

DISCONTINUOUS GALERKIN AND C^0 -IP FINITE ELEMENT APPROXIMATION OF PERIODIC HAMILTON–JACOBI–BELLMAN–ISAACS PROBLEMS WITH APPLICATION TO NUMERICAL HOMOGENIZATION

ELLYA L. KAWECKI¹  AND TIMO SPREKELER^{2,*} 

Abstract. In the first part of the paper, we study the discontinuous Galerkin (DG) and C^0 interior penalty (C^0 -IP) finite element approximation of the periodic strong solution to the fully nonlinear second-order Hamilton–Jacobi–Bellman–Isaacs (HJBI) equation with coefficients satisfying the Cordes condition. We prove well-posedness and perform abstract *a posteriori* and *a priori* analyses which apply to a wide family of numerical schemes. These periodic problems arise as the corrector problems in the homogenization of HJBI equations. The second part of the paper focuses on the numerical approximation to the effective Hamiltonian of ergodic HJBI operators *via* DG/ C^0 -IP finite element approximations to approximate corrector problems. Finally, we provide numerical experiments demonstrating the performance of the numerical schemes.

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1. INTRODUCTION

In the first part of this paper we study the periodic boundary value problem for the fully nonlinear second-order Hamilton–Jacobi–Bellman–Isaacs (HJBI) equation

$$\begin{cases} \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \{-A^{\alpha\beta} : \nabla^2 u - b^{\alpha\beta} \cdot \nabla u + c^{\alpha\beta} u - f^{\alpha\beta}\} = 0 & \text{in } Y, \\ u \text{ is } Y\text{-periodic,} \end{cases} \quad (1.1)$$

where \mathcal{A} and \mathcal{B} are compact metric spaces, and $Y := (0, 1)^n \subset \mathbb{R}^n$ denotes the unit cell in dimension $n \geq 2$. Here, we use the notation

$$\varphi^{\alpha\beta} := \varphi(\cdot, \alpha, \beta), \quad \varphi \in \{A, b, c, f\},$$

and assume that the functions

$$A = (a_{ij}) : \mathbb{R}^n \times \mathcal{A} \times \mathcal{B} \rightarrow \mathbb{R}_{\text{sym}}^{n \times n}, \quad b = (b_i) : \mathbb{R}^n \times \mathcal{A} \times \mathcal{B} \rightarrow \mathbb{R}^n, \quad c, f : \mathbb{R}^n \times \mathcal{A} \times \mathcal{B} \rightarrow \mathbb{R}$$

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¹ Haas F1 Team, Vehicle Sciences, Overthorpe Rd, Banbury OX16 4PN, UK.

² Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076, Singapore.

*Corresponding author: timo.sprekeler@nus.edu.sg

are uniformly continuous and Y -periodic in their first argument $y \in \mathbb{R}^n$. Further, we assume that A is uniformly elliptic (see (2.2)), that $\inf_{\mathbb{R}^n \times \mathcal{A} \times \mathcal{B}} c > 0$, and that the coefficients A, b, c satisfy the Cordes condition

$$|A|^2 + \frac{|b|^2}{2\lambda} + \frac{c^2}{\lambda^2} \leq \frac{1}{n + \delta} \left(\operatorname{tr}(A) + \frac{c}{\lambda} \right)^2$$

in $\mathbb{R}^n \times \mathcal{A} \times \mathcal{B}$ for some constants $\lambda > 0$ and $\delta \in (0, 1)$. These assumptions guarantee the existence and uniqueness of a periodic strong solution $u \in H_{\text{per}}^2(Y)$ to the HJBI problem (1.1); see Section 2.2.

The goal of the first part of the paper is the construction of discontinuous Galerkin (DG) and C^0 interior penalty (C^0 -IP) finite element schemes for the periodic HJBI problem (1.1) and their rigorous *a posteriori* and *a priori* error analysis; see Section 2.

The fully nonlinear HJBI equation is a very general elliptic PDE arising in many contexts, such as stochastic differential games and optimal control problems. In the case that one of the metric spaces \mathcal{A}, \mathcal{B} is a singleton set, the HJBI equation becomes the HJB equation arising in stochastic optimal control theory, with applications in finance, engineering, and renewable energies. Interestingly, the HJBI equation is capable of capturing other famous nonlinear PDEs, such as the fully nonlinear Monge–Ampère (MA) equation arising in illumination optics, optimal transport (see [38]), and differential geometry. The MA equation is conditionally elliptic, with classical examples exhibiting a lack of uniqueness. The HJBI formulation of the MA equation is uniquely solvable and has been used in Feng and Jensen [19], Brenner and Kawecki [9] to overcome this lack of uniqueness.

The HJBI problem is well understood in the framework of viscosity solutions (see *e.g.*, [15, 23, 29]), and there have been several numerical advances based on methods that enjoy a numerical analogue of the comparison principle used in the theory of viscosity solutions. Such methods include finite difference and semi-Lagrangian schemes such as Feng and Jensen [19], and also integro-differential finite element methods; see Camilli and Jakobsen [12], Salgado and Zhang [49]. However, enforcing a discrete maximum principle can be restrictive in practice and can lead to the requirement for large, or even unbounded stencils.

There is not a lot of work on finite element methods for periodic HJB/HJBI problems in the numerical analysis literature, and we refer to Gallistl *et al.* [26] for a mixed finite element scheme for periodic HJB problems. In recent years, there have been several advances in finite element methods for the Dirichlet problem based on the theory of the concept of strong solutions to HJBI equations. Such methods are typically more flexible than the finite difference method and allow one to capture complex geometries and to obtain higher order convergence rates. The existence and uniqueness of strong solutions to linear nondivergence-form PDEs (arising in the linearization of HJBI problems) and to the HJB equation was established in Smears and Süli [51–53], along with the well-posedness of optimal *hp*-finite element methods. These methods involved additional stabilizing forms that enforced a numerical analogue of the Miranda–Talenti estimate which is key to the well-posedness of the strong PDE. Other primal finite element methods that tackle the HJB problem are Neilan and Wu [44] and Brenner and Kawecki [9]. Here the authors use a discrete analogue of the Miranda–Talenti estimate, based on the theory of $H^2(\Omega) \cap H_0^1(\Omega)$ enrichment operators (see [9, 36, 37, 44]), to prove strong monotonicity of the scheme without the need for an additional stabilizing bilinear form.

Following on from these approaches, these ideas have been extended from the HJB problem to the HJBI problem in Kawecki and Smears [36], and have been analyzed under a general framework that incorporates *a priori* and *a posteriori* error analysis for a wide family of finite element methods that encompasses the aforementioned schemes [9, 44, 51, 52]. In Kawecki and Smears [37], the convergence of a family of adaptive finite element schemes for HJBI problems was proven. More recently, a virtual element method for the approximation of linear nondivergence-form PDEs and HJBI problems has been proposed and analyzed in Kawecki and Pryer [35].

Alongside this, we refer the reader to the papers [11, 12, 25, 26, 30, 31] by various authors for finite element approaches allowing the use of H^1 -conforming finite elements for HJB problems.

We refer to Kawecki [33] for finite element methods for linear nondivergence-form elliptic PDEs on curved domains, and to Gallistl [24], Kawecki [34] for those with oblique boundary conditions. For a survey on recent developments of numerical methods for fully nonlinear PDEs see Feng *et al.* [20] and Neilan *et al.* [45].

Periodic HJBI problems of the form (1.1) arise naturally as corrector problems in the periodic homogenization of HJBI equations, which is the focus of the second part of this paper. More precisely, we are interested in the numerical approximation of the effective Hamiltonian corresponding to HJBI operators $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$ of the form

$$F(x, y, p, R) := \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \{-A^{\alpha\beta}(x, y) : R - b^{\alpha\beta}(x, y) \cdot p - f^{\alpha\beta}(x, y)\}$$

with sufficiently regular coefficients which are Y -periodic in $y \in \mathbb{R}^n$.

To any fixed point $(x, p, R) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\text{sym}}^{n \times n}$ we associate the *approximate correctors* $\{v^\sigma(\cdot; x, p, R)\}_{\sigma>0} \subset C(\mathbb{R}^n)$, defined as the unique viscosity solutions to the *cell σ -problem* (see [5]) for parameters $\sigma > 0$, that is,

$$\begin{cases} \sigma v^\sigma(y; x, p, R) + F(x, y, p, R + \nabla_y^2 v^\sigma(y; x, p, R)) = 0 & \text{for } y \in Y, \\ y \mapsto v^\sigma(y; x, p, R) \text{ is } Y\text{-periodic.} \end{cases}$$

The operator F is called *ergodic* (in the y -variable) at the point (x, p, R) if there exists a constant $H(x, p, R)$ such that

$$-\sigma v^\sigma(\cdot; x, p, R) \xrightarrow{\sigma \searrow 0} H(x, p, R) \quad \text{uniformly,}$$

and we say F is ergodic if F is ergodic at every point (x, p, R) and call the function

$$H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}, \quad (x, p, R) \mapsto H(x, p, R)$$

the effective Hamiltonian corresponding to F ; see Alvarez and Bardi [5].

The cell σ -problem is an approximation to the *true cell problem* familiar to the reader coming from periodic homogenization (see [16, 17]), that is, for fixed (x, p, R) there exists at most one constant $\mu \in \mathbb{R}$ such that there exists a viscosity solution $v(\cdot; x, p, R) \in C(\mathbb{R}^n)$, a *corrector*, to the problem

$$\begin{cases} F(x, y, p, R + \nabla^2 v(y; x, p, R)) = \mu & \text{for } y \in Y, \\ y \mapsto v(y; x, p, R) \text{ is } Y\text{-periodic,} \end{cases} \quad (1.2)$$

and when such a μ exists, F is ergodic at (x, p, R) and we have that $H(x, p, R) = \mu$. However, to a given ergodic operator there may be no corrector in general, and we refer to Alvarez and Bardi [2–5], Alvarez *et al.* [6], and Arisawa and Lions [7] for a detailed overview.

The goal of this second part of the paper is the construction of a numerical scheme for the approximation of the effective Hamiltonian to ergodic HJBI operators which is based on discontinuous Galerkin or C^0 -IP finite element approximations to the approximate correctors; see Section 3.

The literature on numerical effective Hamiltonians to second-order HJB and HJBI operators is quite sparse. For the numerical homogenization of linear equations in nondivergence-form we refer the reader to Capdeboscq *et al.* [14] (see also [54]). The numerical homogenization of HJB equations *via* a mixed finite element approximation of the approximate correctors has been proposed and analyzed in Gallistl *et al.* [26]. A finite difference approach for numerical effective Hamiltonians to HJB operators can be found in Camilli and Marchi [13], and some exact formulas and numerical simulations for effective Hamiltonians to certain types of HJB operators are available in Finlay and Oberman [21, 22].

It seems that there are no finite element schemes for the numerical approximation of effective Hamiltonians to HJBI operators in the current literature. Let us note that there is significantly more work (see *e.g.*, [1, 18, 27, 28, 41, 46–48]) on numerical effective Hamiltonians to first-order Hamilton–Jacobi and Hamilton–Jacobi–Isaacs equations.

This paper is organized as follows: Section 2 is focused on the DG and C^0 -IP finite element approximation to the periodic HJBI problem (1.1). After proving existence and uniqueness of a periodic strong solution in Section 2.2, we discuss discretization and notation aspects in Section 2.3. We perform an *a posteriori* analysis independent of the choice of numerical scheme in Section 2.4, which is based on periodic enrichment and a mixed *a posteriori* bound. In Section 2.5, we perform an *a priori* error analysis for an abstract numerical scheme under natural assumptions, and present a family of numerical schemes in Section 2.5.2.

Section 3 is focused on the numerical approximation of the effective Hamiltonian to ergodic HJBI operators. We recall the definition of ergodicity and introduce the effective Hamiltonian in Section 3.1. Thereafter, in Sections 3.2 and 3.3, we present the approximation scheme for the effective Hamiltonian based on DG/ C^0 -IP finite element approximations to the cell σ -problem.

In Section 4, we present numerical experiments demonstrating the performance of the numerical scheme for a periodic HJBI problem (Sect. 4.1) and the approximation of the effective Hamiltonian to an ergodic HJBI operator (Sect. 4.2).

2. DISCONTINUOUS GALERKIN AND C^0 -IP FEM FOR PERIODIC HJBI PROBLEMS

2.1. Setting

Throughout this work, we work in dimension $n \in \{2, 3\}$ and write $Y := (0, 1)^n$ to denote the unit cell in \mathbb{R}^n . We are interested in Hamilton–Jacobi–Bellman–Isaacs (HJBI) equations posed in a periodic setting, *i.e.*, problems of the form

$$\begin{cases} F[u] := \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \{-A^{\alpha\beta} : \nabla^2 u - b^{\alpha\beta} \cdot \nabla u + c^{\alpha\beta} u - f^{\alpha\beta}\} = 0 & \text{in } Y, \\ u \text{ is } Y\text{-periodic}, \end{cases} \quad (2.1)$$

with \mathcal{A} and \mathcal{B} denoting compact metric spaces, and uniformly continuous functions

$$A = (a_{ij}) : \mathbb{R}^n \times \mathcal{A} \times \mathcal{B} \rightarrow \mathbb{R}_{\text{sym}}^{n \times n}, \quad b = (b_i) : \mathbb{R}^n \times \mathcal{A} \times \mathcal{B} \rightarrow \mathbb{R}^n, \quad c, f : \mathbb{R}^n \times \mathcal{A} \times \mathcal{B} \rightarrow \mathbb{R}$$

satisfying the assumptions specified below. Here, we use the notation

$$\varphi^{\alpha\beta}(y) := \varphi(y, \alpha, \beta), \quad y \in \mathbb{R}^n, \quad (\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$$

for scalar, vector-valued or matrix-valued functions $\varphi \in C(\mathbb{R}^n \times \mathcal{A} \times \mathcal{B}; \mathcal{R})$ with $\mathcal{R} \in \{\mathbb{R}, \mathbb{R}^n, \mathbb{R}_{\text{sym}}^{n \times n}\}$.

We assume that $A^{\alpha\beta}, b^{\alpha\beta}, c^{\alpha\beta}, f^{\alpha\beta}$ are Y -periodic in $y \in \mathbb{R}^n$ and that

$$\inf_{\mathbb{R}^n \times \mathcal{A} \times \mathcal{B}} c > 0.$$

We further require A to be uniformly elliptic, *i.e.*,

$$\exists \zeta_1, \zeta_2 > 0 : \quad \zeta_1 |\xi|^2 \leq A(y, \alpha, \beta) \xi \cdot \xi \leq \zeta_2 |\xi|^2 \quad \forall y, \xi \in \mathbb{R}^n, (\alpha, \beta) \in \mathcal{A} \times \mathcal{B}, \quad (2.2)$$

and that the coefficients satisfy the Cordes condition (see [52]), *i.e.*, that there holds

$$|A|^2 + \frac{|b|^2}{2\lambda} + \frac{c^2}{\lambda^2} \leq \frac{1}{n + \delta} \left(\text{tr}(A) + \frac{c}{\lambda} \right)^2 \quad (2.3)$$

in $\mathbb{R}^n \times \mathcal{A} \times \mathcal{B}$ for some constants $\delta \in (0, 1)$ and $\lambda > 0$ (note $|M| := \sqrt{M : M}$ for $M \in \mathbb{R}^{n \times n}$).

2.2. Well-posedness

In this section, we show that the periodic HJBI problem (2.1) is well-posed in the sense that there exists a unique periodic strong solution, *i.e.*, a unique function $u \in H_{\text{per}}^2(Y)$ satisfying $F[u] = 0$ almost everywhere in Y . Recall that the space $H_{\text{per}}^2(Y) \subset H^2(Y)$ is defined as the closure of $C_{\text{per}}^\infty(Y) := \{v|_Y : v \in C^\infty(\mathbb{R}^n) \text{ is } Y\text{-periodic}\}$ with respect to the H^2 -norm.

2.2.1. The renormalized problem

Let us introduce the function $\gamma = \gamma(y, \alpha, \beta) \in C(\mathbb{R}^n \times \mathcal{A} \times \mathcal{B})$ defined by

$$\gamma := \left(|A|^2 + \frac{|b|^2}{2\lambda} + \frac{c^2}{\lambda^2} \right)^{-1} \left(\operatorname{tr}(A) + \frac{c}{\lambda} \right) \quad (2.4)$$

and note that, by the assumptions on the coefficients A, b, c from Section 2.1, we have

$$\inf_{\mathbb{R}^n \times \mathcal{A} \times \mathcal{B}} \gamma > 0. \quad (2.5)$$

We then consider the renormalized HJBI problem

$$\begin{cases} F_\gamma[u] := \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \{ \gamma^{\alpha\beta} (-A^{\alpha\beta} : \nabla^2 u - b^{\alpha\beta} \cdot \nabla u + c^{\alpha\beta} u - f^{\alpha\beta}) \} = 0 & \text{in } Y, \\ u \text{ is } Y\text{-periodic.} \end{cases} \quad (2.6)$$

It is easily checked that the renormalized problem (2.6) is equivalent to the original problem (2.1) in the sense that they have the same set of periodic strong solutions. More precisely, we can characterize strong solutions to (2.1) as follows:

Remark 2.1. For $u \in H_{\text{per}}^2(Y)$, the following assertions are equivalent:

- (i) $F[u] = 0$ a.e. in Y , i.e., u is a periodic strong solution to the HJBI problem (2.1).
- (ii) $F_\gamma[u] = 0$ a.e. in Y , i.e., u is a periodic strong solution to the renormalized problem (2.6).
- (iii) There holds

$$\int_Y F_\gamma[u] L_\lambda v = 0 \quad \forall v \in H_{\text{per}}^2(Y),$$

where $L_\lambda v := \lambda v - \Delta v$ for functions $v \in H_{\text{per}}^2(Y)$.

Indeed, the equivalence (i) \Leftrightarrow (ii) follows from (2.5) and the compactness of the metric spaces \mathcal{A} and \mathcal{B} (see also [36], Lem. 2.2), and (ii) \Leftrightarrow (iii) is a consequence of the surjectivity of the linear differential operator

$$L_\lambda : H_{\text{per}}^2(Y) \rightarrow L^2(Y), \quad L_\lambda v := \lambda v - \Delta v. \quad (2.7)$$

Note that the surjectivity of the operator L_λ from (2.7) enables us to obtain a variational formulation as for the Dirichlet setting.

2.2.2. Consequences of the Cordes condition

We point out a crucial estimate for the nonlinear operator F_γ . This is a direct consequence of the Cordes condition (2.3) and can be found in [36]. A short proof is provided for demonstrating how the Cordes condition comes into play.

Lemma 2.2. Let $\omega \subset \mathbb{R}^n$ be a bounded open set. For any $u_1, u_2 \in H^2(\omega)$, writing $\delta_u := u_1 - u_2$, we have that

$$|F_\gamma[u_1] - F_\gamma[u_2] - L_\lambda \delta_u| \leq \sqrt{1 - \delta} \sqrt{|\nabla^2 \delta_u|^2 + 2\lambda |\nabla \delta_u|^2 + \lambda^2 \delta_u^2} \quad (2.8)$$

almost everywhere in ω .

Proof. Let $u_1, u_2 \in H^2(\omega)$ and set $\delta_u := u_1 - u_2$. Note that for any bounded sets $\{x^{\alpha\beta}\}_{(\alpha,\beta) \in \mathcal{A} \times \mathcal{B}} \subset \mathbb{R}$ and $\{y^{\alpha\beta}\}_{(\alpha,\beta) \in \mathcal{A} \times \mathcal{B}} \subset \mathbb{R}$ we have that

$$\left| \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} x^{\alpha\beta} - \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} y^{\alpha\beta} \right| \leq \sup_{(\alpha,\beta) \in \mathcal{A} \times \mathcal{B}} |x^{\alpha\beta} - y^{\alpha\beta}|.$$

This yields

$$\begin{aligned}
|F_\gamma[u_1] - F_\gamma[u_2] - L_\lambda \delta_u|^2 &\leq \sup_{(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}} \left| \gamma^{\alpha\beta} (-A^{\alpha\beta} : \nabla^2 \delta_u - b^{\alpha\beta} \cdot \nabla \delta_u + c^{\alpha\beta} \delta_u) + \Delta \delta_u - \lambda \delta_u \right|^2 \\
&\leq \sup_{(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}} \left\{ \left| -\gamma^{\alpha\beta} A^{\alpha\beta} + I \right|^2 + \frac{|\gamma^{\alpha\beta} b^{\alpha\beta}|^2}{2\lambda} + \frac{|\gamma^{\alpha\beta} c^{\alpha\beta} - \lambda|^2}{\lambda^2} \right\} \\
&\quad \times \left(|\nabla^2 \delta_u|^2 + 2\lambda |\nabla \delta_u|^2 + \lambda^2 \delta_u^2 \right) \\
&= \sup_{(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}} \left\{ n + 1 - \frac{\left(\text{tr}(A^{\alpha\beta}) + \frac{c^{\alpha\beta}}{\lambda} \right)^2}{|A^{\alpha\beta}|^2 + \frac{|b^{\alpha\beta}|^2}{2\lambda} + \frac{|c^{\alpha\beta}|^2}{\lambda^2}} \right\} \left(|\nabla^2 \delta_u|^2 + 2\lambda |\nabla \delta_u|^2 + \lambda^2 \delta_u^2 \right) \\
&\leq (1 - \delta) \left(|\nabla^2 \delta_u|^2 + 2\lambda |\nabla \delta_u|^2 + \lambda^2 \delta_u^2 \right)
\end{aligned}$$

almost everywhere in ω , where we have used the Cauchy–Schwarz inequality, simple calculation and the Cordes condition (2.3). \square

Observe that by the triangle and Cauchy–Schwarz inequalities, we can eliminate the term $L_\lambda \delta_u$ from the left-hand side of (2.8). We thus find that, in the situation of Lemma 2.2, we have the Lipschitz-type estimate

$$|F_\gamma[u_1] - F_\gamma[u_2]| \leq \left(\sqrt{1 - \delta} + \sqrt{n + 1} \right) \sqrt{|\nabla^2 \delta_u|^2 + 2\lambda |\nabla \delta_u|^2 + \lambda^2 \delta_u^2} \quad (2.9)$$

almost everywhere in ω . Let us note that this Lipschitz bound holds for any $u_1, u_2 \in H^2(\omega)$ with $\omega \subset \mathbb{R}^n$ being some bounded open set, and is not assuming any periodicity.

2.2.3. Existence and uniqueness of solutions

We are now in a position to prove the existence and uniqueness of periodic strong solutions to the HJBI problem (2.1). In view of Remark 2.1, let us define

$$B : H_{\text{per}}^2(Y) \times H_{\text{per}}^2(Y) \rightarrow \mathbb{R}, \quad B(u, v) := \int_Y F_\gamma[u] L_\lambda v.$$

We can now proceed as in [36] in showing that the Browder–Minty theorem applies and we obtain the following theorem:

Theorem 2.3 (Well-posedness). *In the situation of Section 2.1, there exists a unique periodic strong solution $u \in H_{\text{per}}^2(Y)$ to the HJBI problem (2.1).*

Proof. Note that it is enough to show that B satisfies the Lipschitz property

$$|B(u_1, v) - B(u_2, v)| \lesssim \|u_1 - u_2\|_{H^2(Y)} \|v\|_{H^2(Y)} \quad \forall u_1, u_2, v \in H_{\text{per}}^2(Y), \quad (2.10)$$

and strong monotonicity, i.e.,

$$\|u_1 - u_2\|_{H^2(Y)}^2 \lesssim B(u_1, u_1 - u_2) - B(u_2, u_1 - u_2) \quad \forall u_1, u_2 \in H_{\text{per}}^2(Y). \quad (2.11)$$

The Browder–Minty theorem then yields that there exists a unique $u \in H_{\text{per}}^2(Y)$ such that

$$B(u, v) = 0 \quad \forall v \in H_{\text{per}}^2(Y),$$

which proves the theorem in view of Remark 2.1.

Before we show (2.10) and (2.11), let us note that integration by parts and a density argument yields $\|\Delta v\|_{L^2(Y)} = \|\nabla^2 v\|_{L^2(Y)}$ for any $v \in H_{\text{per}}^2(Y)$, and hence, using integration by parts again, we have

$$\|L_\lambda v\|_{L^2(Y)}^2 = \|\nabla^2 v\|_{L^2(Y)}^2 + 2\lambda \|\nabla v\|_{L^2(Y)}^2 + \lambda^2 \|v\|_{L^2(Y)}^2 \geq C_\lambda \|v\|_{H^2(Y)}^2 \quad \forall v \in H_{\text{per}}^2(Y). \quad (2.12)$$

The Lipschitz property (2.10) now immediately follows from (2.9) and it remains to show strong monotonicity. To this end, let $u_1, u_2 \in H_{\text{per}}^2(Y)$ and write $\delta_u := u_1 - u_2$. Using Lemma 2.2, we find

$$B(u_1, \delta_u) - B(u_2, \delta_u) = \|L_\lambda \delta_u\|_{L^2(Y)}^2 + \int_Y (F_\gamma[u_1] - F_\gamma[u_2] - L_\lambda \delta_u) L_\lambda \delta_u \geq (1 - \sqrt{1 - \delta}) \|L_\lambda \delta_u\|_{L^2(Y)}^2$$

and hence, by (2.12), there holds (2.11) and the claim is proved. \square

Let us note that the identity $\|\Delta v\|_{L^2(Y)} = \|\nabla^2 v\|_{L^2(Y)}$ for any $v \in H_{\text{per}}^2(Y)$ takes the role of a periodic counterpart to the Miranda–Talenti estimate used in the Dirichlet framework; see the proof of Theorem 2 from [52]. We can now obtain a bound on a H^2 -type norm of the unique strong solution in terms of the data.

Remark 2.4. For the unique periodic strong solution $u \in H_{\text{per}}^2(Y)$ to the HJBI problem (2.1), we have the bound

$$\|L_\lambda u\|_{L^2(Y)} = \sqrt{\|\nabla^2 u\|_{L^2(Y)}^2 + 2\lambda \|\nabla u\|_{L^2(Y)}^2 + \lambda^2 \|u\|_{L^2(Y)}^2} \leq \frac{\|F_\gamma[0]\|_{L^2(Y)}}{1 - \sqrt{1 - \delta}}.$$

Proof. Note that we have already obtained the first equality (see (2.12)). We use Lemma 2.2 and the solution property $F_\gamma[u] = 0$ to find

$$(1 - \sqrt{1 - \delta}) \|L_\lambda u\|_{L^2(Y)}^2 \leq \int_Y (F_\gamma[u] - F_\gamma[0]) L_\lambda u = - \int_Y F_\gamma[0] L_\lambda u.$$

We conclude the proof by using Hölder's inequality to obtain

$$\|L_\lambda u\|_{L^2(Y)}^2 \leq \frac{1}{1 - \sqrt{1 - \delta}} \left| \int_Y F_\gamma[0] L_\lambda u \right| \leq \frac{\|F_\gamma[0]\|_{L^2(Y)}}{1 - \sqrt{1 - \delta}} \|L_\lambda u\|_{L^2(Y)},$$

which yields the desired bound. \square

2.3. Discretization

This section is devoted to discretization aspects. We introduce DG and C^0 -IP finite element spaces V_T^0 and V_T^1 for an appropriate partition \mathcal{T} of the computational domain, and define jump and average operators.

2.3.1. The partition \mathcal{T}

We consider a finite conforming partition \mathcal{T} of the closed unit cell \bar{Y} consisting of closed simplices that can be periodically extended in a Y -periodic fashion to \mathbb{R}^n , *i.e.*, we require the discretization to be consistent with the identification of opposite faces by periodicity. We introduce the following mathematical objects associated with the partition \mathcal{T} :

- (i) Set of faces \mathcal{F} and associated unit normal n_F :

We let $\mathcal{F} := \mathcal{F}^I \cup \mathcal{F}^{\text{BP}}$ denote the set of $(n - 1)$ -dimensional faces, where \mathcal{F}^I is the set of all interior faces of \mathcal{T} , and \mathcal{F}^{BP} the set of all boundary face-pairs of \mathcal{T} , *i.e.*, the boundary faces upon a periodic identification of opposite faces. For each face $F \in \mathcal{F}$, we associate a fixed choice of unit normal n_F , where we often only write n for simplicity; see Figure 1.

(ii) Shape-regularity parameter $\theta_{\mathcal{T}}$ and mesh-size function $h_{\mathcal{T}}$:

We let $\theta_{\mathcal{T}} := \max\{\rho_K^{-1} \text{diam}(K) : K \in \mathcal{T}\}$ with ρ_K being the diameter of the largest ball that can be inscribed in the element $K \in \mathcal{T}$. We further introduce $h_{\mathcal{T}} : \bar{Y} \rightarrow \mathbb{R}$ defined via $h_{\mathcal{T}}|_{\text{int}(K)} := h_K := (\mathcal{L}^n(K))^{\frac{1}{n}}$ for all $K \in \mathcal{T}$ and $h_{\mathcal{T}}|_F := h_F := (\mathcal{H}^{n-1}(F))^{\frac{1}{n-1}}$ for all $F \in \mathcal{F}$.

Let us note that the concept of boundary face-pairs was introduced in [55] in the context of discontinuous Galerkin methods for linear elliptic periodic boundary value problems.

2.3.2. Finite element spaces $V_{\mathcal{T}}^s$

For fixed $P \geq 2$, we define the discontinuous Galerkin finite element space $V_{\mathcal{T}}^0$ and the C^0 -IP finite element space $V_{\mathcal{T}}^1$ by

$$V_{\mathcal{T}}^0 := \{v_{\mathcal{T}} \in L^2(Y) : v_{\mathcal{T}}|_K \in \mathbb{P}_P \ \forall K \in \mathcal{T}\} \quad \text{and} \quad V_{\mathcal{T}}^1 := V_{\mathcal{T}}^0 \cap H_{\text{per}}^1(Y),$$

where \mathbb{P}_P denotes the space of polynomials of degree at most P .

Let us make some comments about the derivatives of functions in the finite element spaces. For a function $v \in V_{\mathcal{T}}^0$, we define $\nabla v \in L^1(Y; \mathbb{R}^n)$ to be the piecewise gradient and $\nabla^2 v \in L^1(Y; \mathbb{R}^{n \times n})$ to be the piecewise Hessian over the elements of the partition. We then define $\Delta v := \text{tr}(\nabla^2 v) \in L^1(Y)$.

We equip the finite element spaces $V_{\mathcal{T}}^s$, $s \in \{0, 1\}$, with the norm

$$\|v_{\mathcal{T}}\|_{\mathcal{T}, \lambda}^2 := \int_Y \left(|\nabla^2 v_{\mathcal{T}}|^2 + 2\lambda |\nabla v_{\mathcal{T}}|^2 + \lambda^2 v_{\mathcal{T}}^2 \right) + |v_{\mathcal{T}}|_{J, \mathcal{T}}^2, \quad |v_{\mathcal{T}}|_{J, \mathcal{T}}^2 := \int_{\mathcal{F}} \left(h_{\mathcal{T}}^{-1} \|\llbracket \nabla v_{\mathcal{T}} \rrbracket\|^2 + h_{\mathcal{T}}^{-3} \|\llbracket v_{\mathcal{T}} \rrbracket\|^2 \right)$$

for functions $v_{\mathcal{T}} \in V_{\mathcal{T}}^s$. In order to simplify the presentation, throughout this work we write $\int_{\mathcal{E}} := \sum_{K \in \mathcal{E}} \int_K$ for collections $\mathcal{E} \subset \mathcal{T}$ of elements and $\int_{\mathcal{G}} := \sum_{F \in \mathcal{G}} \int_F$ for collections $\mathcal{G} \subset \mathcal{F}$ of faces. The jump operator $\llbracket \cdot \rrbracket$ is defined in the following paragraph.

2.3.3. Jump and average operators

For elements $K \in \mathcal{T}$, we write $\tau_{\partial K} : \text{BV}(K) \rightarrow L^1(\partial K)$ to denote the trace operator. Further, for $v \in \text{BV}(Y)$ we define $\tau_{\partial K} v := \tau_{\partial K}(v|_K)$ for elements $K \in \mathcal{T}$. We then introduce the jump $\llbracket v \rrbracket_F$ and the average $\{v\}_F$ of a function $v \in \text{BV}(Y)$ over a face $F = \partial K \cap \partial K' \in \mathcal{F}$ shared by the elements $K, K' \in \mathcal{T}$ by

$$\begin{aligned} \llbracket v \rrbracket_F &:= \tau_{\partial K} v|_F - \tau_{\partial K'} v|_F \in L^1(F), \\ \{v\}_F &:= \frac{\tau_{\partial K} v|_F + \tau_{\partial K'} v|_F}{2} \in L^1(F), \end{aligned}$$

where K, K' are labeled such that the unit normal n_F is the outward normal to K on the face F ; see Figure 1. To simplify the presentation, we will often simply write $\llbracket \cdot \rrbracket$ and $\{ \cdot \}$, and drop the subscript.

2.4. A posteriori analysis

Let $u \in H_{\text{per}}^2(Y)$ denote the unique solution to the HJBI problem (2.1) and let $v_{\mathcal{T}} \in V_{\mathcal{T}}^0$ be arbitrary. The goal of this section is to estimate the $\|\cdot\|_{\mathcal{T}, \lambda}$ -distance between u and $v_{\mathcal{T}}$, i.e.,

$$\|u - v_{\mathcal{T}}\|_{\mathcal{T}, \lambda}^2 = \int_Y \left(|\nabla^2(u - v_{\mathcal{T}})|^2 + 2\lambda |\nabla(u - v_{\mathcal{T}})|^2 + \lambda^2 (u - v_{\mathcal{T}})^2 \right) + |u - v_{\mathcal{T}}|_{J, \mathcal{T}}^2,$$

in terms of a computable quantity not depending on the solution u . We start by introducing periodic enrichment operators which are an important tool in establishing the *a posteriori* bound.

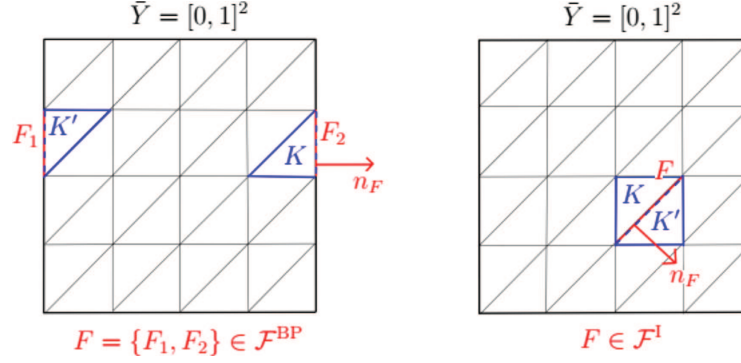


FIGURE 1. Illustration of a boundary face-pair $F \in \mathcal{F}^{\text{BP}}$ (left) and an interior face $F \in \mathcal{F}^{\text{I}}$ (right) in dimension $n = 2$.

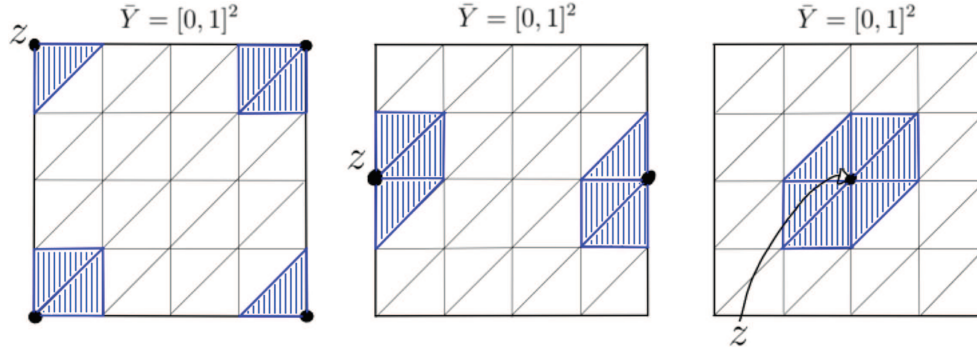


FIGURE 2. Illustration of the periodic neighborhood $N(z) \subset \mathcal{T}$ in dimension $n = 2$. Left: $z \in \mathcal{Z} \cap \partial Y$ corner point, middle: $z \in \mathcal{Z} \cap \partial Y$ non-corner boundary point, right: $z \in \mathcal{Z} \cap Y$ interior point.

2.4.1. Periodic enrichment

We let \mathcal{Z} be the set of points in \bar{Y} corresponding to the Lagrange degrees of freedom for the function space $V_{\mathcal{T}}^1 = V_{\mathcal{T}}^0 \cap H_{\text{per}}^1(Y)$, where boundary nodes on ∂Y are identified with all their Y -periodic counterparts. For $z \in \mathcal{Z}$, we then define the *periodic neighborhood* $N(z) \subset \mathcal{T}$ to be the set of all elements $K \in \mathcal{T}$ that contain z or any periodically identical point to z ; see Figure 2.

Let us introduce an operator

$$E_1 : V_{\mathcal{T}}^0 \rightarrow V_{\mathcal{T}}^0 \cap H_{\text{per}}^1(Y),$$

which we call the H_{per}^1 -enrichment operator, defined through averaging of the function values in periodic neighborhoods of points in \mathcal{Z} . That is, for $v_{\mathcal{T}} \in V_{\mathcal{T}}^0$, we define the function $E_1 v_{\mathcal{T}} \in V_{\mathcal{T}}^1$ by prescribing

$$E_1 v_{\mathcal{T}}(z) := \frac{1}{|N(z)|} \sum_{K \in N(z)} v_{\mathcal{T}}|_K(z)$$

at points $z \in \mathcal{Z}$. Denoting the collection of interior faces and boundary face-pairs neighboring an element $K \in \mathcal{T}$ by $\mathcal{F}_K := \{F \in \mathcal{F} : F \cap K \neq \emptyset\}$, there exists a constant $C_{E_1} = C_{E_1}(n, \theta_T, P) > 0$ such that

$$\sum_{m=0}^2 \int_K h_T^{2m-4} |\nabla^m(v_T - E_1 v_T)|^2 \leq C_{E_1} \int_{\mathcal{F}_K} h_T^{-3} \|\llbracket v_T \rrbracket\|^2 \quad \forall K \in \mathcal{T} \quad (2.13)$$

for all $v_T \in V_T^0$. This bound follows from the arguments in [32].

Let us also discuss the periodic enrichment of vector fields. To this end, we define the space containing potential gradients of functions in the finite element spaces by

$$W_T := \{v_T \in L^2(Y; \mathbb{R}^n) : v_T|_K \in \mathbb{P}_{P-1}^n \ \forall K \in \mathcal{T}\}.$$

Indeed, observe that $\nabla v_T \in W_T$ for any $v_T \in V_T^s$, $s \in \{0, 1\}$. Analogously to E_1 , we can then construct a linear operator

$$E_1^g : W_T \rightarrow W_T \cap H_{\text{per}}^1(Y; \mathbb{R}^n)$$

with the property that there exists a constant $C_{E_1^g} = C_{E_1^g}(n, \theta_T, P) > 0$ such that

$$\int_K \left(|\nabla(w_T - E_1^g w_T)|^2 + h_T^{-2} |w_T - E_1^g w_T|^2 \right) \leq C_{E_1^g} \int_{\mathcal{F}_K} h_T^{-1} \|\llbracket w_T \rrbracket\|^2 \quad \forall K \in \mathcal{T} \quad (2.14)$$

for all $w_T \in W_T$. With the enrichment operators at hand we can proceed with the *a posteriori* analysis, independent of the choice of the numerical scheme.

2.4.2. The *a posteriori* bound

It will be useful to introduce some notation from the mixed finite element theory developed in [26]. Let us consider the function space

$$X := W_{\text{per}}(Y; \mathbb{R}^n) \times H_{\text{per}}^1(Y),$$

which we equip with the $\|\cdot\|_\lambda$ -norm given by

$$\|(w', u')\|_\lambda^2 := \|\nabla w'\|_{L^2(Y)}^2 + 2\lambda \|\nabla u'\|_{L^2(Y)}^2 + \lambda^2 \|u'\|_{L^2(Y)}^2, \quad (w', u') \in X.$$

We recall that the spaces $W_{\text{per}}(Y) \subset H_{\text{per}}^1(Y)$ and $W_{\text{per}}(Y; \mathbb{R}^n) \subset H_{\text{per}}^1(Y; \mathbb{R}^n)$ are defined as

$$W_{\text{per}}(Y) := \left\{ v \in H_{\text{per}}^1(Y) : \int_Y v = 0 \right\}, \quad W_{\text{per}}(Y; \mathbb{R}^n) := (W_{\text{per}}(Y))^n.$$

We further define the mixed analogue F_γ^M to the nonlinear operator F_γ by

$$F_\gamma^M[(w', u')] := \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \{ \gamma^{\alpha\beta} (-A^{\alpha\beta} : \nabla w' - b^{\alpha\beta} \cdot \nabla u' + c^{\alpha\beta} u' - f^{\alpha\beta}) \}$$

for pairs $(w', u') \in X$, and observe that the solution $u \in H_{\text{per}}^2(Y)$ to (2.1) satisfies

$$F_\gamma^M[(\nabla u, u)] = F_\gamma[u] = 0 \quad \text{a.e. in } Y. \quad (2.15)$$

We can use the arguments from [26] to prove an *a posteriori* bound on the $\|\cdot\|_\lambda$ -distance between the solution pair $(\nabla u, u)$ and an arbitrary pair $(w', u') \in X$.

Lemma 2.5. *Let $u \in H_{\text{per}}^2(Y)$ denote the unique solution to the HJBI problem (2.1). Then, for any $(w', u') \in X$ there holds*

$$\|(\nabla u - w', u - u')\|_\lambda^2 \leq C_{\text{post}}^M \left(\|F_\gamma^M[(w', u')]\|_{L^2(Y)}^2 + \|\text{rot}(w')\|_{L^2(Y)}^2 + \|\nabla u' - w'\|_{L^2(Y)}^2 \right),$$

where the constant $C_{\text{post}}^M = C_{\text{post}}^M(\delta, \lambda) > 0$ is given by

$$C_{\text{post}}^M := 2 \left(1 - \sqrt{1 - \delta} \right)^{-2} \max \left\{ \lambda + 2\lambda \left(1 - \sqrt{1 - \delta} \right)^2, 16 \right\}.$$

Proof. We define the semilinear form $a^M : X \times X \rightarrow \mathbb{R}$ by

$$\begin{aligned} a^M((w_1, u_1), (w_2, u_2)) \\ := \int_Y F_\gamma^M[(w_1, u_1)](\lambda u_2 - \nabla \cdot w_2) + \sigma_1 \int_Y \text{rot}(w_1) \cdot \text{rot}(w_2) + \sigma_2 \int_Y (\nabla u_1 - w_1) \cdot (\nabla u_2 - w_2) \end{aligned}$$

with $\sigma_1, \sigma_2 > 0$ given by

$$\sigma_1 := 1 - \frac{1}{2}\sqrt{1 - \delta}, \quad \sigma_2 := \frac{\lambda}{2} \left(1 - \sqrt{1 - \delta} \right) + \frac{\lambda}{4} \left(1 - \sqrt{1 - \delta} \right)^{-1}.$$

A straightforward adaptation of the proof of Lemma 2.3 from [26] yields the monotonicity estimate

$$C_\delta \| (w_1 - w_2, u_1 - u_2) \|_\lambda^2 \leq a^M((w_1, u_1), (w_1 - w_2, u_1 - u_2)) - a^M((w_2, u_2), (w_1 - w_2, u_1 - u_2))$$

for all $(w_1, u_1), (w_2, u_2) \in X$, where $C_\delta := \frac{1}{4}(1 - \sqrt{1 - \delta}) > 0$. In particular, in view of (2.15), we find that

$$C_\delta \| (\nabla u - w', u - u') \|_\lambda^2 \leq -a^M((w', u'), (\nabla u - w', u - u')) \quad \forall (w', u') \in X. \quad (2.16)$$

Let $(w', u') \in X$ be arbitrary and write $(\delta_w, \delta_u) := (\nabla u - w', u - u')$. Using the Cauchy–Schwarz and Young inequalities to bound the right-hand side of (2.16), we have

$$\begin{aligned} C_\delta \| (\delta_w, \delta_u) \|_\lambda^2 &\leq |a^M((w', u'), (\delta_w, \delta_u))| \\ &\leq \frac{1}{C_\delta} \|F_\gamma^M[(w', u')]\|_{L^2(Y)}^2 + \frac{C_\delta}{4} \|\lambda \delta_u - \nabla \cdot \delta_w\|_{L^2(Y)}^2 + \sigma_1 \|\text{rot}(w')\|_{L^2(Y)}^2 + \sigma_2 \|\nabla u' - w'\|_{L^2(Y)}^2 \end{aligned}$$

and we can conclude that

$$\|(\delta_w, \delta_u)\|_\lambda^2 \leq \frac{1}{C_\delta^2} \|F_\gamma^M[(w', u')]\|_{L^2(Y)}^2 + \frac{1}{2} \|(\delta_w, \delta_u)\|_\lambda^2 + \frac{\sigma_1}{C_\delta} \|\text{rot}(w')\|_{L^2(Y)}^2 + \frac{\sigma_2}{C_\delta} \|\nabla u' - w'\|_{L^2(Y)}^2$$

upon noting $\|\nabla \cdot \delta_w\|_{L^2(Y)} \leq \|\nabla \delta_w\|_{L^2(Y)}$ as $\delta_w \in H_{\text{per}}^1(Y; \mathbb{R}^n)$; see [26]. Finally, absorbing the term $\frac{1}{2} \|(\delta_w, \delta_u)\|_\lambda^2$ into the left-hand side of the above inequality, we obtain that

$$\|(\delta_w, \delta_u)\|_\lambda^2 \leq \frac{2}{C_\delta^2} \max\{1, \sigma_1 C_\delta, \sigma_2 C_\delta\} \left(\|F_\gamma^M[(w', u')]\|_{L^2(Y)}^2 + \|\text{rot}(w')\|_{L^2(Y)}^2 + \|\nabla u' - w'\|_{L^2(Y)}^2 \right).$$

Noting that $\max\{1, \sigma_1 C_\delta, \sigma_2 C_\delta\} = \max\{1, \sigma_2 C_\delta\}$ as $\delta \in (0, 1)$, we obtain the claimed result. \square

We can use Lemma 2.5 and the H_{per}^1 -enrichment operators to prove the following *a posteriori* error bound:

Theorem 2.6 (*A posteriori error bound*). *Let $u \in H_{\text{per}}^2(Y)$ denote the unique solution to the HJBI problem (2.1). Then, there holds*

$$\|u - v_{\mathcal{T}}\|_{T, \lambda}^2 \leq 4C_{\text{post}}^M \int_Y |F_\gamma[v_{\mathcal{T}}]|^2 + C |v_{\mathcal{T}}|_{J, \mathcal{T}}^2 \quad \forall v_{\mathcal{T}} \in V_{\mathcal{T}}^0,$$

where $C_{\text{post}}^M = C_{\text{post}}^M(\delta, \lambda) > 0$ is the constant defined in Lemma 2.5 and $C > 0$ is some constant depending only on $\delta, \lambda, n, \theta_{\mathcal{T}}, P$ which can be read off from the proof.

Proof. Let $v_T \in V_T^0$ be arbitrary and set

$$\begin{aligned} v &:= E_1 v_T \in V_T^0 \cap H_{\text{per}}^1(Y), \\ w &:= E_1^g(\nabla v_T) - \int_Y E_1^g(\nabla v_T) \in W_T \cap W_{\text{per}}(Y; \mathbb{R}^n). \end{aligned}$$

By the triangle inequality, we have

$$\|u - v_T\|_{T,\lambda}^2 \leq 2 \|(\nabla u - w, u - v)\|_\lambda^2 + 2 \int_Y (|\nabla(w - \nabla v_T)|^2 + 2\lambda |\nabla(v - v_T)|^2 + \lambda^2 (v - v_T)^2) + |v_T|_{J,T}^2,$$

which we can further bound, using the properties of the enrichment operators (2.13) and (2.14), to obtain that

$$\|u - v_T\|_{T,\lambda}^2 \leq 2 \|(\nabla u - w, u - v)\|_\lambda^2 + 2C_0 |v_T|_{J,T}^2 + |v_T|_{J,T}^2,$$

where $C_0 = C_0(\lambda, n, \theta_T, P) > 0$ denotes the constant $C_0 := 2C_{E_1^g} + 2C_{E_1} \lambda \max\{\lambda, 2\}$. We can apply Lemma 2.5 to find

$$\|u - v_T\|_{T,\lambda}^2 \leq 2C_{\text{post}}^M \left(\|F_\gamma^M[(w, v)]\|_{L^2(Y)}^2 + \|\text{rot}(w)\|_{L^2(Y)}^2 + \|\nabla v - w\|_{L^2(Y)}^2 \right) + (2C_0 + 1) |v_T|_{J,T}^2. \quad (2.17)$$

Note that, using the triangle and Hölder inequalities, and the enrichment bounds (2.13) and (2.14), we have

$$\|\text{rot}(w)\|_{L^2(Y)}^2 = \int_Y |\text{rot}(w - \nabla v_T)|^2 \leq 2 \int_Y |\nabla(w - \nabla v_T)|^2 \leq 4C_{E_1^g} |v_T|_{J,T}^2$$

for the second term on the right-hand side of (2.17), and

$$\begin{aligned} \|\nabla v - w\|_{L^2(Y)}^2 &= \left\| \nabla v - E_1^g(\nabla v_T) - \int_Y (\nabla v - E_1^g(\nabla v_T)) \right\|_{L^2(Y)}^2 \\ &\leq 4 \|\nabla v - E_1^g(\nabla v_T)\|_{L^2(Y)}^2 \\ &\leq 8 \left(\int_Y |\nabla(v - v_T)|^2 + \int_Y |\nabla v_T - E_1^g(\nabla v_T)|^2 \right) \\ &\leq 16 (C_{E_1} + C_{E_1^g}) |v_T|_{J,T}^2 \end{aligned}$$

for the third term on the right-hand side of (2.17) (note that $\int_Y \nabla v = 0$ since $v \in H_{\text{per}}^1(Y)$). Finally, for the first term on the right-hand side of (2.17), we successively use the triangle inequality together with $F_\gamma[v_T] = F_\gamma^M[(\nabla v_T, v_T)]$, a Lipschitz property of F_γ^M which is shown analogously to (2.9), and the enrichment bounds (2.13) and (2.14) to obtain

$$\begin{aligned} \|F_\gamma^M[(w, v)]\|_{L^2(Y)}^2 &\leq 2 \int_Y |F_\gamma[v_T]|^2 + 2 \int_Y |F_\gamma^M[(w, v)] - F_\gamma^M[(\nabla v_T, v_T)]|^2 \\ &\leq 2 \int_Y |F_\gamma[v_T]|^2 + 2C_l^2 \int_Y (|\nabla(w - \nabla v_T)|^2 + 2\lambda |\nabla(v - v_T)|^2 + \lambda^2 (v - v_T)^2) \\ &\leq 2 \int_Y |F_\gamma[v_T]|^2 + 2C_0 C_l^2 |v_T|_{J,T}^2, \end{aligned}$$

where $C_l = C_l(\delta, n)$ denotes the constant $C_l := \sqrt{1 - \delta} + \sqrt{n + 1} > 0$. Altogether, in view of (2.17), we have proved that

$$\|u - v_T\|_{T,\lambda}^2 \leq 4C_{\text{post}}^M \int_Y |F_\gamma[v_T]|^2 + \left(2C_{\text{post}}^M (2C_0 C_l^2 + 4C_{E_1^g} + 16(C_{E_1} + C_{E_1^g})) + 2C_0 + 1 \right) |v_T|_{J,T}^2,$$

which is as required. \square

Let us emphasize that the presented *a posteriori* analysis was based on the construction of periodic enrichment operators of H^1 -type as well as an *a posteriori* bound derived from a mixed formulation of the problem; thus bypassing the need for the construction of H^2 -type enrichment operators as done in [36]. This concludes the *a posteriori* analysis and we proceed with an abstract *a priori* analysis for a wide class of numerical schemes in the next section.

2.5. Numerical scheme and *a priori* analysis

Let us consider an abstract numerical scheme written in the following form: For chosen $s \in \{0, 1\}$, find a function $u_T \in V_T^s$ satisfying

$$a_T(u_T, v_T) = 0 \quad \forall v_T \in V_T^s. \quad (2.18)$$

2.5.1. Abstract *a priori* analysis

Here, we assume that the nonlinear form $a_T : V_T^s \times V_T^s \rightarrow \mathbb{R}$ satisfies the assumptions listed below:

(A1) Linearity in second argument: $a_T(w_T, \cdot) : V_T^s \rightarrow \mathbb{R}$ is linear for any fixed $w_T \in V_T^s$.

(A2) Strong monotonicity: There exists a constant $C_M > 0$ such that

$$\|w_T - v_T\|_{T,\lambda}^2 \leq C_M(a_T(w_T, w_T - v_T) - a_T(v_T, w_T - v_T)) \quad \forall w_T, v_T \in V_T^s.$$

(A3) Lipschitz continuity: There exists a constant $C_L > 0$ such that

$$|a_T(w_T, v_T) - a_T(w'_T, v_T)| \leq C_L \|w_T - w'_T\|_{T,\lambda} \|v_T\|_{T,\lambda} \quad \forall w_T, w'_T, v_T \in V_T^s.$$

(A4) Discrete consistency: There exists a linear operator $L_T : V_T^s \rightarrow L^2(Y)$ such that, for some constant $C_1 > 0$, we have

$$\|L_T v_T\|_{L^2(Y)} \leq C_1 \|v_T\|_{T,\lambda} \quad \forall v_T \in V_T^s,$$

and, for some constant $C_2 > 0$, we have

$$\left| a(w_T, v_T) - \int_Y F_\gamma[w_T] L_T v_T \right| \leq C_2 |w_T|_{J,T} \|v_T\|_{T,\lambda} \quad \forall w_T, v_T \in V_T^s.$$

Observe that the assumptions (A1)–(A4) guarantee well-posedness of the numerical scheme, that is, there exists a unique solution $u_T \in V_T^s$ satisfying (2.18). We can show an *a priori* bound in this general setting similarly to [36]. A short proof is provided for completeness.

Theorem 2.7 (*A priori* error bound). *For chosen $s \in \{0, 1\}$, let $a_T : V_T^s \times V_T^s \rightarrow \mathbb{R}$ be a nonlinear form satisfying the assumptions (A1)–(A4). Further, let $u \in H_{\text{per}}^2(Y)$ denote the unique solution to the HJBI problem (2.1). Then, there exists a unique solution $u_T \in V_T^s$ to (2.18) and we have the near-best approximation bound*

$$\|u - u_T\|_{T,\lambda} \leq C_e \inf_{v_T \in V_T^s} \|u - v_T\|_{T,\lambda}, \quad (2.19)$$

where the constant $C_e > 0$ is given by

$$C_e := 1 + C_M \left(C_1 \left(\sqrt{1 - \delta} + \sqrt{n + 1} \right) + C_2 \right). \quad (2.20)$$

Proof. As we have already noted, the existence and uniqueness of a solution $u_T \in V_T^s$ to (2.18) follows from the assumptions on the nonlinear form a_T , and it only remains to show the near-best approximation bound (2.19). To this end, let $v_T \in V_T^s$ be arbitrary and observe that

$$\|v_T - u_T\|_{T,\lambda}^2 \leq C_M(a_T(v_T, v_T - u_T) - a_T(u_T, v_T - u_T)) = C_M a_T(v_T, v_T - u_T) \quad (2.21)$$

by strong monotonicity (A2) and the solution property (2.18) of $u_{\mathcal{T}}$. In order to further bound the right-hand side, we successively use the discrete consistency (A4), the solution property and regularity of u , and the Lipschitz property (2.9) of F_{γ} to obtain

$$\begin{aligned} a_{\mathcal{T}}(v_{\mathcal{T}}, v_{\mathcal{T}} - u_{\mathcal{T}}) &\leq \left| \int_Y F_{\gamma}[v_{\mathcal{T}}] L_{\mathcal{T}}(v_{\mathcal{T}} - u_{\mathcal{T}}) \right| + C_2 |v_{\mathcal{T}}|_{J, \mathcal{T}} \|v_{\mathcal{T}} - u_{\mathcal{T}}\|_{\mathcal{T}, \lambda} \\ &\leq \left(C_1 \|F_{\gamma}[v_{\mathcal{T}}] - F_{\gamma}[u]\|_{L^2(Y)} + C_2 |v_{\mathcal{T}} - u|_{J, \mathcal{T}} \right) \|v_{\mathcal{T}} - u_{\mathcal{T}}\|_{\mathcal{T}, \lambda} \\ &\leq \left(C_1 \left(\sqrt{1 - \delta} + \sqrt{n + 1} \right) + C_2 \right) \|v_{\mathcal{T}} - u\|_{\mathcal{T}, \lambda} \|v_{\mathcal{T}} - u_{\mathcal{T}}\|_{\mathcal{T}, \lambda}. \end{aligned}$$

Combination with the previous estimate (2.21) yields

$$\|v_{\mathcal{T}} - u_{\mathcal{T}}\|_{\mathcal{T}, \lambda} \leq C_M \left(C_1 \left(\sqrt{1 - \delta} + \sqrt{n + 1} \right) + C_2 \right) \|u - v_{\mathcal{T}}\|_{\mathcal{T}, \lambda},$$

which in turn implies

$$\|u - u_{\mathcal{T}}\|_{\mathcal{T}, \lambda} \leq \|u - v_{\mathcal{T}}\|_{\mathcal{T}, \lambda} + \|v_{\mathcal{T}} - u_{\mathcal{T}}\|_{\mathcal{T}, \lambda} \leq C_e \|u - v_{\mathcal{T}}\|_{\mathcal{T}, \lambda}$$

with $C_e > 0$ given by (2.20). We conclude the proof by taking the infimum over $v_{\mathcal{T}} \in V_{\mathcal{T}}^s$. \square

We conclude this section by noting that Theorem 2.7 implies convergence of the numerical approximation under mesh-refinement. While convergence together with optimal rates follow immediately from standard approximation arguments in the case that the exact solution satisfies additional regularity assumptions, it is not that clear when we only have a minimal regularity solution $u \in H_{\text{per}}^2(Y)$. For the latter case, we can argue as in Corollary 4.7 of [36] and obtain the following result.

Remark 2.8 (Convergence of the numerical approximation). For a sequence of conforming simplicial meshes $\{\mathcal{T}_k\}_k$ with $\max_{K \in \mathcal{T}_k} h_K \rightarrow 0$ as $k \rightarrow \infty$, we have that

$$\inf_{v_{\mathcal{T}_k} \in V_{\mathcal{T}_k}^s} \|u - v_{\mathcal{T}_k}\|_{\mathcal{T}_k, \lambda} \xrightarrow[k \rightarrow \infty]{} 0.$$

In particular, in view of (2.19), given $a_{\mathcal{T}_k} : V_{\mathcal{T}_k}^s \times V_{\mathcal{T}_k}^s \rightarrow \mathbb{R}$ satisfying (A1)–(A4) with constants uniformly bounded in k , we have that

$$\|u - u_{\mathcal{T}_k}\|_{\mathcal{T}_k, \lambda} \xrightarrow[k \rightarrow \infty]{} 0$$

for the sequence of numerical approximations $\{u_{\mathcal{T}_k}\}_k \subset V_{\mathcal{T}_k}^s$.

2.5.2. The family of numerical schemes

For chosen $s \in \{0, 1\}$ and a parameter $\theta \in [0, 1]$, we now consider the numerical scheme of finding $u_{\mathcal{T}} \in V_{\mathcal{T}}^s$ satisfying (2.18) with

$$a_{\mathcal{T}} : V_{\mathcal{T}}^s \times V_{\mathcal{T}}^s \rightarrow \mathbb{R}, \quad a_{\mathcal{T}}(w_{\mathcal{T}}, v_{\mathcal{T}}) := \int_Y F_{\gamma}[w_{\mathcal{T}}] L_{\lambda, \mathcal{T}} v_{\mathcal{T}} + \theta S_{\mathcal{T}}(w_{\mathcal{T}}, v_{\mathcal{T}}) + J_{\mathcal{T}}(w_{\mathcal{T}}, v_{\mathcal{T}}),$$

where we define the linear operator $L_{\lambda, \mathcal{T}} v_{\mathcal{T}} := \lambda v_{\mathcal{T}} - \Delta v_{\mathcal{T}}$ for $v_{\mathcal{T}} \in V_{\mathcal{T}}^s$, the stabilization bilinear form $S_{\mathcal{T}} : V_{\mathcal{T}}^s \times V_{\mathcal{T}}^s \rightarrow \mathbb{R}$ via

$$\begin{aligned} S_{\mathcal{T}}(w_{\mathcal{T}}, v_{\mathcal{T}}) &:= \int_Y (\nabla^2 w_{\mathcal{T}} : \nabla^2 v_{\mathcal{T}} - \Delta w_{\mathcal{T}} \Delta v_{\mathcal{T}}) + \int_{\mathcal{F}} (\{\Delta_{\mathcal{T}} w_{\mathcal{T}}\} \llbracket \nabla v_{\mathcal{T}} \cdot n \rrbracket + \{\Delta_{\mathcal{T}} v_{\mathcal{T}}\} \llbracket \nabla w_{\mathcal{T}} \cdot n \rrbracket) \\ &\quad - \int_{\mathcal{F}} (\nabla_{\mathcal{T}} \{\nabla w_{\mathcal{T}} \cdot n\} \cdot \llbracket \nabla_{\mathcal{T}} v_{\mathcal{T}} \rrbracket + \nabla_{\mathcal{T}} \{\nabla v_{\mathcal{T}} \cdot n\} \cdot \llbracket \nabla_{\mathcal{T}} w_{\mathcal{T}} \rrbracket), \end{aligned}$$

and, for chosen parameters $\eta_1, \eta_2 > 0$, the jump penalization form $J_T : V_T^s \times V_T^s \rightarrow \mathbb{R}$ via

$$J_T(w_T, v_T) := \eta_1 \int_{\mathcal{F}} h_T^{-1} \llbracket \nabla w_T \rrbracket \cdot \llbracket \nabla v_T \rrbracket + \eta_2 \int_{\mathcal{F}} h_T^{-3} \llbracket w_T \rrbracket \llbracket v_T \rrbracket.$$

Here, the tangential gradient and Laplacian on mesh faces are denoted by ∇_T and Δ_T . Note that the difference to the numerical scheme for the Dirichlet framework presented in Section 5 of [36] is due to the fact that the periodic setting requires to treat boundary faces as boundary face-pairs and thus, analogously to interior faces.

This scheme is an adaptation of the method presented in [36] for the homogeneous Dirichlet problem. The analysis of this method, *i.e.*, the verification of the assumptions (A1)–(A4), is analogous to [36] and hence omitted. The main result is the following:

Theorem 2.9. *There exist constants $\bar{\eta}_1, \bar{\eta}_2 > 0$, depending only on n, θ_T, θ, P and the Cordes parameters δ, λ , such that, for any $\theta \in [0, 1]$, if $\eta_1 \geq \bar{\eta}_1$ and $\eta_2 \geq \bar{\eta}_2$, the properties (A1)–(A4) are satisfied and Theorem 2.7 applies.*

Remark 2.10. The constants $\bar{\eta}_1, \bar{\eta}_2$ and the constant C_e in the near-best approximation bound (2.19) remain bounded as $\lambda \searrow 0$. This can be seen similarly to the Dirichlet setting by regarding the λ -dependence of the constants in the proofs of Theorem 5.5 from [36] and Lemma 6 of [52].

2.5.3. Rates of convergence

In this paragraph, we briefly discuss rates of convergence for the convergence of the numerical approximation $u_{\mathcal{T}_k} \in V_{\mathcal{T}_k}^s$ to the true solution $u \in H_{\text{per}}^2(Y)$ of the HJBI problem (2.1) in the $\|\cdot\|_{\mathcal{T}_k, \lambda}$ -norm as $k \rightarrow \infty$, where $\{\mathcal{T}_k\}_k$ is a sequence of conforming simplicial meshes with the property that $\max_{K \in \mathcal{T}_k} h_K \rightarrow 0$ as $k \rightarrow \infty$.

We distinguish the cases of (i) minimal regularity of u , *i.e.*, no additional regularity is assumed on the true solution, and (ii) higher regularity of u .

- (i) Minimal regularity of u : In view of Remark 2.8, if the true solution is merely of regularity $u \in H_{\text{per}}^2(Y)$, we have the convergence

$$\|u - u_{\mathcal{T}_k}\|_{\mathcal{T}_k, \lambda} \xrightarrow[k \rightarrow \infty]{} 0$$

for the sequence of numerical approximations $\{u_{\mathcal{T}_k}\}_k \subset V_{\mathcal{T}_k}^s$ to u , but we do not have a rate.

- (ii) Higher regularity of u : In view of Theorem 2.7 and standard interpolation inequalities, if the true solution $u \in H_{\text{per}}^2(Y)$ satisfies $u \in H^{2+r_K}(K)$ with $r_K > 0$ for all $K \in \mathcal{T}_k$, then we have that

$$\|u - u_{\mathcal{T}_k}\|_{\mathcal{T}_k, \lambda} \lesssim \inf_{v_{\mathcal{T}_k} \in V_{\mathcal{T}_k}^s} \|u - v_{\mathcal{T}_k}\|_{\mathcal{T}_k, \lambda} \lesssim \sqrt{\sum_{K \in \mathcal{T}_k} h_K^{2 \min\{r_K, P-1\}} \|\nabla u\|_{H^{1+r_K}(K)}^2}.$$

In particular, if $u \in H^{2+r}(Y)$ for some $r > 0$, and writing $h_k := \max_{K \in \mathcal{T}_k} h_K$, there holds

$$\|u - u_{\mathcal{T}_k}\|_{\mathcal{T}_k, \lambda} = \mathcal{O}\left(h_k^{\min\{r, P-1\}}\right) \quad \text{as } k \rightarrow \infty.$$

Higher-order Sobolev regularity of strong solutions to HJBI equations is a delicate question and the authors are not aware of conditions which would guarantee such property. Regarding regularity in Hölder spaces, it is known that the maximal regularity one can assert for the unique viscosity solution is $C^{1, \alpha}$; see [42, 43].

3. APPROXIMATION OF EFFECTIVE HAMILTONIANS TO HJBI OPERATORS

3.1. The effective Hamiltonian

We start by recalling the definition of the effective Hamiltonian based on the *cell σ -problem*; see [2, 5, 6]. Let us consider an HJBI operator $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$ given by

$$F(x, y, p, R) := \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \{-A^{\alpha\beta}(y) : R - b^{\alpha\beta}(x, y) \cdot p - f^{\alpha\beta}(x, y)\} \quad (3.1)$$

with \mathcal{A} and \mathcal{B} denoting compact metric spaces, and functions

$$\begin{aligned} A &= (a_{ij})_{1 \leq i, j \leq n} : \mathbb{R}^n \times \mathcal{A} \times \mathcal{B} \rightarrow \mathbb{R}_{\text{sym}}^{n \times n}, & (y, \alpha, \beta) &\mapsto A(y, \alpha, \beta) =: A^{\alpha\beta}(y), \\ b &= (b_i)_{1 \leq i \leq n} : \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{A} \times \mathcal{B} \rightarrow \mathbb{R}^n, & (x, y, \alpha, \beta) &\mapsto b(x, y, \alpha, \beta) =: b^{\alpha\beta}(x, y), \\ f &: \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{A} \times \mathcal{B} \rightarrow \mathbb{R}, & (x, y, \alpha, \beta) &\mapsto f(x, y, \alpha, \beta) =: f^{\alpha\beta}(x, y) \end{aligned}$$

satisfying the assumptions stated below in paragraph 3.1.1.

To the HJBI operator (3.1), we associate the corresponding *cell σ -problem*: for fixed $(x, p, R) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\text{sym}}^{n \times n}$ and a positive parameter $\sigma > 0$, there exists a unique viscosity solution $v^\sigma = v^\sigma(\cdot; x, p, R) \in C(\mathbb{R}^n)$ to the problem

$$\begin{cases} \sigma v^\sigma + F(x, y, p, R + \nabla_y^2 v^\sigma) = 0 & \text{for } y \in Y, \\ y \mapsto v^\sigma(y; x, p, R) \text{ is } Y\text{-periodic.} \end{cases} \quad (3.2)$$

The function $v^\sigma(\cdot; x, p, R)$ is called an *approximate corrector*.

Definition 3.1 (Ergodicity and effective Hamiltonian). Let $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$ be an HJBI operator of the form (3.1).

- (i) We say F is ergodic (in the y -variable) at a point $(x, p, R) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\text{sym}}^{n \times n}$ if there exists a constant $H(x, p, R) \in \mathbb{R}$ such that

$$-\sigma v^\sigma(\cdot; x, p, R) \xrightarrow{\sigma \searrow 0} H(x, p, R) \quad \text{uniformly.} \quad (3.3)$$

Further, we call F ergodic if it is ergodic at every $(x, p, R) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\text{sym}}^{n \times n}$.

- (ii) If F is ergodic, we call the function

$$H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}, \quad (x, p, R) \mapsto H(x, p, R)$$

defined *via* (3.3) the effective Hamiltonian corresponding to F .

The assumptions on the coefficients made in paragraph 3.1.1 are such that the HJBI operator (3.1) fits into the framework considered in [6], which guarantees ergodicity. The corresponding effective Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$ is automatically continuous and degenerate elliptic, that is,

$$R_1 - R_2 \geq 0 \implies H(x, p, R_1) \leq H(x, p, R_2)$$

for any $x, p \in \mathbb{R}^n$, $R_1, R_2 \in \mathbb{R}_{\text{sym}}^{n \times n}$.

Remark 3.2. In the periodic homogenization of elliptic and parabolic HJBI equations

$$u_\varepsilon^e + F\left(x, \frac{x}{\varepsilon}, \nabla u_\varepsilon^e, \nabla^2 u_\varepsilon^e\right) = 0, \quad \partial_t u_\varepsilon^p + F\left(x, \frac{x}{\varepsilon}, \nabla_x u_\varepsilon^p, \nabla_x^2 u_\varepsilon^p\right) = 0,$$

posed in a suitable Dirichlet/Cauchy setting, the effective Hamiltonian determines the homogenized equation

$$u_0^e + H(x, \nabla u_0^e, \nabla^2 u_0^e) = 0, \quad \partial_t u_0^p + H(x, \nabla_x u_0^p, \nabla_x^2 u_0^p) = 0;$$

see [6, 16, 17].

In this setting, having $A = A(y, \alpha, \beta)$ being independent of the state variable x , it can be shown that

$$|H(x_1, p, R) - H(x_2, p, R)| \leq C|x_1 - x_2|(1 + |p|) + \omega(|x_1 - x_2|) \quad \forall x_1, x_2, p \in \mathbb{R}^n, R \in \mathbb{R}_{\text{sym}}^{n \times n},$$

for some constant $C > 0$ and modulus of continuity ω , which guarantees a comparison principle for the effective problem and implies homogenization; see [6]. In the convex case, when the coefficients are independent of α , we have the following relation between approximate correctors and correctors.

Remark 3.3. If the coefficients A, b, f are independent of α , and f satisfies the same assumptions as the components of b , then for fixed (x, p, R) the sequence $\{v^\sigma(\cdot; x, p, R) - v^\sigma(0; x, p, R)\}_{\sigma > 0}$ converges in $C^2(\mathbb{R}^n)$ as $\sigma \searrow 0$ to a corrector $v \in C^2(\mathbb{R}^n)$, *i.e.*, a solution to the true cell problem (1.2); see [13].

3.1.1. Assumptions on the coefficients

We assume $A = \frac{1}{2}GG^T \in C(\mathbb{R}^n \times \mathcal{A} \times \mathcal{B}; \mathbb{R}^{n \times n})$, $b \in C(\mathbb{R}^n \times \mathbb{R}^n \times \mathcal{A} \times \mathcal{B}; \mathbb{R}^n)$ and $f \in C(\mathbb{R}^n \times \mathbb{R}^n \times \mathcal{A} \times \mathcal{B}; \mathbb{R})$ satisfy the assumptions listed below.

- G, b, f are bounded continuous functions on their respective domains.
- $G = G(y, \alpha, \beta), b = b(x, y, \alpha, \beta)$ are Lipschitz continuous in (x, y) , uniformly in (α, β) .
- $f = f(x, y, \alpha, \beta)$ is uniformly continuous in (x, y) , uniformly in (α, β) .
- G, b, f are Y -periodic in the fast variable y .
- Uniform ellipticity: $\exists \zeta_1, \zeta_2 > 0 : \zeta_1 |\xi|^2 \leq A^{\alpha\beta}(y) \xi \cdot \xi \leq \zeta_2 |\xi|^2 \quad \forall y, \xi \in \mathbb{R}^n, (\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$.
- Cordes condition: There exist constants $\lambda > 0$ and $\delta \in (0, 1)$ such that

$$|A^{\alpha\beta}(y)|^2 + \frac{|b^{\alpha\beta}(x, y)|^2}{2\lambda} + \frac{1}{\lambda^2} \leq \frac{1}{n + \delta} \left(\text{tr}(A^{\alpha\beta}(y)) + \frac{1}{\lambda} \right)^2 \quad (3.4)$$

for all $(x, y, \alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{A} \times \mathcal{B}$.

3.2. Approximation of the cell σ -problem

For fixed $(x, p, R) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\text{sym}}^{n \times n}$ and a positive parameter $\sigma \in (0, \bar{\sigma})$ with fixed $\bar{\sigma} > 0$, let us consider the cell σ -problem (3.2) in the rewritten form

$$\begin{cases} \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \left\{ -A^{\alpha\beta} : \nabla^2 v^\sigma + \sigma v^\sigma - g_{x,p,R}^{\alpha\beta} \right\} = 0 & \text{for } y \in Y, \\ y \mapsto v^\sigma(y; x, p, R) \text{ is } Y\text{-periodic,} \end{cases} \quad (3.5)$$

where $g_{x,p,R}^{\alpha\beta} : \mathbb{R}^n \rightarrow \mathbb{R}$ is the Y -periodic function given by

$$g_{x,p,R}^{\alpha\beta}(y) := g_{x,p,R}(y, \alpha, \beta) := A^{\alpha\beta}(y) : R + b^{\alpha\beta}(x, y) \cdot p + f^{\alpha\beta}(x, y)$$

for $y \in \mathbb{R}^n$ and $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$. The following lemma shows that, for any $\sigma > 0$, the problem (3.5) admits a unique strong solution $v^\sigma \in H_{\text{per}}^2(Y)$ and that we have a uniform bound on $|v^\sigma|_{H^2(Y)}$.

Lemma 3.4. *Assume that the assumptions of Section 3.1.1 hold and let $(x, p, R) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\text{sym}}^{n \times n}$ be fixed. Then, for any $\sigma > 0$, there exists a unique periodic strong solution $v^\sigma \in H_{\text{per}}^2(Y)$ to the cell σ -problem (3.5). Furthermore, we have the bound*

$$|v^\sigma|_{H^2(Y)} \leq C \quad (3.6)$$

with $C > 0$ independent of σ .

Proof. It is straightforward to check that all assumptions of Theorem 2.3 are satisfied. In particular, the problem (3.5) satisfies the Cordes condition

$$|A|^2 + \frac{\sigma^2}{\lambda_\sigma^2} \leq \frac{1}{n + \delta} \left(\text{tr}(A) + \frac{\sigma}{\lambda_\sigma} \right)^2 \quad \text{in } \mathbb{R}^n \times \mathcal{A} \times \mathcal{B},$$

where $\lambda_\sigma > 0$ is defined by $\lambda_\sigma := \sigma\lambda$. Therefore, we find that there exists a unique periodic strong solution $v^\sigma \in H_{\text{per}}^2(Y)$ to (3.5). Note that the corresponding renormalization function $\gamma^\sigma \in C(\mathbb{R}^n \times \mathcal{A} \times \mathcal{B})$ (see (2.4)) is given by

$$\gamma^\sigma := \frac{\text{tr}(A) + \frac{\sigma}{\lambda_\sigma}}{|A|^2 + \frac{\sigma^2}{\lambda_\sigma^2}} = \frac{\text{tr}(A) + \frac{1}{\lambda}}{|A|^2 + \frac{1}{\lambda^2}}$$

and hence, $\gamma := \gamma^\sigma$ is independent of σ . The uniform bound (3.6) now follows from Remark 2.4. \square

The discontinuous Galerkin ($s = 0$) or the C^0 -IP ($s = 1$) finite element method from Section 2 yields an approximation $v_T^\sigma \in V_T^s$ to the problem (3.5) satisfying

$$\|v^\sigma - v_T^\sigma\|_{T, \lambda_\sigma} \leq C \inf_{z_T \in V_T^s} \|v^\sigma - z_T\|_{T, \lambda_\sigma} \leq C \inf_{z_T \in V_T^s} \|v^\sigma - z_T\|_{T, \bar{\sigma}\lambda}, \quad (3.7)$$

where the constant $C > 0$ can be chosen to be independent of σ ; see Section 2.5. In view of (3.7) and Section 2.5.3, we immediately obtain the following result.

Lemma 3.5 (Approximation of the cell σ -problem). *We assume that the assumptions of Section 3.1.1 hold. Let $\{\mathcal{T}_k\}_k$ be a sequence of conforming simplicial meshes with the property that $h_k := \max_{K \in \mathcal{T}_k} h_K \rightarrow 0$ as $k \rightarrow \infty$. For $\sigma \in (0, \bar{\sigma})$, we denote the periodic strong solution to (3.5) by $v^\sigma = v^\sigma(\cdot; x, p, R) \in H_{\text{per}}^2(Y)$ and, for a chosen $s \in \{0, 1\}$, the sequence of numerical approximations obtained by the finite element method from Section 2 by $\{v_{T_k}^\sigma\}_k \subset V_{T_k}^s$. Then, we have the convergence*

$$\|v^\sigma - v_{T_k}^\sigma\|_{T_k, \lambda_\sigma} \xrightarrow{k \rightarrow \infty} 0.$$

Further, the following assertions hold:

- (i) If $v^\sigma \in H^{2+r_K}(K)$ with $r_K > 0$ for all $K \in \mathcal{T}_k$, then we have the error bound

$$\|v^\sigma - v_{T_k}^\sigma\|_{T_k, \lambda_\sigma} \lesssim \inf_{z_{T_k} \in V_{T_k}^s} \|v^\sigma - z_{T_k}\|_{T, \bar{\sigma}\lambda} \lesssim \sqrt{\sum_{K \in \mathcal{T}_k} h_K^{2 \min\{r_K, P-1\}} \|\nabla v^\sigma\|_{H^{1+r_K}(K)}^2}$$

with constants independent of σ and the choice of (x, p, R) .

- (ii) In particular, if $v^\sigma \in H^{2+r}(Y)$ for some $r > 0$, and σ, x, p, R are fixed, we have the convergence rate

$$\|v^\sigma - v_{T_k}^\sigma\|_{T_k, \lambda_\sigma} = \mathcal{O}\left(h_k^{\min\{r, P-1\}}\right) \quad \text{as } k \rightarrow \infty.$$

Let us observe that without any additional regularity assumptions on v^σ , we have that $\|\nabla v^\sigma\|_{H^1(Y)} \leq C$ is uniformly bounded in σ . Indeed, this follows from (3.6) and Poincaré's inequality.

3.3. Approximation of the effective Hamiltonian

Let us note that the effective Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$ given by (3.3) is defined via the viscosity solutions $v^\sigma \in C(\mathbb{R}^n)$ to the cell σ -problem (3.2). Therefore, we make the technical assumption that

$$v^\sigma \in W_{\text{loc}}^{2,n}(\mathbb{R}^n), \quad (3.8)$$

so that the strong solution coincides with the unique viscosity solution to (3.5); see [10, 39, 40]. This is no further restriction when $n = 2$ or when we have an HJB problem; see [26].

Let us define the approximate effective Hamiltonian H_T^σ for $\sigma > 0$ via

$$H_T^\sigma : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}, \quad H_T^\sigma(x, p, R) := -\sigma \int_Y v_T^\sigma(\cdot; x, p, R). \quad (3.9)$$

We note that this definition is quite natural as we have from (3.3) that

$$Q_{x,p,R}^\sigma := \|- \sigma v^\sigma(\cdot; x, p, R) - H(x, p, R)\|_{L^\infty(Y)} \xrightarrow{\sigma \searrow 0} 0$$

for any $(x, p, R) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\text{sym}}^{n \times n}$.

Theorem 3.6 (Approximation of the effective Hamiltonian). *Assume that the assumptions of Section 3.1.1 and (3.8) hold. Let $\{\mathcal{T}_k\}_k$ be a sequence of conforming simplicial meshes with the property that $h_k := \max_{K \in \mathcal{T}_k} h_K \rightarrow 0$ as $k \rightarrow \infty$. Let $H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$ denote the effective Hamiltonian given by (3.3) and $H_{\mathcal{T}_k}^\sigma : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$ its numerical approximation (3.9). Then, for $\sigma \in (0, \bar{\sigma})$ and $(x, p, R) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\text{sym}}^{n \times n}$, we have the error bound*

$$|H_{\mathcal{T}_k}^\sigma(x, p, R) - H(x, p, R)| \lesssim Q_{x,p,R}^\sigma + \inf_{z_{\mathcal{T}_k} \in V_{\mathcal{T}_k}^s} \|v^\sigma(\cdot; x, p, R) - z_{\mathcal{T}_k}\|_{\mathcal{T}_k, \bar{\sigma}\lambda}. \quad (3.10)$$

In particular, there holds $\lim_{\sigma \rightarrow 0} \lim_{k \rightarrow \infty} |H_{\mathcal{T}_k}^\sigma(x, p, R) - H(x, p, R)| = 0$ for any (x, p, R) . Further,

- (i) if there exist $\{r_K\}_{K \in \mathcal{T}_k} \subset (0, \infty)$ such that $\sup_{K \in \mathcal{T}_k} \|\nabla v^\sigma(\cdot; x, p, R)\|_{H^{1+r_K}(K)} \leq C_{x,p,R} |K|^{\frac{1}{2}}$ holds uniformly in σ , then $\lim_{\frac{1}{\sigma}, k \rightarrow \infty} |H_{\mathcal{T}_k}^\sigma(x, p, R) - H(x, p, R)| = 0$ for any (x, p, R) with convergence rate

$$|H_{\mathcal{T}_k}^\sigma(x, p, R) - H(x, p, R)| \lesssim Q_{x,p,R}^\sigma + C_{x,p,R} \sqrt{\sum_{K \in \mathcal{T}_k} h_K^{2 \min\{r_K, P-1\}} |K|}. \quad (3.11)$$

- (ii) if there exists $r > 0$ such that $\|\nabla v^\sigma(\cdot; x, p, R)\|_{H^{1+r}(Y)} \leq C_{x,p,R}$ holds uniformly in σ , then we have that $\lim_{\frac{1}{\sigma}, k \rightarrow \infty} |H_{\mathcal{T}_k}^\sigma(x, p, R) - H(x, p, R)| = 0$ for any (x, p, R) with convergence rate

$$|H_{\mathcal{T}_k}^\sigma(x, p, R) - H(x, p, R)| \lesssim Q_{x,p,R}^\sigma + C_{x,p,R} h_k^{\min\{r, P-1\}}. \quad (3.12)$$

The constants absorbed in \lesssim in (3.10), (3.11) and (3.12) are independent of σ and (x, p, R) .

Proof. Let $\sigma \in (0, \bar{\sigma})$ and $(x, p, R) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\text{sym}}^{n \times n}$. We observe that by (3.7), and recalling $\lambda_\sigma = \sigma\lambda$, we have

$$\begin{aligned} \|\sigma v^\sigma(\cdot; x, p, R) - \sigma v_{\mathcal{T}_k}^\sigma(\cdot; x, p, R)\|_{L^2(Y)} &\lesssim \|v^\sigma(\cdot; x, p, R) - v_{\mathcal{T}_k}^\sigma(\cdot; x, p, R)\|_{\mathcal{T}_k, \lambda_\sigma} \\ &\lesssim \inf_{z_{\mathcal{T}_k} \in V_{\mathcal{T}_k}^s} \|v^\sigma(\cdot; x, p, R) - z_{\mathcal{T}_k}\|_{\mathcal{T}_k, \bar{\sigma}\lambda} \end{aligned} \quad (3.13)$$

with constants independent of σ and (x, p, R) . Further, we note that

$$\|-\sigma v^\sigma(\cdot; x, p, R) - H(x, p, R)\|_{L^2(Y)} \leq Q_{x,p,R}. \quad (3.14)$$

We can now conclude, using Hölder and triangle inequalities together with (3.13) and (3.14), that we have

$$\begin{aligned} |H_{\mathcal{T}_k}^\sigma(x, p, R) - H(x, p, R)| &= \left| -\sigma \int_Y v_{\mathcal{T}_k}^\sigma(\cdot; x, p, R) - H(x, p, R) \right| \\ &= \left| \int_Y (-\sigma v_{\mathcal{T}_k}^\sigma(\cdot; x, p, R) - H(x, p, R)) \right| \\ &\leq \|-\sigma v_{\mathcal{T}_k}^\sigma(\cdot; x, p, R) - H(x, p, R)\|_{L^2(Y)} \\ &\lesssim Q_{x,p,R} + \inf_{z_{\mathcal{T}_k} \in V_{\mathcal{T}_k}^s} \|v^\sigma(\cdot; x, p, R) - z_{\mathcal{T}_k}\|_{\mathcal{T}_k, \bar{\sigma}\lambda}, \end{aligned}$$

where the constant absorbed in \lesssim is independent of σ and (x, p, R) . This completes the proof of (3.10). The assertions (i) and (ii) are immediate consequences of (3.10) in view of Lemma 3.5. \square

Remark 3.7 (Improvement for HJB operators). Let us assume that the coefficients A, b, f from the HJBI operator (3.1) are such that the operator simplifies to an HJB operator

$$F(x, y, p, R) := \sup_{\beta \in \mathcal{B}} \{-A(y, \beta) : R - b(x, y, \beta) \cdot p - f(x, y, \beta)\}$$

with f satisfying the same assumptions as the components of b . We then have for $\sigma \in (0, \bar{\sigma})$ with $\bar{\sigma}$ sufficiently small and $(x, p, R) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\text{sym}}^{n \times n}$ that $Q_{x,p,R}^\sigma \leq C\sigma(1 + |p| + |R|)$ and $\|\nabla v^\sigma\|_{H^{1+r}(Y)} \leq C(1 + |p| + |R|)$, uniformly in σ , for some $r > 0$; see [13, 26]. Therefore, by Theorem 3.6 (ii), we have the error bound

$$|H_{T_k}^\sigma(x, p, R) - H(x, p, R)| \lesssim \left(\sigma + h_k^{\min\{r, P-1\}}\right)(1 + |p| + |R|),$$

where the constant absorbed in \lesssim is independent of σ and (x, p, R) . In particular, for any fixed (x, p, R) , there holds $\lim_{\sigma, k \rightarrow \infty} |H_{T_k}^\sigma(x, p, R) - H(x, p, R)| = 0$ with convergence rate

$$|H_{T_k}^\sigma(x, p, R) - H(x, p, R)| = \mathcal{O}\left(\sigma + h_k^{\min\{r, P-1\}}\right) \quad \text{as } \sigma^{-1}, k \rightarrow \infty. \quad (3.15)$$

Comparing (3.15) with the convergence rate obtained for the approximation of the effective Hamiltonian by the mixed finite element scheme presented in [26], we see that the threshold regarding the convergence rate in h_k is for both schemes $\mathcal{O}(h_k^r)$, and this maximum rate of convergence can be reached by choosing large polynomial degrees.

4. NUMERICAL EXPERIMENTS

4.1. Numerical solution of a periodic HJBI problem

In this numerical experiment, we consider the periodic HJBI problem

$$\begin{cases} \inf_{\alpha \in [0, \frac{1}{2}]} \sup_{\beta \in [0, 2\pi]} \{-A^{\alpha\beta} : \nabla^2 u + c^{\alpha\beta} u - f^{\alpha\beta}\} = 0 & \text{in } Y, \\ u \text{ is } Y\text{-periodic}, \end{cases} \quad (4.1)$$

where we define the diffusion coefficient by

$$A^{\alpha\beta} := Q(\beta) \begin{pmatrix} \frac{\cos(\alpha) + \sin(\alpha)}{\sqrt{2}} & 0 \\ 0 & \frac{\cos(\alpha) - \sin(\alpha)}{\sqrt{2}} \end{pmatrix} Q(\beta)^T, \quad Q(\beta) := \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix},$$

and set $c^{\alpha\beta} := \frac{\sec(\alpha)}{\sqrt{2}}$ and $f^{\alpha\beta} := \frac{\sec(\alpha)}{\sqrt{2}} \tilde{f}$ for $(\alpha, \beta) \in [0, \frac{1}{2}] \times [0, 2\pi]$. Here, we choose $\tilde{f} \in C_{\text{per}}(Y)$ such that the solution to (4.1) is given by

$$u : [0, 1]^2 \rightarrow \mathbb{R}, \quad u(y_1, y_2) = \cos(2\pi y_1) \cos(2\pi y_2).$$

We leave it to the reader to check that this problem fits into the setting of Section 2.1. In particular, we have that the Cordes condition (2.3) holds with $\lambda = 1$.

Remark 4.1. The renormalized HJBI problem (2.6) corresponding to (4.1) is given by

$$\begin{cases} \inf_{\alpha \in [0, \frac{1}{2}]} \sup_{\beta \in [0, 2\pi]} \{-\gamma^{\alpha\beta} A^{\alpha\beta} : \nabla^2 u + u\} = \tilde{f} & \text{in } Y, \\ u \text{ is } Y\text{-periodic}, \end{cases}$$

where $\gamma^{\alpha\beta} := \sqrt{2} \cos(\alpha)$ for $(\alpha, \beta) \in [0, \frac{1}{2}] \times [0, 2\pi]$.

We apply the C^0 -IP and discontinuous Galerkin finite element schemes from Section 2.5.2 to the HJBI problem (4.1). Under uniform mesh-refinement, we illustrate the behavior of the error

$$\|u - u_T\|_T := \sqrt{\int_Y (|\nabla^2(u - u_T)|^2 + 2|\nabla(u - u_T)|^2 + (u - u_T)^2) + |u - u_T|_{J,T}^2} \quad (4.2)$$

and of the *a posteriori* error estimator (see Theorem 2.6), i.e.,

$$\eta_T(u_T) := \sqrt{\int_Y |F_\gamma[u_T]|^2 + |u_T|_{J,T}^2} \quad (4.3)$$

for the numerical approximation $u_T \in V_T^s$. For the implementation, we have used the software package NGSolve [50] and the discrete nonlinear problems are solved using a Howard-type algorithm as in [36]. Since the Isaacs operator is nonconvex, the discrete nonlinear problems are solved using a Howard-type algorithm similar to Algorithm Ho-4 from [8]. To the authors' knowledge, it is not known whether Algorithm Ho-4 of [8] converges in general for Isaacs type problems (though it is empirically observed in our experiments). However, see [52] for a proof of the superlinear convergence of the semismooth Newton (or Howard) algorithm for the HJB problem.

Figure 3 presents the performance of the C^0 interior penalty and discontinuous Galerkin finite element methods using polynomial degrees $P \in \{2, 3\}$ and parameters $\theta \in \{0, \frac{1}{2}\}$. We observe optimal rates of convergence for both schemes, that is, order $\mathcal{O}(N^{-\frac{1}{2}})$ for $P = 2$ and order $\mathcal{O}(N^{-1})$ for $P = 3$, where we denote the number of degrees of freedom by N .

4.2. Numerical approximation of the effective Hamiltonian

In this numerical experiment, we demonstrate the numerical scheme for the approximation of the effective Hamiltonian corresponding to the HJBI operator

$$F : \mathbb{R}^2 \times \mathbb{R}_{\text{sym}}^{2 \times 2} \rightarrow \mathbb{R}, \quad F(y, R) := \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \{-A^{\alpha\beta}(y) : R - 1\} \quad (4.4)$$

with $\mathcal{A} := [1, 2]$, $\mathcal{B} := [0, 1]$, and the coefficient $A = A(y, \alpha, \beta) : \mathbb{R}^2 \times \mathcal{A} \times \mathcal{B} \rightarrow \mathbb{R}_{\text{sym}}^{2 \times 2}$ given by

$$A^{\alpha\beta}(y) := (a_0(y) + \alpha\beta a_1(y))B,$$

where we choose positive scalar functions $a_0, a_1 : \mathbb{R}^2 \rightarrow (0, \infty)$ and a symmetric positive definite matrix $B \in \mathbb{R}_{\text{sym}}^{2 \times 2}$ defined by

$$B := \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}, \quad a_0 \equiv 1, \quad a_1(y) := \sin^2(2\pi y_1) \cos^2(2\pi y_2) + 1.$$

It is straightforward to check that this problem fits into the framework of Section 3.1.1 and in particular we have that the Cordes condition (3.4) holds with $\lambda = \frac{1}{4}$. This HJBI operator is chosen so that we know the effective Hamiltonian explicitly.

Remark 4.2. It can be checked that the HJBI operator (4.4) can be rewritten as HJB operator

$$F(y, R) = \sup_{\beta \in \mathcal{B}} \{-(a_0(y) + \beta a_1(y))B : R - 1\}, \quad (y, R) \in \mathbb{R}^2 \times \mathbb{R}_{\text{sym}}^{2 \times 2},$$

for which the effective Hamiltonian $H : \mathbb{R}_{\text{sym}}^{2 \times 2} \rightarrow \mathbb{R}$ is known explicitly and given by

$$H(R) := \max \left\{ -\left(\int_Y \frac{1}{a_0} \right)^{-1} B : R - 1, -\left(\int_Y \frac{1}{a_0 + a_1} \right)^{-1} B : R - 1 \right\}$$

for $R \in \mathbb{R}_{\text{sym}}^{2 \times 2}$; see [21].

We make it our goal to approximate the effective Hamiltonian $H(R)$ at the point

$$R := \begin{pmatrix} -2 & 1 \\ 1 & -3 \end{pmatrix},$$

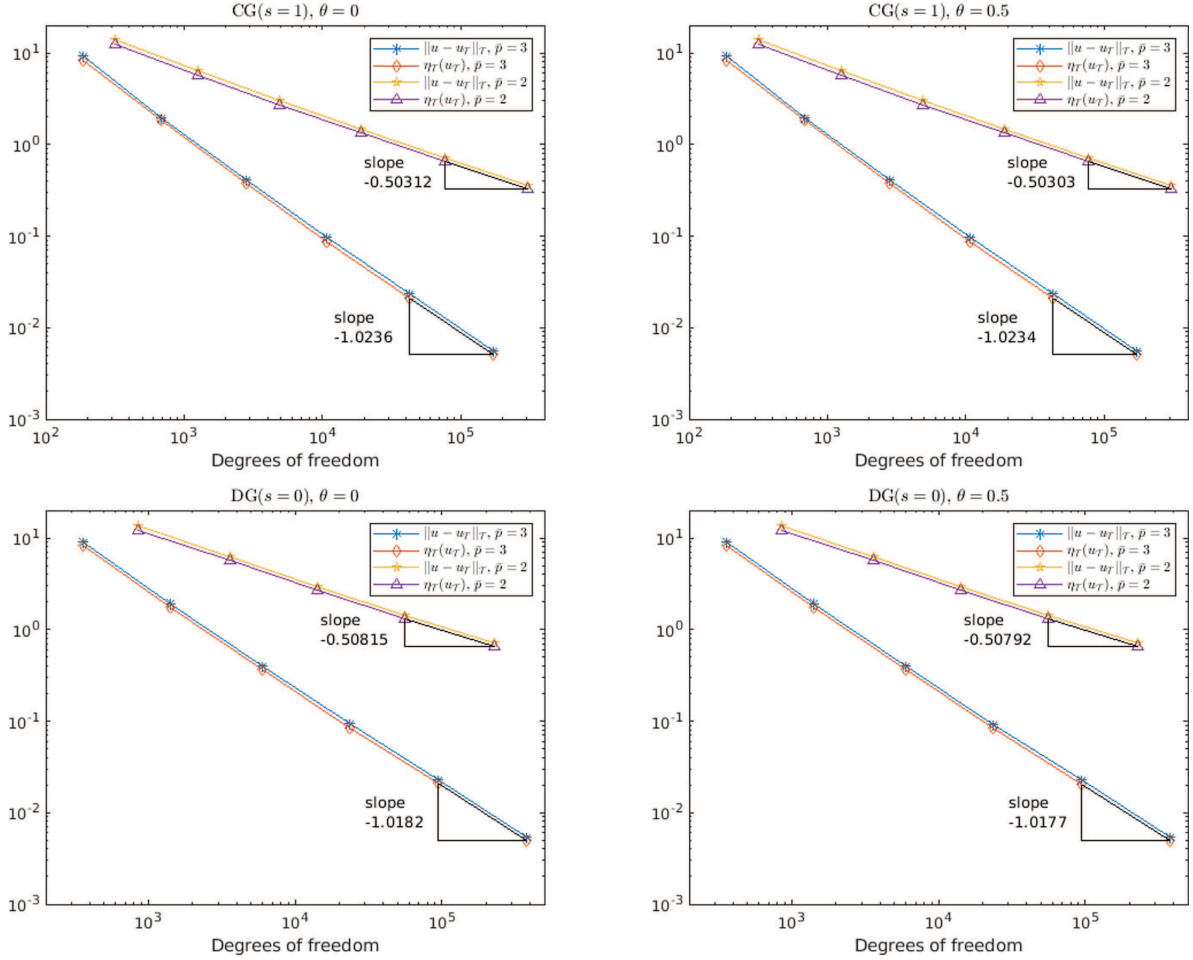


FIGURE 3. Approximation of the solution u to the HJBI problem (4.1) via the C^0 -IP (top) and DG (bottom) schemes under mesh-refinement with polynomial degrees $P \in \{2, 3\}$. We illustrate the error (4.2) and the *a posteriori* estimator (4.3) for the approximation $u_T \in V_T^s$ to the solution u , using $\theta = 0$ (left) and $\theta = \frac{1}{2}$ (right).

noting that the same problem was already used for the numerical experiments in [26]. As we have $B : R = -18 < 0$, the true effective Hamiltonian at this chosen point can be computed as

$$H(R) = -\left(\int_Y \frac{1}{a_0 + a_1}\right)^{-1} B : R - 1 = \frac{9\sqrt{6}\pi}{K(\frac{1}{3})} - 1 \approx 38.9429127, \quad (4.5)$$

where $K(\frac{1}{3}) := \int_0^{\frac{\pi}{2}} (1 - \frac{1}{3} \sin^2(t))^{-\frac{1}{2}} dt$ is the complete elliptic integral of the first kind with parameter $m = \frac{1}{3}$.

In our numerical experiments, we approximate the true value of the effective Hamiltonian $H(R)$ from (4.5) by $H_T^\sigma(R)$ as defined in (3.9), where we use the C^0 -IP finite element method ($s = 1$) with $\theta = \frac{1}{2}$ to obtain the approximation $v_T^\sigma(\cdot; R)$ to the solution $v^\sigma(\cdot; R)$ of the cell σ -problem as described in Section 3.2. We denote the relative approximation error by

$$E_T^\sigma := \frac{|H_T^\sigma(R) - H(R)|}{|H(R)|}, \quad H_T^\sigma(R) := -\sigma \int_Y v_T^\sigma(\cdot; R)$$

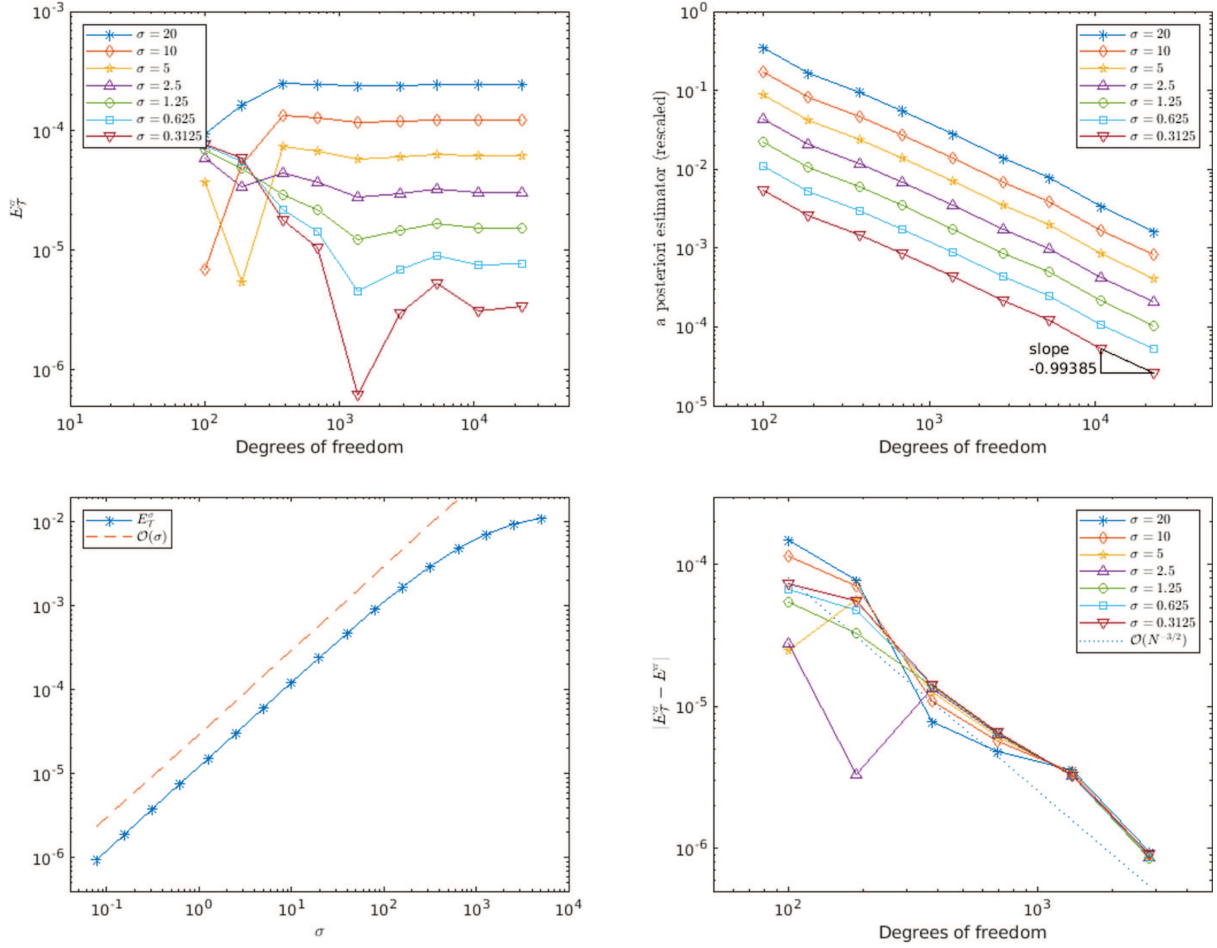


FIGURE 4. *Top*: relative error E_T^σ (left) and rescaled *a posteriori* error estimator (right) for fixed σ under mesh-refinement using $P = 3$. *Bottom*: relative error E_T^σ using a fixed discretization with $P = 20$ (left), and illustration of the speed of convergence of E_T^σ to E^σ for fixed σ under mesh-refinement using $P = 3$ (right).

and further write

$$E^\sigma := \frac{|H^\sigma(R) - H(R)|}{|H(R)|}, \quad H^\sigma(R) := -\sigma \int_Y v^\sigma(\cdot; R).$$

Let us point out that the approximate corrector $v^\sigma(\cdot; R)$ and consequently the value of E^σ is not known exactly, but we expect that $E^\sigma = \mathcal{O}(\sigma)$ from Remark 3.7.

Figure 4 (top) shows the behavior of the relative approximation error E_T^σ under uniform mesh-refinement for fixed values of σ , and the corresponding *a posteriori* error estimator $\eta_T(v_T^\sigma)$ (re-scaled by a multiplicative constant C_σ for illustration purposes) using polynomial degree $P = 3$. We observe that E_T^σ converges to a constant, namely E^σ , and that the *a posteriori* estimator is of order $\mathcal{O}(N^{-1})$ as expected, where N denotes the degrees of freedom. In particular, let us emphasize that this is the expected behavior and that the relative error for large numbers of degrees of freedom is entirely dominated by the σ -error E^σ .

Figure 4 (bottom) illustrates accurate approximations to the unknown values E^σ for various values of the parameter σ , and the convergence rate for the convergence of E_T^σ to the value E^σ . The accurate approximations to the values E^σ are obtained using high polynomial degree $P = 20$ and a fixed triangulation (longest edge $\sqrt{2} \times 2^{-3}$), and we observe convergence of order $\mathcal{O}(\sigma)$ as σ tends to zero, as expected. Let us note that it is difficult to obtain accurate approximations for extremely small values of σ as those lead to poorly conditioned discrete problems. We further observe that $|E_T^\sigma - E^\sigma|$ is of order $\mathcal{O}(N^{-\frac{3}{2}})$ for fixed σ , where we take the unknown value E^σ to be the previously obtained accurate approximation. This rate is higher than predicted by Remark 3.7, which is based on an error estimate in the $\|\cdot\|_{T,\lambda_\sigma}$ -norm and is therefore indeed expected to overestimate the error between $H_T^\sigma(R)$ and $H(R)$ related to the weaker integral functional from (3.9).

5. CONCLUSION

In this work we introduced discontinuous Galerkin and C^0 interior penalty finite element schemes for the numerical approximation of periodic HJBI problems with an application to the approximation of effective Hamiltonians to ergodic HJBI operators. The first part of this paper was focused on periodic HJBI cell problems and we have performed rigorous *a posteriori* and *a priori* error analyses for a wide class of numerical schemes. In particular, the *a posteriori* analysis was independent of the choice of numerical scheme. The second part of this paper was focused on the approximation of the effective Hamiltonian corresponding to ergodic HJBI operators. An approximation scheme for the effective Hamiltonian *via* a DG/ C^0 -IP approximation to approximate correctors was presented and rigorously analyzed. Finally, we presented numerical experiments illustrating the theoretical results and the performance of the numerical schemes.

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