*

(1) *Assume that $\boldsymbol{u}\in\boldsymbol{H}^{s-1}(\Omega)$, $\nabla\times\boldsymbol{u}\in\boldsymbol{H}^s(\Omega)$, $s\geqslant 3/2+\delta$ with $\delta>0$. Then we have the following error estimates for the interpolation Π_h^k,*

$$\left\|\boldsymbol{u}-\Pi_h^k\boldsymbol{u}\right\|_{H(\mathrm{curl}^2;\Omega)}\leqslant Ch^{\min\{s-1,k-1\}}(\|\boldsymbol{u}\|_{s-1}+\|\nabla\times\boldsymbol{u}\|_s).$$

(2) *Assume that $\boldsymbol{u}\in\boldsymbol{H}^{s+1}(\Omega)$, $s\geqslant 3/2+\delta$ with $\delta>0$. Then we have the following error estimates for the interpolation Π_h^k,*

$$\left\|\boldsymbol{u}-\Pi_h^k\boldsymbol{u}\right\|+h\left\|\nabla(\boldsymbol{u}-\Pi_h^k\boldsymbol{u})\right\|+h^2\left\|\nabla\nabla\times(\boldsymbol{u}-\Pi_h^k\boldsymbol{u})\right\|\leqslant Ch^{\min\{s+1,k+1\}}\left\|\boldsymbol{u}\right\|_{s+1}.$$

(3) *Assume that $\boldsymbol{u}\in\boldsymbol{H}^{1/2+\delta}(T)$ with $\delta>0$ and $\nabla\times\boldsymbol{u}|_T\in\boldsymbol{P}_{k-1}^+(T)$, then*

$$\left\|\boldsymbol{u}-\Pi_h^k\boldsymbol{u}\right\|_T\leqslant C\big(h_T^{1/2+\delta}\left\|\boldsymbol{u}\right\|_{1/2+\delta,T}+h_T\|\nabla\times\boldsymbol{u}\|_T\big).$$

For 2D $H(\mathrm{curl}^2)$-conforming elements, we refer to the family $r=k+1$ in [23].

2.3. An $\boldsymbol{H}(\mathrm{curl}^2)$-type Clément interpolation

Let ω_v be the union of elements sharing the vertex v and $R_v\phi$ be the L^2 projection of $\phi\in L^2(\omega_v)$ on ω_v, *i.e.*, $R_v\phi\in P_1(\omega_v)$ such that

$$\int_{\omega_v}(\phi-R_v\phi)\,p\mathrm{d}V=0,\quad\forall p\in P_1(\omega_v).$$

For $\boldsymbol{u} \in \boldsymbol{H}^{1/2+\delta}(\Omega)$ with $\nabla \times \boldsymbol{u} \in \boldsymbol{H}^{3/2+\delta}(\Omega)$, the lowest-order $H(\mathrm{curl}^2;\Omega)$ interpolation $\Pi_h^2 \boldsymbol{u}$ defined by the DOFs (2.1)–(2.4) can be written as

$$\Pi_h^2 \boldsymbol{u} = \sum_{v\in\mathcal{N}_h} \sum_{i=1}^{3} \alpha_v^i(\boldsymbol{u})\boldsymbol{\phi}_v^i + \sum_{e\in\mathcal{E}_h} \sum_{i=1}^{3} \alpha_e^i(\boldsymbol{u})\boldsymbol{\phi}_e^i + \sum_{f\in\mathcal{F}_h} \alpha_f(\boldsymbol{u})\boldsymbol{\phi}_f,$$

where

$$\begin{aligned}
\alpha_v^i(\boldsymbol{u}) &= \text{ the } i\text{th component of } \nabla \times \boldsymbol{u}(v),\\
\alpha_e^i(\boldsymbol{u}) &= \int_e \boldsymbol{u}\cdot\boldsymbol{\tau}_e q_i \mathrm{d}s \text{ for any } q_i \in P_2(e),\\
\alpha_f(\boldsymbol{u}) &= \int_f \boldsymbol{u}\cdot B_T(\hat{\boldsymbol{n}}_{\hat f} \times \hat{\boldsymbol{x}}|_{\hat f} \times \hat{\boldsymbol{n}}_{\hat f})\mathrm{d}A \text{ with } f \in \partial T,
\end{aligned}$$

and the functions $\boldsymbol{\phi}_v^i$, $\boldsymbol{\phi}_e^i$, and $\boldsymbol{\phi}_f$ are the corresponding dual basis functions. Now we define an $H(\mathrm{curl}^2)$-type Clément interpolation $\widetilde{\Pi}_h^2$ for $\boldsymbol{u} \in \boldsymbol{H}^{1/2+\delta}(\Omega)$ with $\nabla \times \boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ by replacing $\alpha_v^i(\boldsymbol{u})$ with $\widetilde{\alpha}_v^i(\boldsymbol{u}) = R_v^1\big((\nabla\times\boldsymbol{u})_i\big)(v)$. The interpolation is well-defined and the following error estimate holds.

Theorem 2.3. *For any $T \in \mathcal{T}_h$, let $\omega_T = \cup_{v_i\in\mathcal{N}_h(T)}\omega_{v_i}$. Then, for $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$, it holds that*

$$\|\boldsymbol{u} - \widetilde{\Pi}_h^2\boldsymbol{u}\|_T + h_T\|\nabla(\boldsymbol{u} - \widetilde{\Pi}_h^2\boldsymbol{u})\|_T + h_T^2\|\nabla\big(\nabla\times(\boldsymbol{u} - \widetilde{\Pi}_h^2\boldsymbol{u})\big)\|_T \leqslant Ch^2\|\boldsymbol{u}\|_{2,\omega_T}.$$

The theorem can be obtained by combining the approximation properties of Π_h^k and the classic Clément interpolation.

2.4. A decomposition of $\boldsymbol{H_0}(\mathbf{curl}^2;\boldsymbol{\Omega})$

Motivated by the decomposition of $H_0(\mathrm{curl};\Omega)$ in Proposition 5.1 of [19], we obtain a decomposition of the space $H_0(\mathrm{curl}^2;\Omega)$, which plays a critical role in the analysis.

Lemma 2.4. *Let $\nabla H_0^1(\Omega)$ be the set of gradients of functions in $H_0^1(\Omega)$. Then $\nabla H_0^1(\Omega)$ is a closed subspace of $H_0(\mathrm{curl}^2;\Omega)$ and*

$$H_0(\mathrm{curl}^2;\Omega) = X \oplus \nabla H_0^1(\Omega), \tag{2.5}$$

where $X = \big\{\boldsymbol{u}\in H_0(\mathrm{curl}^2;\Omega)\big|(\boldsymbol{u},\nabla p) = 0,\ \ \forall p \in H_0^1(\Omega)\big\}$. Namely, for any $\boldsymbol{u} \in H_0(\mathrm{curl}^2;\Omega)$, $\boldsymbol{u} = \boldsymbol{u}^0 + \boldsymbol{u}^\perp$ with $\boldsymbol{u}^0 \in \nabla H_0^1(\Omega)$ and $\boldsymbol{u}^\perp \in X$. Furthermore, $\boldsymbol{u}^\perp$ admits the splitting

$$\boldsymbol{u}^\perp = \nabla\phi + \boldsymbol{v}, \tag{2.6}$$

where $\phi \in H_0^1(\Omega)$ and $\boldsymbol{v} \in \boldsymbol{H}^2(\Omega)$ satisfying

$$\|\boldsymbol{v}\|_2 \leqslant C\|\nabla\times\boldsymbol{u}^\perp\|_1. \tag{2.7}$$

$$\|\nabla\phi\| \leqslant C\left(\|\nabla\times\boldsymbol{u}^\perp\|_1 + \|\boldsymbol{u}^\perp\|\right). \tag{2.8}$$

Proof. The proof of (2.5) can be found in [44]. To prove (2.6)–(2.8), let $\mathcal{O}$ be a bounded, smooth, contractible open set with $\bar\Omega \subset \mathcal{O}$. For any $\boldsymbol{u}^\perp \in X$, we can extend $\boldsymbol{u}^\perp$ in the following way:

$$\widetilde{\boldsymbol{u}} = \begin{cases} \boldsymbol{u}^\perp, & \Omega,\\ 0, & \mathcal{O}/\bar\Omega.\end{cases}$$

Obviously, $\widetilde{\boldsymbol{u}} \in H_0(\mathrm{curl};\mathcal{O})$ and $\nabla\times\widetilde{\boldsymbol{u}} \in \boldsymbol{H}_0^1(\mathcal{O})$. According to Proposition 4.1 in [16], there exists a $\boldsymbol{w} \in \boldsymbol{H}^2(\mathcal{O})$ such that

$$\nabla\times(\boldsymbol{w}-\widetilde{\boldsymbol{u}}) = 0 \text{ and } \|\boldsymbol{w}\|_{2,\mathcal{O}} \leqslant C\|\nabla\times\widetilde{\boldsymbol{u}}\|_{1,\mathcal{O}}. \tag{2.9}$$

Based on Theorem 2.9 of [20], there exists a unique function p of $H^1(\mathcal{O})/\mathbb{R}$ such that

$$\boldsymbol{w}-\widetilde{\boldsymbol{u}} = \nabla p. \tag{2.10}$$

Now, we restrict (2.10) to the domain $\mathcal{O}/\bar{\Omega}$ and obtain

$$\nabla p = \boldsymbol{w} \in \boldsymbol{H}^2(\mathcal{O}/\bar{\Omega}). \tag{2.11}$$

Using the extension theorem [11], we can extend $p \in H^3(\mathcal{O}/\bar{\Omega})$ to $\widetilde{p}$, which is defined on $\mathcal{O}$ and satisfies

$$\|\widetilde{p}\|_{3,\mathcal{O}} \leqslant C\,\|p\|_{3,\mathcal{O}/\bar{\Omega}} \leqslant C\,\|\nabla p\|_{2,\mathcal{O}/\bar{\Omega}} = C\,\|\boldsymbol{w}\|_{2,\mathcal{O}/\bar{\Omega}}\,, \tag{2.12}$$

where we have used Poincaré–Friedrichs inequality for $p \in H^3(\mathcal{O}/\bar{\Omega})$ since we can choose p for which $\int_{\mathcal{O}/\bar{\Omega}} p = 0$. Restricting on Ω, we have

$$\boldsymbol{u}^{\perp} = \underbrace{\boldsymbol{w}-\nabla\widetilde{p}}_{\in\boldsymbol{H}^2(\Omega)} + \nabla\underbrace{(\widetilde{p}-p)}_{\in H^1(\Omega)} \triangleq \boldsymbol{v}+\nabla\phi.$$

Note that $\phi = \widetilde{p}-p \in H_0^1(\Omega)$ since $\widetilde{p}$ is the extension of p. Therefore, (2.6) is proved. By virtue of (2.12) and (2.9), we obtain

$$\|\boldsymbol{v}\|_2 = \|\boldsymbol{w}-\nabla\widetilde{p}\|_2 \leqslant \|\boldsymbol{w}-\nabla\widetilde{p}\|_{2,\mathcal{O}} \leqslant C\|\boldsymbol{w}\|_{2,\mathcal{O}} \leqslant C\|\nabla\times\widetilde{\boldsymbol{u}}\|_{1,\mathcal{O}} = C\|\nabla\times\boldsymbol{u}^{\perp}\|_1$$

and

$$\|\nabla\phi\| = \|\boldsymbol{u}^{\perp}-\boldsymbol{v}\| \leqslant \|\boldsymbol{u}^{\perp}\| + \|\boldsymbol{v}\| \leqslant \|\boldsymbol{u}^{\perp}\| + \|\boldsymbol{v}\|_2 \leqslant C\left(\|\boldsymbol{u}^{\perp}\| + \|\nabla\times\boldsymbol{u}^{\perp}\|_1\right).$$

□

3. An *a priori* error estimate for the eigenvalue problem

Following [33], the quad-curl eigenvalue problem is to seek λ and $\boldsymbol{u}$ such that

$$\begin{aligned} (\nabla\times)^4\boldsymbol{u} &= \lambda\boldsymbol{u} && \text{in } \Omega,\\ \nabla\cdot\boldsymbol{u} &= 0 && \text{in } \Omega,\\ \boldsymbol{u}\times\boldsymbol{n} &= 0 && \text{on } \partial\Omega,\\ \nabla\times\boldsymbol{u}\times\boldsymbol{n} &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

where $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega$. The assumption that Ω is contractible implies $\lambda \neq 0$. The variational form of the quad-curl eigenvalue problem is to find $\lambda \in \mathbb{R}$ and $\boldsymbol{u} \in X$ such that

$$((\nabla\times)^2\boldsymbol{u},(\nabla\times)^2\boldsymbol{v}) = \lambda(\boldsymbol{u},\boldsymbol{v}), \quad \forall\boldsymbol{v} \in X. \tag{3.2}$$

We define some discrete spaces.

$$\begin{aligned} V_h &= \{\boldsymbol{v}_h \in H(\mathrm{curl}^2;\Omega):\ \boldsymbol{v}_h|_T \in V_h^k(T)\},\\ V_h^0 &= \{\boldsymbol{v}_h \in V_h:\ \boldsymbol{n}\times\boldsymbol{v}_h = 0 \text{ and } \boldsymbol{n}\times\nabla\times\boldsymbol{v}_h = 0 \text{ on } \partial\Omega\},\\ S_h &= \{w_h \in H^1(\Omega):\ w_h|_T \in P_k(T)\},\\ S_h^0 &= \{w_h \in S_h,\ w_h|_{\partial\Omega} = 0\},\\ X_h &= \{\boldsymbol{u}_h \in V_h^0 \mid (\boldsymbol{u}_h,\nabla q_h) = 0,\ \text{for all } q_h \in S_h^0\}. \end{aligned}$$

The discrete problem for (3.2) is to find $\lambda_h \in \mathbb{R}$ and $\boldsymbol{u}_h \in X_h$ such that

$$((\nabla\times)^2\boldsymbol{u}_h,(\nabla\times)^2\boldsymbol{v}_h) = \lambda_h(\boldsymbol{u}_h,\boldsymbol{v}_h), \quad \forall\boldsymbol{v} \in X_h. \tag{3.3}$$

3.1. The source problem

We start with the associated source problem. Given $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, find $\boldsymbol{u} \in H_0(\text{curl}^2;\Omega)$ and $p \in H_0^1(\Omega)$ such that

$$\begin{aligned} (\nabla\times)^4\boldsymbol{u} + \boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad && \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 && \text{in } \Omega, \\ \boldsymbol{u}\times\boldsymbol{n} &= 0 && \text{on } \partial\Omega, \\ \nabla\times\boldsymbol{u}\times\boldsymbol{n} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.4}$$

Note that $p = 0$ for $\boldsymbol{f} \in H(\text{div}^0;\Omega)$.

The weak formulation is to find $(\boldsymbol{u};p) \in H_0(\text{curl}^2;\Omega) \times H_0^1(\Omega)$ such that

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) &= (\boldsymbol{f},\boldsymbol{v}), \quad && \forall \boldsymbol{v} \in H_0(\text{curl}^2;\Omega), \\ b(\boldsymbol{u},q) &= 0, && \forall q \in H_0^1(\Omega), \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) &= \big((\nabla\times)^2\boldsymbol{u},(\nabla\times)^2\boldsymbol{v}\big) + (\boldsymbol{u},\boldsymbol{v}), \\ b(\boldsymbol{v},p) &= (\boldsymbol{v},\nabla p). \end{aligned}$$

Define $Y := \big\{\boldsymbol{w} \in H_0(\text{curl};\Omega) : (\boldsymbol{w},\nabla q) = 0, \ \ \forall q \in H_0^1(\Omega)\big\}$, then $\nabla\times\boldsymbol{u} \in Y$. By applying Friedrichs inequality on $\nabla\times\boldsymbol{u}$, we get $a(\cdot,\cdot)$ is coercive on $H_0(\text{curl}^2;\Omega)$, *i.e.*,

$$a(\boldsymbol{u},\boldsymbol{u}) \geqslant \|\boldsymbol{u}\|^2_{H(\text{curl}^2;\Omega)}.$$

In addition, the following Babuška–Brezzi condition holds,

$$\sup_{\boldsymbol{v}\in H_0(\text{curl}^2;\Omega)} \frac{b(\boldsymbol{v},p)}{\|\boldsymbol{v}\|_{H(\text{curl}^2;\Omega)}} \geqslant \frac{b(\nabla p,p)}{\|\nabla p\|_{H(\text{curl}^2;\Omega)}} = \|\nabla p\| \geqslant C\|p\|_1.$$

The well-posedness of (3.5) then follows from Theorem 1.3.2 of [34]. Consequently, we can define a bounded solution operator $A : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ such that, for $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, $A\boldsymbol{f} = \boldsymbol{u} \in X \subset \boldsymbol{L}^2(\Omega)$ satisfies

$$a(A\boldsymbol{f},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}), \quad \forall \boldsymbol{v} \in X.$$

The operator A is selfadjoint since

$$(A\boldsymbol{\phi},\boldsymbol{\psi}) = a(A\boldsymbol{\psi},A\boldsymbol{\phi}) = a(A\boldsymbol{\phi},A\boldsymbol{\psi}) = (\boldsymbol{\phi},A\boldsymbol{\psi}), \quad \forall \boldsymbol{\phi},\boldsymbol{\psi} \in \boldsymbol{L}^2(\Omega).$$

A is also compact due to the following result.

Lemma 3.1. *X processes the continuous compactness property.*

Proof. Since $X \subset Y \hookrightarrow\hookrightarrow \boldsymbol{L}^2(\Omega)$ [26], then $X \hookrightarrow\hookrightarrow \boldsymbol{L}^2(\Omega)$. □

The $H(\text{curl}^2)$-conforming finite element method seeks $\boldsymbol{u}_h \in V_h^0$ and $p_h \in S_h^0$ such that

$$\begin{aligned} a(\boldsymbol{u}_h,\boldsymbol{v}_h) + b(\boldsymbol{v}_h,p_h) &= (\boldsymbol{f},\boldsymbol{v}_h), \quad && \forall \boldsymbol{v}_h \in V_h^0, \\ b(\boldsymbol{u}_h,q_h) &= 0, && \forall q_h \in S_h^0. \end{aligned} \tag{3.6}$$

Since $\nabla\times\boldsymbol{u}_h \in Y$, by the Friedrichs inequality on $\nabla\times\boldsymbol{u}_h$, there exists a constant C independent of h such that

$$a(\boldsymbol{u}_h,\boldsymbol{u}_h) = \|\boldsymbol{u}_h\|^2 + \|(\nabla\times)^2\boldsymbol{u}_h\|^2 \geqslant C\|\boldsymbol{u}_h\|_{H(\text{curl}^2;\Omega)}.$$

The well-posedness of problem (3.6) is then due to the discrete Babuška–Brezzi condition,

$$\sup_{\boldsymbol{v}\in V_h^0}\frac{b(\boldsymbol{v}_h,p_h)}{\|\boldsymbol{v}\|_{H(\mathrm{curl}^2;\Omega)}}\geqslant\frac{b(\nabla p_h,p_h)}{\|\nabla p_h\|_{H(\mathrm{curl}^2;\Omega)}}=\|\nabla p_h\|\geqslant C\|p_h\|_1.$$

Consequently, we can define a discrete solution operator $A_h:\boldsymbol{L}^2(\Omega)\to\boldsymbol{L}^2(\Omega)$ such that $\boldsymbol{u}_h=A_h\boldsymbol{f}\in X_h$ is the solution of (3.6). We will use the standard finite element framework and the approximation property of the interpolation to obtain the approximation property of the numerical solution. To this end, we first introduce a new space:

$$H(\mathrm{gradcurl};\Omega)=\{\boldsymbol{u}\in H(\mathrm{curl};\Omega):\nabla\times\boldsymbol{u}\in\boldsymbol{H}^1(\Omega)\},$$

and the associated space with vanishing trace

$$H_0(\mathrm{gradcurl};\Omega)=\{\boldsymbol{u}\in H(\mathrm{gradcurl};\Omega):\boldsymbol{n}\times\boldsymbol{u}=0\text{ and }\nabla\times\boldsymbol{u}=0\text{ on }\partial\Omega\}.$$

Equip the space $H(\mathrm{gradcurl};\Omega)$ with norm $\|\boldsymbol{u}\|^2_{H(\mathrm{gradcurl};\Omega)}=\|\boldsymbol{u}\|^2+\|\nabla\nabla\times\boldsymbol{u}\|^2$. We can show that the space $H_0(\mathrm{gradcurl};\Omega)$ is equivalent to $H_0(\mathrm{curl}^2;\Omega)$.

Lemma 3.2. *The space $H_0(curl^2;\Omega)$ coincides with $H_0(gradcurl;\Omega)$. Moreover, for $\boldsymbol{u}\in H_0(curl^2;\Omega)$,*

$$\|\boldsymbol{u}\|_{H(curl^2;\Omega)}\leqslant C\|\boldsymbol{u}\|_{H(gradcurl;\Omega)}. \tag{3.7}$$

Proof. To prove $H_0(\mathrm{curl}^2;\Omega)=H_0(\mathrm{gradcurl};\Omega)$, it suffices to show $H_0(\mathrm{curl}^2;\Omega)\subset H_0(\mathrm{gradcurl};\Omega)$ since $H_0(\mathrm{gradcurl};\Omega)\subset H_0(\mathrm{curl}^2;\Omega)$ is trivial. For $\boldsymbol{u}\in H_0(\mathrm{curl}^2;\Omega)$, we have $\nabla\times\boldsymbol{u}\in H_0(\mathrm{div};\Omega)=\{\boldsymbol{u}\in\boldsymbol{L}^2(\Omega):\nabla\cdot\boldsymbol{u}\in L^2(\Omega),\ \boldsymbol{u}\cdot\boldsymbol{n}=0\text{ on }\partial\Omega\}$ since $\nabla\cdot(\nabla\times\boldsymbol{u})=0$ and $(\nabla\times\boldsymbol{u})\cdot\boldsymbol{n}=\nabla_{\partial\Omega}\cdot(\boldsymbol{u}\times\boldsymbol{n})=0$ on $\partial\Omega$. It then follows from $H_0(\mathrm{curl};\Omega)\cap H_0(\mathrm{div};\Omega)=\boldsymbol{H}_0^1(\Omega)$ ([20], Lem. 2.5) that $\nabla\times\boldsymbol{u}\in\boldsymbol{H}_0^1(\Omega)$, and hence $\boldsymbol{u}\in H_0(\mathrm{gradcurl};\Omega)$. The inequality (3.7) follows from the Poncaré inequality. □

Lemma 3.3. *$\boldsymbol{C}_0^\infty(\Omega)$ is dense in $H_0(gradcurl;\Omega)$.*

Proof. The density of $\boldsymbol{C}_0^\infty(\Omega)$ in $H_0(\mathrm{gradcurl};\Omega)$ has been proved in Theorem 3.15 of [43]. The proof uses a similar argument to the one used to prove the density of $\boldsymbol{C}_0^\infty(\Omega)$ in $H_0(\mathrm{curl};\Omega)$ [26]. □

With the density of $\boldsymbol{C}_0^\infty(\Omega)$ in $H_0(\mathrm{gradcurl};\Omega)$, we can obtain the following approximation property.

Theorem 3.4. *For $\boldsymbol{f}\in\boldsymbol{L}^2(\Omega)$, it holds*

$$\|A\boldsymbol{f}-A_h\boldsymbol{f}\|_{H(\mathrm{curl}^2;\Omega)}\to 0\text{ as }h\to 0.$$

Proof. The usual theory of mixed method shows that

$$\|A\boldsymbol{f}-A_h\boldsymbol{f}\|_{H(\mathrm{curl}^2;\Omega)}\leqslant C\left\{\inf_{\boldsymbol{v}_h\in V_h^0}\|A\boldsymbol{f}-\boldsymbol{v}_h\|_{H(\mathrm{curl}^2;\Omega)}+\inf_{q_h\in S_h^0}\|p-q_h\|_1\right\}. \tag{3.8}$$

It follows from the approximation property of the canonical interpolations, Lemma 3.2, the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$, and Lemma 3.3 that

$$\inf_{\boldsymbol{v}_h\in V_h^0}\|A\boldsymbol{f}-\boldsymbol{v}_h\|_{H(\mathrm{curl}^2;\Omega)}+\inf_{q_h\in S_h^0}\|p-q_h\|_1\to 0\text{ as }h\to 0.$$

□

If p and $A\boldsymbol{f}$ are smoother, then from the approximation property of Π_h^k, we have the following error estimate with a convergence order.

Theorem 3.5. *Assume that $A\boldsymbol{f}\in\boldsymbol{H}^{s-1}(\Omega),\nabla\times A\boldsymbol{f}\in\boldsymbol{H}^s(\Omega)$, and $p\in H^s(\Omega)$ ($s\geqslant 3/2+\delta$ with $\delta>0$). It holds that*

$$\|A\boldsymbol{f}-A_h\boldsymbol{f}\|_{H(\mathrm{curl}^2;\Omega)}\leqslant Ch^{s-1}\left(\|A\boldsymbol{f}\|_{s-1}+\|\nabla\times A\boldsymbol{f}\|_s+\|p\|_s\right).$$

3.2. An *a priori* error estimate of the eigenvalue problem

We first rewrite the eigenvalue problem (3.2) by adding a low-order term $\boldsymbol{u}$. Find $\lambda \in \mathbb{R}$ and $(\boldsymbol{u}; p) \in H_0(\mathrm{curl}^2; \Omega) \times H_0^1(\Omega)$ such that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= (\lambda + 1)(\boldsymbol{u}, \boldsymbol{v}), \quad && \forall \boldsymbol{v} \in H_0(\mathrm{curl}^2; \Omega), \\ b(\boldsymbol{u}, q) &= 0, && \forall q \in H_0^1(\Omega). \end{aligned} \tag{3.9}$$

Due to the fact that $\nabla \cdot \boldsymbol{u} = 0$, we have $p = 0$. Then (3.9) can be written as an operator eigenvalue problem of finding $\mu := 1/(\lambda + 1) \in \mathbb{R}$ and $\boldsymbol{u} \in X$ such that

$$A\boldsymbol{u} = \mu \boldsymbol{u}. \tag{3.10}$$

The discrete eigenvalue problem (3.3) is equivalent to seeking $\lambda_h \in \mathbb{R}$ and $(\boldsymbol{u}_h; p_h) \in V_h^0 \times S_h^0$ such that

$$\begin{aligned} a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) &= (\lambda_h + 1)(\boldsymbol{u}_h, \boldsymbol{v}_h), \quad && \forall \boldsymbol{v}_h \in V_h^0, \\ b(\boldsymbol{u}_h, q_h) &= 0, && \forall q_h \in S_h^0. \end{aligned} \tag{3.11}$$

Using the operator A_h, the eigenvalue problem is to find $\mu_h \in \mathbb{R}$ and $\boldsymbol{u}_h \in X_h$ such that

$$A_h \boldsymbol{u}_h = \mu_h \boldsymbol{u}_h, \tag{3.12}$$

where $\mu_h = 1/(\lambda_h + 1)$.

Let $\Lambda = \{h_n, n = 0, 1, 2, \ldots\}$ be a sequence of mesh sizes such that

$$h_0 > h_1 > h_2 > \cdots > 0 \text{ and } h_n \to 0 \text{ as } n \to \infty.$$

Define a collection of operators,

$$\mathcal{A} = \{A_h : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega),\ h \in \Lambda\}.$$

To apply the abstract convergence theory [29], we need to verify:

(1) $\mathcal{A}$ is collectively compact, *i.e.*, for each bounded set $U \subset \boldsymbol{L}^2(\Omega)$, $\mathcal{A}(U) = \{A_h \boldsymbol{u} : \boldsymbol{u} \in U, h \in \Lambda\}$ is relatively compact.
(2) $\mathcal{A}$ is point-wise convergent, *i.e.*, for $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, $A_h \boldsymbol{f} \to A\boldsymbol{f}$ strongly in $\boldsymbol{L}^2(\Omega)$ as $h \to 0$.

Theorem 3.4 verifies (2). It remains to verify (1). We first show $\{X_h\}_{h\in\Lambda}$ has discrete compactness property.

Theorem 3.6. *$\{X_h\}_{h\in\Lambda}$ processes the discrete compactness property, i.e., for every $\{\boldsymbol{w}_n\}_{n=1}^{\infty}$ such that*

- *$\boldsymbol{w}_n \in X_{h_n}$ for each n and $h_n \to 0$ as $n \to \infty$,*
- *there is a constant C independent of $\boldsymbol{w}_n$ such that $\|\boldsymbol{w}_n\|_{H(curl^2;\Omega)} \leqslant C$,*

then there exists a subsequence, still denoted $\{\boldsymbol{w}_n\}$, and a function $\boldsymbol{w} \in X$ such that

$$\boldsymbol{w}_n \to \boldsymbol{w} \text{ strongly in } \boldsymbol{L}^2(\Omega) \text{ and weekly in } X \text{ as } n \to \infty.$$

Proof. Let $\boldsymbol{w}_n \in X_{h_n}, n = 1, 2, \ldots$ and $h_n \to 0$ as $n \to \infty$. Suppose $\|\boldsymbol{w}_n\|_{H(\mathrm{curl}^2;\Omega)} \leqslant C < \infty$ for all n. Seek $p^n \in H_0^1(\Omega)$ such that $(\nabla p^n, \nabla \xi) = (\boldsymbol{w}_n, \nabla \xi)$ for all $\xi \in H_0^1(\Omega)$. Set $\boldsymbol{w}^n = \boldsymbol{w}_n - \nabla p^n$, clearly, $\boldsymbol{w}^n$ satisfies

$$\begin{aligned} &(\nabla\times)^2 \boldsymbol{w}^n = (\nabla\times)^2 \boldsymbol{w}_n, \quad \nabla \times \boldsymbol{w}^n = \nabla \times \boldsymbol{w}_n, \quad \text{and} \quad \nabla \cdot \boldsymbol{w}^n = 0 \quad \text{in} \quad \Omega, \\ &\boldsymbol{n} \times \boldsymbol{w}^n = \boldsymbol{n} \times \boldsymbol{w}_n \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Hence $\boldsymbol{w}^n \in X$ and $\|\boldsymbol{w}^n\|_{H(\mathrm{curl}^2;\Omega)} \leqslant C$. By Lemma 3.1, there is a subsequence, still denoted by $\{\boldsymbol{w}^n\}_{n=1}^{\infty}$, and a function $\boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$ such that $\boldsymbol{w}^n \to \boldsymbol{w}$ as $n \to \infty$ strongly in $\boldsymbol{L}^2(\Omega)$. Furthermore, we can prove $\boldsymbol{w} \in X$. In fact, for any $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$, $\lim_{n\to\infty}(\nabla \times \boldsymbol{w}^n, \boldsymbol{v}) \leqslant C\|\boldsymbol{v}\|$, which implies the limit of $(\nabla \times \boldsymbol{w}^n, \boldsymbol{v})$ is a bounded linear functional on $\boldsymbol{L}^2(\Omega)$. By Riesz representation theorem, there exists a unique element $\boldsymbol{z} \in \boldsymbol{L}^2(\Omega)$ such that

$$\lim_{n\to\infty}(\nabla \times \boldsymbol{w}^n, \boldsymbol{v}) = (\boldsymbol{z}, \boldsymbol{v}).$$

Picking $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, we have

$$(\boldsymbol{w}, \nabla \times \boldsymbol{v}) = \lim_{n\to\infty}(\boldsymbol{w}^n, \nabla \times \boldsymbol{v}) = \lim_{n\to\infty}(\nabla \times \boldsymbol{w}^n, \boldsymbol{v}) = (\boldsymbol{z}, \boldsymbol{v}),$$

which implies $\boldsymbol{z} = \nabla \times \boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$. Moreover, it holds that

$$(\boldsymbol{w}, \nabla \times \boldsymbol{v}) = (\nabla \times \boldsymbol{w}, \boldsymbol{v}) + \langle \boldsymbol{w} \times \boldsymbol{n}, \boldsymbol{v}\rangle = (\boldsymbol{z}, \boldsymbol{v}) + \langle \boldsymbol{w} \times \boldsymbol{n}, \boldsymbol{v}\rangle,$$

which leads to $\langle \boldsymbol{w} \times \boldsymbol{n}, \boldsymbol{v}\rangle = 0$ for all $\boldsymbol{v} \in H^{1/2}(\partial\Omega)$ because of the surjectivity from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$. Therefore, we arrive at $\boldsymbol{w} \times \boldsymbol{n} = 0$ on $\partial\Omega$, and hence $\boldsymbol{w} \in H_0(\mathrm{curl};\Omega)$. We can then prove $\nabla \times \boldsymbol{w} \in H_0(\mathrm{curl};\Omega)$ by replacing $\boldsymbol{w}$ with $\nabla \times \boldsymbol{w}$. Finally, $\boldsymbol{w} \in X$ since

$$(\boldsymbol{w}, \nabla q) = \lim_{n\to\infty}(\boldsymbol{w}^n, \nabla q) = 0, \quad \forall q \in H_0^1(\Omega).$$

The weak convergence of $\boldsymbol{w}^n \rightarrow \boldsymbol{w}$ in X then follows. By Lemma 7.15 of [26], $\boldsymbol{w}^n \in \boldsymbol{H}^{1/2+s}(\Omega)$ with $s > 0$, and it holds

$$\|\boldsymbol{w}^n\|_{1/2+s} \leqslant C\|\nabla \times \boldsymbol{w}_n\|. \tag{3.13}$$

Since $\boldsymbol{w}^n \in \boldsymbol{H}^{1/2+s}(\Omega)$ and $\nabla \times \boldsymbol{w}^n = \nabla \times \boldsymbol{w}_n \in C^0(\Omega)$, we know by Lemma 4.1 of [23] that the interpolation $\Pi_{h_n}\boldsymbol{w}^n$ is well-defined. Since $\Pi_{h_n}\boldsymbol{w}_n = \boldsymbol{w}_n$, $\Pi_{h_n}\nabla p^n$ is well-defined, and it holds $\boldsymbol{w}_n - \Pi_{h_n}\boldsymbol{w}^n = \Pi_{h_n}\nabla p^n = \nabla\pi_{h_n}p^n$ with π_{h_n} the Lagrange interpolation (see [24], Lem. 5.3 and [23], Lem. 4.5). Hence, using the fact that $\boldsymbol{w} \in H(\mathrm{curl}^2;\Omega)$ and $\boldsymbol{w}_n \in X_{h_n}$,

$$\begin{aligned}\|\boldsymbol{w} - \boldsymbol{w}_n\|^2 &= (\boldsymbol{w} - \boldsymbol{w}_n, \boldsymbol{w} - \Pi_{h_n}\boldsymbol{w}^n) + (\boldsymbol{w} - \boldsymbol{w}_n, \Pi_{h_n}\boldsymbol{w}^n - \boldsymbol{w}_n)\\ &= (\boldsymbol{w} - \boldsymbol{w}_n, \boldsymbol{w} - \Pi_{h_n}\boldsymbol{w}^n) + (\boldsymbol{w} - \boldsymbol{w}_n, -\nabla\pi_{h_n}p^n)\\ &= (\boldsymbol{w} - \boldsymbol{w}_n, \boldsymbol{w} - \Pi_{h_n}\boldsymbol{w}^n) \leqslant \|\boldsymbol{w} - \boldsymbol{w}_n\|\|\boldsymbol{w} - \Pi_{h_n}\boldsymbol{w}^n\|,\end{aligned}$$

which implies

$$\|\boldsymbol{w} - \boldsymbol{w}_n\| \leqslant \|\boldsymbol{w} - \Pi_{h_n}\boldsymbol{w}^n\| \leqslant \|\boldsymbol{w} - \boldsymbol{w}^n\| + \|\boldsymbol{w}^n - \Pi_{h_n}\boldsymbol{w}^n\|. \tag{3.14}$$

To estimate the second term on the right-hand side, we apply Theorem 2.2 and (3.13) to obtain

$$\|\boldsymbol{w}^n - \Pi_{h_n}\boldsymbol{w}^n\| \leqslant Ch_n^{1/2+s}(\|\boldsymbol{w}^n\|_{1/2+s} + \|\nabla \times \boldsymbol{w}^n\|) \leqslant Ch_n^{1/2+s}\|\nabla \times \boldsymbol{w}^n\|. \tag{3.15}$$

Combining (3.14) and (3.15) leads to

$$\|\boldsymbol{w} - \boldsymbol{w}_n\| \leqslant \|\boldsymbol{w} - \boldsymbol{w}^n\| + Ch_n^{1/2+s}\|\boldsymbol{w}_n\|_{H(\mathrm{curl}^2;\Omega)}.$$

Since the right-hand side converges to zero, we have proved that $\boldsymbol{w}_n \rightarrow \boldsymbol{w}$ in $\boldsymbol{L}^2(\Omega)$ as $n \rightarrow \infty$. The weak convergence $\boldsymbol{w}_n \rightarrow \boldsymbol{w}$ in X follows from the strong convergence $\boldsymbol{w}_n \rightarrow \boldsymbol{w}$ in $\boldsymbol{L}^2(\Omega)$, the weak convergence $\boldsymbol{w}^n \rightarrow \boldsymbol{w}$ in X, and the fact that $\nabla \times \boldsymbol{w}_n = \nabla \times \boldsymbol{w}^n$.

$\square$

Theorem 3.7. *$\mathcal{A}$ is collectively compact.*

Proof. Suppose $U \subset \boldsymbol{L}^2(\Omega)$ is a bounded set. For any $\boldsymbol{u} \in U$, according to the wellposedness of (3.6), $A_h\boldsymbol{u} \in X_h$ satisfies

$$\|(\nabla\times)^2 A_h\boldsymbol{u}\| + \|A_h\boldsymbol{u}\| \leqslant C\|\boldsymbol{u}\|.$$

By the Friedrichs inequality, we have

$$\|A_h\boldsymbol{u}\|_{H(\mathrm{curl}^2;\Omega)} \leqslant C\|\boldsymbol{u}\|,$$

which implies $\{A_h\boldsymbol{u} : \boldsymbol{u} \in U, h \in \Lambda\} \subset W := \cup_{h\in\Lambda} X_h$ is bounded in $H(\mathrm{curl}^2;\Omega)$.

To prove that the set $\{A_h\boldsymbol{u} : \boldsymbol{u} \in U, h \in \Lambda\}$ is relatively compact, it suffices to show that $W \hookrightarrow\hookrightarrow \boldsymbol{L}^2(\Omega)$. Suppose $\{\boldsymbol{w}_n\}_{n=1}^{\infty} \subset W$ is bounded in $H(\mathrm{curl}^2;\Omega)$. Then $\boldsymbol{w}_n \in X_{h_n}$ for some h_n. If $h_n \to 0$ as $n \to \infty$, there exists a convergent subspace in $\boldsymbol{L}^2(\Omega)$ according to the discrete compactness of X_h (Thm. 3.6). If $h_n \geqslant \delta > 0$, then $\{\boldsymbol{w}_n\}_{n=1}^{\infty}$ is contained in a finite dimensional space, and hence there exists a convergent subsequence. □

Theorem 3.8. *Let μ be an eigenvalue of A with multiplicity m and $E(\mu)$ be the associated eigenspace. Let $\{\boldsymbol{\phi}_j\}_{j=1}^m$ be an orthonormal basis for $E(\mu)$. There exist exactly m discrete eigenvalues $\mu_{j,h}$ and the associated eigenfunctions $\boldsymbol{\phi}_{j,h}, j = 1, 2, \ldots, m,$ of A_h such that*

$$|\mu - \mu_{j,h}| \to 0, \text{ as } h \to 0, \tag{3.16}$$

and

$$|\mu - \mu_{j,h}| \leqslant C \max_{1\leqslant i\leqslant m} a(\boldsymbol{\phi}_i - \boldsymbol{\phi}_{i,h}, \boldsymbol{\phi}_i - \boldsymbol{\phi}_{i,h}). \tag{3.17}$$

Moreover, if $\boldsymbol{\phi} \in \boldsymbol{H}^{s-1}(\Omega)$ and $\nabla\times\boldsymbol{\phi} \in \boldsymbol{H}^s(\Omega)$ for any $\boldsymbol{\phi} \in E(\mu)$, then, for h small enough,

$$|\mu - \mu_{j,h}| = O(h^{2(s-1)}). \tag{3.18}$$

Proof. According to Theorem 4 of [29], it holds that

$$|\mu - \mu_{j,h}| \leqslant C\left\{\sum_{i,k=1}^{m} |((A - A_h)\boldsymbol{\phi}_i, \boldsymbol{\phi}_k)| + \|(A - A_h)|_{E(\mu)}\|^2\right\}.$$

Since $\boldsymbol{\phi}_i, A\boldsymbol{\phi}_i \in X$, we have

$$\begin{aligned}((A - A_h)\boldsymbol{\phi}_i, \boldsymbol{\phi}_k) &= (\nabla\times\nabla\times(A - A_h)\boldsymbol{\phi}_i, \nabla\times\nabla\times A\boldsymbol{\phi}_k) + ((A - A_h)\boldsymbol{\phi}_i, A\boldsymbol{\phi}_k)\\ &= (\nabla\times\nabla\times(A - A_h)\boldsymbol{\phi}_i, \nabla\times\nabla\times(A - A_h)\boldsymbol{\phi}_k) + ((A - A_h)\boldsymbol{\phi}_i, (A - A_h)\boldsymbol{\phi}_k)\\ &\leqslant |||(A - A_h)\boldsymbol{\phi}_i|||\,|||(A - A_h)\boldsymbol{\phi}_k|||,\end{aligned}$$

which together with the fact that $E(\mu)$ is finite dimensional leads to

$$\begin{aligned}|\mu - \mu_{j,h}| &\leqslant C\left\{\max_{1\leqslant i\leqslant m} |||(A - A_h)\boldsymbol{\phi}_i|||^2 + \|(A - A_h)|_{E(\mu)}\|^2\right\}\\ &\leqslant C\max_{1\leqslant i\leqslant m} \|(A - A_h)\boldsymbol{\phi}_i\|^2_{H(\mathrm{curl}^2;\Omega)}.\end{aligned}$$

Then (3.16) follows from the pointwise convergence of A_h to A in $H(\mathrm{curl}^2;\Omega)$ (Thm. 3.4). Since $\nabla\cdot\boldsymbol{\phi}_i = 0$, it follows from (3.8) that

$$\begin{aligned}\|(A - A_h)\boldsymbol{\phi}_i\|_{H(\mathrm{curl}^2;\Omega)} &\leqslant \inf_{\boldsymbol{v}_h\in X_h} \|A\boldsymbol{\phi}_i - \boldsymbol{v}_h\|_{H(\mathrm{curl}^2;\Omega)}\\ &= \inf_{\boldsymbol{v}_h\in X_h} \mu\|\boldsymbol{\phi}_i - (1/\mu)\boldsymbol{v}_h\|_{H(\mathrm{curl}^2;\Omega)} \leqslant \mu\|\boldsymbol{\phi}_i - \boldsymbol{\phi}_{i,h}\|_{H(\mathrm{curl}^2;\Omega)}\end{aligned}$$

$$\leqslant C\mu\sqrt{a(\boldsymbol{\phi}_i-\boldsymbol{\phi}_{i,h},\boldsymbol{\phi}_i-\boldsymbol{\phi}_{i,h})} \quad \text{(Poincaré inequality)},$$

which proves (3.17). Let $\boldsymbol{\phi}\in E(\mu)$. Furthermore, if $\boldsymbol{\phi}\in \boldsymbol{H}^{s-1}(\Omega)$ and $\nabla\times\boldsymbol{\phi}\in\boldsymbol{H}^s(\Omega)$, according to Theorem 3.5, we have that

$$\|(A-A_h)\boldsymbol{\phi}\|_{H(\mathrm{curl}^2;\Omega)}\leqslant Ch^{s-1}(\|A\boldsymbol{\phi}\|_{s-1}+\|\nabla\times A\boldsymbol{\phi}\|_s)\leqslant C\mu h^{s-1}(\|\boldsymbol{\phi}\|_{s-1}+\|\nabla\times\boldsymbol{\phi}\|_s).$$

Since $E(\mu)$ is finite dimensional, we obtain (3.18). $\square$

Remark 3.9. The estimate $|\mu-\mu_{j,h}|\leqslant C\max_{1\leqslant i\leqslant m} a(\boldsymbol{\phi}_i-\boldsymbol{\phi}_{i,h},\boldsymbol{\phi}_i-\boldsymbol{\phi}_{i,h})$ will be applied to obtain (4.2) in the next section. The reason that we consider the quad-curl problem (3.4) with the low-order term $\boldsymbol{u}$ is to make $\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{H(\mathrm{curl}^2;\Omega)}\leqslant C\mu\sqrt{a(\boldsymbol{\phi}-\boldsymbol{\phi}_h,\boldsymbol{\phi}-\boldsymbol{\phi}_h)}$ hold.

4. *A posteriori* error estimates for the eigenvalue problem

Assume that $(\lambda;\boldsymbol{u};p)\in\mathbb{R}\times H_0(\mathrm{curl}^2;\Omega)\times H_0^1(\Omega)$ is a simple eigenpair of (3.9) with $\|\boldsymbol{u}\|_0=1$ and $(\lambda_h;\boldsymbol{u}_h;p_h)\in\mathbb{R}\times V_h^0\times S_h^0$ is the associated finite element eigenpair of (3.11) with $\|\boldsymbol{u}_h\|_0=1$. According to Theorem 3.8 and (3.28a) of [1], the following inequalities hold:

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|\leqslant C\rho_\Omega(h)\|\!|\boldsymbol{u}-\boldsymbol{u}_h|\!\|, \tag{4.1}$$

$$|\lambda_h-\lambda|\leqslant C\|\!|\boldsymbol{u}-\boldsymbol{u}_h|\!\|^2, \tag{4.2}$$

where

$$\rho_\Omega(h)=\sup_{\boldsymbol{f}\in\boldsymbol{L}^2(\Omega),\|\boldsymbol{f}\|=1}\inf_{\boldsymbol{v}\in V_h^0}\|A\boldsymbol{f}-\boldsymbol{v}\|_{H(\mathrm{curl}^2;\Omega)}.$$

It is obvious that $\rho_\Omega(h)\to 0$ as $h\to 0$.

Define two projection operators R_h, Q_h as follows. For $\boldsymbol{u}\in H_0(\mathrm{curl}^2;\Omega)$ and $p\in H_0^1(\Omega)$, find $R_h\boldsymbol{u}\in V_h^0, Q_hp\in S_h^0$, such that

$$\begin{aligned} a(\boldsymbol{u}-R_h\boldsymbol{u},\boldsymbol{v}_h)+b(\boldsymbol{v}_h,p-Q_hp)&=0, \quad \forall \boldsymbol{v}_h\in V_h^0,\\ b(\boldsymbol{u}-R_h\boldsymbol{u},q_h)&=0,\quad \forall q_h\in S_h^0. \end{aligned}$$

According to the orthogonality and the uniqueness of the discrete eigenvalue problem,

$$\boldsymbol{u}_h=(\lambda_h+1)R_hA\boldsymbol{u}_h.$$

Let $(\boldsymbol{u}^h;p^h)$ be the solution of (3.5) with $\boldsymbol{f}=(\lambda_h+1)\boldsymbol{u}_h$. Then

$$\boldsymbol{u}^h=(\lambda_h+1)A\boldsymbol{u}_h \text{ and } \boldsymbol{u}_h=R_h\boldsymbol{u}^h. \tag{4.3}$$

The following theorem relates the eigenvalue problem to the source problem (3.5) with $\boldsymbol{f}=(\lambda_h+1)\boldsymbol{u}_h$.

Theorem 4.1. *Let $r(h)=\rho_\Omega(h)+\|\boldsymbol{u}-\boldsymbol{u}_h\|$. It holds that*

$$\|\!|\boldsymbol{u}^h-R_h\boldsymbol{u}^h|\!\|-Cr(h)\|\!|\boldsymbol{u}-\boldsymbol{u}_h|\!\|\leqslant\|\!|\boldsymbol{u}-\boldsymbol{u}_h|\!\|\leqslant\|\!|\boldsymbol{u}^h-R_h\boldsymbol{u}^h|\!\|+Cr(h)\|\!|\boldsymbol{u}-\boldsymbol{u}_h|\!\|. \tag{4.4}$$

Furthermore, for h small enough, there exist two constants c and C such that

$$c\|\!|\boldsymbol{u}^h-R_h\boldsymbol{u}^h|\!\|\leqslant\|\!|\boldsymbol{u}-\boldsymbol{u}_h|\!\|\leqslant C\|\!|\boldsymbol{u}^h-R_h\boldsymbol{u}^h|\!\|. \tag{4.5}$$

Proof. Since $\boldsymbol{u}_h = R_h \boldsymbol{u}^h$, by the triangle inequality, we have that

$$-\|\boldsymbol{u} - \boldsymbol{u}^h\| + \|\boldsymbol{u}^h - R_h \boldsymbol{u}^h\| \leqslant \|\boldsymbol{u} - \boldsymbol{u}_h\| \leqslant \|\boldsymbol{u} - \boldsymbol{u}^h\| + \|\boldsymbol{u}^h - R_h \boldsymbol{u}^h\|.$$

Using the fact $\boldsymbol{u} = (\lambda + 1)A\boldsymbol{u}$ and (4.3), we obtain that

$$\|\boldsymbol{u} - \boldsymbol{u}^h\| = \|(\lambda + 1)A\boldsymbol{u} - (\lambda_h + 1)A\boldsymbol{u}_h\| \leqslant |\lambda + 1|\|A(\boldsymbol{u} - \boldsymbol{u}_h)\| + |\lambda - \lambda_h|\|A\boldsymbol{u}_h\|. \tag{4.6}$$

Due to the well-posedness of (3.5), it holds that

$$\|A(\boldsymbol{u} - \boldsymbol{u}_h)\| \leqslant C\|\boldsymbol{u} - \boldsymbol{u}_h\|,$$

which, together with (4.1) and (4.2), leads to

$$\|\boldsymbol{u} - \boldsymbol{u}^h\| \leqslant Cr(h)\|\boldsymbol{u} - \boldsymbol{u}_h\|. \tag{4.7}$$

Then (4.4) follows immediately. Note that $r(h) \to 0$ as $h \to 0$. For h small enough, (4.4) implies (4.5). $\square$

Now to obtain an *a posteriori* error estimate for the eigenvalue problem, it suffices to derive an *a posteriori* error estimate for the source problem with $\boldsymbol{f} = (\lambda_h + 1)\boldsymbol{u}_h$. The exact solution and numerical solution are $(\boldsymbol{u}^h; p^h)$ and $(\boldsymbol{u}_h; 0)$, respectively.

Denote the total errors by $\boldsymbol{e} := \boldsymbol{u}^h - \boldsymbol{u}_h$ and $\varepsilon := p^h - 0 = p^h$. Then $\boldsymbol{e} \in H_0(\mathrm{curl}^2; \Omega)$ and $\varepsilon \in H_0^1(\Omega)$ satisfy the defect equations

$$a(\boldsymbol{e}, \boldsymbol{v}) + b(\boldsymbol{v}, \varepsilon) = r_1(\boldsymbol{v}), \qquad \forall \boldsymbol{v} \in H_0(\mathrm{curl}^2; \Omega), \tag{4.8}$$
$$b(\boldsymbol{e}, q) = r_2(\nabla q), \qquad \forall q \in H_0^1(\Omega), \tag{4.9}$$

where

$$\begin{aligned} r_1(\boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}) - \big((\nabla\times)^2 \boldsymbol{u}_h, (\nabla\times)^2 \boldsymbol{v}\big) - (\boldsymbol{u}_h, \boldsymbol{v}), \\ r_2(\nabla q) &= -(\boldsymbol{u}_h, \nabla q). \end{aligned}$$

We have the following Galerkin orthogonality

$$r_1(\boldsymbol{v}_h) = 0, \qquad \forall \boldsymbol{v}_h \in V_h^0, \tag{4.10}$$
$$r_2(\nabla q_h) = 0, \qquad \forall q_h \in S_h^0. \tag{4.11}$$

The error estimator will be constructed by employing Lemma 2.4. Writing $\boldsymbol{e} = \boldsymbol{e}^0 + \boldsymbol{e}^\perp$ and $\boldsymbol{v} = \boldsymbol{v}^0 + \boldsymbol{v}^\perp$ with $\boldsymbol{e}^0, \boldsymbol{v}^0 \in \nabla H_0^1(\Omega)$ and $\boldsymbol{e}^\perp, \boldsymbol{v}^\perp \in X$, we obtain that

$$\big(\boldsymbol{e}^0, \boldsymbol{v}^0\big) + \big(\boldsymbol{v}^0, \nabla\varepsilon\big) = r_1(\boldsymbol{v}^0), \qquad \forall \boldsymbol{v}^0 \in \nabla H_0^1(\Omega), \tag{4.12}$$
$$\big((\nabla\times)^2 \boldsymbol{e}^\perp, (\nabla\times)^2 \boldsymbol{v}^\perp\big) + \big(\boldsymbol{e}^\perp, \boldsymbol{v}^\perp\big) = r_1(\boldsymbol{v}^\perp), \qquad \forall \boldsymbol{v}^\perp \in X, \tag{4.13}$$
$$(\boldsymbol{e}^0, \nabla q) = r_2(\nabla q), \qquad \forall q \in H_0^1(\Omega). \tag{4.14}$$

The estimators for the irrotational part $\boldsymbol{e}^0$, the solenoidal part $\boldsymbol{e}^\perp$, and $\nabla\varepsilon$ will be derived separately. Firstly, we consider the irrotational part $\boldsymbol{e}^0$ and $\nabla\varepsilon$. For a $\vartheta \in H_0^1(\Omega)$, we have

$$\begin{aligned} r_1(\nabla\vartheta) = \sum_{T\in\mathcal{T}_h} (\boldsymbol{f} - \boldsymbol{u}_h, \nabla\vartheta)_T &= \sum_{T\in\mathcal{T}_h} -(\nabla\cdot\boldsymbol{f}, \vartheta)_T + \sum_{f\in\mathcal{F}_h^{\mathrm{int}}} \big\langle [\![\boldsymbol{n}_f\cdot\boldsymbol{f}]\!]_f, \vartheta\big\rangle_f \\ &+ \sum_{T\in\mathcal{T}_h} (\nabla\cdot\boldsymbol{u}_h, \vartheta)_T - \sum_{f\in\mathcal{F}_h^{\mathrm{int}}} \big\langle [\![\boldsymbol{n}_f\cdot\boldsymbol{u}_h]\!]_f, \vartheta\big\rangle_f, \end{aligned}$$

where $f \in \mathcal{F}_h^{\rm int}$ is the common face of two adjacent elements $T_1, T_2 \in \mathcal{T}_h$, $\boldsymbol{n}_f$ is the unit normal vector of f directed towards the interior of T_1, and the jump

$$[\![\boldsymbol{n}_f \cdot \boldsymbol{u}_h]\!]_f = \boldsymbol{n}_f \cdot \boldsymbol{u}_h|_{T_2} - \boldsymbol{n}_f \cdot \boldsymbol{u}_h|_{T_1}.$$

We also have

$$r_2(\nabla\vartheta) = \sum_{T\in\mathcal{T}_h} -(\boldsymbol{u}_h, \nabla\vartheta)_T = \sum_{T\in\mathcal{T}_h} (\nabla\cdot\boldsymbol{u}_h, \vartheta)_T - \sum_{f\in\mathcal{F}_h^{\rm int}} \langle [\![\boldsymbol{n}_f\cdot\boldsymbol{u}_h]\!]_f, \vartheta\rangle_f .$$

We introduce the error terms which are related to the upper and lower bounds for $\boldsymbol{e}^0$ and $\nabla\varepsilon$:

$$\eta_0^2 := \sum_{T\in\mathcal{T}_h} \left(\eta_0^T\right)^2 + \sum_{f\in\mathcal{F}_h^{\rm int}} \left(\eta_0^f\right)^2, \tag{4.15}$$

$$\eta_3^2 := \sum_{T\in\mathcal{T}_h} \left(\eta_3^T\right)^2 + \sum_{f\in\mathcal{F}_h^{\rm int}} \left(\eta_3^f\right)^2, \tag{4.16}$$

where

$$\begin{aligned}
\eta_0^T &:= h_T \left\|\nabla\cdot\boldsymbol{f}\right\|_T, && T\in\mathcal{T}_h,\\
\eta_0^f &:= h_f^{1/2} \left\|[\![\boldsymbol{n}_f\cdot\boldsymbol{f}]\!]_f\right\|_f, && f\in\mathcal{F}_h^{\rm int},\\
\eta_3^T &:= h_T \left\|\nabla\cdot\boldsymbol{u}_h\right\|_T, && T\in\mathcal{T}_h,\\
\eta_3^f &:= h_f^{1/2} \left\|[\![\boldsymbol{n}_f\cdot\boldsymbol{u}_h]\!]_f\right\|_f, && f\in\mathcal{F}_h^{\rm int}.
\end{aligned}$$

Next, we consider the bounds for $\boldsymbol{e}^\perp$. For $\boldsymbol{w}\in X$, the residual $r_1(\boldsymbol{w})$ can be expressed as

$$\begin{aligned}
r_1(\boldsymbol{w}) &= \sum_{T\in\mathcal{T}_h} \left(\boldsymbol{f}-\boldsymbol{u}_h, \boldsymbol{w}\right)_T - \left((\nabla\times)^2\boldsymbol{u}_h, (\nabla\times)^2\boldsymbol{w}\right)_T\\
&= \sum_{T\in\mathcal{T}_h} \left(\boldsymbol{f}-(\nabla\times)^4\boldsymbol{u}_h-\boldsymbol{u}_h, \boldsymbol{w}\right)_T + \sum_{f\in\mathcal{F}_h^{\rm int}} \left\langle [\![(\nabla\times)^2\boldsymbol{u}_h\times\boldsymbol{n}_f]\!]_f, \nabla\times\boldsymbol{w}\right\rangle_f\\
&\quad + \sum_{f\in\mathcal{F}_h^{\rm int}} \left\langle [\![(\nabla\times)^3\boldsymbol{u}_h\times\boldsymbol{n}_f]\!]_f, \boldsymbol{w}\right\rangle_f,
\end{aligned}$$

where $[\![(\nabla\times)^2\boldsymbol{u}_h\times\boldsymbol{n}_f]\!]_f$ and $[\![(\nabla\times)^3\boldsymbol{u}_h\times\boldsymbol{n}_f]\!]_f$ stand for the jump of the tangential component of $(\nabla\times)^2\boldsymbol{u}_h$ and $(\nabla\times)^3\boldsymbol{u}_h$, respectively. The bounds for $\|\boldsymbol{e}^\perp\|$ contain the error terms

$$\eta_1^2 := \sum_{T\in\mathcal{T}_h} \left(\eta_1^T\right)^2 + \sum_{f\in\mathcal{F}_h^{\rm int}} \left(\eta_{1;1}^f\right)^2 + \sum_{f\in\mathcal{F}_h^{\rm int}} \left(\eta_{1;2}^f\right)^2, \tag{4.17}$$

$$\eta_2^2 := \sum_{T\in\mathcal{T}_h} \left(\eta_2^T\right)^2, \tag{4.18}$$

where

$$\begin{aligned}
\eta_1^T &:= h_T^2 \left\|\pi_h\boldsymbol{f}-(\nabla\times)^4\boldsymbol{u}_h-\boldsymbol{u}_h\right\|_T, && T\in\mathcal{T}_h,\\
\eta_2^T &:= h_T^2 \left\|\boldsymbol{f}-\pi_h\boldsymbol{f}\right\|_T, && T\in\mathcal{T}_h,\\
\eta_{1;1}^f &:= h_f^{1/2} \left\|[\![\boldsymbol{n}_f\times(\nabla\times)^2\boldsymbol{u}_h]\!]_f\right\|_f, && f\in\mathcal{F}_h^{\rm int},\\
\eta_{1;2}^f &:= h_f^{3/2} \left\|[\![\boldsymbol{n}_f\times(\nabla\times)^3\boldsymbol{u}_h]\!]_f\right\|_f, && f\in\mathcal{F}_h^{\rm int},
\end{aligned}$$

and $\pi_h \boldsymbol{f}$ denotes the $\boldsymbol{L}^2$-projection of $\boldsymbol{f}$ onto $\{\boldsymbol{v}_h \in \boldsymbol{L}^2(\Omega) : \boldsymbol{v}_h|_T \in \boldsymbol{P}_k(T),\ \forall T \in \mathcal{T}_h\}$. For $T \in \mathcal{T}_h$, we define a local error indicator $\eta_h(\boldsymbol{u}_h, T)$ by

$$\eta_h^2(\boldsymbol{u}_h, T) = (\eta_0^T)^2 + (\eta_1^T)^2 + (\eta_3^T)^2 + \sum_{f \in \partial T} \left((\eta_0^f)^2 + (\eta_3^f)^2 + (\eta_{1;1}^f)^2 + (\eta_{1;2}^f)^2 \right),$$

and a global *a posteriori* error estimator by

$$\eta_h^2(\boldsymbol{u}_h, \Omega) = \eta_0^2 + \eta_1^2 + \eta_3^2.$$

Now we state the *a posteriori* estimate for $\boldsymbol{e}$ and ε in the energy norm.

Theorem 4.2. *Let η_0, η_1, η_2, and η_3 be defined in (4.15), (4.17), (4.18), and (4.16), respectively. Then if $h < 1$,*

$$\gamma_1(\eta_0 + \eta_1 + \eta_3) - \gamma_2 \eta_2 \leqslant |||\boldsymbol{e}||| + \|\nabla \varepsilon\| \leqslant \Gamma_1(\eta_0 + \eta_1 + \eta_3) + \Gamma_2 \eta_2,$$

and, if h is small enough,

$$\gamma_3(\eta_1 + \eta_3) - \gamma_4(\eta_2 + h^2 \eta_0) \leqslant |||\boldsymbol{e}||| \leqslant \Gamma_3(\eta_0 + \eta_1 + \eta_3) + \Gamma_4 \eta_2,$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4 are some constants independent of h.

Since $\boldsymbol{f} = (\lambda_h + 1)\boldsymbol{u}_h$, according to the definition of η_0, η_2, and η_3, we have that $\eta_0 = \lambda_h \eta_3$ and $\eta_2 = 0$. Then by Theorems 4.1, 4.2, and (4.2), we can obtain the following *a posteriori* error estimates for the eigenvectors and eigenvalues.

Theorem 4.3. *For h small enough, there exist constants c_1, C_1, and C_2 such that*

$$c_1(\eta_1 + \eta_3) \leqslant |||\boldsymbol{u} - \boldsymbol{u}_h||| \leqslant C_1(\eta_1 + (\lambda_h + 1)\eta_3),$$

and

$$|\lambda - \lambda_h| \leqslant C_2(\eta_1 + (\lambda_h + 1)\eta_3)^2,$$

where η_1 and η_3 are respectively defined in (4.17) and (4.16) with $\boldsymbol{f} = (\lambda_h + 1)\boldsymbol{u}_h$.

The proof of Theorem 4.2. Since $\boldsymbol{e} = \boldsymbol{e}^0 + \boldsymbol{e}^\perp$, the proof is split into three parts corresponding to $\boldsymbol{e}^0$, $\boldsymbol{e}^\perp$, and ε, respectively.

(i) Estimation of the irrotational part $\boldsymbol{e}^0$. Based on (4.14), we can rewrite $\boldsymbol{e}^0 = \nabla \varphi$ with φ solving the following uniformly positive definite variational problem on $H_0^1(\Omega)$. Seek $\varphi \in H_0^1(\Omega)$ such that

$$(\nabla \varphi, \nabla q) = r_2(\nabla q), \quad \forall q \in H_0^1(\Omega). \tag{4.19}$$

Note that $r_2(\nabla q_h) = 0, \forall q_h \in S_h^0$. Define a projection operator $P_h^k : H_0^1(\Omega) \longrightarrow S_h^0$ such that (see, *e.g.*, [3, 29, 31])

$$P_h^k \phi = \phi, \quad \forall \phi \in S_h^0, \tag{4.20}$$

$$\|\phi - P_h^k \phi\|_T \leqslant C h_T \|\nabla \phi\|_{\omega_T}, \tag{4.21}$$

$$\|\phi - P_h^k \phi\|_f \leqslant C \sqrt{h_f} \|\nabla \phi\|_{\omega_f}, \tag{4.22}$$

$$\|\nabla P_h^k \phi\|_T \leqslant C \|\nabla \phi\|_{\omega_T}, \tag{4.23}$$

where ω_T is defined in Theorem 2.3 and ω_f is the union of elements sharing at least one vertex with $f \in \mathcal{F}_h$. Due to (4.14) and the orthogonal property (4.11), we have that

$$\|\boldsymbol{e}^0\|^2 = r_2(\boldsymbol{e}^0) = r_2(\nabla\varphi - \nabla P_h^k\varphi) = -(\boldsymbol{u}_h, \nabla(\varphi - P_h^k\varphi)). \tag{4.24}$$

Using integration by parts, (4.21), and (4.22), we obtain that

$$\begin{aligned}
\Big(\boldsymbol{u}_h, \nabla(\varphi - P_h^k\varphi)\Big) &= \sum_{T\in\mathcal{T}_h} -\Big(\nabla\cdot\boldsymbol{u}_h, \varphi - P_h^k\varphi\Big)_T + \sum_{f\in\mathcal{F}_h^{\text{int}}} \big\langle [\![\boldsymbol{n}_f\cdot\boldsymbol{u}_h]\!]_f, \varphi - P_h^k\varphi\big\rangle_f \\
&\leqslant C\sum_{T\in\mathcal{T}_h} \|\nabla\cdot\boldsymbol{u}_h\|_T\, h_T\, \|\nabla\varphi\|_{\omega_T} + C\sum_{f\in\mathcal{F}_h^{\text{int}}} \|[\![\boldsymbol{n}_f\cdot\boldsymbol{u}_h]\!]_f\|_f \sqrt{h_f}\, \|\nabla\varphi\|_{\omega_f} \\
&\leqslant C\Big(\sum_{T\in\mathcal{T}_h} \|\nabla\cdot\boldsymbol{u}_h\|_T^2\, h_T^2\Big)^{1/2}\|\boldsymbol{e}^0\| + C\Big(\sum_{f\in\mathcal{F}_h^{\text{int}}} \|[\![\boldsymbol{n}_f\cdot\boldsymbol{u}_h]\!]_f\|_f^2\, h_f\Big)^{1/2}\|\boldsymbol{e}^0\|.
\end{aligned}$$

Therefore, we have

$$\|\boldsymbol{e}^0\| \leqslant C\eta_3. \tag{4.25}$$

We now derive lower bounds for $\boldsymbol{e}^0$ using the bubble functions.
Denote by $\lambda_1^T, \lambda_2^T, \lambda_3^T, \lambda_4^T$ the barycentric coordinates of $T \in \mathcal{T}_h$ and define the bubble function b_T by

$$b_T = \begin{cases} 256\lambda_1^T\lambda_2^T\lambda_3^T\lambda_4^T, & \text{on } T,\\ 0, & \Omega\backslash T.\end{cases}$$

Given $f \in \mathcal{F}_h$, a common edge of T_1 and T_2, let $\varpi_f = T_1 \cup T_2$ and enumerate the vertices of T_1 and T_2 such that the vertices of f are numbered first. Define the face-bubble function b_f by

$$b_f = \begin{cases} 27\lambda_1^{T_i}\lambda_2^{T_i}\lambda_3^{T_i}, & \text{on } T_i,\ i = 1, 2,\\ 0, & \Omega\backslash\varpi_f.\end{cases}$$

Using the technique of Lemma 3.3 in [37], we have the following norm equivalences.

$$\|b_T\phi_h\|_T \leqslant \|\phi_h\|_T \leqslant C\|b_T^{1/2}\phi_h\|_T, \qquad \forall\phi_h \in P_k(T), \tag{4.26}$$

$$\|b_f\phi_h\|_f \leqslant \|\phi_h\|_f \leqslant C\|b_f^{1/2}\phi_h\|_f, \qquad \forall\phi_h \in P_k(f). \tag{4.27}$$

Using (4.26), integration by parts, the inverse inequality, and the fact that $b_T\nabla\cdot\boldsymbol{u}_h \in H_0^1(T) \subset H_0^1(\Omega)$, we have that

$$\begin{aligned}
\frac{(\eta_3^T)^2}{h_T^2} &= \|\nabla\cdot\boldsymbol{u}_h\|_T^2 \leqslant C\|b_T^{1/2}\nabla\cdot\boldsymbol{u}_h\|_T^2 = C\big(\nabla\cdot\boldsymbol{u}_h, b_T\nabla\cdot\boldsymbol{u}_h\big)_T \\
&= -C\big(\boldsymbol{u}_h, \nabla(b_T\nabla\cdot\boldsymbol{u}_h)\big)_T = Cr_2\big(\nabla(b_T\nabla\cdot\boldsymbol{u}_h)\big) \\
&= C\left(\boldsymbol{e}^0, \nabla(b_T\nabla\cdot\boldsymbol{u}_h)\right)_T \leqslant C\|\boldsymbol{e}^0\|_T\|\nabla(b_T\nabla\cdot\boldsymbol{u}_h)\|_T \\
&\leqslant \frac{C}{h_T}\|\boldsymbol{e}^0\|_T\|\nabla\cdot\boldsymbol{u}_h\|_T \leqslant \frac{C}{h_T^2}\|\boldsymbol{e}^0\|_T\eta_3^T,
\end{aligned}$$

which implies that

$$\eta_3^T \leqslant C\|\boldsymbol{e}^0\|_T. \tag{4.28}$$

We define a continuous operator $P_T : L^\infty(f) \to L^\infty(T)$ as in [37]. According to Lemma 3.3 of [37],

$$\|P_T\sigma\|_T \leqslant Ch_T^{1/2}\|\sigma\|_f.$$

Denote $[\![\boldsymbol{n}_f \cdot \boldsymbol{u}_h]\!]_{f;T_i} = P_{T_i}[\![\boldsymbol{n}_f \cdot \boldsymbol{u}_h]\!]_f$, then

$$\|[\![\boldsymbol{n}_f \cdot \boldsymbol{u}_h]\!]_{f;T_i}\|_{T_i} \leqslant Ch_{T_i}^{1/2}\|[\![\boldsymbol{n}_f \cdot \boldsymbol{u}_h]\!]_f\|_f. \tag{4.29}$$

The estimate of the local upper bound for η_3^f can be obtained similarly:

$$\begin{aligned}
\frac{(\eta_3^f)^2}{h_f} &= \|[\![\boldsymbol{n}_f \cdot \boldsymbol{u}_h]\!]_f\|_f^2 \leqslant C\big\langle[\![\boldsymbol{n}_f \cdot \boldsymbol{u}_h]\!]_f b_f, [\![\boldsymbol{n}_f \cdot \boldsymbol{u}_h]\!]_f\big\rangle_f \\
&= C\sum_{i=1}^{2}\Big(\Big(\boldsymbol{u}_h, \nabla\left(b_f[\![\boldsymbol{n}_f \cdot \boldsymbol{u}_h]\!]_{f;T_i}\right)\Big)_{T_i} + \Big(\nabla\cdot\boldsymbol{u}_h, b_f[\![\boldsymbol{n}_f \cdot \boldsymbol{u}_h]\!]_{f;T_i}\Big)_{T_i}\Big) \\
&= C\sum_{i=1}^{2}\Big(\Big(\boldsymbol{e}^0, \nabla\left(b_f[\![\boldsymbol{n}_f \cdot \boldsymbol{u}_h]\!]_{f;T_i}\right)\Big)_{T_i} + \Big(\nabla\cdot\boldsymbol{u}_h, b_f[\![\boldsymbol{n}_f \cdot \boldsymbol{u}_h]\!]_{f;T_i}\Big)_{T_i}\Big) \\
&\leqslant C\sum_{i=1}^{2}\left(h_{T_i}^{-1}\|\boldsymbol{e}^0\|_{T_i} + \|\nabla\cdot\boldsymbol{u}_h\|_{T_i}\right)\eta_3^f,
\end{aligned}$$

where we have used

$$\|\nabla(b_f[\![\boldsymbol{n}_f \cdot \boldsymbol{u}_h]\!]_{f;T_i})\|_{T_i} \leqslant Ch_{T_i}^{-1}\|b_f[\![\boldsymbol{n}_f \cdot \boldsymbol{u}_h]\!]_{f;T_i}\|_{T_i} \leqslant Ch_{T_i}^{-1/2}\|[\![\boldsymbol{n}_f \cdot \boldsymbol{u}_h]\!]_f\|_f.$$

Consequently,

$$\eta_3^f \leqslant C\big(\|\boldsymbol{e}^0\|_{\varpi_f} + \eta_3^{T_1} + \eta_3^{T_2}\big) \leqslant C\|\boldsymbol{e}^0\|_{\varpi_f}. \tag{4.30}$$

Now collecting (4.25), (4.28), and (4.30), we have that

$$c\eta_3 \leqslant \|\boldsymbol{e}^0\| \leqslant C\eta_3. \tag{4.31}$$

(ii) Estimation of $\nabla\varepsilon$. Similar to the upper estimate of $\boldsymbol{e}^0$, we can obtain an upper bound for $\|\nabla\varepsilon\|$. Due to (4.12) and (4.14) , we have

$$\|\nabla\varepsilon\|^2 = r_1\big(\nabla\varepsilon\big) - r_2\big(\nabla\varepsilon\big) = r_1\big(\nabla(\varepsilon - P_h^k\varepsilon)\big) - r_2\big(\nabla(\varepsilon - P_h^k\varepsilon)\big) = \big(\boldsymbol{f}, \nabla(\varepsilon - P_h^k\varepsilon)\big).$$

From integration by parts, (4.21), and (4.22),

$$\begin{aligned}
\|\nabla\varepsilon\|^2 &= \sum_{T\in\mathcal{T}_h} -\big(\nabla\cdot\boldsymbol{f}, \varepsilon - P_h^k\varepsilon\big)_T + \sum_{f\in\mathcal{F}_h^{\mathrm{int}}}\big\langle[\![\boldsymbol{n}_f \cdot \boldsymbol{f}]\!]_f, \varepsilon - P_h^k\varepsilon\big\rangle_f \\
&\leqslant \sum_{T\in\mathcal{T}_h}\|\nabla\cdot\boldsymbol{f}\|_T h_T\|\nabla\varepsilon\|_{\omega_T} + \|[\![\boldsymbol{n}_f \cdot \boldsymbol{f}]\!]_f\|_f\sqrt{h_f}\|\nabla\varepsilon\|_{\omega_f} \\
&\leqslant C\|\nabla\varepsilon\|\bigg(\Big(\sum_{T\in\mathcal{T}_h}\|\nabla\cdot\boldsymbol{f}\|_T^2 h_T^2\Big)^{1/2} + \Big(\sum_{f\in\mathcal{F}_h^{\mathrm{int}}}\|[\![\boldsymbol{n}_f \cdot \boldsymbol{f}]\!]_f\|_f^2 h_f\Big)^{1/2}\bigg).
\end{aligned}$$

Therefore, we have that

$$\|\nabla\varepsilon\| \leqslant C\eta_0. \tag{4.32}$$

By a similar argument to the lower estimate of $\boldsymbol{e}^0$, we have the lower bounds for $\nabla\varepsilon$:

$$\eta_0^T \leqslant \|\nabla\varepsilon\|_T, \tag{4.33}$$

$$\eta_0^f \leqslant C\big(\|\nabla\varepsilon\|_{\varpi_f} + \eta_0^{T_1} + \eta_0^{T_2}\big) \leqslant C\|\nabla\varepsilon\|_{\varpi_f}. \tag{4.34}$$

Combining (4.32), (4.33), and (4.34), we arrive at

$$c\eta_0 \leqslant \|\nabla\varepsilon\| \leqslant C\eta_0. \tag{4.35}$$

(iii) Estimation of the solenoidal part $\boldsymbol{e}^\perp$. We start with proving the upper bound for η_1^T by using b_T again. Employing the similar technique of Lemma 3.3 in [37], we have the following estimates for any $\boldsymbol{v}$ in finite dimensional spaces:

$$\|\boldsymbol{v}\|_T^2 \leqslant C\|b_T\boldsymbol{v}\|_T^2, \tag{4.36}$$

$$\|b_T^2\boldsymbol{v}\|_T^2 \leqslant C\|\boldsymbol{v}\|_T^2. \tag{4.37}$$

Let $\boldsymbol{\phi}_h = \pi_h\boldsymbol{f} - (\nabla\times)^4\boldsymbol{u}_h - \boldsymbol{u}_h$, we then have

$$\begin{aligned}
\left(\frac{\eta_1^T}{h_T^2}\right)^2 &= \left\|\pi_h\boldsymbol{f} - (\nabla\times)^4\boldsymbol{u}_h - \boldsymbol{u}_h\right\|_T^2\\
&\leqslant C\left(\boldsymbol{f} - (\nabla\times)^4\boldsymbol{u}_h - \boldsymbol{u}_h, b_T^2\boldsymbol{\phi}_h\right)_T + C\left(\pi_h\boldsymbol{f} - \boldsymbol{f}, b_T^2\boldsymbol{\phi}_h\right)_T \quad \big(\text{by } (4.36)\big)\\
&= Cr_1(b_T^2\boldsymbol{\phi}_h) + C\left(\pi_h\boldsymbol{f} - \boldsymbol{f}, b_T^2\boldsymbol{\phi}_h\right)_T \quad \big(b_T^2\boldsymbol{\phi}_h \in H_0(\mathrm{curl}^2;\Omega)\big)\\
&= Ca(\boldsymbol{e}, b_T^2\boldsymbol{\phi}_h) + Cb(b_T^2\boldsymbol{\phi}_h, \varepsilon) + C\left(\pi_h\boldsymbol{f} - \boldsymbol{f}, b_T^2\boldsymbol{\phi}_h\right)_T \quad \big(\text{by } (4.8)\big)\\
&\leqslant C|\!|\!|\boldsymbol{e}|\!|\!|_T |\!|\!|b_T^2\boldsymbol{\phi}_h|\!|\!|_T + C\left\|\nabla\varepsilon\right\|_T \left\|b_T^2\boldsymbol{\phi}_h\right\|_T + C\eta_2^T h_T^{-2}\left\|b_T^2\boldsymbol{\phi}_h\right\|_T.
\end{aligned}$$

Due to the inverse inequality and (4.37), it holds that

$$|\!|\!|b_T^2\boldsymbol{\phi}_h|\!|\!|_T^2 = \|b_T^2\boldsymbol{\phi}_h\|_T^2 + \|(\nabla\times)^2 b_T^2\boldsymbol{\phi}_h\|_T^2 \leqslant Ch_T^{-4}\|b_T^2\boldsymbol{\phi}_h\|_T^2 \leqslant Ch_T^{-4}\|\boldsymbol{\phi}_h\|_T^2.$$

Thus we obtain that

$$\left(\frac{\eta_1^T}{h_T^2}\right)^2 \leqslant C\frac{\eta_1^T}{h_T^2}\left(h_T^{-2}|\!|\!|\boldsymbol{e}|\!|\!|_T + \|\nabla\varepsilon\|_T + h_T^{-2}\eta_2^T\right).$$

Dividing the above inequality by $\frac{\eta_1^T}{h_T^2}$ and multiplying by h_T^2, we obtain

$$\eta_1^T \leqslant C\left(|\!|\!|\boldsymbol{e}|\!|\!|_T + h_T^2\|\nabla\varepsilon\|_T + \eta_2^T\right). \tag{4.38}$$

Next we estimate the upper bound for $\eta_{1;1}^f$ by using the bubble functions b_T and b_f. Let T_1 and T_2 be two elements sharing the face f. Denote $\boldsymbol{\psi}_h|_{T_i} = P_{T_i}[\![\boldsymbol{n}_f \times (\nabla\times)^2\boldsymbol{u}_h]\!]_f$ for $i = 1, 2$, then

$$\|\boldsymbol{\psi}_h\|_{T_i} \leqslant Ch_{T_i}^{1/2}\|[\![\boldsymbol{n}_f \times (\nabla\times)^2\boldsymbol{u}_h]\!]_f\|_f. \tag{4.39}$$

Denote $\boldsymbol{\omega}_{f,1} = (b_{T_1} - b_{T_2})b_f\boldsymbol{n}_f \times \boldsymbol{\psi}_h$. A simple calculation shows that

$$(\nabla\times\boldsymbol{\omega}_{f,1})|_f = \frac{256}{27}\left(\frac{S_f}{3|T_1|} + \frac{S_f}{3|T_2|}\right)b_f^2\boldsymbol{\psi}_h,$$

where S_f stands for the area of the face f. Similar to (4.36) and (4.37), the following inequalities hold

$$\|v\|_f \leqslant C\|b_f v\|_f, \tag{4.40}$$

$$\|(b_{T_1} - b_{T_2})b_f v\|_{\varpi_f} \leqslant C\|v\|_{\varpi_f}. \tag{4.41}$$

Now we are ready to construct the upper bound for $\eta_{1;1}^f$:

$$\begin{aligned}
&h_f^{-1}\|[\![\boldsymbol{n}_f \times (\nabla\times)^2\boldsymbol{u}_h]\!]_f\|_f^2 \\
&\quad\leqslant C\big\langle[\![\boldsymbol{n}_f \times (\nabla\times)^2\boldsymbol{u}_h]\!]_f, \nabla\times\boldsymbol{\omega}_{f,1}\big\rangle_f \quad \big(\text{by } (4.40)\big) \\
&\quad= C\big((\nabla\times)^4\boldsymbol{u}_h, \boldsymbol{\omega}_{f,1}\big) - C\big((\nabla\times)^2\boldsymbol{u}_h, (\nabla\times)^2\boldsymbol{\omega}_{f,1}\big) \\
&\quad= Cr_1(\boldsymbol{\omega}_{f,1}) - C\big(\boldsymbol{f} - \boldsymbol{u}_h - (\nabla\times)^4\boldsymbol{u}_h, \boldsymbol{\omega}_{f,1}\big) \quad \big(\boldsymbol{\omega}_{f,1} \in H_0(\mathrm{curl}^2;\Omega)\big) \\
&\quad\leqslant C\|\!|\boldsymbol{e}|\!\|_{\varpi_f}\|\!|\boldsymbol{\omega}_{f,1}|\!\|_{\omega_f} + C\|\boldsymbol{\omega}_{f,1}\|_{\varpi_f}\left(\|\nabla\varepsilon\|_{\varpi_f} + \sum_{i=1}^{2} h_{T_i}^{-2}\big(\eta_1^{T_i} + \eta_2^{T_i}\big)\right).
\end{aligned}$$

By applying the inverse inequality, (4.39), and (4.41), we get

$$\|\!|\boldsymbol{\omega}_{f,1}|\!\|_{\varpi_f} \leqslant h_f^{-2}\|\boldsymbol{\omega}_{f,1}\|_{\varpi_f} \leqslant h_f^{-3/2}\|[\![\boldsymbol{n}_f \times (\nabla\times)^2\boldsymbol{u}_h]\!]_f\|_f,$$

which, together with (4.38), leads to

$$\eta_{1;1}^f \leqslant C\Big(\|\!|\boldsymbol{e}|\!\|_{\varpi_f} + \eta_2^{T_1} + \eta_2^{T_2} + h_{T_1}^2\|\nabla\varepsilon\|_{T_1} + h_{T_2}^2\|\nabla\varepsilon\|_{T_2}\Big). \tag{4.42}$$

The upper bound for $\eta_{1;2}^f$ can be constructed in a similar way. Extend $[\![\boldsymbol{n}_f \times (\nabla\times)^3\boldsymbol{u}_h]\!]_f$ to $[\![\boldsymbol{n}_f \times (\nabla\times)^3\boldsymbol{u}_h]\!]_{f;T_i}$ on T_i such that

$$\|[\![\boldsymbol{n}_f \times (\nabla\times)^3\boldsymbol{u}_h]\!]_{f;T_i}\|_{T_i} \leqslant Ch_{T_i}^{1/2}\|[\![\boldsymbol{n}_f \times (\nabla\times)^3\boldsymbol{u}_h]\!]_f\|_f. \tag{4.43}$$

Denote $\boldsymbol{\omega}_{f,2}|_{T_i} = b_f^2[\![\boldsymbol{n}_f \times (\nabla\times)^3\boldsymbol{u}_h]\!]_{f;T_i}$, then

$$\begin{aligned}
\|[\![\boldsymbol{n}_f \times (\nabla\times)^3\boldsymbol{u}_h]\!]_f\|_f^2 &\leqslant C\big\langle[\![\boldsymbol{n}_f \times (\nabla\times)^3\boldsymbol{u}_h]\!]_f, \boldsymbol{\omega}_{f,2}\big\rangle_f \\
&= \big((\nabla\times)^4\boldsymbol{u}_h, \boldsymbol{\omega}_{f,2}\big) - \big((\nabla\times)^2\boldsymbol{u}_h, (\nabla\times)^2\boldsymbol{\omega}_{f,2}\big) - \big\langle[\![\boldsymbol{n}_f \times (\nabla\times)^2\boldsymbol{u}_h]\!]_f, \nabla\times\boldsymbol{\omega}_{f,2}\big\rangle_f \\
&= r_1(\boldsymbol{\omega}_{f,2}) - \big(\boldsymbol{f} - \boldsymbol{u}_h - (\nabla\times)^4\boldsymbol{u}_h, \boldsymbol{\omega}_{f,2}\big) - \big\langle[\![\boldsymbol{n}_f \times (\nabla\times)^2\boldsymbol{u}_h]\!]_f, \nabla\times\boldsymbol{\omega}_{f,2}\big\rangle_f \\
&\leqslant Ch_f^{-3/2}\Big(\sum_{i=1}^{2}\big(\eta_1^{T_i} + \eta_2^{T_i} + h_{T_i}^2\|\nabla\varepsilon\|_{T_i}\big) + \eta_{1;1}^f + \|\!|\boldsymbol{e}|\!\|_{\varpi_f}\Big)\left\|[\![\boldsymbol{n}_f \times (\nabla\times)^3\boldsymbol{u}_h]\!]_f\right\|_f.
\end{aligned}$$

Dividing the above inequality by $\left\|[\![\boldsymbol{n}_f \times (\nabla\times)^3\boldsymbol{u}_h]\!]_f\right\|_f$ and applying (4.38) and (4.42), we obtain

$$\eta_{1,2}^f \leqslant C\left(\eta_2^{T_1} + \eta_2^{T_2} + \|\!|\boldsymbol{e}|\!\|_{T_1\cup T_2} + h_{T_1}^2\|\nabla\varepsilon\|_{T_1} + h_{T_2}^2\|\nabla\varepsilon\|_{T_2}\right). \tag{4.44}$$

Collecting (4.38), (4.42), and (4.44), we have that

$$\eta_1 \leqslant C\left(\eta_2 + \|\!|\boldsymbol{e}|\!\| + h^2\|\nabla\varepsilon\|\right). \tag{4.45}$$

It remains to construct the upper bound of $\boldsymbol{e}^{\perp}$. For $\boldsymbol{e}^{\perp} \in X \subset H_0(\mathrm{curl}^2;\Omega)$, according to Lemma 2.4, $\boldsymbol{e}^{\perp} = \boldsymbol{w} + \nabla\psi$ with $\boldsymbol{w} \in \boldsymbol{H}^2(\Omega)$ and $\psi \in H_0^1(\Omega)$. Then we have

$$|||\boldsymbol{e}^{\perp}|||^2 = r_1(\boldsymbol{e}^{\perp}) = r_1(\boldsymbol{w}) + r_1(\nabla\psi).$$

Due to the Galerkin orthogonality (4.10), for any $\boldsymbol{w}_h \in V_h^0$,

$$\begin{aligned}
r_1(\boldsymbol{w}) &= r_1(\boldsymbol{w} - \boldsymbol{w}_h)\\
&= \sum_{T\in\mathcal{T}_h}\Big(\big(\boldsymbol{f} - \boldsymbol{u}_h - (\nabla\times)^4\boldsymbol{u}_h, \boldsymbol{w} - \boldsymbol{w}_h\big)_T + \sum_{f\in\mathcal{F}_h(T)}\big\langle \boldsymbol{n}_f \times (\nabla\times)^3\boldsymbol{u}_h, \boldsymbol{w} - \boldsymbol{w}_h\big\rangle_f\\
&\quad - \sum_{f\in\mathcal{F}_h(T)}\big\langle \boldsymbol{n}_f \times (\nabla\times)^2\boldsymbol{u}_h, \nabla\times(\boldsymbol{w} - \boldsymbol{w}_h)\big\rangle_f\Big)\\
&\leqslant \sum_{T\in\mathcal{T}_h}\Big(\|\boldsymbol{\pi}_h\boldsymbol{f} - \boldsymbol{u}_h - (\nabla\times)^4\boldsymbol{u}_h\|_T\|\boldsymbol{w} - \boldsymbol{w}_h\|_T + \|\boldsymbol{\pi}_h\boldsymbol{f} - \boldsymbol{f}\|_T\|\boldsymbol{w} - \boldsymbol{w}_h\|_T\Big)\\
&\quad + \sum_{f\in\mathcal{F}_h^{\mathrm{int}}}\|[\![\boldsymbol{n}_f \times (\nabla\times)^2\boldsymbol{u}_h]\!]_f\|_f\|\nabla\times(\boldsymbol{w} - \boldsymbol{w}_h)\|_f\\
&\quad + \sum_{f\in\mathcal{F}_h^{\mathrm{int}}}\|[\![\boldsymbol{n}_f \times ((\nabla\times)^3\boldsymbol{u}_h)]\!]_f\|_f\|\boldsymbol{w} - \boldsymbol{w}_h\|_f\\
&\leqslant C(\eta_1 + \eta_2)\Big(\sum_{T\in\mathcal{T}_h}\Big(h_T^{-4}\|\boldsymbol{w} - \boldsymbol{w}_h\|_T^2 + \sum_{f\in\mathcal{F}_h(T)} h_f^{-1}\|\nabla\times(\boldsymbol{w} - \boldsymbol{w}_h)\|_f^2\\
&\quad + \sum_{f\in\mathcal{F}_h(T)} h_f^{-3}\|\boldsymbol{w} - \boldsymbol{w}_h\|_f^2\Big)\Big)^{1/2}.
\end{aligned}$$

Let $\boldsymbol{w}_h = \widetilde{\Pi}_h^2\boldsymbol{w}$. According to the trace inequality and Theorem 2.3, we obtain

$$\sum_{T\in\mathcal{T}_h}\Big(h_T^{-4}\|\boldsymbol{w} - \boldsymbol{w}_h\|_T^2 + \sum_{f\in\mathcal{F}_h(T)} h_f^{-1}\|\nabla\times(\boldsymbol{w} - \boldsymbol{w}_h)\|_f^2 + \sum_{f\in\mathcal{F}_h(T)} h_f^{-3}\|\boldsymbol{w} - \boldsymbol{w}_h\|_f^2\Big) \leqslant C\|\boldsymbol{w}\|_2^2.$$

Furthermore, we use (2.6), (2.7), and the Poincaré inequality to obtain

$$r_1(\boldsymbol{w}) \leqslant C(\eta_1 + \eta_2)\|\boldsymbol{w}\|_2 \leqslant C(\eta_1 + \eta_2)\|\nabla\times\boldsymbol{e}^{\perp}\|_1 \leqslant C(\eta_1 + \eta_2)|||\boldsymbol{e}^{\perp}|||.$$

Similar to the proof of (4.32), using (2.8), it holds that

$$r_1(\nabla\psi) \leqslant C(\eta_0 + \eta_3)\|\nabla\psi\| \leqslant C(\eta_0 + \eta_3)|||\boldsymbol{e}^{\perp}|||.$$

Hence,

$$|||\boldsymbol{e}^{\perp}||| \leqslant C(\eta_0 + \eta_1 + \eta_2 + \eta_3). \tag{4.46}$$

Combining (4.32), (4.35), (4.45), and (4.46), we obtain Theorem 4.2.

5. Numerical examples

In this section, we will present some numerical results in 2D. The quad-curl problem in 2D is

$$\begin{aligned}
(\nabla\times)^4\boldsymbol{u} &= \lambda\boldsymbol{u} && \text{in } \Omega,\\
\nabla\cdot\boldsymbol{u} &= 0 && \text{in } \Omega,\\
\boldsymbol{u}\times\boldsymbol{n} &= 0 && \text{on } \partial\Omega,\\
\nabla\times\boldsymbol{u} &= 0 && \text{on } \partial\Omega,
\end{aligned} \tag{5.1}$$

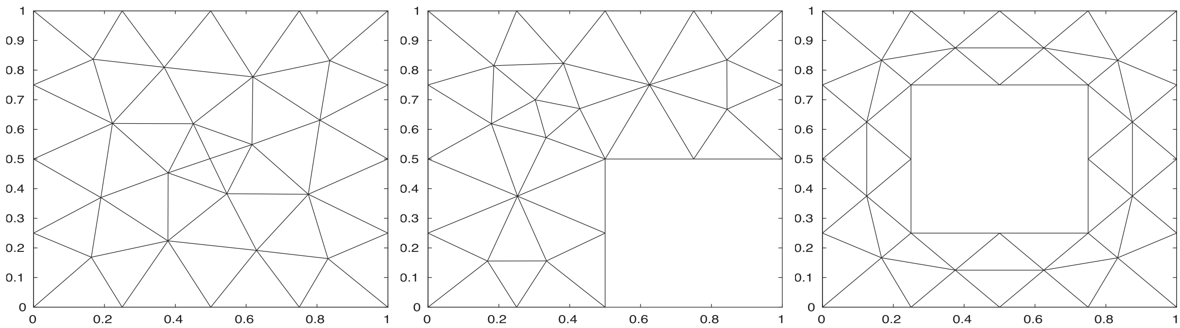


FIGURE 1. Sample meshes for Ω_1 (*left*), Ω_2 (*middle*), and Ω_3 (*right*).

where $\nabla \times \boldsymbol{u} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$ for $\boldsymbol{u} = (u_1, u_2)$ and $\nabla \times u = (\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1})$ for a scalar u. Since the outermost $\nabla\times$ is acting on a scalar, the function $\sigma = \nabla \times \boldsymbol{u}$ satisfies

$$\begin{aligned} \Delta\sigma &\in H^1(\Omega) && \text{in } \Omega, \\ \sigma &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{5.2}$$

When Ω is a polygon, according to Theorem 14.6 of [18], $\sigma \in H^{1+\pi/\omega-\epsilon}(\Omega)$ for any $\epsilon > 0$. Here ω is the largest interior angle at the corners of Ω.

Since $\boldsymbol{u} \in X$, according to Lemma 2.4, $\boldsymbol{u} = \nabla\phi + \boldsymbol{v}$ with $\phi \in H_0^1(\Omega)$ and $\boldsymbol{v} \in \boldsymbol{H}^2(\Omega)$. In addition, according to the proof of Lemma 2.4, ϕ actually satisfies $\Delta\phi \in H^1(\Omega)$, and hence $\phi \in H^{1+\pi/\omega-\epsilon}(\Omega)$. Therefore $\boldsymbol{u} \in \boldsymbol{H}^{\min\{\pi/\omega-\epsilon,2\}}(\Omega)$.

5.1. *A priori* error estimates

We consider three different domains:

- Ω_1: the unit square given by $(0,1) \times (0,1)$,
- Ω_2: the L-shaped domain given by $(0,1) \times (0,1)/\big([1/2,1) \times (0,1/2]\big)$,
- Ω_3: given by $(0,1) \times (0,1)/\big([1/4,3/4] \times [1/4,3/4]\big)$.

The eigenvectors on Ω_1 are in $\{\boldsymbol{u} \in \boldsymbol{H}^{2-\epsilon}(\Omega) : \nabla\times\boldsymbol{u} \in H^{3-\epsilon}(\Omega)\}$. The eigenvectors on Ω_2 are in $\{\boldsymbol{u} \in \boldsymbol{H}^{2/3-\epsilon}(\Omega) : \nabla\times\boldsymbol{u} \in H^{5/3-\epsilon}(\Omega)\}$. According to Theorem 3.8, the convergence orders for Ω_1 and Ω_2 are 4 and 4/3, respectively.

The initial meshes of the domains are shown in Figure 1. In Tables 1–3, we list the first five eigenvalues. Since the exact eigenvalues are unknown, the relative error is adopted:

$$\text{Error} = \left| \frac{\lambda_i^h - \lambda_i^{h/2}}{\lambda_i^{h/2}} \right|.$$

TABLE 1. The first 5 eigenvalues of Ω_1 with the fourth-order elements.

h	λ_1^h	λ_2^h	λ_3^h	λ_4^h	λ_5^h
1/4	7.08101988e+02	7.08102390e+02	2.35145718e+03	4.25922492e+03	5.02522026e+03
1/8	7.07978763e+02	7.07978786e+02	2.35006082e+03	4.25597055e+03	5.02401495e+03
1/16	7.07971973e+02	7.07971975e+02	2.34999027e+03	4.25582307e+03	5.02399272e+03
1/32	7.07971564e+02	7.07971564e+02	2.34998613e+03	4.25581473e+03	5.02399235e+03
1/64	7.07971528e+02	7.07971555e+02	2.34998587e+03	4.25581421e+03	5.02399235e+03

Table 2. The first 5 eigenvalues of Ω_2 with the fourth-order elements.

h	λ_1^h	λ_2^h	λ_3^h	λ_4^h	λ_5^h
1/4	5.34885649e+02	1.57586875e+03	6.10288551e+03	6.40711482e+03	1.09459861e+04
1/8	5.35061810e+02	1.57477474e+03	6.09556539e+03	6.37916246e+03	1.09184358e+04
1/16	5.35222062e+02	1.57468831e+03	6.09528577e+03	6.37104166e+03	1.09152964e+04
1/32	5.35292267e+02	1.57467206e+03	6.09528045e+03	6.36787675e+03	1.09143027e+04
1/64	5.35320748e+02	1.57466664e+03	6.09528434e+03	6.36661570e+03	1.09139180e+04

Table 3. The first 5 non-zero eigenvalues of Ω_3 with the fourth-order elements.

h	λ_1^h	λ_2^h	λ_3^h	λ_4^h	λ_5^h
1/4	9.43570924e+02	9.43570924e+02	3.35118080e+03	5.10757870e+03	1.03672699e+04
1/8	9.40543704e+02	9.40543704e+02	3.33230800e+03	5.11255084e+03	1.03470233e+04
1/16	9.39507116e+02	9.39507116e+02	3.32612997e+03	5.11519580e+03	1.03445476e+04
1/32	9.39103168e+02	9.39103168e+02	3.32373447e+03	5.11630255e+03	1.03438189e+04
1/64	9.38943028e+02	9.38943036e+02	3.32278551e+03	5.11674950e+03	1.03435487e+04

Table 4. Convergence rate for Ω_1 with the fourth-order elements (relative error).

h	λ_1^h	Error	Order
1/4	7.08101988e+02	1.74021691e-04	–
1/8	7.07978763e+02	9.59045415e-06	4.1815
1/16	7.07971973e+02	5.77922813e-07	4.0527
1/32	7.07971564e+02	5.08588883e-08	3.5063
1/64	7.07971528e+02	–	–

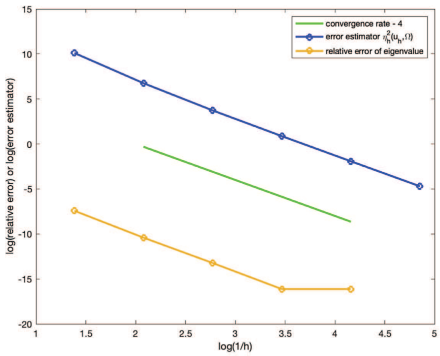
Table 5. Convergence rate for Ω_2 with the fourth-order elements (relative error).

h	λ_1^h	Error	Order
1/4	5.34885649e+02	3.29341761e-04	–
1/8	5.35061810e+02	2.99502830e-04	0.1370
1/16	5.35222062e+02	1.31169871e-04	1.1911
1/32	5.35292267e+02	5.32057764e-05	1.3018
1/64	5.35320748e+02	–	–

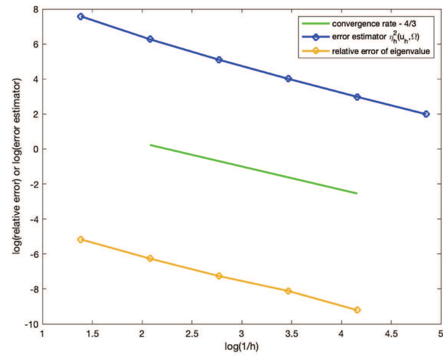
Table 6. Convergence rate for Ω_3 with the fourth-order elements (relative error).

h	λ_1^h	Error	Order
1/4	9.43570924e+02	3.20825910e-03	–
1/8	9.40543704e+02	1.10211572e-03	1.5415
1/16	9.39507116e+02	4.29957430e-04	1.3580
1/32	9.39103168e+02	1.70524522e-04	1.3342
1/64	9.38943028e+02	–	–

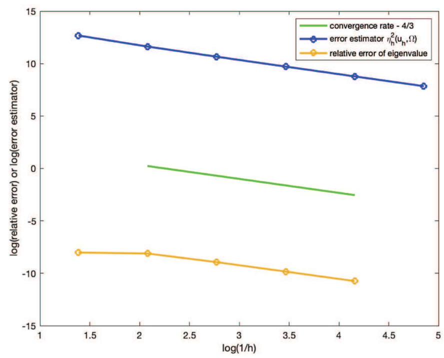
Tables 4–6 show the convergence rates of the relative errors for the first eigenvalues. We can observe a convergence order 4 for Ω_1 and 4/3 for Ω_2, which agrees with the theoretical results. We can also observe a convergence order 4/3 for Ω_3 even if this case is not covered in the theoretical analysis.



(a)



(b)



(c)

FIGURE 2. The convergence rates of error estimators and the relative errors. (a) The third eigenvalue of Ω_1. (b) The first eigenvalue on Ω_2. (c) The third eigenvalue on Ω_3.

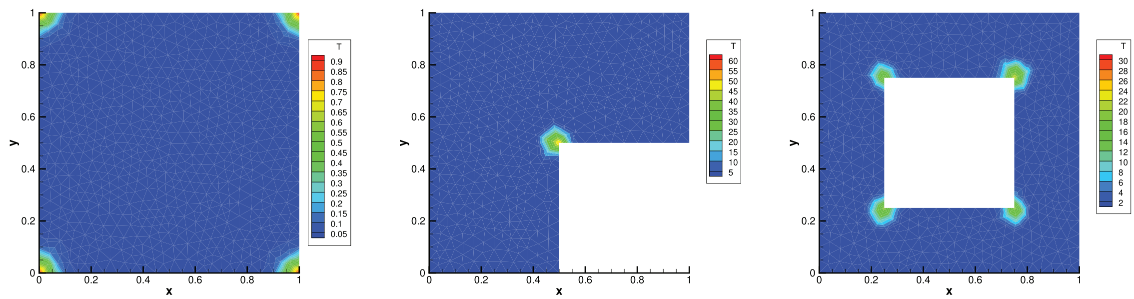


FIGURE 3. The local indicators $\eta_h(\boldsymbol{u}_h, T)$.

5.2. *A posteriori* error estimates

Figure 2 shows global error estimators $\eta_h^2(\boldsymbol{u}_h, \Omega)$ and the relative errors of some simple eigenvalues for the three domains. It can be observed that the relative errors and the estimators have the same convergence rates, which confirms the upper bound estimate for the simple eigenvalues. Figure 3 shows the distribution of the local indicators $\eta_h(\boldsymbol{u}_h, T)$. The estimators are large at corners and catch the singularities effectively.

6. Conclusion

We proved *a priori* and robust *a posteriori* error estimates for the $H(\mathrm{curl}^2)$-conforming finite element method when solving the quad-curl eigenvalue problem. Due to a new decomposition of the function in $H(\mathrm{curl}^2;\Omega)$, the theory assumes no extra regularity of the eigenfunctions. The *a posteriori* error estimator is essential for the adaptive finite element method. It can also be applied to test spurious eigenvalues.

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References

[1] I. Babuška and J. Osborn, Finite element-Galerkin approximation of the eigenvalues and eigenvectors of selfadjoint problems. *Math. Comput.* **52** (1989) 275–297.

[2] I. Babuška and J. Osborn, Eigenvalue Problems. Elsevier (1991) 641–787.

[3] R. Beck, R. Hiptmair, R. Hoppe and B. Wohlmuth, Residual based a posteriori error estimators for eddy current computation. *ESAIM: M2AN* **34** (2000) 159–182.

[4] D. Boffi, L. Gastaldi, R. Rodríguez and I. Šebestová, Residual-based a posteriori error estimation for the Maxwell's eigenvalue problem. *IMA J. Numer. Anal.* **37** (2017) 1710–1732.

[5] D. Boffi, L. Gastaldi, R. Rodríguez and I. Šebestová, A posteriori error estimates for Maxwell's eigenvalue problem. *J. Sci. Comput.* **78** (2019) 1250–1271.

[6] S.C. Brenner, J. Sun and L. Sung, Hodge decomposition methods for a quad-curl problem on planar domains. *J. Comput. Sci.* **73** (2017) 495–513.

[7] S.C. Brenner, J. Cui and L. Sung, Multigrid methods based on Hodge decomposition for a quad-curl problem. *Comput. Methods Appl. Math.* **19** (2019) 215–232.

[8] F. Cakoni and H. Haddar, A variational approach for the solution of the electromagnetic interior transmission problem for anisotropic media. *Inverse Prob. Imaging* **1** (2017) 443–456.

[9] F. Cakoni, D. Colton, P. Monk and J. Sun, The inverse electromagnetic scattering problem for anisotropic media. *Inverse Prob.* **26** (2010) 074004.

[10] S. Cao, L. Chen and X. Huang, Error analysis of a decoupled finite element method for quad-curl problems. *J. Sci. Comput.* **90** (2022) 1–25.

[11] L. Chen, Sobolev Spaces and Elliptic Equations. *Course Notes* (2016).

[12] H. Chen, J. Li, W. Qiu and C. Wang, A mixed finite element scheme for quad-curl source and eigenvalue problems. *Commun. Comput. Phys.* **29** (2021) 1125–1151.

[13] G. Chen, W. Qiu and L. Xu, Analysis of an interior penalty DG method for the quad-curl problem. *IMA J. Numer. Anal.* **41** (2021) 2990–3023.

[14] S. Christiansen and K. Hu, Generalized finite element systems for smooth differential forms and stokes' problem. *Numer. Math.* **140** (2018) 327–371.

[15] S. Cochez-Dhondt and S. Nicaise, Robust a posteriori error estimation for the Maxwell equations. *Comput. Methods Appl. Mech. Eng.* **196** (2007) 2583–2595.

[16] M. Costabel and A. McIntosh, On bogovskiĭ and regularized poincaré integral operators for de rham complexes on lipschitz domains. *Math. Z.* **265** (2010) 297–320.

[17] X. Dai, J. Xu and A. Zhou, Convergence and optimal complexity of adaptive finite element eigenvalue computations. *Numer. Math.* **110** (2008) 313–355.

[18] M. Dauge, Elliptic Boundary Value Problems on Corner Domains: Smoothness and Asymptotics of Solutions. Vol. 1341. Springer (2006).

[19] A.-S.B.-B. Dhia, C. Hazard and S. Lohrengel, A singular field method for the solution of Maxwell's equations in polyhedral domains. *SIAM J. Appl. Math.* **59** (1999) 2028–2044.

[20] V. Girault and P. Raviart, Finite Element Methods for Navier–Stokes Equations: Theory and Algorithms. Vol. 5. Springer Science & Business Media (2012).

[21] J. Han, Shifted inverse iteration based multigrid methods for the quad-curl eigenvalue problem. *Appl. Math. Comput.* **367** (2020) 124770.

[22] Q. Hong, J. Hu, S. Shu and J. Xu, A discontinuous Galerkin method for the fourth-order curl problem. *J. Comput. Math.* **30** (2012) 565–578.

[23] K. Hu, Q. Zhang and Z. Zhang, Simple curl-curl-conforming finite elements in two dimensions. *SIAM J. Sci. Comput.* **42** (2020) A3859–A3877.

[24] K. Hu, Q. Zhang and Z. Zhang, A family of finite element stokes complexes in three dimensions. *SIAM J. Numer. Anal.* **60** (2022) 222–243.

[25] P. Monk, A posteriori error indicators for Maxwell's equations. *J. Comput. Appl. Math.* **100** (1998) 173–190.

[26] P. Monk, Finite Element Methods for Maxwell's Equations. Oxford University Press (2003).
[27] P. Monk and J. Sun, Finite element methods for Maxwell's transmission eigenvalues. *SIAM J. Sci. Comput.* **34** (2012) B247–B264.
[28] S. Nicaise, Singularities of the quad-curl problem. *J. Differ. Equ.* **264** (2018) 5025–5069.
[29] J. Osborn, Spectral approximation for compact operators. *Math. Comput.* **29** (1975) 712–725.
[30] J. Schöberl, A posteriori error estimates for Maxwell equations. *Math. Comput.* **77** (2008) 633–649.
[31] L. Scott and S. Zhang, Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comput.* **54** (1990) 483–493.
[32] J. Sun, Iterative methods for transmission eigenvalues. *SIAM J. Numer. Anal.* **49** (2011) 1860–1874.
[33] J. Sun, A mixed FEM for the quad-curl eigenvalue problem. *Numer. Math.* **132** (2016) 185–200.
[34] J. Sun and A. Zhou, Finite Element Methods for Eigenvalue Problems. Chapman and Hall/CRC, Boca Raton, FL (2016).
[35] Z. Sun, J. Cui, F. Gao and C. Wang, Multigrid methods for a quad-curl problem based on C^0 interior penalty method. *Comput. Math. App.* **76** (2018) 2192–2211.
[36] J. Sun, Q. Zhang and Z. Zhang, A curl-conforming weak Galerkin method for the quad-curl problem. *BIT Numer. Math.* **59** (2019) 1093–1114.
[37] R. Verfürth, A review of a posteriori error estimation techniques for elasticity problems. *Comput. Methods Appl. Mech. Eng.* **176** (1999) 419–440.
[38] C. Wang, Z. Sun and J. Cui, A new error analysis of a mixed finite element method for the quad-curl problem. *Appl. Math. Comput.* **349** (2019) 23–38.
[39] L. Wang, H. Li and Z. Zhang, H(curl^2)-conforming spectral element method for quad-curl problems. *Comput. Methods Appl. Math.* **21** (2021) 661–681.
[40] L. Wang, W. Shan, H. Li and Z. Zhang, H(curl^2)-conforming quadrilateral spectral element method for quad-curl problems. *Math. Models Methods Appl. Sci.* **31** (2021) 1951–1986.
[41] S. Zhang, Mixed schemes for quad-curl equations. *ESAIM: M2AN* **52** (2018) 147–161.
[42] S. Zhang, Regular decomposition and a framework of order reduced methods for fourth order problems. *Numer. Math.* **138** (2018) 241–271.
[43] Q. Zhang, *New conforming finite elements based on the de rham complexes for some fourth-order problems*, Ph.D. dissertation (2021).
[44] Q. Zhang, L. Wang and Z. Zhang, H(curl^2)-conforming finite elements in 2 dimensions and applications to the quad-curl problem. *SIAM J. Sci. Comput.* **41** (2019) A1527–A1547.
[45] B. Zheng and J. Xu, A nonconforming finite element method for fourth order curl equations in $\mathbb{R}^3$. *Math. Comput.* **80** (2011) 1871–1886.