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A HYBRIDIZABLE DISCONTINUOUS GALERKIN METHOD FOR SECOND ORDER ELLIPTIC EQUATIONS WITH DIRAC DELTA SOURCE

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Abstract. In this paper, we investigate a hybridizable discontinuous Galerkin method for second order elliptic equations with Dirac measures. Under assumption that the domain is convex and the mesh is quasi-uniform, *a priori* error estimate for the error in L^2-norm is proved. By duality argument and Oswald interpolation, *a posteriori* error estimates for the errors in L^2-norm and $W^{1,p}$-seminorm are also obtained. Finally, numerical examples are provided to validate the theoretical analysis.

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1. Introduction

In this article, we consider the following problem

$$-\Delta u = \delta_{x_0} \quad \text{in } \Omega, \tag{1.1a}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.1b}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is an open, bounded, polygonal or polyhedral domain with Lipschitz boundary $\partial\Omega$, and δ_{x_0} is a Dirac measure concentrated at the interior point $x_0 \in \Omega$. An instance of system (1.1) can be found in the electric field generated by a point charge. Another instance appears in the PDE-constrained optimal control problem [11, 12] and in the acoustic monopoles or pollutant transport and degradation in an aquatic media [3].

Since the Dirac function δ_{x_0} does not belong to $H^{-1}(\Omega)$, the solution of problem (1.1) is not in $H^1(\Omega)$. In spite of the fact that the solution of problem (1.1) has a very low regularity, it still can be numerically approximated by standard finite element methods. For $d = 2$, Babuška [8] obtained the convergence rate $O(h^{1-\epsilon})$ $(\epsilon > 0)$ for the error in L^2-norm. In [33], Scott removed ϵ and yielded the convergence rate $O(h^{2-d/2})$. For an arbitrary Borel measure, Casas derived in [11] a similar result by using different techniques. As for the interior maximum norm error estimates, it has been proved in [34] by Schatz and Wahlbin. It is worth noting that Eriksson [19] proved *a priori* error estimates for L^1- and $W^{1,1}$-errors on adequately refined meshes.

The singular nature of problem (1.1) suggests that the meshes adequately refined around the delta support should be used to boost the accuracy of approximation. In [6], Apel *et al.* proved the L^2-error estimates of almost optimal order by using graded meshes on a convex and polygonal domain. In [2], adaptive finite element

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methods based on *a posteriori* error estimators were considered for $d = 2$. The efficient and reliable *a posteriori* error estimators for L^p-norm ($p \in (1, \infty)$) and $W^{1,p}$-seminorm ($p \in (p_0, 2)$) were obtained by using duality argument, where $p_0 \in [1, 2)$ is a real number depending on the largest inner angle of the domain Ω. For the error in fractional Sobolev space $H^s(\Omega)$ ($s \in (\frac{1}{2}, 1)$), the residual type *a posteriori* error estimators with specifically tailored oscillation were derived by Gaspoz *et al.* [25]. In [7], Agnelli *et al.* developed a reliable and efficient *a posteriori* error estimator for the weighted H^1_α-norm ($\alpha \in \mathbb{I} \subset (\frac{d}{2} - 1, \frac{d}{2})$).

In 2012, Houston and Wihler [26] studied the discontinuous Galerkin (DG) methods for problem (1.1) with $d = 2$. The convergence rate $O(h)$ for L^2-error was proved under a constraint that x_0 lies in the interior of an element. In addition, *a posteriori* error estimator, which is efficient and reliable for an extended L^2-norm, was also shown. It is well known that the DG method is very flexible when it is applied to solve the partial differential equations, however too many globally coupled degrees of freedom are always used and the discrete system is large in particular when the high order polynomial is utilized. Relatively, the HDG methods, proposed by Cockburn *et al.* [15], not only can keep the advantage of DG methods, but also can get a system with significantly reduced degrees of freedom by introducing the Lagrange multipliers. Currently, it has been used for many problems such as elliptic [13, 14], convection diffusion [21], fluid flow [16, 28, 32], and optimal control [18, 23, 29], etc.. To the best of our knowledge, there still has no work on error analysis of HDG methods for problems with Dirac measures.

Therefore, in this paper, we investigate error analysis of HDG methods for problem (1.1). In particular, *a priori* error estimate with convergence rate $O(h^{2-d/2})$ is obtained for the error in L^2-norm. On the other hand, *a posteriori* error estimator, which provides an upper and a lower bounds for the error in L^2-norm, is proved in a convex domain. Moreover, *a posteriori* error estimator that is efficient and reliable for the error in $W^{1,p}$-seminorm is also derived in a non-convex Lipschitz polygon, where $p \in (P^\Omega, 2)$ and $P^\Omega > 0$ is a real number depending on the largest inner angle of the domain Ω. Finally, some numerical examples are presented to validate the numerical analysis.

Compared with [2], we need introducing the Oswald interpolation to prove *a posteriori* error estimator for $W^{1,p}$-seminorm, moreover the *a posteriori* error estimators obtained in this paper incorporate the term $\|u_h - \widehat{u}_h\|_{0,p,\partial K}$ that does not appear in [2], see Lemma 4.4 and Section 4 for more detail, where $(u_h, \widehat{u}_h)$ is the HDG solution of problem (1.1). It is worth noting that Oswald interpolation is a very important tool in *a posteriori* error analysis of HDG methods [4, 5, 14, 17], because it provides a continuous approximation for a discontinuous piecewise polynomial function. Compared with [26], we not only prove *a priori* and *a posteriori* error estimates for L^2-norm in two- and three-dimensional cases, but also derive *a posteriori* error estimate for $W^{1,p}$-seminorm in two dimensional case.

The rest of this article is arranged as follows: In Section 2, notation and definition corresponding to Sobolev spaces and meshes are provided. In addition, the HDG scheme and some known results are also presented in this section. In Section 3, *a priori* error estimates are proved. In Sections 4 and 5, the reliability and efficiency of *a posteriori* error estimators are shown respectively. In Section 6, some numerical examples are presented to validate the numerical analysis. Finally, we end this paper by some conclusions in Section 7.

Throughout this paper, let c with or without subscript be a generic positive constant independent of the mesh size, which may be different in different places. For ease of exposition, we denote $A \le cB$ by $A \lesssim B$ and $A \approx B$ by $A \lesssim B \lesssim A$.

2. Weak formulation and HDG discretization

For any bounded and open set $D \subset \mathbb{R}^d$ or $D \subset \mathbb{R}^{d-1}$, $W^{s,q}(D)$ denotes the standard Sobolev space with norm $\|\cdot\|_{s,q,D}$ and seminorm $|\cdot|_{s,q,D}$. When $q = 2$, the Sobolev space $W^{s,2}(D)$ is denoted by $H^s(D)$ with norm $\|\cdot\|_{s,D}$ and seminorm $|\cdot|_{s,D}$. If we further have $s = 0$, then $H^0(D)$ coincides with $L^2(D)$, and the inner product is described by $(\cdot,\cdot)_D$ for $D \subset \mathbb{R}^d$ or $\langle\cdot,\cdot\rangle_D$ for $D \subset \mathbb{R}^{d-1}$. For $q \in (1, \infty)$, let q' be the conjugate number of q such that $\frac{1}{q} + \frac{1}{q'} = 1$, then the dual space to $W^{s,q}_0(\Omega)$ is denoted by $W^{-s,q'}(\Omega)$, where $W^{s,q}_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in the space $W^{s,q}(\Omega)$ [1]. If no confusion induced, we also use $\langle\cdot,\cdot\rangle$ to denote the duality pairing between spaces $W^{-s,q'}(\Omega)$ and $W_0^{s,q}(\Omega)$. Finally, we define $H(\mathrm{div},\Omega) := \{\mathbf{v} \in (L^2(\Omega))^d : \nabla\cdot\mathbf{v} \in L^2(\Omega)\}$, where $\nabla\cdot$ is the divergence operator.

The weak formulation of problem (1.1) is to find $u \in W_0^{1,p}(\Omega)$, $p \in [1, \frac{d}{d-1})$, such that

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx = v(x_0) \quad \forall v \in W_0^{1,p'}(\Omega). \tag{2.1}$$

From the embedding theorem, we know that $W_0^{1,p'}(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$, hence $v(x_0)$ is well-defined.

Remark 2.1. If the domain Ω is convex, we know from ([11], Thm. 2) and ([22], Thm. 2.1) that the weak formulation (2.1) has a unique weak solution $u \in W_0^{1,p}(\Omega)$ with $1 \le p < \frac{d}{d-1}$. If the domain Ω is only a polygonal domain with Lipschitz boundary, the aforementioned result is also correct for $d = 2$, see ([2], Sect. 2) for more detail.

In order to describe the HDG discretization of problem (2.1), we consider a conforming and shape-regular triangulation $\mathcal{T}_h$ of the domain Ω such that $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} \overline{K}$. Denote $\mathcal{E}_h^o$ the set of all interior faces of $\mathcal{T}_h$ and $\mathcal{E}_h^\partial$ the set of all boundary faces. Then we define $\mathcal{E}_h = \mathcal{E}_h^o \cup \mathcal{E}_h^\partial$ and $\partial\mathcal{T}_h = \{\partial K : K \in \mathcal{T}_h\}$, where ∂K denotes the boundary of K. For any $K \in \mathcal{T}_h$ and $F \in \mathcal{E}_h$, let h_K and h_F be the diameters of K and F. Moreover we set $h = \max_{K\in\mathcal{T}_h} h_K$. For the interior face $F \in \mathcal{E}_h^o$, we define the jumps $[v]|_F$ and $[\nabla v\cdot\mathbf{n}]|_F$ by

$$[v]|_F = v^+ - v^-, \quad [\nabla v\cdot\mathbf{n}]|_F = \nabla v^+\cdot\mathbf{n}^+ + \nabla v^-\cdot\mathbf{n}^-$$

where v^+ and v^- are the traces of v on $F = \overline{K^+}\cap\overline{K^-}$, and $\mathbf{n}^+$ and $\mathbf{n}^-$ denote the unit vector normal to the face F. As for $F \in \mathcal{E}_h^\partial$, we formally set $[v]|_F = v|_F$ and $[\nabla v\cdot\mathbf{n}]|_F = (\nabla v)|_F\cdot\mathbf{n}$. Next, we define the mesh-dependent inner products and norms:

$$(v_1,v_2)_D = \sum_{K\in D}(v_1,v_2)_K, \quad \langle\mu_1,\mu_2\rangle_{\partial D} = \sum_{K\in D}\langle\mu_1,\mu_2\rangle_{\partial K}, \quad \forall D \subset \mathcal{T}_h,$$
$$\|v\|_{0,p,\mathcal{T}_h} = \Big(\sum_{K\in\mathcal{T}_h}\|v\|_{0,p,K}^p\Big)^{1/p}, \quad 1 \le p < \infty$$
$$|||(v,\mu)|||_{1,h}^2 = \|\nabla v\|_{0,\mathcal{T}_h}^2 + \sum_{K\in\mathcal{T}_h} h_K^{-1}\|v-\mu\|_{L^2(\partial K)}^2,$$

For $k \ge 1$, the discontinuous finite element spaces corresponding to the partition $\mathcal{T}_h$ are given as follows:

$$\begin{aligned} W_h^k &= \{w \in L^2(\Omega) : w|_K \in \mathcal{P}^k(K),\ \forall K \in \mathcal{T}_h\},\\ M_h^k &= \{\mu \in L^2(\mathcal{E}_h) : \mu|_F \in \mathcal{P}^k(F),\ \forall F \in \mathcal{E}_h\},\\ M_{h,0}^k &= \{\mu \in M_h^k : \mu|_F = 0,\ \forall F \in \mathcal{E}_h^\partial\}, \end{aligned}$$

where $\mathcal{P}^k(D)$ denotes the set of polynomials of degree no larger than k on the domain D. Then the HDG discretization of problem (2.1), that approximates the exact solution $(u, u|_{\mathcal{E}_h})$, reads as follows: Find $(u_h,\widehat{u}_h) \in W_h^k \times M_{h,0}^k$ such that

$$a_h(u_h,\widehat{u}_h;v_h,\mu_h) = \frac{1}{\sharp T_{x_0}}\sum_{K\in T_{x_0}} v_h|_K(x_0) \quad \forall (v_h,\mu_h) \in W_h^k \times M_{h,0}^k, \tag{2.2}$$

where $T_{x_0} = \{K \in \mathcal{T}_h : x_0 \in \overline{K}\}$, $\sharp T_{x_0}$ denotes the number of elements in T_{x_0}, and a_h is defined by

$$\begin{aligned} a_h(u_h,\widehat{u}_h;v_h,\mu_h) = {} & (\nabla u_h,\nabla v_h)_{\mathcal{T}_h} - \langle\nabla v_h\cdot\mathbf{n}, u_h - \widehat{u}_h\rangle_{\partial\mathcal{T}_h}\\ & - \langle\nabla u_h\cdot\mathbf{n}, v_h - \mu_h\rangle_{\partial\mathcal{T}_h} + \langle\tau(u_h - \widehat{u}_h), v_h - \mu_h\rangle_{\partial\mathcal{T}_h}. \end{aligned}$$

Here τ is the stabilization parameter. According to the inverse estimate and trace inequality, we have

$$\begin{aligned} a_h(v_h,\mu_h;v_h,\mu_h) &= \|\nabla v_h\|_{0,\Omega}^2 + \|\tau^{1/2}(v_h-\mu_h)\|_{L^2(\partial\mathcal{T}_h)}^2 - 2\langle \nabla v_h\cdot\mathbf{n}, v_h-\mu_h\rangle_{\partial\mathcal{T}_h} \\ &\geq \frac{1}{2}\|\nabla v_h\|_{0,\mathcal{T}_h}^2 + \sum_{K\in\mathcal{T}_h}\sum_{F\in\partial K}(\tau-\frac{c}{h_K})\|v_h-\mu_h\|_{0,F}^2, \end{aligned} \tag{2.3}$$

for any $(v_h,\mu_h)\in W_h^k\times M_h^k$, where c is a constant depending on the polynomial degree and shape-regularity of the mesh. Therefore the bilinear form a_h is coercive with respect to the norm $|||(\cdot,\cdot)|||_{1,h}$ for $\tau=\frac{\tau_0}{h_F}$ on each face $F\in\mathcal{E}_h$ with τ_0 sufficiently large. So the HDG formulation (2.2) has an unique solution $(u_h,\widehat{u}_h)\in W_h^k\times M_{h,0}^k$ for τ_0 sufficiently large. Notice that the effect of τ_0 on the error will be discussed by numerical experiments in Section 6.

Now we end this section by introducing some known results that will play an important role in the subsequent proofs.

Lemma 2.2. *For each element $K\in\mathcal{T}_h$ and any given nonnegative integer j, the following estimates hold*

$$|v|_{i,p,K}\lesssim h_K^{-i+\frac{d}{p}-\frac{d}{q}}\|v\|_{q,K},\quad \forall v\in\mathcal{P}^j(K),\ i=0,1,\ 1\le p,q\le\infty \tag{2.4}$$

$$\|v\|_{0,p,\partial K}\lesssim \|v\|_{0,p,K}^{1-1/p}\|v\|_{1,p,K}^{1/p},\quad \forall v\in W^{1,p}(K),\ 1\le p\le\infty. \tag{2.5}$$

Proof. The approximation results (2.4) and (2.5) can be found in Lemma 4.5.3, Theorem 1.6.6 from [9]. □

Lemma 2.3. *Let $I:\mathcal{C}(\overline{\Omega})\to V_h^1$ be the Lagrange interpolation operator, where V_h^1 denotes the space of continuous piecewise linear polynomials. Then we have*

$$|v-Iv|_{i,p,K}\lesssim h_K^{2-i}|v|_{2,p,K},\quad \forall v\in W^{2,p}(K),\ i=0,1,2,\ \frac{d}{2}<p<\infty, \tag{2.6}$$

$$|v-Iv|_{i,p,K}\lesssim h_K^{1-i}|v|_{1,p,K},\quad \forall v\in W^{1,p}(K),\ i=0,1,\ d<p<\infty, \tag{2.7}$$

$$|v-Iv|_{0,\infty,K}\lesssim h_K^{2-\frac{d}{p}}|v|_{2,p,K},\quad \forall v\in W^{2,p}(K),\ \frac{d}{2}<p<\infty, \tag{2.8}$$

$$|v-Iv|_{0,\infty,K}\lesssim h_K^{1-\frac{d}{p}}|v|_{1,p,K},\quad \forall v\in W^{1,\infty}(\Omega),\ d<p<\infty. \tag{2.9}$$

Proof. The Lagrange interpolation error estimates can be found in Theorem 4.4.4, Corollary 4.4.7 from [9]. □

Lemma 2.4. *Let $\widetilde{I}:W^{s,p}\to V_h^1$ be the Scott-Zhang interpolation. Then we have*

$$\sum_{K\in\mathcal{T}_h} h_K^{p(r-s)}\|v-\widetilde{I}v\|_{r,p,K}^p\lesssim |v|_{s,p,\Omega}^p,\quad \forall v\in W^{s,p}(\Omega), \tag{2.10}$$

where $0\le r\le s\le 2$ and $1\le p\le\infty$.

Proof. The result can be found in Theorem 4.6.12 from [9]. □

The next result shows that the polynomial $v_h\in W_h^k$ can be approximated by a continuous function $\widetilde{v}_h\in W_h^k\cap H_0^1(\Omega)$. It is well-known that this is the so-called Oswald interpolation.

Lemma 2.5. *For any $v_h\in W_h^k$, there exists a function $\widetilde{v}_h\in W_h^k\cap H_0^1(\Omega)$ such that*

$$\sum_{K\in\mathcal{T}_h}\|v_h-\widetilde{v}_h\|_{0,p,K}^p\lesssim \sum_{F\in\mathcal{E}_h^o} h_F\|[v_h]\|_{0,p,F}^p + \sum_{F\in\mathcal{E}_h^\partial} h_F\|v_h\|_{0,p,F}^p, \tag{2.11}$$

$$\sum_{K\in\mathcal{T}_h}\|\nabla(v_h-\widetilde{v}_h)\|_{0,p,K}^p\lesssim \sum_{F\in\mathcal{E}_h^o} h_F^{1-p}\|[v_h]\|_{0,p,F}^p + \sum_{F\in\mathcal{E}_h^\partial} h_F^{1-p}\|v_h\|_{0,p,F}^p, \tag{2.12}$$

for $1\le p<\infty$.

Proof. Here we only prove the error estimate (2.12). As for the error estimate (2.11), it can be proved similarly.

For the case of $p = 2$, the approximation (2.12) has been proved in Theorem 2.2 from [27]. As for other cases, the method of proof is similar.

Let $\mathcal{N}_K = \{x_K^{(j)},\ j = 1, \cdots, m\}$ be the Lagrange nodes of K and $\{\phi_K^{(j)},\ j = 1, \cdots, m\}$ the corresponding Lagrange basis functions. Then we have $v_h = \sum_{K\in\mathcal{T}_h}\sum_{j=1}^m \alpha_K^{(j)}\phi_K^{(j)}$ and $\widetilde{v}_h = \sum_{\nu\in\mathcal{N}}\beta^{(\nu)}\phi^{(\nu)}$, where $\mathcal{N} = \bigcup_{K\in\mathcal{T}_h}\mathcal{N}_K$ and $\phi^{(\nu)}$ is the Lagrange basis function of the node ν. Here $\beta^{(\nu)}$ is defined as that in Theorem 2.2 from [27]. Hence

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}\|\nabla(v_h-\widetilde{v}_h)\|_{0,p,K}^p &= \sum_{K\in\mathcal{T}_h}\int_K |\sum_{j=1}^m(\alpha_K^{(j)}-\beta_K^{(j)})\nabla\phi_K^{(j)}|^p\mathrm{d}x\\ &\lesssim \sum_{K\in\mathcal{T}_h}\sum_{j=1}^m|\alpha_K^{(j)}-\beta_K^{(j)}|^p\|\nabla\phi_K^{(j)}\|_{0,p,K}^p,\end{aligned}$$

where $\beta_K^{(j)} = \beta^{(\nu)}$ whenever $x_K^{(j)} = \nu$. Since $\|\nabla\phi_K^{(j)}\|_{0,p,K} \lesssim h_K^{\frac{d}{p}-1}$, the above inequality and inverse estimate (2.4) yield

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}\|\nabla(v_h-\widetilde{v}_h)\|_{0,p,K}^p &\lesssim \sum_{K\in\mathcal{T}_h} h_K^{d-p}\sum_{j=1}^m|\alpha_K^{(j)}-\beta_K^{(j)}|^p\\ &\lesssim \sum_{F\in\mathcal{E}_h^o} h_F^{d-p}\|[v_h]\|_{0,\infty,F}^p + \sum_{F\in\mathcal{E}_h^\partial} h_F^{d-p}\|v_h\|_{0,\infty,F}^p\\ &\lesssim \sum_{F\in\mathcal{E}_h^o} h_F^{1-p}\|[v_h]\|_{0,p,F}^p + \sum_{F\in\mathcal{E}_h^\partial} h_F^{1-p}\|v_h\|_{0,p,F}^p.\end{aligned}$$

□

3. *A priori* error analysis

This section mainly focuses on *a priori* error analysis for the error $\|u-u_h\|_{0,\Omega}$. To this end, we introduce the following auxiliary problem:

$$-\Delta\phi_f = f \quad \text{in } \Omega, \tag{3.1a}$$
$$\phi_f = 0 \quad \text{on } \partial\Omega. \tag{3.1b}$$

Theorem 3.1. *The problem (3.1) has an unique weak solution which holds the following regularities from the different cases:*

(i) If $f \in L^2(\Omega)$ and the domain Ω is convex, we have

$$\phi_f \in H^2(\Omega)\cap H_0^1(\Omega) \quad \textit{and} \quad \|\phi_f\|_{2,\Omega} \lesssim \|f\|_{0,\Omega}. \tag{3.2}$$

(ii) If $f \in L^{p'}(\Omega)$, $d = 2$ and the domain Ω is a non-convex Lipschtiz polygon, we have

$$\phi_f \in W_0^{2,p'}(\Omega) \quad \textit{and} \quad |\phi_f|_{2,p',\Omega} \lesssim \|f\|_{0,p',\Omega}, \tag{3.3}$$

where $(2-\frac{\pi}{\theta})p' < 2$ and $\theta > \pi$ is the largest inner angle of the domain Ω.

Proof. The result of case (i) is well-known [10]. The result of case (ii) can be found in Section 4 from [2] and Theorem 4.4.4.13 from [24]. □

Following the approach in [26,33], we define $\delta_h \in W_h^k$ such that $\delta_h = 0$ on $\mathcal{T}_h \backslash T_{x_0}$, and

$$\int_K \delta_h v_h dx = v_h|_K(x_0) \quad \forall v_h \in \mathcal{P}^k(K),\ K \in T_{x_0}. \tag{3.4}$$

Then with ([26], Subsect. 3.1) and inverse estimate (2.4), we can obtain

$$\|\delta_h\|_{0,K} \approx h_K^{-d/2}, \quad \forall K \in T_{x_0}. \tag{3.5}$$

Furthermore, we define ϕ_{δ_h} as the solution of problem (3.1) with $f = \frac{1}{\sharp T_{x_0}}\delta_h$, and the weak formulation is to find $\phi_{\delta_h} \in H_0^1(\Omega)$ such that

$$a(\phi_{\delta_h}, v) = \frac{1}{\sharp T_{x_0}} \int_\Omega \delta_h v \mathrm{dx}, \quad \forall v \in H_0^1(\Omega). \tag{3.6}$$

From now on, we assume in the rest of this section that the domain Ω is convex and the partition $\mathcal{T}_h$ is quasi-uniform.

By a simple calculation, we can find $u - u_h = u - \phi_{\delta_h} + \phi_{\delta_h} - u_h$, hence *a priori* error analysis for the error $\|u - u_h\|_{0,\Omega}$ can be divided into two parts. Firstly, we are going to deal with the error $\|u - \phi_{\delta_h}\|_{0,\Omega}$.

Lemma 3.2. *Let $u \in W_0^{1,p}(\Omega)$ ($p \in [1, \frac{d}{d-1})$) and $\phi_{\delta_h} \in H_0^1(\Omega)$ be the solutions of problems (2.1) and (3.6). If the domain Ω is convex and the mesh is quasi-uniform, it holds that*

$$\|u - \phi_{\delta_h}\|_{0,\Omega} \lesssim h^{2-d/2}.$$

Proof. Let ϕ_f be the solution of problem (3.1) with $f \in L^2(\Omega)$. Let $\phi_{f,h} \in V_h^1$ and $\phi_{\delta_h,h} \in V_h^1$ be the standard finite element approximations of ϕ_f and ϕ_{δ_h}. Then we have the following standard error estimates (see, *e.g.*, [10]):

$$\|\phi_f - \phi_{f,h}\|_{L^\infty(\Omega)} \lesssim h^{2-d/2}\|f\|_{0,\Omega}, \tag{3.7}$$

$$\|\phi_f - \phi_{f,h}\|_{0,\Omega} + h\|\phi_f - \phi_{f,h}\|_{1,\Omega} \lesssim h^2\|\phi_f\|_{2,\Omega} \lesssim h^2\|f\|_{0,\Omega}. \tag{3.8}$$

Moreover, integration by parts can yield

$$\begin{aligned}
\int_\Omega (u - \phi_{\delta_h}) f &= \int_\Omega \nabla(u - \phi_{\delta_h}) \cdot \nabla(\phi_f - \phi_{f,h}) + \int_\Omega \nabla(u - \phi_{\delta_h}) \cdot \nabla\phi_{f,h} \\
&= \int_\Omega \nabla(u - \phi_{\delta_h}) \cdot \nabla(\phi_f - \phi_{f,h}) \\
&= \phi_f(x_0) - \phi_{f,h}(x_0) + \int_\Omega \nabla(\phi_{\delta_h,h} - \phi_{\delta_h}) \cdot \nabla(\phi_f - \phi_{f,h}) \\
&\lesssim h^{2-d/2}\|f\|_{0,\Omega} + h^{2-d/2}\|f\|_{0,\Omega},
\end{aligned}$$

by (3.5), (3.7) and (3.8), which concludes the proof. □

Now the remaining task is to estimate the error $\|\phi_{\delta_h} - u_h\|_{0,\Omega}$. Obviously, according to the definitions of ϕ_{δ_h} and δ_h, we know that $(u_h, \widehat{u}_h)$ is an HDG approximation of ϕ_{δ_h}. Hence from ([30], Thm. 2), we have

$$\|\phi_{\delta_h} - u_h\|_{0,\Omega} \lesssim h^2|\phi_{\delta_h}|_{2,\Omega}, \tag{3.9}$$

which, together with (3.5) and Lemma 3.2, yields the following *a priori* error estimate.

Theorem 3.3. *Let u and $(u_h, \widehat{u}_h)$ be the solutions of problems (2.1) and (2.2). If the domain Ω is convex and the mesh is quasi-uniform, we have*

$$\|u - u_h\|_{0,\Omega} \lesssim h^{2-d/2}.$$

4. Reliability of *a posteriori* error estimators

In this section, we consider a posteriori error analysis for the HDG scheme (2.2). Specifically, we will prove a posteriori error estimators for the errors $\|u-u_h\|_{0,\Omega}$ and $\|\nabla(u-u_h)\|_{s,\Omega}$.

If x_0 is a node of the partition $\mathcal{T}_h$, we define the local error estimators η_K and $\zeta_{K,s}$ by

$$\eta_K^2 = h_K^4\|\Delta u_h\|_{0,K}^2 + \tau_0^2 h_K\|u_h-\widehat{u}_h\|_{0,\partial K}^2,$$
$$\zeta_{K,s}^s = h_K^s\|\Delta u_h\|_{0,s,K}^s + \tau_0^s h_K^{1-s}\|u_h-\widehat{u}_h\|_{0,s,\partial K}^s,$$

otherwise,

$$\eta_K^2 = \begin{cases} h_K^{4-d} + h_K^4\|\Delta u_h\|_{0,K}^2 + \tau_0^2 h_K\|u_h-\widehat{u}_h\|_{0,\partial K}^2 & \text{if } x_0\in\overline{K},\\ h_K^4\|\Delta u_h\|_{0,K}^2 + \tau_0^2 h_K\|u_h-\widehat{u}_h\|_{0,\partial K}^2 & \text{otherwise,}\end{cases}$$

$$\zeta_{K,s}^s = \begin{cases} h_K^{2-s} + h_K^s\|\Delta u_h\|_{0,s,K}^s + \tau_0^s h_K^{1-s}\|u_h-\widehat{u}_h\|_{0,s,\partial K}^s & \text{if } x_0\in\overline{K},\\ h_K^s\|\Delta u_h\|_{0,s,K}^s + \tau_0^s h_K^{1-s}\|u_h-\widehat{u}_h\|_{0,s,\partial K}^s & \text{otherwise.}\end{cases}$$

Before proving a posteriori error estimates, we first show the following properties for the HDG scheme (2.2).

Lemma 4.1. *Let $(u_h,\widehat{u}_h)$ be the solution of problem (2.2), then we have*

$$[\nabla u_h\cdot\boldsymbol{n}]|_F = \frac{\tau_0}{h_F}(u_h^+-\widehat{u}_h) + \frac{\tau_0}{h_F}(u_h^- -\widehat{u}_h),\quad \mathcal{E}_h^o\ni F=\overline{K^+}\cap\overline{K^-}, \tag{4.1}$$

$$(\Delta u_h, v_h)_{\mathcal{T}_h} = -\langle\nabla v_h\cdot\boldsymbol{n}, u_h-\widehat{u}_h\rangle_{\partial\mathcal{T}_h} + \sum_{K\in\mathcal{T}_h}\sum_{F\in\partial K}\frac{\tau_0}{h_F}\langle u_h-\widehat{u}_h, v_h\rangle_F$$
$$-\frac{1}{\sharp T_{x_0}}\sum_{K\in T_{x_0}} v_h|_K(x_0), \tag{4.2}$$

for any $v_h\in W_h^k$.

Proof. By setting $v_h=0$ in (2.2), we arrive at

$$\langle\nabla u_h\cdot\mathbf{n},\mu_h\rangle_{\partial\mathcal{T}_h} - \langle\tau(u_h-\widehat{u}_h),\mu_h\rangle_{\partial\mathcal{T}_h} = 0,\quad \forall\mu_h\in M_{h,0}^k,$$

where $\tau=\frac{\tau_0}{h_F}$ on each face $F\in\mathcal{E}_h$. Hence the result (4.1) can be obtained by the above equality. Moreover the equality (4.2) can be derived directly by using (4.1), (2.2) and integration by parts. □

Now we are going to prove a posteriori error estimate for the error $\|u-u_h\|_{0,\Omega}$ by the duality argument.

Lemma 4.2. *Let u and $(u_h,\widehat{u}_h)$ be the solutions of problems (2.1) and (2.2). If the domain Ω is convex, we have*

$$\|u-u_h\|_{0,\Omega} \lesssim \Big(\sum_{K\in\mathcal{T}_h}\eta_K^2\Big)^{1/2}. \tag{4.3}$$

Proof. Let ϕ_f be the solution of problem (3.1) with $f\in L^2(\Omega)$. From the case (i) of Theorem 3.1, we know that the solution ϕ_f satisfies the regularity (3.2). Hence by integration by parts, we arrive at

$$\begin{aligned}(u-u_h,f)_\Omega &= (\nabla u,\nabla\phi_f)_\Omega + (u_h,\Delta\phi_f)_{\mathcal{T}_h}\\ &= \phi_f(x_0) + (\Delta u_h,\phi_f)_{\mathcal{T}_h} + \langle\nabla\phi_f\cdot\mathbf{n},u_h\rangle_{\partial\mathcal{T}_h} - \langle\nabla u_h\cdot\mathbf{n},\phi_f\rangle_{\partial\mathcal{T}_h}.\end{aligned}$$

Insert (4.2) in the above equality to yield

$$
\begin{aligned}
(u-u_h,f)_\Omega &= \phi_f(x_0) - \frac{1}{\sharp T_{x_0}} \sum_{K\in T_{x_0}} v_h|_K(x_0) + (\Delta u_h, \phi_f - v_h)_{\mathcal{T}_h} \\
&\quad + \langle \nabla\phi_f\cdot\mathbf{n}, u_h\rangle_{\partial\mathcal{T}_h} - \langle \nabla v_h\cdot\mathbf{n}, u_h-\widehat{u}_h\rangle_{\partial\mathcal{T}_h} \\
&\quad - \langle \nabla u_h\cdot\mathbf{n}, \phi_f\rangle_{\partial\mathcal{T}_h} + \langle \tau(u_h-\widehat{u}_h), v_h\rangle_{\partial\mathcal{T}_h} \\
&= \Big(\phi_f(x_0) - \frac{1}{\sharp T_{x_0}} \sum_{K\in T_{x_0}} v_h|_K(x_0)\Big) + (\Delta u_h, \phi_f - v_h)_{\mathcal{T}_h} \\
&\quad + \langle \nabla(\phi_f - v_h)\cdot\mathbf{n}, u_h-\widehat{u}_h\rangle_{\partial\mathcal{T}_h} + \langle \tau(u_h-\widehat{u}_h), v_h-\phi_f\rangle_{\partial\mathcal{T}_h} \\
&= I_1+I_2+I_3+I_4,
\end{aligned}
\tag{4.4}
$$

for any $v_h \in W_h^k$. Note that we have used (4.1) and the fact that $\nabla\phi_f \in H(\mathrm{div},\Omega)$ in the derivation of (4.4). Let $v_h = I\phi_f$ in (4.4), we can get by (2.5) and (2.6)

$$
I_2 \lesssim \sum_{K\in\mathcal{T}_h} h_K^2 \|\Delta u_h\|_{0,K} |\phi_f|_{2,K}, \tag{4.5}
$$

and

$$
\begin{aligned}
I_3+I_4 &\lesssim \sum_{K\in\mathcal{T}_h} \big(\|\nabla(\phi_f - I\phi_f)\|_{0,K}^{1/2} \|\nabla(\phi_f - I\phi_f)\|_{1,K}^{1/2}\big) \|u_h-\widehat{u}_h\|_{0,\partial K} \\
&\quad + \sum_{K\in\mathcal{T}_h} \frac{\tau_0}{h_K} \big(\|\phi_f - I\phi_f\|_{0,K}^{1/2} \|\phi_f - I\phi_f\|_{1,K}^{1/2}\big) \|u_h-\widehat{u}_h\|_{0,\partial K} \\
&\lesssim \sum_{K\in\mathcal{T}_h} \tau_0 h_K^{1/2} \|u_h-\widehat{u}_h\|_{0,\partial K} |\phi_f|_{2,K}.
\end{aligned}
\tag{4.6}
$$

As for the term I_1, if x_0 is a node of the partition $\mathcal{T}_h$, we have $I_1 = 0$, otherwise, the Lagrange interpolation error estimate (2.8) can get

$$
I_1 = \frac{1}{\sharp T_{x_0}} \sum_{K\in T_{x_0}} (\phi_f(x_0) - I\phi_f|_K(x_0)) \le \frac{1}{\sharp T_{x_0}} \sum_{K\in T_{x_0}} \|\phi_f - I\phi_f\|_{0,\infty,K} \lesssim \frac{1}{\sharp T_{x_0}} \sum_{K\in T_{x_0}} h_K^{2-\frac{d}{2}} |\phi_f|_{2,K}. \tag{4.7}
$$

Hence the result (4.3) can be achieved by using (3.2) and (4.4)-(4.7). □

Remark 4.3. Since the function $\nabla\phi_f$ does not belong to $H(div,\Omega)$ for any $\phi_f \in W_0^{2,p'}(\Omega)$ $(p'<2)$, the additional term $\langle \nabla\phi_f\cdot\mathbf{n}, \widehat{u}_h\rangle_{\partial\mathcal{T}_h}$ can not be eliminated when *a posteriori* error estimate of $\|u-u_h\|_{0,p,\Omega}$ $(P_\Omega < p < \infty)$ is proved by using the result introduced in case (ii) of Theorem 3.1, where $P_\Omega = \frac{2\theta}{\pi} > 2$. This is why we have not provided *a posteriori* error analysis for $\|u-u_h\|_{0,p,\Omega}$ $(P_\Omega < p < \infty)$.

In order to prove *a posteriori* error estimate for $\|\nabla(u-u_h)\|_{0,p,\Omega}$ in two-dimensional case, we consider a problem that is the same with that in [2]: Find $w \in W_0^{1,p'}(\Omega)$ such that

$$
a(w,v) = \int_\Omega \Psi\cdot\nabla v, \quad \forall v \in W_0^{1,p}(\Omega), \tag{4.8}
$$

for $\Psi \in (L^{p'}(\Omega))^2$. From [2], we know that if $d=2$ and the domain Ω is a Lipschtiz polygon, the problem (4.8) has an unique weak solution $w \in W_0^{1,p'}(\Omega)$ satisfying

$$
|w|_{1,p',\Omega} \lesssim \|\Psi\|_{0,p',\Omega}, \tag{4.9}
$$

for $p \in (P^\Omega, 2)$, where $P^\Omega := \max\{1, 2/(1+\frac{\pi}{\theta})\}$, θ is the largest inner angle of the domain Ω and $\frac{1}{p}+\frac{1}{p'} = 1$.

Lemma 4.4. *Let u and $(u_h, \widehat{u}_h)$ be the solutions of problems (2.1) and (2.2). If $d = 2$ and the domain Ω is a Lipschitz polygon, the following error estimate holds*

$$\|\nabla(u - u_h)\|_{0,p,\mathcal{T}_h} \lesssim \Big(\sum_{K\in\mathcal{T}_h} \zeta_{K,p}^p\Big)^{1/p},$$

where $p \in (P^\Omega, 2)$.

Proof. Since u_h does not belong to the space $W^{1,p}(\Omega)$, we can not apply the duality argument directly for $u-u_h$ by (4.8). Here, let $\widetilde{u}_h$ be the Oswald interpolation of u_h, then using duality argument for $u - \widetilde{u}_h$ we have

$$\begin{aligned}\int_\Omega \nabla(u-\widetilde{u}_h)\cdot\Psi \mathrm{dx} &= (\nabla(u-\widetilde{u}_h), \nabla w)_\Omega \\ &= w(x_0) + (\nabla(u_h - \widetilde{u}_h), \nabla w)_{\mathcal{T}_h} - (\nabla u_h, \nabla w)_{\mathcal{T}_h} \\ &= w(x_0) + (\nabla(u_h - \widetilde{u}_h), \nabla w)_{\mathcal{T}_h} + (\Delta u_h, w)_{\mathcal{T}_h} - \langle \nabla u_h\cdot \mathbf{n}, w\rangle_{\partial\mathcal{T}_h}.\end{aligned}$$

Insert (4.2) in the above equality to infer that

$$\begin{aligned}\int_\Omega \nabla(u-\widetilde{u}_h)\cdot\Psi \mathrm{dx} =& \Big(w(x_0) - \frac{1}{\sharp T_{x_0}} \sum_{K\in T_{x_0}} v_h|_K(x_0)\Big) + (\Delta u_h, w - v_h)_{\mathcal{T}_h} \\ &+ (\nabla(u_h-\widetilde{u}_h), \nabla w)_{\mathcal{T}_h} - \langle \nabla v_h\cdot\mathbf{n}, u_h - \widehat{u}_h\rangle_{\partial\mathcal{T}_h} \\ &+ \langle \tau(u_h-\widehat{u}_h), v_h - w\rangle_{\partial\mathcal{T}_h} \\ =& E_1 + E_2 + E_3 + E_4 + E_5.\end{aligned} \tag{4.10}$$

for any $v_h \in W_h^k$.

Let $v_h = Iw \in V_h^1$ in (4.10), the trace inequality (2.5) and Lagrange interpolation error estimate (2.7) can get

$$E_2 \lesssim \sum_{K\in\mathcal{T}_h} h_K \|\Delta u_h\|_{0,p,K} |w|_{1,p',K}, \tag{4.11}$$

and

$$\begin{aligned}E_4 + E_5 \lesssim& \sum_{K\in\mathcal{T}_h} \|u_h - \widehat{u}_h\|_{0,p,\partial K}\Big(\|\nabla Iw\|_{0,p',K}^{\frac{1}{p}} \|\nabla Iw\|_{1,p',K}^{\frac{1}{p'}} \\ &+ \frac{\tau_0}{h_K}\|w - Iw\|_{0,p',K}^{\frac{1}{p}}\|w-Iw\|_{1,p',K}^{\frac{1}{p'}}\Big) \\ \lesssim& \sum_{K\in\mathcal{T}_h} \tau_0 h_K^{\frac{1}{p}-1}\|u_h-\widehat{u}_h\|_{0,p,\partial K}|w|_{1,p',K}.\end{aligned} \tag{4.12}$$

If x_0 is a node of the partition $\mathcal{T}_h$, $E_1 = 0$, otherwise, we have

$$E_1 \le \frac{1}{\sharp T_{x_0}} \sum_{K\in T_{x_0}} \|w - Iw\|_{0,\infty,K} \lesssim \sum_{K\in T_{x_0}} h_K^{\frac{2}{p}-1}|w|_{1,p',K}. \tag{4.13}$$

by the Lagrange interpolation error estimate (2.9). As for the term E_3, the Oswald interpolation error estimate (2.12) can yield

$$\begin{aligned} E_3 \lesssim & \Big(\sum_{K\in\mathcal{T}_h} \|\nabla(u_h - \widetilde{u}_h)\|_{0,p,K}^p \Big)^{1/p} |w|_{1,p',\Omega} \\ \lesssim & \Big(\sum_{F\in\mathcal{E}_h^o} h_F^{1-p} \|[u_h]\|_{0,p,F}^p + \sum_{F\in\mathcal{E}_h^\partial} h_F^{1-p}\|u_h\|_{0,p,F}^p \Big)^{1/p} |w|_{1,p',\Omega} \\ \lesssim & \Big(\sum_{K\in\mathcal{T}_h} h_K^{1-p} \|u_h - \widehat{u}_h\|_{0,p,\partial K}^p \Big)^{1/p} |w|_{1,p',\Omega}. \end{aligned} \tag{4.14}$$

Hence combining (4.9) and (4.10)–(4.14), we have

$$\|\nabla(u - \widetilde{u}_h)\|_{0,p,\Omega} \lesssim \Big(\sum_{K\in\mathcal{T}_h} \zeta_{K,p}^p \Big)^{1/p}, \tag{4.15}$$

which, together with the Oswald interpolation error estimate (2.12) and the triangle inequality, concludes the proof. □

5. Efficiency of *a posteriori* error estimators

In this section, we mainly prove the efficiency of *a posteriori* error estimators. Before this, we first introduce the element and face bubble functions B_K and B_F as that in [35]. Following the approach in [2, 20], we define

$$\psi_K(x) = \begin{cases} B_K^2(x)\frac{|x-x_0|^2}{h_K^2} & \text{if } x_0 \in K, \\ B_K^2(x) & \text{otherwise,} \end{cases}$$

for any $K \in \mathcal{T}_h$, and

$$\psi_F(x) = \begin{cases} B_F^2(x)\frac{|x-x_0|^2}{h_F^2} & \text{if } x_0 \in w_F, \\ B_F^2(x) & \text{otherwise,} \end{cases}$$

for $F \in \mathcal{E}_h^o$, where $w_F := \bigcup\{K : F \subset \partial K,\ K \in \mathcal{T}_h\}$.

Lemma 5.1. *For each $K \in \mathcal{T}_h$ and $F \in \mathcal{E}_h^o$, let ψ_K and ψ_F be defined as above. Then*

$$\psi_K = \nabla\psi_K \cdot \boldsymbol{n} = 0 \quad \text{on } \partial K, \quad \psi_F = \nabla\psi_F \cdot \boldsymbol{n} = 0 \quad \text{on the boundary of } w_F, \tag{5.1}$$

$$\|v\|_{0,p,K} \lesssim \|v\psi_K^{\frac{1}{p}}\|_{0,p,K}, \quad \|w\|_{0,p,F} \lesssim \|w\psi_F^{\frac{1}{p}}\|_{0,p,F}, \tag{5.2}$$

for $p \in (1,\infty)$, $v \in \mathcal{P}^j(K)$ and $w \in \mathcal{P}^j(F)$, where j is a nonnegative integer.

Proof. The result (5.1) can be obtained directly by the definitions of ψ_K and ψ_F, and the approximation result (5.2) can be derived by Lemma 3 from [20]. □

Lemma 5.2. *Let u and $(u_h, \widehat{u}_h)$ be the solutions of problems (2.1) and (2.2). Then we have*

$$h_K^2\|\Delta u_h\|_{0,K} \lesssim \|u - u_h\|_{0,K}, \quad \forall K \in \mathcal{T}_h, \tag{5.3}$$

and

$$h_K\|\Delta u_h\|_{0,p,K} \lesssim \|\nabla(u - u_h)\|_{0,p,K}, \tag{5.4}$$

for $p \in (P^\Omega, 2)$.

Proof. Let $v_h = \psi_K \Delta u_h$, by (5.1), (5.2) and inverse estimate (2.4) we arrive at

$$
\begin{aligned}
\|\Delta u_h\|_{0,K}^2 &\lesssim (\Delta u_h, v_h)_K = (\Delta(u_h - u), v_h)_K \\
&= (u_h - u, \Delta v_h)_K \lesssim \|u - u_h\|_{0,K} h_K^{-2} \|v_h\|_{0,K} \\
&\lesssim h_K^{-2} \|u - u_h\|_{0,K} \|\Delta u_h\|_{0,K}.
\end{aligned}
\tag{5.5}
$$

Hence the estimate (5.3) can be obtained by (5.5).

Similarly, let $v_h = \psi_K \Delta u_h$, we have

$$
\begin{aligned}
\|\Delta u_h\|_{0,K}^2 &\lesssim (\Delta(u_h - u), v_h)_K \\
&= (\nabla(u_h - u), \nabla v_h)_K \lesssim \|\nabla(u - u_h)\|_{0,p,K} h_K^{-1} \|v_h\|_{0,p',K} \\
&\lesssim h_K^{-1} h_K^{\frac{d}{p'} - \frac{d}{2}} \|\nabla(u - u_h)\|_{0,p,K} \|\Delta u_h\|_{0,K}.
\end{aligned}
\tag{5.6}
$$

So we can obtain the error estimate (5.4) by (5.6) and inverse estimate (2.4). $\square$

For the case of x_0 is not a node of the partition $\mathcal{T}_h$, we know that the error estimators η_K and $\zeta_{K,s}$ include an additional term. In order to control this term, we define a cutoff function B_{x_0} for the element $K \in T_{x_0}$ such that

$$0 \le B_{x_0} \le 1 \quad \forall x \in \Omega, \tag{5.7}$$

$$B_{x_0} = 1 \quad \forall x \in \Omega: \ |x - x_0| \le \frac{t}{4}, \tag{5.8}$$

$$B_{x_0} = 0 \quad \forall x \in \Omega: \ |x - x_0| \ge \frac{3t}{4}, \tag{5.9}$$

$$|B_{x_0}|_{m,\infty,w_K} \lesssim t^{-m} \quad m = 1, 2, \tag{5.10}$$

where $w_K := \bigcup\{K' \in \mathcal{T}_h : K \cap K' \neq \emptyset\}$, and t denotes the distance of x_0 to the boundary of w_K. Note that the cutoff function has been used in [2]. Then using the fact that $h_K \lesssim t$ and (5.10), we have

$$|B_{x_0}|_{m,p',w_K} \lesssim t^{-m} h_K^{d/p'} \lesssim h_K^{d/p'-m}, \quad m = 1, 2, \ 1 \le p' \le \infty. \tag{5.11}$$

Lemma 5.3. *For any $K \in T_{x_0}$, let B_{x_0} and w_K be defined as above. Then we have*

$$h_K^{2-\frac{d}{2}} \lesssim \|u - u_h\|_{0,w_K} + \sum_{K' \in w_K} \tau_0 h_{K'}^{\frac{1}{2}} \|u_h - \widehat{u}_h\|_{0,\partial K'}, \tag{5.12}$$

and

$$h_K^{\frac{2}{p}-1} \lesssim \|\nabla(u - u_h)\|_{0,p,w_K} + \sum_{K' \in w_K} \tau_0 h_{K'}^{\frac{1}{p}-1} \|u_h - \widehat{u}_h\|_{0,p,\partial K'}, \tag{5.13}$$

for $p \in (P^\Omega, 2)$.

Proof. Using (4.1), the definition of B_{x_0} and (5.11), we can infer that

$$
\begin{aligned}
1 =& B_{x_0}(x_0) = (\nabla(u - u_h), \nabla B_{x_0})_{\mathcal{T}_h} + (\nabla u_h, \nabla B_{x_0})_{\mathcal{T}_h} \\
=& -(u - u_h, \Delta B_{x_0})_{w_K} - (\Delta u_h, B_{x_0})_{w_K} \\
& - \langle u_h, \nabla B_{x_0} \cdot \mathbf{n} \rangle_{\partial w_K} + \langle \nabla u_h \cdot \mathbf{n}, B_{x_0} \rangle_{\partial w_K} \\
\lesssim & \sum_{K' \in w_K} \Big(\|u - u_h\|_{0,K'} h_K^{d/2-2} + \|\Delta u_h\|_{0,K'} h_K^{d/2} \\
& + \tau_0 \|u_h - \widehat{u}_h\|_{0,\partial K'} h_K^{(d-1)/2-1} \Big),
\end{aligned}
\tag{5.14}
$$

where $\partial w_K := \{\partial K' : K' \in w_K\}$. Hence we can get (5.12) by using (5.14) and (5.3).

Similarly, we have

$$\begin{aligned} 1 =& (\nabla(u-u_h), \nabla B_{x_0})_{w_K} - (\Delta u_h, B_{x_0})_{w_K} + \langle \nabla u_h \cdot \mathbf{n}, B_{x_0}\rangle_{\partial w_K} \\ \lesssim& \sum_{K' \in w_K} \Big(\|\nabla(u-u_h)\|_{0,p,K'} h_K^{2/p'-1} + \|\Delta u_h\|_{0,p,K'} h_K^{2/p'} \\ &+ \tau_0 \|u_h - \widehat{u}_h\|_{0,p,\partial K'} h_K^{1/p'-1}\Big). \end{aligned} \tag{5.15}$$

So we can obtain (5.13) by combining (5.15) and (5.4). □

Obviously, the remaining task is to bound the error estimator $\|u_h - \widehat{u}_h\|_{0,p,\partial K}$. But, seemingly, it is not an easy task. First of all, we show the following relationships.

Lemma 5.4. *Let $(u_h, \widehat{u}_h)$ be the solution of problem (2.2). Then the following relationships hold*

$$\|[\nabla u_h \cdot \boldsymbol{n}]\|_{0,p,F}^p + \tau_0^p h_F^{-p} \|[u_h]\|_{0,p,F}^p \lesssim 2^{p-1}\Big(\|\frac{\tau_0}{h_F}(u_h^+ - \widehat{u}_h)\|_{0,p,F}^p + \|\frac{\tau_0}{h_F}(u_h^- - \widehat{u}_h)\|_{0,p,F}^p \Big), \tag{5.16}$$

and

$$\|\frac{\tau_0}{h_F}(u_h^+ - \widehat{u}_h)\|_{0,p,F}^p + \|\frac{\tau_0}{h_F}(u_h^- - \widehat{u}_h)\|_{0,p,F}^p \lesssim 2^{p-1}\Big(\|[\nabla u_h \cdot \boldsymbol{n}]\|_{0,p,F}^p + \tau_0^p h_F^{-p} \|[u_h]\|_{0,p,F}^p \Big), \tag{5.17}$$

for $1 \le p < \infty$ and $\mathcal{E}_h^o \ni F = \overline{K^+} \cap \overline{K^-}$.

Proof. To prove (5.16) and (5.17), we only need to prove the following results:

$$|[\nabla u_h \cdot \mathbf{n}]|^p + \tau_0^p h_F^{-p} |[u_h]|^p \lesssim 2^{p-1}\Big(|\frac{\tau_0}{h_F}(u_h^+ - \widehat{u}_h)|^p + |\frac{\tau_0}{h_F}(u_h^- - \widehat{u}_h)|^p \Big), \tag{5.18}$$

$$|\frac{\tau_0}{h_F}(u_h^+ - \widehat{u}_h)|^p + |\frac{\tau_0}{h_F}(u_h^- - \widehat{u}_h)|^p \lesssim 2^{p-1}\Big(|[\nabla u_h \cdot \mathbf{n}]|^p + \tau_0^p h_F^{-p} |[u_h]|^p \Big). \tag{5.19}$$

By using (4.1) and the definition of the jump $[u_h]$, we yield

$$|[\nabla u_h \cdot \mathbf{n}]| + \tau_0 h_F^{-1} |[u_h]| \lesssim |\frac{\tau_0}{h_F}(u_h^+ - \widehat{u}_h)| + |\frac{\tau_0}{h_F}(u_h^- - \widehat{u}_h)|.$$

Hence we can get (5.18) by the above inequality.

On the other hand, by using the triangle inequality, we yield

$$\begin{aligned} |\frac{\tau_0}{h_F}(u_h^+ - \widehat{u}_h)| &= \frac{1}{2} |\tau_0 h_F^{-1}[u_h] + \frac{\tau_0}{h_F}(u_h^+ - \widehat{u}_h) + \frac{\tau_0}{h_F}(u_h^- - \widehat{u}_h)| \\ &\le \frac{1}{2}\tau_0 h_F^{-1} |[u_h]| + \frac{1}{2} |[\nabla u_h \cdot \mathbf{n}]|, \end{aligned}$$

and

$$|\frac{\tau_0}{h_F}(u_h^- - \widehat{u}_h)| \le \frac{1}{2}\tau_0 h_F^{-1} |[u_h]| + \frac{1}{2} |[\nabla u_h \cdot \mathbf{n}]|.$$

Therefore the estimate (5.19) can be derived by the above two inequalities. □

Now we are going to bound the error estimator $\|[\nabla u_h \cdot \mathbf{n}]\|_{0,p,F}^p$ for $F \in \mathcal{E}_h^o$.

Lemma 5.5. *Let u and $(u_h, \widehat{u}_h)$ be the solutions of problems (1.1) and (2.2). Then for each $F \in \mathcal{E}_h^o$, we have*

$$h_F^{\frac{3}{2}}\|[\nabla u_h \cdot \boldsymbol{n}]\|_{0,F} \lesssim \|u - u_h\|_{0,w_F} + h_F^{\frac{1}{2}}\|[u_h]\|_{0,F}, \tag{5.20}$$

and

$$h_F^{\frac{1}{p}}\|[\nabla u_h \cdot \boldsymbol{n}]\|_{0,p,F} \lesssim \|\nabla(u - u_h)\|_{0,p,w_F}, \tag{5.21}$$

for $p \in (P^\Omega, 2)$.

Proof. Let $P_F : L^\infty(F) \to L^\infty(w_F)$ be a continuation operator [35] such that

$$\|P_F w\|_{0,w_F} \lesssim h_F^{\frac{1}{2}}\|w\|_{0,F}, \quad \forall w \in \mathcal{P}^j(F), \tag{5.22}$$

for any nonnegative integer j. Let $v = \psi_F(P_F[\nabla u_h \cdot \mathbf{n}])$.

By the trace inequality, the inverse estimate (2.4), Lemma 5.1 and (5.22), we can get

$$\begin{aligned}\|[\nabla u_h \cdot \mathbf{n}]\|_{0,F}^2 &\lesssim \langle [\nabla u_h \cdot \mathbf{n}], v\rangle_F = \int_{w_F} \nabla \cdot (v \nabla u_h)\\ &= (\Delta u_h, v)_{w_F} + (\nabla v, \nabla u_h)_{w_F}\\ &\leq \|\Delta u_h\|_{0,w_F}\|v\|_{0,w_F} - (\Delta v, u_h - u)_{w_F} + \langle \nabla v \cdot \mathbf{n}, [u_h]\rangle_F\\ &\lesssim \|[\nabla u_h \cdot \mathbf{n}]\|_{0,F}\Big(\|\Delta u_h\|_{0,w_F} h_F^{\frac{1}{2}} + h_F^{-\frac{3}{2}}\|u - u_h\|_{0,w_F}\\ &\quad + h_F^{-1}\|[u_h]\|_{0,F}\Big).\end{aligned}$$

Hence the approximation (5.20) can be obtained by (5.3) and the above estimate.

Similarly, we have

$$\begin{aligned}\|[\nabla u_h \cdot \mathbf{n}]\|_{0,F}^2 &\lesssim (\Delta u_h, v)_{w_F} + (\nabla v, \nabla(u_h - u))_{w_F}\\ &\lesssim \|[\nabla u_h \cdot \mathbf{n}]\|_{0,F}\Big(h_F^{\frac{d}{p'}-\frac{d}{2}+\frac{1}{2}}\|\Delta u_h\|_{0,p,w_F} + h_F^{\frac{d}{p'}-\frac{d}{2}-\frac{1}{2}}\|\nabla(u - u_h)\|_{0,p,w_F}\Big).\end{aligned} \tag{5.23}$$

Therefore, we can derive (5.21) by (5.4), the inverse estimate (2.4) and (5.23). ∎

Combining Lemmas (4.2), (4.4), (5.2), (5.3), (5.4) and (5.5) we can get the following theorem.

Theorem 5.6. *Let u and $(u_h, \widehat{u}_h)$ be the solutions of problems (2.1) and (2.2). Let $\widetilde{u}_h$ be the Oswald interpolation of u_h.*

(i) If the domain Ω is convex, we have

$$\|u - u_h\|_{0,\Omega} + \|u - \widetilde{u}_h\|_{0,\Omega} \lesssim \Big(\sum_{K\in\mathcal{T}_h} \eta_K^2\Big)^{1/2}, \tag{5.24}$$

$$\Big(\sum_{K\in\mathcal{T}_h} \eta_K^2\Big)^{1/2} \lesssim \tau_0(\|u - u_h\|_{0,\Omega} + \|u - \widetilde{u}_h\|_{0,\Omega}), \tag{5.25}$$

(ii) If d=2 and the domain Ω is a Lipschitz polygon, we have

$$\|\nabla(u - u_h)\|_{0,p,\mathcal{T}_h} + \Big(\sum_{F\in\mathcal{E}_h} \tau_0^p h_F^{1-p}\|[u_h - \widetilde{I}u]\|_{0,p,F}^p\Big)^{1/p} \lesssim \Big(\sum_{K\in\mathcal{T}_h} \zeta_{K,p}^p\Big)^{1/p}, \tag{5.26}$$

$$\Big(\sum_{K\in\mathcal{T}_h} \zeta_{K,p}^p\Big)^{1/p} \lesssim \|\nabla(u - u_h)\|_{0,p,\Omega} + \Big(\sum_{F\in\mathcal{E}_h} \tau_0^p h_F^{1-p}\|[u_h - \widetilde{I}u]\|_{0,p,F}^p\Big)^{1/p}, \tag{5.27}$$

for $p \in (P^\Omega, 2)$, where $P^\Omega = \max\{1, 2/(1+\frac{\pi}{\theta})\}$ and θ is the largest inner angle of the domain Ω.

Proof. By using Oswald interpolation error estimate (2.11), we infer that

$$\begin{aligned}\|u-\widetilde{u}_h\|_{0,\Omega} &\le \|u-u_h\|_{0,\Omega} + \Big(\sum_{K\in\mathcal{T}_h}\|u_h-\widetilde{u}_h\|_{0,K}^2\Big)^{1/2}\\ &\lesssim \|u-u_h\|_{0,\Omega} + \Big(\sum_{F\in\mathcal{E}_h^o} h_F\|[u_h]\|_{0,F}^2 + \sum_{F\in\mathcal{E}_h^\partial} h_F\|u_h\|_{0,F}^2\Big)^{1/2}\\ &\lesssim \|u-u_h\|_{0,\Omega} + \Big(\sum_{K\in\mathcal{T}_h} h_K\|u_h-\widehat{u}_h\|_{0,\partial K}^2\Big)^{1/2}.\end{aligned} \tag{5.28}$$

Hence we can get (5.24) by combining (4.3) and (5.28).

In virtue of the results introduced in Lemmas 5.2, 5.3, 5.4 and 5.5, we yield

$$\Big(\sum_{K\in\mathcal{T}_h}\eta_K^2\Big)^{1/2} \lesssim \|u-u_h\|_{0,\Omega} + \Big(\sum_{F\in\mathcal{E}_h^o}\tau_0^2 h_F\|[u_h]\|_{0,F}^2 + \sum_{F\in\mathcal{E}_h^\partial}\tau_0^2 h_F\|u_h\|_{0,F}^2\Big)^{1/2}, \tag{5.29}$$

and

$$\begin{aligned}\Big(\sum_{K\in\mathcal{T}_h}\zeta_{K,p}^p\Big)^{1/p} &\lesssim \|\nabla(u-u_h)\|_{0,p,\mathcal{T}_h} + \Big(\sum_{F\in\mathcal{E}_h^o}\tau_0^p h_F^{1-p}\|[u_h]\|_{0,p,F}^p + \sum_{F\in\mathcal{E}_h^\partial}\tau_0^p h_F^{1-p}\|u_h\|_{0,p,F}^p\Big)^{1/p}\\ &= \|\nabla(u-u_h)\|_{0,p,\mathcal{T}_h} + \Big(\sum_{F\in\mathcal{E}_h}\tau_0^p h_F^{1-p}\|[u_h-\widetilde{I}u]\|_{0,p,F}^p\Big)^{1/p}.\end{aligned} \tag{5.30}$$

Obviously, the approximation result (5.30) is the same as (5.27). Moreover the inverse estimate (2.4) and trace inequality (2.5) result in

$$\begin{aligned}h_F^{\frac12}\|[u_h]\|_{0,F} = h_F^{\frac12}\|[u_h-\widetilde{u}_h]\|_{0,F} &\le h_F^{\frac12}\|u_h^+-\widetilde{u}_h\|_{0,F} + h_F^{\frac12}\|u_h^- -\widetilde{u}_h\|_{0,F}\\ &\lesssim \|u_h^+-\widetilde{u}_h\|_{0,K^+} + \|u_h^- -\widetilde{u}_h\|_{0,K^-},\end{aligned} \tag{5.31}$$

for any $\mathcal{E}_h^o \ni F = \overline{K^+}\cap\overline{K^-}$. Hence

$$\begin{aligned}\Big(\sum_{F\in\mathcal{E}_h^o}\tau_0^2 h_F\|[u_h]\|_{0,F}^2 + \sum_{F\in\mathcal{E}_h^\partial}\tau_0^2 h_F\|u_h\|_{0,F}^2\Big)^{1/2} &\lesssim \tau_0\|u_h-\widetilde{u}_h\|_{0,\Omega}\\ &\lesssim \tau_0(\|u-u_h\|_{0,\Omega} + \|u-\widetilde{u}_h\|_{0,\Omega}),\end{aligned} \tag{5.32}$$

which, together with (5.29), derives the approximation result (5.25).

Finally the approximation result (5.26) can be obtained by Lemma 4.4 and the fact that

$$\Big(\sum_{F\in\mathcal{E}_h}\tau_0^p h_F^{1-p}\|[u_h-\widetilde{I}u]\|_{0,p,F}^p\Big)^{1/p} \lesssim \Big(\sum_{K\in\mathcal{T}_h}\tau_0^p h_K^{1-p}\|u_h-\widehat{u}_h\|_{0,p,K}^p\Big)^{1/p}.$$

□

6. Numerical experiments

In this section, some numerical experiments are provided to validate the theoretical analysis. For the adaptive HDG algorithm designed by the obtained *a posteriori* error estimators, we use the following marking strategy

$$\sum_{K\in\mathcal{M}}\xi_K \ge \gamma\xi$$

TABLE 1. The convergence history of $\|u-u_h\|_{0,\Omega}$ for HDG scheme (2.2) for $k=1$.

h	$\|e_u\|_{0,\Omega}$ ($\tau_0=15$)	Order	$\|e_u\|_{0,\Omega}$ ($\tau_0=25$)	Order	$\|e_u\|_{0,\Omega}$ ($\tau_0=100$)	Order
$\frac{1}{4}$	1.4376e-2		1.3591e-2		1.4455e-2	
$\frac{1}{8}$	7.1722e-3	1.003	6.7986e-3	0.999	7.3180e-3	0.982
$\frac{1}{16}$	3.5832e-3	1.001	3.3979e-3	1.001	3.6712e-3	0.995
$\frac{1}{32}$	1.7912e-3	1.000	1.6987e-3	1.000	1.8371e-3	0.999
$\frac{1}{64}$	8.9554e-4	1.000	8.4934e-4	1.000	9.1873e-4	1.000
$\frac{1}{128}$	4.4777e-4	1.000	4.2467e-4	1.000	4.5939e-4	1.000
$\frac{1}{256}$	2.2388e-4	1.000	2.1233e-4	1.000	2.2970e-4	1.000

TABLE 2. The convergence history of $\|u-u_h\|_{0,\Omega}$ for HDG scheme (2.2) for $k=2$.

h	$\|e_u\|_{0,\Omega}$ ($\tau_0=15$)	Order	$\|e_u\|_{0,\Omega}$ ($\tau_0=25$)	Order	$\|e_u\|_{0,\Omega}$ ($\tau_0=100$)	Order
$\frac{1}{4}$	3.6253e-2		9.1561e-3		6.9934e-3	
$\frac{1}{8}$	1.8224e-2	0.992	4.5782e-3	1.000	3.4990e-3	0.999
$\frac{1}{16}$	9.1121e-3	1.000	2.2891e-3	1.000	1.7495e-3	1.000
$\frac{1}{32}$	4.5560e-3	1.000	1.1446e-3	1.000	8.7477e-4	1.000
$\frac{1}{64}$	2.2780e-3	1.000	5.7228e-4	1.000	4.3738e-4	1.000
$\frac{1}{128}$	1.1390e-3	1.000	2.8614e-4	1.000	2.1869e-4	1.000
$\frac{1}{256}$	5.6950e-4	1.000	1.4307e-4	1.000	1.0935e-4	1.000

to select the marking set $\mathcal{M}$, where $\gamma\in(0,1]$, $\xi=\eta_2^2$ or ζ_p^p, and ξ_K is the restriction of ξ on the element $K\in\mathcal{T}_h$. Here $\eta_2^2=\sum_{K\in\mathcal{T}_h}\eta_K^2$ and $\zeta_p^p=\sum_{K\in\mathcal{T}_h}\zeta_K^p$. We refine the marking set $\mathcal{M}$ by bisections to generate a new mesh.

Note here that the figure of convergence history is plotted in log-log coordinates in this section. Moreover we set $e_u=u-u_h$ and

$$E=\|e_u\|_{0,\Omega}+\|u-\tilde{u}_h\|_{0,\Omega},$$
$$E_{2,s}=\|\nabla e_u\|_{0,s,\Omega}+\Big(\sum_{F\in\mathcal{E}_h}\tau_0^s h_F^{1-s}\|[u_h]\|_{0,s,F}^s\Big)^{1/s}.$$

Example 6.1. Based on the domain $\Omega=(0,1)^2$, we consider the problem (1.1) with $x_0=(0.5,0.5)$. In this example, the Dirichlet boundary conditions are imposed so that the exact solution is given by

$$u(\mathbf{x})=-\frac{1}{2\pi}\log|\mathbf{x}-x_0|.$$

We test this example under the constraint that x_0 be a vertex of all partitions.

In Tables 1 and 2, the convergence history and convergence order of the error $\|e_u\|_{0,\Omega}$ for different τ_0 and k are provided on uniform meshes. We find that the convergence rate $O(h)$ can be achieved. From the perspective of error, we find from Figure 1 that the best choice of τ_0 may be 25 and 150 for $k=1$ and $k=2$. Furthermore, in Figure 2 and Table 3, the convergence histories of $\|u-u_h\|_{0,\Omega}$ for finite element methods (FEM) and HDG methods are presented for $k=1$. Obviously, involving a suitable choice of τ_0, we can expect that the HDG methods of this paper are slightly better than the finite element methods.

From now on, we set $\tau_0=25$ and we test this example with adaptive HDG algorithm. The meshes and the corresponding surfaces of u_h, generated by η_2 and $\zeta_{1.5}$, are provided in Figures 3 and 5 for $k=1$ and $\gamma=0.2$.

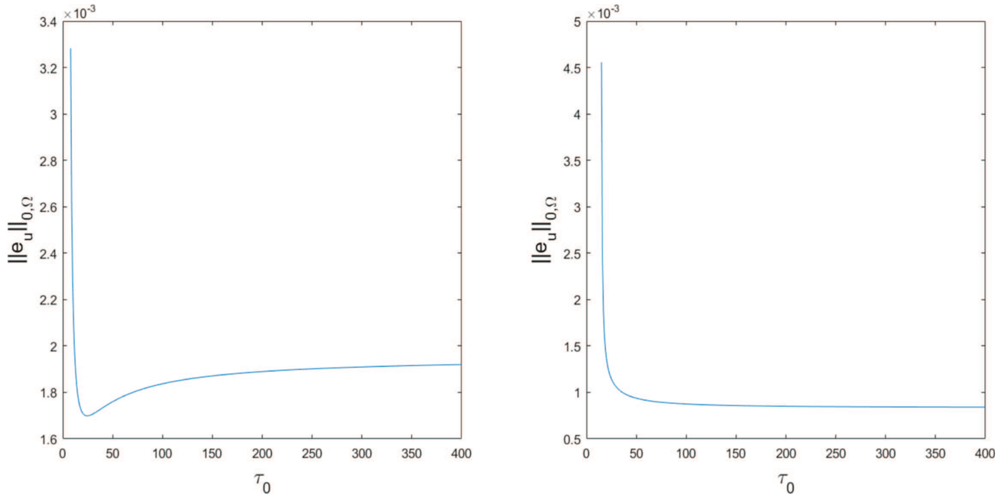

FIGURE 1. Left: The error $\|u - u_h\|_{0,\Omega}$ for different τ_0 for $k = 1$. Right: The error $\|u - u_h\|_{0,\Omega}$ for different τ_0 for $k = 2$. Here we set $h = \frac{1}{32}$.

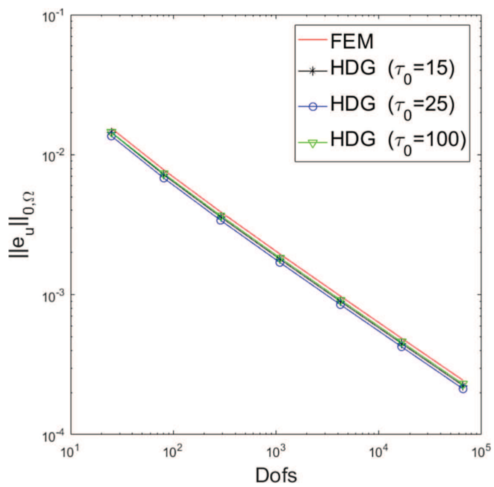

FIGURE 2. The convergence histories of $\|u - u_h\|_{0,\Omega}$ for finite element methods and HDG methods. Here we set $k = 1$.

Obviously, the mesh nodes are concentrated around x_0. In Figure 4, we present the convergence histories of η_2 and $\|e_u\|_{0,\Omega}$ for different k and γ. We observe from the left graph of Figure 4 that the convergence rate $O(N^{-(k+1)/2})$ can be obtained. From the right graph of Figure 4, we can see that for the same numerical accuracy the number of required vertices will be increased as γ becomes large.

By a simple calculation, we have $P^{\Omega} = 1$. Hence in the left graph of Figure 6, we give the convergence histories of ζ_p and $\|\nabla e_u\|_{0,p,\Omega}$ for $k = 1$, $\gamma = 0.2$ and $p = 1.2, 1.5, 1.8$. The results show that these errors and estimators can obtain the convergence rate $O(N^{-1/2})$. In the right graph of Figure 6, we also plot the convergence histories of $\zeta_{1.5}$ and $\|\nabla e_u\|_{0,1.5,\Omega}$ for different γ. Obviously, the convergence rate decreases as γ increases.

TABLE 3. The convergence history of $\|u-u_h\|_{0,\Omega}$ for finite element methods and HDG methods for $k=1$.

h	$\|e_u\|_{0,\Omega}$ (FEM)	Order	$\|e_u\|_{0,\Omega}$ (HDG, $\tau_0=25$)	Order
$\frac{1}{4}$	1.5426e-2		1.3591e-2	
$\frac{1}{8}$	7.7780e-3	0.988	6.7986e-3	0.999
$\frac{1}{16}$	3.9028e-3	0.995	3.3979e-3	1.001
$\frac{1}{32}$	1.9532e-3	0.999	1.6987e-3	1.000
$\frac{1}{64}$	9.7683e-4	1.000	8.4934e-4	1.000
$\frac{1}{128}$	4.8844e-4	1.000	4.2467e-4	1.000
$\frac{1}{256}$	2.4422e-4	1.000	2.2970e-4	1.000

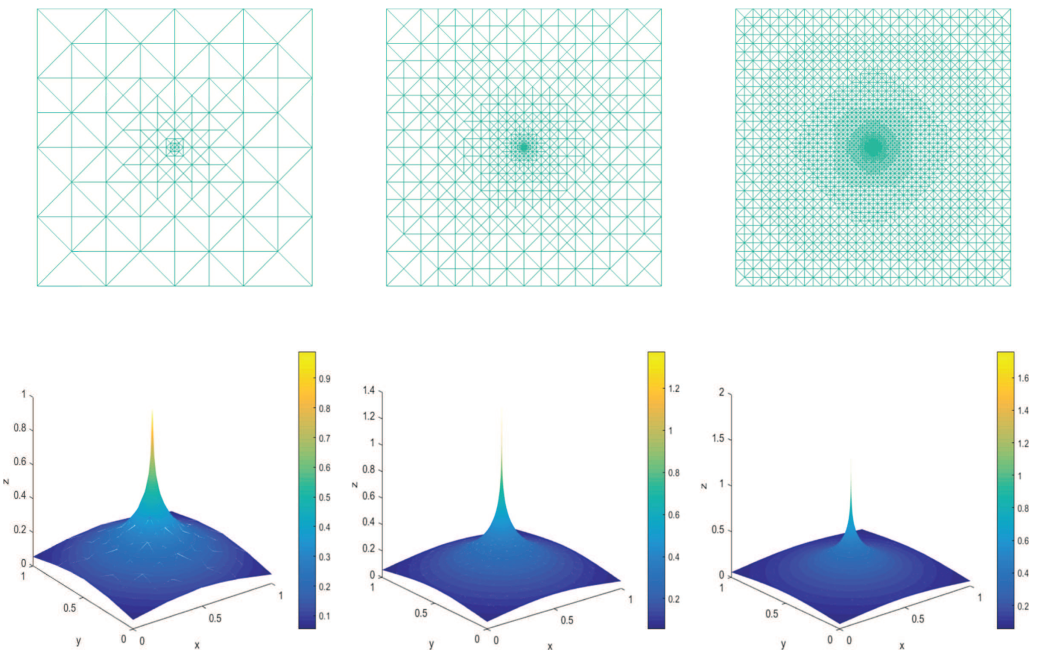

FIGURE 3. Top: The adaptive meshes, generated by η_2, with 123, 636 and 3165 nodes. Bottom: The corresponding surfaces of u_h. Here we set $k=1$ and $\gamma=0.2$.

Example 6.2. In this example, we consider the problem (1.1) in the L-shaped domain $\Omega=(-1,1)^2\backslash[0,1)\times(-1,0]$ with $x_0=(0.5,0.5)$. The Dirichlet boundary conditions are imposed such that the exact solution is given by

$$u(x)=-\frac{1}{2\pi}\log|x-x_0|+|x|^{2/3}\sin(\frac{2\vartheta}{3}),$$

where $\vartheta\in(0,\frac{3\pi}{2})$ denotes the angle.

The initial mesh consists of 12 triangles. According to the definition of P^Ω, we know that $P^\Omega=1.2$. Throughout this section, let $\tau_0=15$ and x_0 be a vertex of all partitions.

In Figure 7, the convergence histories of η_2 and $\|e_u\|_{0,\Omega}$ and the efficiency index η_2/E are given. The convergence rate $O(N^{-(k+1)/2})$ can be obtained. In Figure 8, the adaptive meshes, generated by $\zeta_{1.3}$ and $\zeta_{1.5}$, after

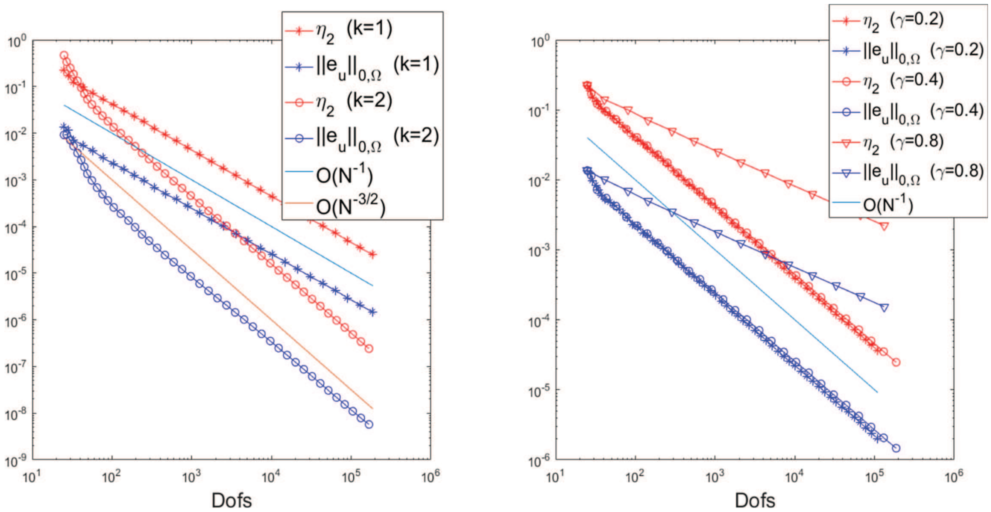

FIGURE 4. Left: The convergence histories of η_2 and $\|e\|_{0,\Omega}$ for $\gamma = 0.4$ and $k = 1, 2$. Right: The convergence histories of η_2 and $\|e_u\|_{0,\Omega}$ for $k = 1$ and $\gamma = 0.2, 0.4, 0.8$.

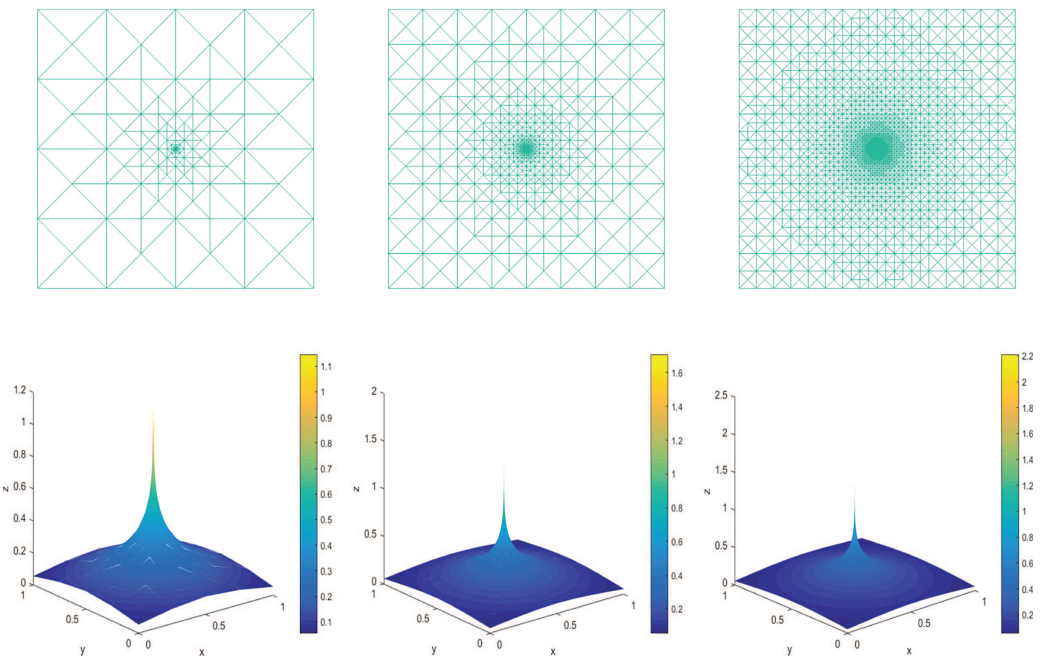

FIGURE 5. Top: The adaptive meshes, generated by $\zeta_{1.5}$, with 124, 686 and 3513 nodes. Bottom: The corresponding surfaces of u_h. Here we set $k = 1$ and $\gamma = 0.2$.

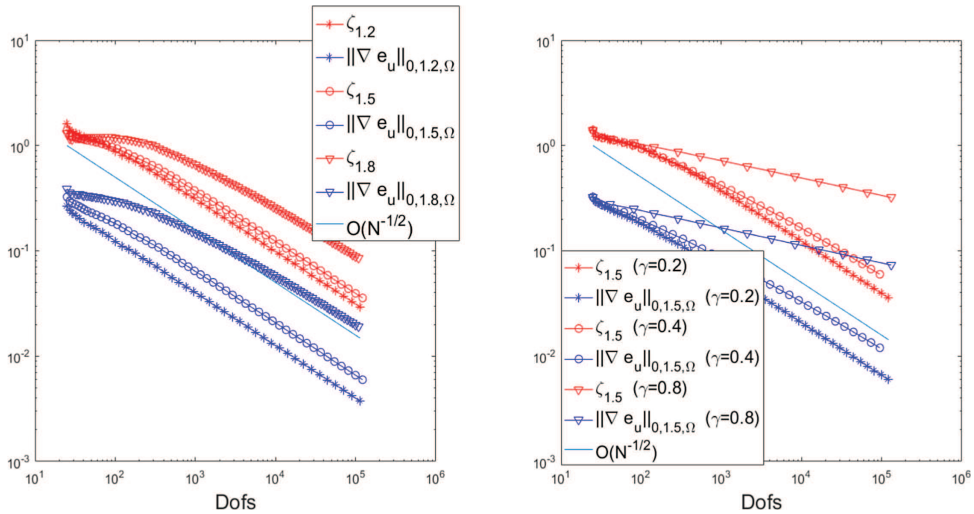

FIGURE 6. Left: The convergence histories for $\gamma = 0.2$ and $k = 1$. Right: The convergence histories for $k = 1$ and $\gamma = 0.2, 0.4, 0.8$.

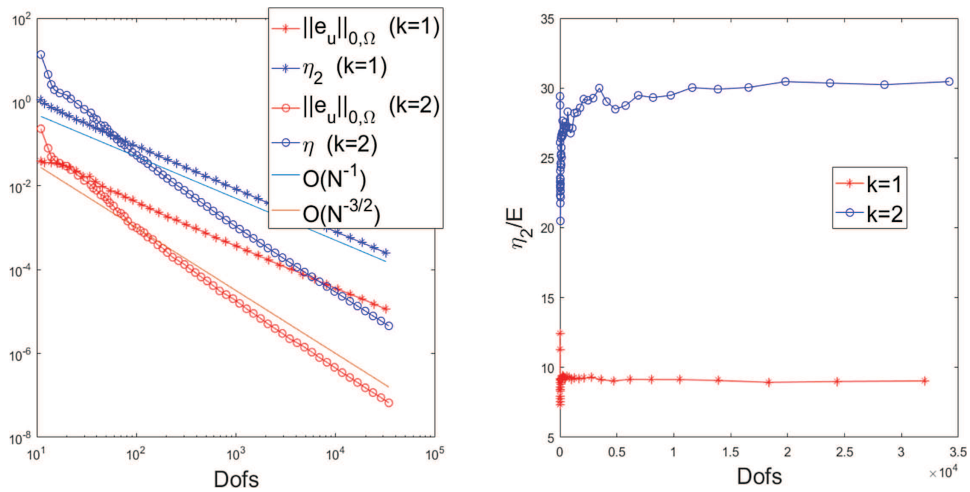

FIGURE 7. Left: The convergence histories of $\|e_u\|_{0,\Omega}$ and η_2 for $k = 1$ and $k = 2$. Right: The efficiency index η_2/E for $k = 1$ and $k = 2$. Here we set $\gamma = 0.3$.

13, 19 and 25 iterations are shown for $k = 1$ and $\gamma = 0.3$. We find that the mesh nodes are concentrated around the point x_0 and the reentrant corner. In Figure 9, the convergence histories of $\|\nabla e_u\|_{0,s,\Omega}$, $E_{2,s}$, and ζ_s are performed for $\gamma = 0.3$, $k = 1, 2$ and $s = 1.3, 1.5$. The errors and estimators can all get the convergence rate $O(N^{-k/2})$.

Finally, we make a comparison between [2] and this paper for $k = 1$ and $\gamma = 0.3$. Note that the estimators and errors obtained in [2] are labeled by ζ_s^{Ref} and e_u^{Ref}. In Figure 10, the convergence histories are performed, and the convergence rate $O(N^{-1/2})$ can be obtained. Obviously, the error estimators and the errors derived in this paper are smaller than that achieved by [2]. Therefore, from the perspective of error, the HDG result of this paper is better than the FEM result of [2] for the suitable choice of τ_0.

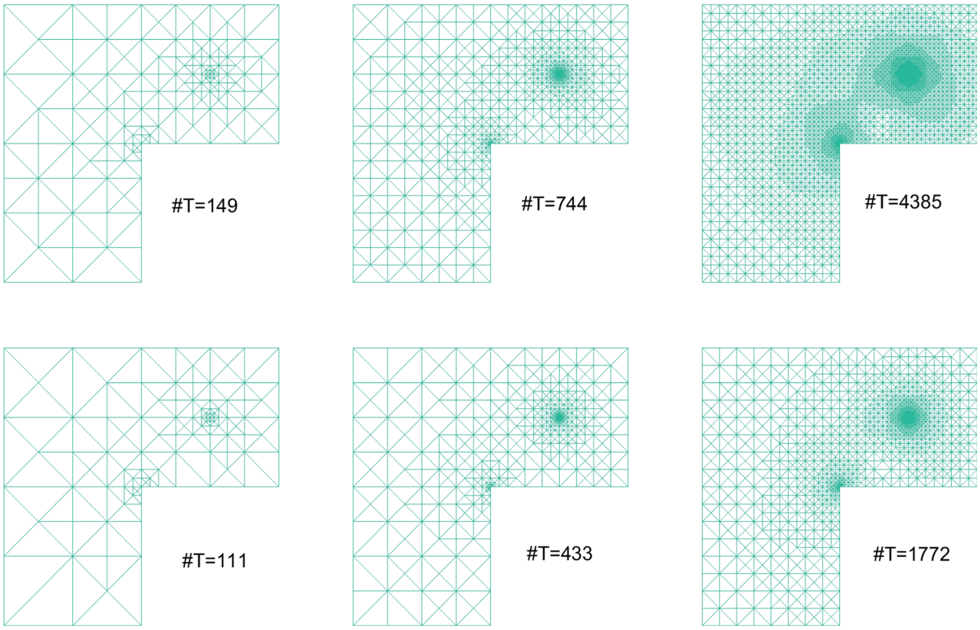

FIGURE 8. Top: The adaptive meshes, generated by $\zeta_{1.3}$, after 13, 19 and 25 iterations. Bottom: The adaptive meshes, generated by $\zeta_{1.5}$, after 13, 19 and 25 iterations. Here, we set $\gamma = 0.3$ and $k = 1$.

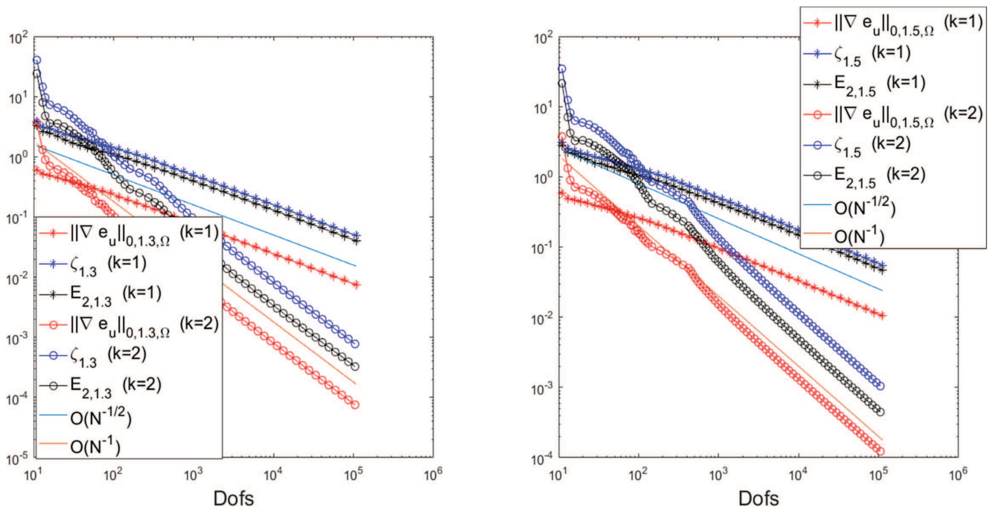

FIGURE 9. Left: The convergence histories of $\|\nabla e_u\|_{0,1.3,\Omega}$, $\zeta_{1.3}$ and $E_{2,1.3}$ for $k = 1$ and $k = 2$. Right: The convergence histories of $\|\nabla e_u\|_{0,1.5,\Omega}$, $\eta_{1.5}$ and $E_{2,1.5}$ for $k = 1$ and $k = 2$. Here we set $\gamma = 0.3$.

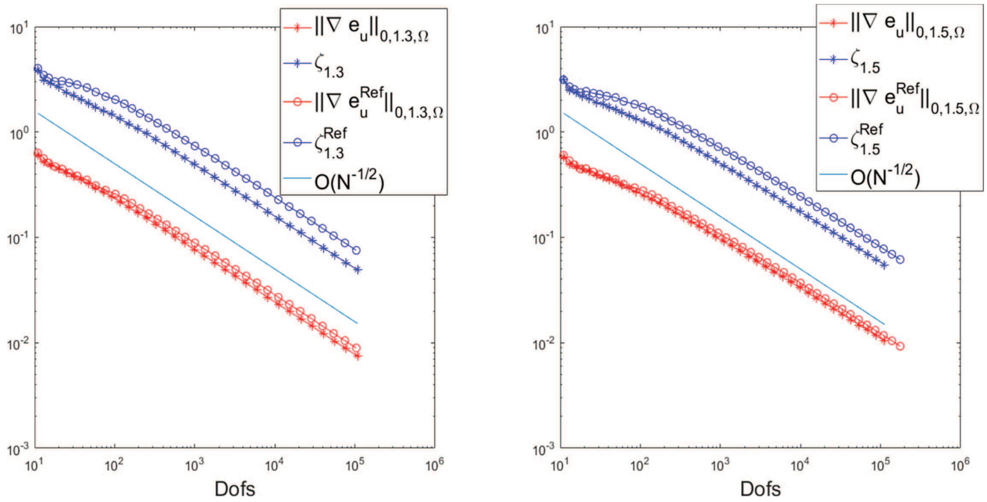

FIGURE 10. The convergence histories for $s = 1.3$ and $s = 1.5$.

7. CONCLUSIONS

In this paper, we investigate HDG methods for elliptic problems with Dirac measures. Firstly, *a priori* error estimate with convergence rate $O(h)$ is proved for the error in L^2-norm. Then, by duality argument and Oswald interpolation, the efficient and reliable *a posteriori* error estimators for the errors in L^2-norm and $W^{1,p}$-seminorm are obtained.

Finally the obtained *a posteriori* error estimators are used to design adaptive HDG algorithm, and some numerical examples are provided to verify the theoretical analysis and show the performance of the obtained *a posteriori* error estimators. By the numerical results, we find that the HDG scheme and error estimators of this paper are slightly better than the finite element discretization and error estimators of [2] based on a suitable choice of τ_0, see Figures 2 and 10.

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